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Abstract: We analyze the accuracy and well-posedness of generalized impedance boundary value problems in the framework of scattering problems from unbounded highly absorbing media. We restrict ourselves in this first work to the scalar problem (E-mode for electromagnetic scattering problems). Compared to earlier works, the unboundedness of the rough absorbing layer introduces severe difficulties in the analysis for the generalized impedance boundary conditions, since classical compactness arguments are no longer possible. Our new analysis is based on the use of Rellich-type estimates and boundedness of $L2$ solution operators. We also discuss numerical approximation of obtained GIBC (up to order 3) and numerically test theoretical convergence rates.

Key-words: Scattering problems, unbounded domains, asymptotic models, generalized impedance boundary conditions, high conductivity

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Résumé : Nous analysons la précision et le cractère bien posé des problèmes aux limites d'impédances généralisées dans le contexte des problèmes de diffraction par des milieux non bornés fortement absorbants. Nous nous restreignons dans ce premier travail au problème scalaire (polarization E). Comparé aux travaux précédents, le caractère non borné du milieu introduit une difficltée supplémentaire puisque les arguments classiques de compacité ne peuvent être utilisés. Notre analyse repose sur l'utilisation d'identités de Rellich et des estimations a priori de type L^2 sur la solution du problème. Nous discutons également l'approximation numérique des conditions obtenues et testons leur ordre de précision.

Mots-clés : Problèmes de diffraction, milieux non bornées, modèles asymptotiques, impédances géneralisées, forte conductivité

1 Introduction

Time harmonic wave scattering from rough layers is an important problem in science and engineering, as it describes for instance scattering of electromagnetic waves from the ground when one models the earth as a rough stratified medium. In such a model, the moisture of soil causes absorption of the electromagnetic wave inside the ground, and thus naturally leads to a scattering problem for a rough absorbing layer. Since waves inside the absorbing part of the medium decay exponentially with respect to the distance to the layer's boundary, a lot of research has been carried out how to replace the wave scattering problem inside the absorbing layer by some easily handable absorbing boundary condition on the interface in between the absorbing layer and free space [1, 7, 8, 13, 14]. The aim of such a boundary condition is to set up an approximate scattering problem merely in the complement of the absorbing object, while still guaranteeing a reliable error bound on the solution of the approximate problem. This error bound depends on what we call the order of the boundary condition as well as on the magnitude of the absorption inside the layer. Indeed, we treat the magnitude of absorption as a parameter and expand the acoustic field in a power series with respect to the inverse of this parameter. Approximate boundary conditions are built after truncation of this series, the order of obtained conditions then corresponds to the truncation index. Truncation at order zero simply leads to a Dirichlet boundary condition, which is naturally the formal limit condition as the absorption tends to infinity; truncation at order one leads to a (usual) impedance boundary condition. This is the reason why we call the conditions arising from truncation at higher order *generalized* impedance boundary conditions (GIBC).

In this paper, we analyse GIBC for rough absorbing layers up to order three and shall restrict ourselves to the scalar problem (which corresponds in 2-D to the E-polarization of electromagnetic waves). While the construction of such conditions is rather analogous to the case of a bounded absorbing inhomogeneity, the error analysis is more complicated. This is already obvious when one considers merely existence and uniqueness of solution for the approximate problems. For instance, for the case of a bounded obstacle, existence of solution for the time harmonic exterior Dirichlet or impedance scattering problem is known for a long time [12]. For the rough surface scattering problem with a Dirichlet boundary condition, corresponding results have only been achieved during the last decade, firstly by using integral equation approach [2, 6, 15], and more recently, by using a variational approach in [3,5]. For scattering from rough infinite layers we refer to recent results in [10]. For the variational theory on rough surface scattering, Rellich identities have been shown to be particularly useful since they provide a-priori bounds on a solution to the scattering problem, thereby establishing existence and uniqueness of solution via an inf-sup condition. Such a-priori bounds are also important for proving existence of solution to rough layer scattering problems involving higher order impedance boundary conditions. Moreover, they permit to construct a bounded L^2 solution operator. By definition, the solution operator maps Dirichlet boundary data on the rough surface to the radiating solution of the Helmholtz equation taking this boundary data. We show that this operator has a bounded extension from square integrable functions on the interface into the space of square integrable functions in a layer of finite height above the interface.

The important role of this L^2 solution operator in our analysis is to replace compactness arguments present in earlier rigorous error analysis of generalized impedance boundary conditions. Since compact embeddings of Sobolev spaces do not hold in our unbounded setting, we cannot use such types of compactness arguments which are always present in earlier works on generalized impedance boundary conditions for bounded objects [8]. The L^2 solution operator does part of this job. The other part is mainly done by a Rellich identity for radiating solutions of the Helmholtz equation over a rough layer. Through our Rellich identity we are able to prove existence and uniqueness of solution to the rough layer scattering problem subject to generalized impedance boundary conditions up to order three. Further, we show optimal error bounds for solutions to the approximate scattering problems involving our generalized impedance boundary conditions compared to the solution of the original scattering problem in the absorbing layer.

The structure of this paper is as follows. The first section is dedicated to the presentation of the mathematical setting of the scattering problem from unbounded rough surfaces and the introduction of used notation. Section 3 serves as a brief review of the main steps in deriving generalized impedance boundary conditions and needed extensions to the case of rough surfaces. Afterwards, we analyse lower order Neumann-to-Dirichlet conditions in Section 4. This cases enables us to present the analysis in a simpler framework. The case of more complicated higher order condition is analysed in Section 4. We indicate in particular how "stablized" conditions can be treated in a similar way as standard impedance boundary conditions. Finally, in Appendix A we provide some auxiliary existence and regularity results on rough surface scattering, some of which might have an interest in its own.

2 Settings of the problem and notation

Let us start with a brief description of the geometrical setting and our notation, such that we can afterwards present the problem mathematical setting in detail. Points in the Euclidean space $\mathbb{R}^m$ ($m = 2, 3$) are denoted by $x = (x_1, x_2, \dots, x_m)^\top$ and sometimes it is convenient to write $x = (\tilde{x}, x_m)^\top$, that is, $\tilde{x}$ are the first $m-1$ coordinates of $x \in \mathbb{R}^m$. In this work, $m = 2$ or 3, although all arguments also work in higher dimension. By $\mathbb{R}^m_\pm := \{x \in \mathbb{R}^m,\, x_m \gtrless 0\}$ we denote the upper and lower half space of $\mathbb{R}^m$ and the plane in between $\mathbb{R}^m_\pm$ is called $\Gamma_0 = \{x \in \mathbb{R}^m,\, x_m = 0\}$. More generally, $\Gamma_a = \{x = (x_1, x_2, \dots, x_m)^\top \in \mathbb{R}^m,\, x_m = a\}$, $a \in \mathbb{R}$. The half space above and below Γ_a is denoted by $U_a^\pm := \{x \in \mathbb{R}^m,\, x_m \gtrless a\}$. The domain $\Omega := \{x \in \mathbb{R}^m,\, -a < x_m < a\}$ is partitioned into two parts $\Omega_\pm := \{x \in \Omega,\, x_n \gtrless f(\tilde{x})\}$ by the interface $\Gamma := \{x \in \mathbb{R}^m,\, f(\tilde{x}) = x_m\}$, given by a twice continuously differentiable function $f : \mathbb{R}^{m-1} \to \mathbb{R}$, $-a < f < a$. For simplicity, we also introduce $\Omega_R := \{x \in \Omega,\, x_1^2 + \dots x_{m-1}^2 < R^2\}$. The exterior unit normal field to Ω and Ω_R is called ν; on Γ we choose the unit normal field ν to point downwards into Ω_-. The boundary of Ω_R is $C_R := \partial\Omega_R$. We split $C_R = C_R^+ \cup C_R^-$ with $C_R^- := \{x \in \Gamma_{-a},\, x_1^2 + \dots x_{m-1}^2 < R^2\}$ and define $M_R := C_R \cap \Omega$. We refer to Figure 1 for a sketch of the geometry of the problem.

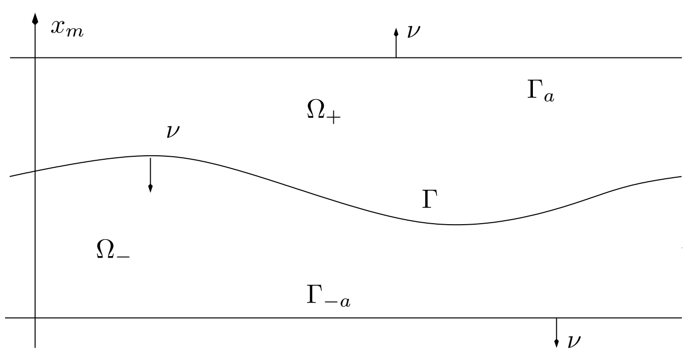


Figure 1: The geometry for the rough layer problem. The domain $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ lies in between the two planes $\Gamma_a = \{x_m = a\}$ and $\Gamma_{-a} = \{x_m = -a\}$. In Ω, the refractive index n^2 varies; while n^2 is Lipschitz continuous in $\Omega_\pm$, the index may jump across Γ. The unit normal ν points out of Ω and on Γ we choose ν to point downwards. The domain $\Omega_R = \{x \in \Omega,\, |\tilde{x}| < R\}$ is obtained from Ω by cut off in the lateral variables $\tilde{x}$

By $[u]_\Gamma$ we will denote the jump of a function u across the interface Γ, that is, $[u]_\Gamma = u|_\Gamma^+ - u|_\Gamma^-$ where $u|_\Gamma^\pm$ is the limit taken from $\Omega_\pm$. The L^2 inner product on Γ is denoted as $\langle \cdot, \cdot \rangle_{L^2(\Gamma)}$. Let us also remark that the square root of the complex unit $-\mathrm{i}$ in the fourth quadrant is denoted by $\alpha = (1-\mathrm{i})/\sqrt{2}$. All fields in this paper are time harmonic, that is, their time dependence is $\exp(-\mathrm{i}\omega t)$ for frequency $\omega > 0$, and this time dependence will always be suppressed. The wave number k is defined as $k = \omega / c$ with $c > 0$ the speed of sound in vacuum.

Now we turn to the mathematical formulation of the rough layer problem. We describe the medium of propagation by a refractive index function n^2, which is assumed to be a real valued function in Ω_+ and of the special form

$$n^2 = 1 + \frac{\mathrm{i}}{k^2 \varepsilon^2} \qquad \text{in } \Omega_- \text{ for } \varepsilon > 0.$$

The (small) parameter ε will control the magnitude of absorption in Ω_- throughout this paper. The refractive index hence jumps across the interface; concerning smoothness, we always require that $n^2 \in C^{0,1}(\overline{\Omega_+})$ and for some statements we even require more smoothness of $n^2\big|_{\Omega_+}$. Also, we always suppose that $\mathrm{Re}(n^2) \geq c_0 > 0$ and $\mathrm{Im}(n^2) \geq 0$ in Ω_+. In the upper half space U_a^+ we suppose n^2 to equal a real positive constant n_+^2 with $\mathrm{Re}(n_+^2) \geq c_0$ such that $n^2\big|_{\Gamma_a} = n_+^2$, while, obviously, n^2 equals $n_-^2 := 1 + \mathrm{i}/(k^2\varepsilon^2)$ in U_{-a}^-. Important for existence of solutions to rough layer scattering problems is the further assumption that $\partial n^2 / \partial x_m \leq 0$ in Ω_+. All these assumptions are supposed to hold throughout the paper. We remark that we could also deal with a refractive index whose real part varies in Ω_-, but for simplicity do not consider this case here.

The total field due to a local time harmonic source g supported in Ω_+ satisfies the Helmholtz equation

$$\Delta u + k^2 n^2 u = g \qquad \text{in } \mathbb{R}^m, \tag{1}$$

subject to the additional assumption that u and its normal derivative be continuous over the interface Γ where the index of refraction n^2 jumps, and a radiation condition in $U_{\pm a}^\pm$. We note that the solution u to the above problem will depend on ε and denote $u = u^\varepsilon$.

Let us now introduce the radiation condition imposed on u^ε. As shown in [3], the Fourier transform

$$\mathcal{F} : L^2(\mathbb{R}^{m-1}) \to L^2(\mathbb{R}^{m-1}), \quad \mathcal{F}\phi(\xi) = (2\pi)^{-(m-1)/2} \int_{\mathbb{R}^{m-1}} e^{-\mathrm{i}\tilde{x}\cdot\xi} \phi(\tilde{x}) \, \mathrm{d}\tilde{x}, \quad \xi \in \mathbb{R}^{m-1},$$

for $\phi \in L^2(\mathbb{R}^{m-1}) \cap L^1(\mathbb{R}^{m-1})$, allows to explicitly compute a Dirichlet-to-Neumann operator, mapping Dirichlet values ϕ on Γ_a to the Neumann boundary values of the unique radiating solution u in U_a^+ taking Dirichlet trace values ϕ on Γ_a. Construction of this operator relies on the following representation formula for u^ε,

$$u^\varepsilon(x) = (2\pi)^{-(m-1)/2} \int_{\mathbb{R}^{m-1}} \exp\left(\mathrm{i} \left((x_m - a)\sqrt{k^2 n_+^2 - \xi^2} + \tilde{x}\cdot\xi \right) \right) \mathcal{F}\left(u|_{\Gamma_a} \right)(\xi) \, \mathrm{d}\xi, \tag{2}$$

for $x \in U_a^+$. Note that this representation is a superposition of upwards propagating plane waves. Computing the normal derivative of the latter expression reveals that the Dirichlet-to-Neumann operator $T_{n_+^2}^+ : H^{1/2}(\Gamma_a) \to H^{-1/2}(\Gamma_a)$ is given by

$$(T_{n_+^2}^+ \phi)(\tilde{x}) = \mathrm{i}(2\pi)^{-(m-1)/2} \int_{\mathbb{R}^{m-1}} \sqrt{k^2 n_+^2 - \xi^2} \exp(\mathrm{i}\tilde{x}\cdot\xi) \, \mathcal{F}\phi(\xi) \, \mathrm{d}\xi$$

and it is shown in [3] that $T^+_{n_+^2}$ is bounded from $H^{1/2}(\Gamma_a)$ to $H^{-1/2}(\Gamma_a)$. A similar analysis shows that the corresponding Dirichlet-to-Neumann operator $T^-_{n_-^2}$ on Γ_{-a} is given by the very same expression (replacing of course n_+^2 by n_-^2). This is due to the fact that the expansion of u^ε in U^-_{-a} consists of downwards propagating evanescent waves. From the representation of $T^+_{n_+^2}$ as a Fourier multiplier we note that

$$\operatorname{Im} \int_{\Gamma_a} \overline{u^{\varepsilon,3}}\, T^+_{n_+^2}(u^{\varepsilon,3})\, \mathrm{d}s = \operatorname{Im}\left(\mathrm{i} \int_{\mathbb{R}^{m-1}} \sqrt{k^2-\xi^2}\, \left| \mathcal{F}(u^{\varepsilon,3}|_{\Gamma_a}) \right|^2 \mathrm{d}s \right)$$
$$= \int_{|\xi|<k} \sqrt{k^2-\xi^2}\, \left| \mathcal{F}(u^{\varepsilon,3}|_{\Gamma_a}) \right|^2 \mathrm{d}s \geq 0 \quad (3)$$

and a similar computation shows that $-\operatorname{Re} \int_{\Gamma_a} \overline{u^{\varepsilon,3}} T^+_{n_+^2}(u^{\varepsilon,3})\, \mathrm{d}s \geq 0$; again, the same inequalities holds for $T^-_{n_-^2}$.

The two Dirichlet-to-Neumann operators allow to frame the acoustic scattering problem under investigation variationally in the domain Ω. Due to the homogeneous jump conditions $[u]_\Gamma = [\partial u/\partial\nu]_\Gamma = 0$ it makes sense to seek a solution $u^\varepsilon \in H^1(\Omega)$. Twice (formally) applying Green's identity in $\Omega_\pm$ shows that

$$\int_\Omega \left(\nabla u^\varepsilon \cdot \nabla \overline{v} - k^2 n^2 u^\varepsilon\right) \overline{v}\, dx - \int_{\Gamma_a} \overline{v}\, T^+_{n_+^2}(u^\varepsilon)\, ds - \int_{\Gamma_{-a}} \overline{v}\, T^-_{n_-^2}(u^\varepsilon)\, ds = -\int_\Omega g\overline{v}\, \mathrm{d}x \qquad (4)$$

for all $v \in H^1(\Omega)$.

We now introduce the Neumann-to-Dirichlet operator D_ε on Γ which maps $\phi \in H^{-1/2}(\Gamma)$ to the Dirichlet boundary values v^ε on Γ of the unique solution to

$$\Delta v^\varepsilon + \left(k^2 + \frac{\mathrm{i}}{\varepsilon^2}\right) v^\varepsilon = g \quad \text{in } \Omega_-, \qquad \partial v^\varepsilon/\partial\nu = T^-_{n_-^2}(v^\varepsilon) \quad \text{on } \Gamma_{-a}, \qquad \partial v^\varepsilon/\partial_\nu = -\phi \quad \text{on } \Gamma.$$

Due to absorption, the map $D_\varepsilon : \phi \mapsto v^\varepsilon \in H^{1/2}(\Gamma)$ is well defined and bounded. Of course, the restriction $u^\varepsilon|_{\Omega_+}$ solves

$$\Delta u^\varepsilon + k^2 u^\varepsilon = g \quad \text{in } \Omega_+, \qquad \partial u^\varepsilon/\partial\nu = T^+_{n_+^2}(u^\varepsilon) \quad \text{on } \Gamma_a, \qquad u^\varepsilon + D_\varepsilon(\partial u^\varepsilon/\partial\nu) = 0 \quad \text{on } \Gamma. \ (5)$$

Roughly speaking, the idea of a generalized impedance boundary condition is now to construct a (formal) expansion of D_ε in terms of ε and to obtain explicitly computable boundary operators replacing D_ε in (5). These approximations to D_ε will introduce a certain error which we will show to be bounded in terms of powers of ε, the actual power depending on the truncation of the asymptotic development of D_ε. To conclude this brief outlook, as an approximation to the restriction $u^\varepsilon|_{\Omega_+}$ we are going to study problems of the following form,

$$\Delta u^{\varepsilon,p} + k^2 n^2 u^{\varepsilon,p} = g \quad \text{in } \Omega_+, \qquad \partial u^{\varepsilon,p}/\partial\nu = T^+_{n_+^2}(u^{\varepsilon,p}) \quad \text{on } \Gamma,$$

together with

$$u^{\varepsilon,p} + D_{\varepsilon,p}(\partial u^{\varepsilon,p}/\partial\nu) = 0 \quad \text{on } \Gamma$$

for certain Neumann-to-Dirichlet operators $D_{\varepsilon,p}$ of order $p \in \{0,1,2,3\}$, which will be constructed and analysed in detail.

3 Formal construction of GIBC

As a brief introduction to the construction of generalized impedance boundary conditions, we recall some basic ideas, definitions and computations from [8]. The formal computation of these conditions for rough layers is the same as for bounded absorbing inclusion and we can skip most of the computations. Let $\Omega_-^\delta := \{x \in \Omega_-, \operatorname{dist}(x,\Gamma) < \delta\}$ for $\delta > 0$ small enough such that each point $x \in \Omega_-^\delta$ has a unique representation $x = x_\Gamma + q\nu$, with $x_\Gamma \in \Gamma$ and $q > 0$. Since we assumed that the function f which defines Γ is of class C^2, such a choice δ is always possible. Recall that the unit normal ν on Γ was defined to point into Ω_-. The parameter δ is fixed from now on and we also fix a cut off function $\chi \in C^\infty(\Omega_-)$ such that $\chi = 1$ in $\Omega_-^{\delta/2}$, $\chi = 0$ in Ω_-^δ as well as $0 \le \chi \le 1$, $|\nabla\chi| \le C$ and $|\Delta\chi| \le C$ in Ω_-.

The starting point for the construction of generalized impedance boundary conditions is the assumption that the exact solution u^ε of the scattering problem (4) with parameter $\varepsilon > 0$ can be written as

$$u^\varepsilon(x) = u_+^0(x) + \varepsilon u_+^1(x) + \varepsilon^2 u_+^2(x) + \dots, \qquad x \in \Omega_+, \tag{6}$$

for functions $u_+^\ell : \Omega_+ \to \mathbb{C}$ and

$$\chi u^\varepsilon(x) = u_-^0(x_\Gamma, q/\varepsilon) + \varepsilon u_-^1(x_\Gamma, q/\varepsilon) + \varepsilon^2 u_-^2(x_\Gamma, q/\varepsilon) + \dots, \qquad x = x_\Gamma + q\nu \in \Omega_-^\delta, \tag{7}$$

with functions $u_-^\ell : \Gamma \times \mathbb{R}_+ \to \mathbb{C}$ such that $\lim_{\eta\to\infty} u_-^\ell(x_\Gamma,\eta) = 0$ for $x_\Gamma \in \Gamma$. From this expansion we will in the sequel construct boundary conditions which allow to truncate problem (4) at Γ, introducing an error that can be controlled in powers of ε. Following [8], we set

$$\tilde{u}_-^\varepsilon(x_\Gamma,\eta) = u_-^0(x_\Gamma,\eta) + \varepsilon u_-^1(x_\Gamma,\eta) + \varepsilon^2 u_-^2(x_\Gamma,\eta) + \dots, \qquad x_\Gamma \in \Gamma,\ \eta > 0.$$

Since u^ε solves the Helmholtz equation, all functions u_+^ℓ also need to satisfy this equation. Moreover, starting from the Helmholtz equation (1) one can compute a differential equation for $\tilde{u}_-^\varepsilon$ of the form

$$\left(-\frac{\partial^2}{\partial\eta^2} - \mathrm{i}\right)\tilde{u}_-^\varepsilon = \sum_{\ell=1}^{8} \varepsilon^\ell A_\ell \tilde{u}_-^\varepsilon \qquad \text{on } \Gamma \times \mathbb{R}_+,$$

where the A_ℓ are differential operators in (x_Γ,η) independent of ε; see [8] for details. Substituting the latter equation into (7) shows that

$$\left(-\frac{\partial^2}{\partial\eta^2} - \mathrm{i}\right)u_-^p = \sum_{\ell=1}^{8} A_\ell u_-^{p-\ell} \qquad \text{in } \Gamma \times \mathbb{R}_+, \tag{8}$$

where $u_-^{p-\ell} = 0$ for $p - \ell < 0$. Let us think of this equation as a family of second order ordinary differential equations in η with parameter x_Γ. Coupling of (8) with the expansion (6) of u^ε in Ω_+ yields a boundary condition at $\eta = 0$ and together with the decay condition for u_-^ℓ we can solve (8). In more detail, equating terms in (6) and (7) which share the same powers of ε either offers the possibility of coupling the Dirichlet traces,

$$u_+^p(x_\Gamma) = u_+^p(x_\Gamma, 0), \qquad \text{for } x_\Gamma \in \Gamma, \tag{9}$$

or coupling of the normal derivatives,

$$\frac{\partial u_+^{p-1}}{\partial\nu}(x_\Gamma) = \frac{\partial u_+^p}{\partial\eta}(x_\Gamma, 0), \qquad \text{for } x_\Gamma \in \Gamma. \tag{10}$$

We concentrate in this paper on the *second* option, which leads to Neumann-to-Dirichlet impedance boundary conditions, and which is somehow more natural due to the shift of the index. The first option (9) results in Dirichlet-to-Neumann impedance boundary conditions which will not be considered here; the interested reader is referred to [8] where formulas for Dirichlet-to-Neumann impedance boundary conditions are provided which can be transferred to the rough surface context by the techniques presented in this paper.

The differential equations in η in (8) together with (10) and the decay condition $u_-^p(x_\Gamma,\eta)\to 0$ as $\eta\to\infty$ can be explicitly solved in the form

$$u_-^p(x_\Gamma,\eta) = P_{x_\Gamma}^p(\eta)e^{-\alpha\eta}, \quad \eta>0,\, p\in\mathbb{N}_0,$$

with a polynomial $P_{x_\Gamma}^p$ of degree p depending on $\partial u_-^\ell/\partial\nu$, $\ell\in\{0,1,\dots,p-1\}$. There holds a recurrence relation of order 8 for the $P_{x_\Gamma}^p$, which we will however not give here, but only give the explicit form of the u_-^p for $p\in\{0,1,2,3\}$. Before citing the result of this computation in [8, Eq. (4.26)–(4.29)], we remark that we denote Gauss and mean curvature on Γ by G and H, respectively; we also assume familiarity of the reader with standard surface differential operators such as the surface gradient ∇_Γ and the surface Laplacian Δ_Γ, see, e.g, [12] for details. Using the abbreviation $\alpha=(1-\mathrm{i})/\sqrt{2}$ there holds

$$u_-^0(x_\Gamma,\eta) = 0, \tag{11}$$

$$u_-^1(x_\Gamma,\eta) = -\frac{1}{\alpha}\frac{\partial u_+^0}{\partial\nu}(x_\Gamma)e^{-\alpha\eta}, \tag{12}$$

$$u_-^2(x_\Gamma,\eta) = \left(-\frac{1}{\alpha}\frac{\partial u_+^1}{\partial\nu}(x_\Gamma)+\frac{H}{\alpha^2}\frac{\partial u_+^0}{\partial\nu}(x_\Gamma)+\eta\frac{H}{\alpha}\frac{\partial u_+^0}{\partial\nu}(x_\Gamma)\right)e^{-\alpha\eta}, \tag{13}$$

$$\begin{aligned} u_-^3(x_\Gamma,\eta) = \Bigg[&-\frac{1}{\alpha}\frac{\partial u_+^2}{\partial\nu}(x_\Gamma)+\frac{H}{\alpha^2}\frac{\partial u_+^1}{\partial\nu}(x_\Gamma)-\frac{1}{2\alpha^3}(3H^2-G+k^2)\frac{\partial u_+^0}{\partial\nu}(x_\Gamma) \\ &-\frac{1}{2\alpha^3}\Delta_\Gamma\left(\frac{\partial u_+^0}{\partial\nu}\right)+\eta\left(\frac{H}{\alpha}\frac{\partial u_+^1}{\partial\nu}(x_\Gamma)-\frac{1}{2\alpha^2}(\Delta_\Gamma+3H^2-G+k^2)\frac{\partial u_+^0}{\partial\nu}(x_\Gamma)\right) \\ &+\eta^2\frac{1}{2\alpha}(G-3H^2)\frac{\partial u_+^0}{\partial\nu}(x_\Gamma)\Bigg]e^{-\alpha\eta}. \end{aligned} \tag{14}$$

In the following lemma, we show that the functions u_+^p for $p\in\{0,1,2,3\}$ are well defined in $H^1(\Omega_+)$ if the right hand side g of the Helmholtz equation (1) is smooth enough.

Lemma 1. *Set $g^0=g\in L^2(\Omega_+)$ and $g^\ell=0$ for $\ell=1,2,\dots$ and let $p\in\mathbb{N}=\{1,2,\dots\}$. For $g\in H^{p-1}(\Omega_+)$, $n^2\in C^{p-1,1}(\overline{\Omega_+})$ and Γ of class $C^{r,1}$, the functions u_+^ℓ are well defined in $H^1(\Omega_+)$ for $\ell=0,1,\dots,p$ through the recursion*

$$\Delta u_+^\ell + k^2n^2u_+^\ell = g^\ell \quad \text{in } \Omega_+, \qquad \frac{\partial u_+^\ell}{\partial\nu} = T_{n_+^2}^+(u_+^\ell) \quad \text{on } \Gamma_a, \qquad u_+^\ell = u_-^\ell(\cdot,0) \quad \text{on } \Gamma, \tag{15}$$

where $u_-^\ell(\cdot,0)$ is determined from

$$\left(-\frac{\partial^2}{\partial\eta^2}-\mathrm{i}\right)u_-^\ell = \sum_{\ell=1}^{8} A_\ell u_-^{\ell-l} \quad \text{in } \Gamma\times\mathbb{R}_+, \qquad \lim_{\eta\to\infty} u_-^\ell(x_\Gamma,\eta)=0 \quad \text{on } \Gamma,$$

together with the coupling condition (10). *Additionally, the bounds $\|u_+^0\|_{H^2(\Omega_+)}\le C\|g\|_{L^2(\Omega_+)}$, $\|\partial u_+^0/\partial\nu\|_{H^{1/2}(\Gamma)}\le C\|g\|_{L^2(\Omega_+)}$ as well as*

$$\|u_+^\ell\|_{H^{p-\ell}(\Omega_+)}\le C\|g\|_{H^{p-1}(\Omega_+)} \qquad \text{and} \qquad \|\partial u_+^\ell/\partial\nu\|_{H^{p-(1+2\ell)/2}(\Gamma)}\le C\|g\|_{H^{p-1}(\Omega_+)}$$

hold as long as $p-(1+2\ell)/2>0$.

Proof. For $\ell = 0$ we know from (11) that $u_-^0(\cdot,0) = 0$. Hence, u_+^0 solves a homogeneous Dirichlet problem for the Helmholtz equation with right hand side $g^0 = g$. From Theorem 11 we conclude that $u_+^0 \in H^2(\Omega)$ for $g \in L^2(\Omega_+)$; thus the normal derivative $\partial u_+^0/\partial\nu$ belongs to $H^{1/2}(\Gamma)$ and the bounds $\|\partial u_+^0/\partial\nu\|_{H^{1/2}(\Omega_+)} \le C\|u_+^0\|_{H^2(\Omega_+)} \le C\|g\|_{L^2(\Omega_+)}$ hold.

More generally, for $g \in H^{p-1}(\Omega_+)$ and n^2 as well as Γ smooth enough we have that $u_+^0 \in H^{p+1}(\Omega)$; thus the normal derivative $\partial u_+^0/\partial\nu$ belongs to $H^{p-1/2}(\Gamma)$ and the bounds

$$\|\partial u_+^0/\partial\nu\|_{H^{p-1/2}(\Gamma)} \le C\|u_+^0\|_{H^{p+1}(\Omega_+)} \le C(p)\|g\|_{H^{p-1}(\Omega_+)}$$

hold for $p = 1, 2, \ldots$.

In consequence, the Dirichlet trace $u_-^1(\cdot,0)$ given in (12) is well defined in $H^{p-1/2}(\Gamma)$ for g in $H^p(\Omega_+)$ and we can solve the Dirichlet problem (15) in $H^p(\Omega_+)$ according to Theorem 11, with $\|\partial u_+^1/\partial\nu\|_{H^{p-3/2}(\Gamma)} \le C\|\partial u_+^1/\partial\nu\|_{H^p(\Gamma)} \le C\|g\|_{H^{p-1}(\Omega_+)}$, as long as $p - 3/2 > 0$. Existence of u_+^ℓ for $\ell = 2, \ldots, p$ follows from an induction argument. Note that existence and boundedness of u_+^ℓ in $H^1(\Omega_+)$ and $\partial u_+^\ell/\partial\nu \in H^{1/2}(\Gamma)$ requires $g \in H^{p-1}(\Omega_+)$ with $p - (1+2\ell)/2 > 0$, as well as $n^2 \in C^{p-1,1}(\overline{\Omega_+})$ and Γ of class $C^{p,1}$. □

Now we can derive a first version of generalized impedance boundary conditions. Except for the case $p = 3$ these will be the ones for which we prove convergence of optimal order in Section 4. For $p = 3$, sophisticated further manipulations of the impedance boundary condition are necessary to derive convergence of optimal order. Therefore we devote the entire section 5 to the analysis of the case $p = 3$.

The basic idea behind an impedance boundary condition of order p is truncation of the expansion (6) of u^ε at $\ell = p$,

$$\tilde{u}^{\varepsilon,p} = \sum_{\ell=0}^{p} \varepsilon^\ell u_+^\ell \quad \text{in } \Omega_+. \tag{16}$$

In view of (15) we get that $\tilde{u}^{\varepsilon,p} = \sum_{\ell=0}^{p} \varepsilon^\ell u_-^\ell(\cdot,0)$ on Γ. Since $u_-^\ell(\cdot,0)$ is explicitly given in (11)–(14) in terms of the normal derivatives $\partial u_+^\ell/\partial\nu$, we can plug in these formulas into the last equation to obtain for $\ell = 0$ that $\tilde{u}^{\varepsilon,0} = 0$ and for $\ell = 1$ that

$$\tilde{u}^{\varepsilon,1} + \varepsilon\frac{1}{\alpha}\frac{\partial u_+^0}{\partial\nu}(x_\Gamma) = 0, \quad \text{that is,} \quad \tilde{u}^{\varepsilon,1} + \varepsilon\frac{1}{\alpha}\frac{\partial\tilde{u}^{\varepsilon,1}}{\partial\nu}(x_\Gamma) = \varepsilon^2\frac{1}{\alpha}\frac{\partial\tilde{u}^{\varepsilon,1}}{\partial\nu}.$$

Generally speaking, we find a tangential differential operator $D_{\varepsilon,p}$ acting on Γ such that

$$\tilde{u}^{\varepsilon,p} + D_{\varepsilon,p}\left(\frac{\partial\tilde{u}^{\varepsilon,p}}{\partial\nu}\right) = \varepsilon^{p+1}r^{\varepsilon,p}, \quad p \in \{0,1,2,3\}, \tag{17}$$

for functions $r^{\varepsilon,p}$ which are explicitly given in [8, Eq. (4.34)] for $p \in \{0,1,2,3\}$ as

$$\begin{aligned}
r^{\varepsilon,0} &= 0, \quad r^{\varepsilon,1} = \frac{1}{\alpha}\frac{\partial u_+^1}{\partial\nu}, \quad r^{\varepsilon,2} = \frac{1}{\alpha}\frac{\partial u_+^2}{\partial\nu} - \mathrm{i}H\frac{\partial}{\partial\nu}\left(u_+^1 + \varepsilon u_+^2\right), \\
r^{\varepsilon,3} &= \frac{1}{\alpha}\frac{\partial u_+^3}{\partial\nu} - \mathrm{i}H\frac{\partial}{\partial\nu}\left(u_+^1 + \varepsilon u_+^2\right) - \frac{1}{2}\left[\Delta_\Gamma + 3H^2 - G + k^2\right]\left(\frac{\partial}{\partial\nu}(u_+^1 + \varepsilon u_+^2 + \varepsilon^2 u_+^3)\right).
\end{aligned} \tag{18}$$

The differential operators $D_{\varepsilon,p}$ are given by [8, Eq. (3.5)–(3.8)]

$$D_{\varepsilon,0} = 0, \quad D_{\varepsilon,1} = \frac{\varepsilon}{\alpha}, \quad D_{\varepsilon,2} = \frac{\varepsilon}{\alpha} - \mathrm{i}\varepsilon^2 H, \quad D_{\varepsilon,3} = \frac{\varepsilon}{\alpha} - \mathrm{i}\varepsilon^2 H - \varepsilon^3\frac{\alpha}{2}(\Delta_\Gamma + 3H^2 - G + k^2).$$

Neglecting the small right hand side in (17), we *define* an approximation $u^{\varepsilon,p}$ of u^ε, solution of (4), by

$$\Delta u^{\varepsilon,p} + k^2 n^2 u^{\varepsilon,p} = g \quad \text{in } \Omega_+, \qquad \frac{\partial u^{\varepsilon,p}}{\partial \nu} = T^+_{n_+^2}(u^{\varepsilon,p}) \quad \text{on } \Gamma_a, \qquad u^{\varepsilon,p} + D_{\varepsilon,p} u^{\varepsilon,p} = 0 \quad \text{on } \Gamma, \tag{19}$$

for $p \in \{0,1,2\}$. For $p = 3$, $u^{\varepsilon,3}$ will be defined in a more refined way in Section 5. We note that this $u^{\varepsilon,p}$ is defined through a boundary value problem merely posed in Ω_+. Well-posedness of this problem, as well as the approximation of u^ε by $u^{\varepsilon,p}$ is subject of the following sections. However, at least morally we already note from (17) that the condition of order p will introduce an error $\|u^{\varepsilon,p} - u^\varepsilon\|$ which is $\mathcal{O}(\varepsilon^p)$.

4 Impedance Boundary Conditions of Lower Order

Our aim in this section is to provide basic tools for the convergence analysis of generalized impedance boundary conditions leading to proofs of convergence of $u^{\varepsilon,p}$ to u^ε with optimal order $p+1$ for $p \in \{0,1,2\}$. Since the case $p = 3$ requires additional manipulations, we postpone the analysis of this case to the next Section 5. Note, however, that some of the technical lemmas contained in this present section will also be used in Section 5 to treat the case $p = 3$. Our main theorem in this section is the following.

Theorem 2. *Let $g \in H^3(\Omega_+)$, $n^2 \in C^{3,1}(\overline{\Omega_+})$ and Γ of class $C^{4,1}$. There are constants $\varepsilon_0 > 0$ and $C(p) > 0$, independent of $\varepsilon \in (0, \varepsilon_0]$, such that*

$$\|u^\varepsilon - u^{\varepsilon,p}\|_{H^1(\Omega_+)} \leq C(p)\varepsilon^{p+1}, \quad \text{for } p \in \{0,1,2\} \text{ and } 0 < \varepsilon \leq \varepsilon_0. \tag{20}$$

The proof of this theorem requires some preparation. We recall the definition of $\tilde{u}^{\varepsilon,3}$ in (16), the cut off function χ and the coordinates $x = (x_\Gamma, \nu)$ in Ω_-, both introduced in Section 3, and define

$$\tilde{u}^{\varepsilon,p}_\chi(x) = \begin{cases} \displaystyle\sum_{\ell=0}^{p} \varepsilon^\ell u^\ell_+(x), & x \in \Omega_+, \\ \chi(x) \sum_{\ell=0}^{p} \varepsilon^\ell u^\ell_-(x_\Gamma, q/\varepsilon), & x \in \Omega_-, \end{cases} \qquad p \in \{0,1,2,3\}. \tag{21}$$

Note that $\tilde{u}^{\varepsilon,p}_\chi$ precisely captures the expansions in (6) and (7). The proof of Theorem 2 is broken into two main steps, since we are not able to show (20) directly, but rather introduce $\tilde{u}^{\varepsilon,3}$ as an intermediate term and prove that

$$\|u^\varepsilon - u^{\varepsilon,p}\|_{H^1(\Omega_+)} \leq \|u^\varepsilon - \tilde{u}^{\varepsilon,p}\|_{H^1(\Omega_+)} + \|\tilde{u}^{\varepsilon,p} - u^{\varepsilon,p}\|_{H^1(\Omega_+)} \leq C\varepsilon^{p+1}. \tag{22}$$

To treat the two differences which we need to estimate in the last equation, let us first introduce a sequel of technical lemmas.

Lemma 3. *There is a constant $C > 0$ such that*

$$\|u\|^2_{L^2(\Gamma)} \leq C\left(\|\nabla u\|_{L^2(\Omega_\pm)}\|u\|_{L^2(\Omega_\pm)} + \|u\|^2_{L^2(\Omega_\pm)}\right) \quad \text{for all } u \in H^1(\Omega_\pm).$$

Proof. Estimates of this kind are well known from earlier works on generalized impedance boundary conditions, however, since $\Omega_\pm$ is unbounded we briefly sketch the proof for the domain Ω_+.

For $u \in C^\infty(\overline{\Omega_+}) \cap H^1(\Omega_+)$, the fundamental theorem of calculus implies

$$|u(\tilde{x}, h)|^2 - |u(\tilde{x}, f(\tilde{x}))|^2 = 2\int_{f(\tilde{x})}^{h} u(\tilde{x}, x_m)\frac{\partial u(\tilde{x}, s)}{\partial x_m}\,\mathrm{d}s.$$

If we define the segment $S_{\tilde{x}} := \{y = (\tilde{x}, s),\, f(\tilde{x}) < s < h\}$, then

$$|u(\tilde{x}, f(\tilde{x}))|^2 \le 2\|u(\tilde{x},\cdot)\|_{L^2(S_{\tilde{x}})} \left\|\frac{\partial u(\tilde{x},\cdot)}{\partial x_m}\right\|_{L^2(S_{\tilde{x}})} + |u(\tilde{x},h)|^2.$$

Integration with respect to h yields

$$|h - f(\tilde{x})|\,|u(\tilde{x}, f(\tilde{x}))|^2 \le 2|h - f(\tilde{x})|\,\|u(\tilde{x},\cdot)\|_{L^2(S_{\tilde{x}})} \left\|\frac{\partial u(\tilde{x},\cdot)}{\partial x_m}\right\|_{L^2(S_{\tilde{x}})} + \|u(\tilde{x},h)\|^2_{L^2(S_{\tilde{x}})}.$$

We integrate with respect to $\tilde{x}$,

$$\int_{\mathbb{R}^{m-1}} |u(\tilde{x}, f(\tilde{x}))|^2 \,\mathrm{d}\tilde{x} \le 2\|u\|_{L^2(\Omega_+)} \left\|\frac{\partial u}{\partial x_m}\right\|_{L^2(\Omega_+)} + \sup_{\tilde{x}\in\mathbb{R}^{m-1}} |h - f(\tilde{x})|^{-1}\,\|u\|^2_{L^2(\Omega_+)},$$

which gives the claim of the lemma, since $\|u\|^2_{L^2(\Gamma)} \le C \int_{\mathbb{R}^{m-1}} |u(\tilde{x}, f(\tilde{x}))|^2 \,\mathrm{d}\tilde{x}$ and since $C^\infty(\overline{\Omega_+}) \cap H^1(\Omega_+)$ is dense in $H^1(\Omega_+)$. □

Lemma 4. *Assume that $v^\varepsilon \in H^1(\Omega)$ satisfies*

$$\Delta v^\varepsilon + k^2 n^2 v^\varepsilon = 0 \quad in\ \Omega, \qquad \frac{\partial v^\varepsilon}{\partial \nu} = T^+_{n_+^2}(v^\varepsilon) \quad on\ \Gamma_a \qquad for\ \varepsilon \in (0, \varepsilon_0]$$

and the a-priori estimate

$$\left|\int_\Omega \left(|\nabla v^\varepsilon|^2 - k^2 n^2 |v^\varepsilon|^2\right) \mathrm{d}x - \int_{\Gamma_a} \overline{v^\varepsilon}\, T^+_{n_+^2}(v^\varepsilon)\,\mathrm{d}s\right| \le C\left(\varepsilon^{s+1/2}\|v^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^s \|v^\varepsilon\|_{L^2(\Omega_-)}\right) \quad (23)$$

for $C, s > 0$ independent of ε and v^ε. Then

$$\|v^\varepsilon\|_{H^1(\Omega)} \le C\varepsilon^{s+1}, \quad \|v^\varepsilon\|_{L^2(\Omega_-)} \le C\varepsilon^{s+2}, \quad and \quad \|v^\varepsilon\|_{L^2(\Gamma)} \le C\varepsilon^{s+3/2}$$

for $\varepsilon \in (0, \varepsilon_0]$.

Proof. Suppose that $\lambda_\varepsilon = \|v^\varepsilon\|_{L^2(\Omega)}/\varepsilon^{s+1}$ is unbounded as $\varepsilon \to 0$. We set $w^\varepsilon = v^\varepsilon/\|v^\varepsilon\|_{L^2(\Omega)}$ and note that (23) implies

$$\left|\int_\Omega \left(|\nabla w^\varepsilon|^2 - k^2 n^2 |w^\varepsilon|^2\right) \mathrm{d}x - \int_{\Gamma_a} \overline{w^\varepsilon}\, T^+_{n_+^2}(w^\varepsilon)\,\mathrm{d}s\right| \le \frac{C}{\lambda_\varepsilon}\left(\varepsilon^{-1/2}\|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_-)}\right). \quad (24)$$

As C/λ_ε is bounded as $\varepsilon \to 0$, the last inequality implies an estimate for the absolute value of the imaginary part of the left hand side. Moreover, as $-k^2\,\mathrm{Im}(n^2) \le 0$ and $-\mathrm{Im}\int_{\Gamma_a} \overline{w^\varepsilon}\, T^+_{n_+^2}(w^\varepsilon)\,\mathrm{d}s \le 0$ as well, we obtain

$$k^2 \int_{\Omega_-} |w^\varepsilon|^2 \,\mathrm{d}x \le \frac{C}{\lambda_\varepsilon}\left(\varepsilon^{3/2}\|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon\|w^\varepsilon\|_{L^2(\Omega_-)}\right).$$

By Lemma 3,

$$\|w^\varepsilon\|^{3/2}_{L^2(\Omega_-)} \le C\varepsilon^{3/2}\left(\|\nabla w^\varepsilon\|^{1/2}_{L^2(\Omega_-)} + \|w^\varepsilon\|^{1/2}_{L^2(\Omega_-)}\right)\|w^\varepsilon\|^{1/2}_{L^2(\Omega_-)} + C\varepsilon\|w^\varepsilon\|_{L^2(\Omega_-)},$$

and

$$\|w^\varepsilon\|_{L^2(\Omega_-)} \leq C\varepsilon^{3/2}\|\nabla w^\varepsilon\|^{1/2}_{L^2(\Omega_-)} + C\varepsilon\|w^\varepsilon\|^{1/2}_{L^2(\Omega_-)} \leq C\varepsilon^{3/2}\|\nabla w^\varepsilon\|^{1/2}_{L^2(\Omega_-)} + C\varepsilon^{3/2} + \frac{1}{2}\|w^\varepsilon\|^{3/2}_{L^2(\Omega_-)},$$

where we used Young's inequality in the last step. We conclude that

$$\|w^\varepsilon\|^{3/2}_{L^2(\Omega_-)} \leq C\varepsilon^{3/2}\left(1 + \|\nabla w^\varepsilon\|^{1/2}_{L^2(\Omega_-)}\right). \tag{25}$$

Since $-\operatorname{Re}\int_{\Gamma_a} \overline{w^\varepsilon}\, T^+_{n_+^2}(w^\varepsilon)\,\mathrm{d}s > 0$, the corresponding estimate for the real part of (24) yields

$$\int_\Omega |\nabla w^\varepsilon|^2\,\mathrm{d}x \leq k^2\int_\Omega \operatorname{Re}(n^2)|w^\varepsilon|^2\,\mathrm{d}x + \frac{C}{\lambda_\varepsilon}\left(\varepsilon^{-1/2}\|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_-)}\right)$$
$$\leq C + C\left(\varepsilon^{-1/2}\|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_-)}\right). \tag{26}$$

The term $\varepsilon^{-1/2}\|w^\varepsilon\|_{L^2(\Gamma)}$ can be estimated by Lemma 3,

$$\varepsilon^{-1/2}\|w^\varepsilon\|_{L^2(\Gamma)} \leq 2\varepsilon^{-1/2}\|w^\varepsilon\|_{L^2(\Omega_-)} + 2\varepsilon^{-1/2}\|w^\varepsilon\|^{1/2}_{L^2(\Omega_-)}\|\nabla w^\varepsilon\|^{1/2}_{L^2(\Omega_-)},$$

which we plug into (26) to obtain

$$\int_\Omega |\nabla w^\varepsilon|^2\,\mathrm{d}x \leq C + C\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_-)}\left(1 + \|\nabla u^\varepsilon\|^{1/2}_{L^2(\Omega_-)}\right), \qquad \text{for } \varepsilon \in (\varepsilon, \varepsilon_0]. \tag{27}$$

By means of (25),

$$\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_-)} \leq C\left(1 + \|\nabla u^\varepsilon\|^{1/3}_{L^2(\Omega_-)}\right). \tag{28}$$

Hence, using the latter bound in (27) we obtain $\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leq C(1+\|\nabla w^\varepsilon\|^{2/3}_{L^2(\Omega_-)})$, which shows that $\|\nabla w^\varepsilon\|^2_{L^2(\Omega)}$ is uniformly bounded in $\varepsilon \in (0, \varepsilon_0]$. From (25) we conclude that $\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_-)}$ is bounded as well. Lemma 3 implies that $\varepsilon^{-1}\|w^\varepsilon\|^2_{L^2(\Gamma)}$ is bounded for $\varepsilon \in (0, \varepsilon_0]$. Then, however, we conclude from Lemma 12 that $\varepsilon^{-1}\|w^\varepsilon\|^2_{L^2(\Omega_+)}$ is bounded for $\varepsilon \in (0, \varepsilon_0]$. Using (28) we see that $\varepsilon^{-1}\|w^\varepsilon\|^2_{L^2(\Omega_-)}$ is bounded as well. However, the conclusion that $\varepsilon^{-1}\|w^\varepsilon\|^2_{L^2(\Omega)}$ is bounded for $\varepsilon \in (0, \varepsilon_0]$ is a contradiction since by construction $\|w^\varepsilon\|^2_{L^2(\Omega)} = 1$. Hence, the bound $\|v^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{s+1}$ holds for $\varepsilon \in (0, \varepsilon_0]$. The three estimates stated in the lemma now follow from the above inequalities precisely as in the proof of [8, Lemma 5.3]. □

Now we proceed with the estimate of the first difference $\|u^\varepsilon - \tilde{u}^{\varepsilon,p}\|_{H^1(\Omega_+)}$ appearing in (22).

Theorem 5. *Assume that $g \in H^3(\Omega_+)$ and $n^2 \in C^{4,1}(\overline{\Omega_+})$. Then there is a constant $C(p)$ independent of ε such that*

$$\|u^\varepsilon - \tilde{u}^{\varepsilon,p}\|_{H^1(\Omega_+)} \leq C(p)\varepsilon^{p+1} \qquad \text{for } \varepsilon \in (0, \varepsilon_0] \text{ and } p = \{0, 1, 2, 3\}.$$

Proof. We will first prove an estimate for the difference $\|u^\varepsilon - \tilde{u}^{\varepsilon,p}_\chi\|_{H^1(\Omega)}$, where $\tilde{u}^{\varepsilon,p}_\chi$ has been defined in (21). Since $\tilde{u}^{\varepsilon,p}_\chi = \tilde{u}^{\varepsilon,p}$ in Ω_+, the estimate stated in the theorem will follow.

In the proof of Lemma 5.1 in [8] the authors show that the error $e^\varepsilon_p := u^\varepsilon - \tilde{u}^{\varepsilon,p}_\chi$ satisfies the following transmission problem,

$$\Delta e^\varepsilon_p + k^2 n^2 e^\varepsilon_p = 0 \quad \text{in } \Omega_+, \qquad \frac{\partial e^\varepsilon_p}{\partial\nu} = T^+_{n_+^2}(e^\varepsilon_p) \quad \text{on } \Gamma_a,$$
$$[u]_\Gamma = 0, \quad \left[\frac{\partial u}{\partial\nu}\right]_\Gamma = \varepsilon^p\left.\frac{\partial u^p_+}{\partial\nu}\right|_\Gamma, \qquad \Delta e^\varepsilon_p + k^2 n^2 e^\varepsilon_p = r^{p,\varepsilon} \quad \text{in } \Omega_-, \tag{29}$$

where

$$r^{\varepsilon,p}(x) = -\varepsilon^{p-1}\chi \sum_{\ell=1}^{8}\sum_{j=0}^{\ell-1} \varepsilon^p A_{\ell-j-1} u_-^{k+j+1-\ell}(x_\Gamma, q/\varepsilon) + 2\nabla\chi \cdot \sum_{\ell=0}^{p} \varepsilon^\ell \nabla u_-^p + \Delta\chi \sum_{\ell=0}^{p} \varepsilon^\ell \nabla u_-^p,$$

for $x = x_\Gamma + q\nu \in \Omega_-$. The computations leading to the form of $r^{\varepsilon,p}$ are literally the same as in [8]. Also, the proof that $\|g^{\varepsilon,p}\|_{L^2(\Omega_-)} \leq C\varepsilon^{p-1/2}$ can be achieved as in that reference. The variational formulation for e_p^ε in (29), obtained with the help of Green's first identity, is

$$\int_\Omega \left(\nabla e_p^\varepsilon \cdot \nabla \overline{v} - k^2 n^2 e_p^\varepsilon \overline{v}\right) \mathrm{d}x - \int_{\Gamma_a} \overline{v}\, T^+_{n_+^2}(e_p^\varepsilon)\, \mathrm{d}s = \varepsilon^p \int_\Gamma \frac{\partial u_+^p}{\partial \nu} \overline{v}\, \mathrm{d}s - \int_{\Omega_-} g^{\varepsilon,p} \overline{v}\, \mathrm{d}s.$$

for all $v \in H^1(\Omega)$. Plugging in $v = e_p^\varepsilon$ we find that

$$\begin{aligned}
&\left| \int_\Omega \left(|\nabla e_p^\varepsilon|^2 - k^2 n^2 |e_p^\varepsilon|^2 \right) \mathrm{d}x - \int_{\Gamma_a} \overline{v}\, T^+_{n_+^2}(e_p^\varepsilon)\, \mathrm{d}s \right| \\
&\leq \varepsilon^p \left\| \frac{\partial u_+^p}{\partial \nu} \right\|_{L^2(\Gamma)} \|e_p^\varepsilon\|_{L^2(\Gamma)} + \|g^{\varepsilon,p}\|_{L^2(\Omega_-)} \|e_p^\varepsilon\|_{L^2(\Omega_+)} \leq C\varepsilon^p \|e_p^\varepsilon\|_{L^2(\Gamma)} + C\varepsilon^{p-1/2} \|e_p^\varepsilon\|_{L^2(\Omega_+)}
\end{aligned}$$

due to Lemma 1. From this estimate, we conclude by Lemma 4 that $\|e_p^\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{p+1}$, $\|e_p^\varepsilon\|_{L^2(\Omega_-)} \leq C\varepsilon^{p+2}$ and $\|e_p^\varepsilon\|_{L^2(\Gamma)} \leq C\varepsilon^{p+3/2}$, for $p \in \{0,1,2,3,4\}$. Therefore we can finally estimate

$$\begin{aligned}
\|u^\varepsilon - \tilde{u}^{\varepsilon,p}\|_{H^1(\Omega_+)} &= \|u^\varepsilon - \tilde{u}^{\varepsilon,p+1} + \varepsilon^{p+1} u_+^{p+1}\|_{H^1(\Omega_+)} = \|u^\varepsilon - \tilde{u}_\chi^{\varepsilon,p+1} + \varepsilon^{p+1} u_+^{p+1}\|_{H^1(\Omega_+)} \\
&\leq \|u^\varepsilon - \tilde{u}_\chi^{\varepsilon,p+1}\|_{H^1(\Omega_+)} + \varepsilon^{p+1} \|u_+^{p+1}\|_{H^1(\Omega_+)} \leq C\varepsilon^{p+3/2} + C\varepsilon^p, \quad p = \{0,1,2,3\}.
\end{aligned}$$

For the latter bound we exploited the bounds given in Lemma 1 for $\|u_+^\ell\|_{H^1(\Omega_+)}$, $\ell \leq 4$, valid due to our assumption that $g \in H^3(\Omega_+)$ and $n^2 \in C^{4,1}(\overline{\Omega_+})$. □

The next theorem proves existence and uniqueness as well as stability of rough surface scattering problems with (ordinary) impedance boundary conditions on Γ. With this result, we will easily be able to prove the error estimate given in Theorem 2.

Proposition 6. *Let $h \in H^{1/2}(\Gamma)$ and $\mu \in C^1(\Gamma)$ such that $-\operatorname{Im}(\mu) \geq c_0|\mu|^2 > 0$ and $\operatorname{Re}(\mu) \geq c_0 > 0$. The impedance problem*

$$\Delta u + k^2 n^2 u = g \quad \text{in } \Omega_+, \qquad \frac{\partial u}{\partial \nu} = T^+_{n_+^2}(u) \quad \text{on } \Gamma_a, \qquad u + \mu \frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma,$$

has a unique variational solution $u \in H^1(\Omega_+)$, that is, u solves

$$\int_{\Omega_+} \left(\nabla u \cdot \nabla \overline{v} - k^2 n^2 u \overline{v}\right) \mathrm{d}x - \int_{\Gamma_a} \overline{v}\, T^+_{n_+^2}(u)\, \mathrm{d}s + \int_\Gamma \frac{1}{\mu} u \overline{v}\, \mathrm{d}s = -\int_{\Omega_+} g \overline{v}\, \mathrm{d}x + \int_\Gamma \frac{1}{\mu} h \overline{v}\, \mathrm{d}s \quad (30)$$

for all $v \in H^1(\Omega_+)$. Additionally, the bound

$$\begin{aligned}
\|u\|_{H^1(\Omega_+)} \leq C &\left(\left(1 + k^2 \|n^2\|_{L^\infty(\Omega_+)}\right) \left\| |\mu|^2 / \operatorname{Im}(\mu) \right\|_{L^\infty(\Gamma)} \|1/\mu\|_{L^\infty(\Gamma)} \right) \\
&\qquad \left(\|h\|_{H^{1/2}(\Gamma)} + \|\mu\|_{C^1(\Gamma)} \|g\|_{L^2(\Omega_+)} \right)
\end{aligned}$$

holds with C independent of μ.

Actually, existence of a solution to the above problem can also be shown under the less restrictive assumption that $h \in L^2(\Gamma)$ and $\mu \in L^\infty(\Gamma)$ with $\mathrm{Im}(\mu) > c_0 > 0$; then, however, we did not arrive at a constant in the stability estimate which is homogeneous in μ.

Proof. Our proof is based on the boundedness of the L^2 solution operator shown in Lemma 12. In the first part of the proof, we assume that $g = 0$. An application of Green's identity (or, plugging in the impedance boundary condition into the variational formulation (30) with $v = u$)

$$\int_{\Omega_+} \left(|\nabla u|^2 - k^2 n^2 |u|^2\right) \,\mathrm{d}x - \int_{\Gamma_a} \overline{u}\, T^+_{n_+^2}(u) \,\mathrm{d}s + \int_\Gamma \overline{\mu} \left|\frac{\partial u}{\partial \nu}\right|^2 \overline{u} \,\mathrm{d}s = \int_\Gamma \overline{h} \frac{\partial u}{\partial \nu} \,\mathrm{d}s. \tag{31}$$

Take the imaginary part of this equation and note that $\mathrm{Im} \int_{\Gamma_a} \overline{u}\, T^+_{n_+^2}(u) \,\mathrm{d}s \geq 0$ as well as $\mathrm{Im}(\overline{\mu}) \leq 0$ to obtain the following estimate,

$$\int_\Gamma \mathrm{Im}(\mu) \left|\frac{\partial u}{\partial \nu}\right|^2 \overline{u} \,\mathrm{d}s \leq \|h\|_{L^2(\Gamma)} \left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\Gamma)},$$

hence, if $\mathrm{Im}(\mu) \geq c_0 > 0$, $\|\partial u/\partial \nu\|_{L^2(\Gamma)} \leq \|1/\,\mathrm{Im}(\mu)\|_{L^\infty(\Gamma)} \|h\|_{L^2(\Gamma)}$.

Further, taking the imaginary part of (30),

$$\mathrm{Im} \int_{\Gamma_a} \overline{u}\, T^+_{n_+^2}(u) \,\mathrm{d}s - \mathrm{Im} \int_\Gamma \frac{1}{\mu} u \overline{u} \,\mathrm{d}s = \mathrm{Im} \int_\Gamma \frac{1}{\mu} h \overline{u} \,\mathrm{d}s.$$

Since $\mathrm{Im} \int_{\Gamma_a} \overline{u}\, T^+_{n_+^2}(u) \,\mathrm{d}s \geq 0$,

$$- \int_\Gamma \mathrm{Im}(\mu^{-1}) |u|^2 \,\mathrm{d}s = \int_\Gamma \frac{\mathrm{Im}(\mu)}{|\mu|^2} |u|^2 \,\mathrm{d}s \leq \|1/\mu\|_{L^\infty(\Gamma)} \|h\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)}$$

and

$$\|u\|^2_{L^2(\Gamma)} \leq \||\mu|^2/\,\mathrm{Im}(\mu)\|_{L^\infty(\Gamma)} \|1/\mu\|_{L^\infty(\Gamma)} \|h\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)}. \tag{32}$$

Lemma 12 implies

$$\|u\|^2_{L^2(\Omega_+)} \leq C \left\||\mu|^2/\,\mathrm{Im}(\mu)\right\|_{L^\infty(\Gamma)} \|1/\mu\|_{L^\infty(\Gamma)} \|h\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)}.$$

Taking the real part of (31) and noting that $-\mathrm{Re} \int_{\Gamma_a} \overline{u}\, T^+_{n_+^2}(u) \,\mathrm{d}s \geq 0$ yields an $H^1(\Omega_+)$ a-priori estimate for u,

$$\begin{aligned}
\|u\|^2_{H^1(\Omega_+)} + \int_\Gamma \mathrm{Re}(\mu) \left|\frac{\partial u}{\partial \nu}\right|^2 \overline{u} \,\mathrm{d}s &\leq \left(1 + k^2 \|n^2\|_{L^\infty(\Omega_+)}\right) \|u\|^2_{L^2(\Omega_+)} + \|h\|_{H^{1/2}(\Gamma)} \left\|\frac{\partial u}{\partial \nu}\right\|^2_{H^{-1/2}(\Gamma)} \\
&\leq C \underbrace{\left(1 + k^2 \|n^2\|_{L^\infty(\Omega_+)}\right) \left\||\mu|^2/\,\mathrm{Im}(\mu)\right\|_{L^\infty(\Gamma)} \|1/\mu\|_{L^\infty(\Gamma)}}_{C(k,n^2,\mu)} \|h\|_{L^2(\Gamma)} \|u\|_{H^1(\Omega_+)} \\
&\qquad + C \|h\|_{H^{1/2}(\Gamma)} \|u\|^2_{H^1(\Omega_+)}
\end{aligned}$$

and implies existence and uniqueness of solution due to [9, Theorem 2.15], as well as the bound on u given in the theorem. Of course, this is up to now only valid for $g = 0$.

For $g \in L^2(\Omega_+)$ arbitrary, let $w \in H^1(\Omega_+)$ be the variational solution of $\Delta w + k^2 n^2 w = g$ in Ω_+ subject to a Dirichlet boundary condition $w = 0$ on Γ and $\partial w/\partial \nu = T^+_{n_+^2}(w)$ on Γ_a. From Theorem 11 we note that $\|w\|_{H^1(\Omega_+)} + \|\partial u/\partial \nu\|_{L^2(\Gamma)} \leq C \|g\|_{L^2(\Omega_+)}$. Due to regularity result contained in the same lemma, we even have $\|w\|_{H^2(\Omega_+)} \le C\|g\|_{L^2(\Omega_+)}$. The difference $v = u - w$ solves $\Delta v + k^2 n^2 v = 0$ in Ω_+, $v + \mu(\partial v/\partial\nu) = h - \mu(\partial w)/(\partial\nu)$ and $\partial v/\partial\nu = T^+_{n_+^2}(w)$ on Γ_a, again in a variational sense. Therefore

$$\begin{aligned}\|v\|_{H^1(\Omega_+)} &\le C\left(1 + C(k,n^2,\mu)\right)\|h - \mu(\partial w/\partial\nu)\|_{H^{1/2}(\Gamma)}\\ &\le C\left(1 + C(k,n^2,\mu)\right)\left(\|h\|_{H^{1/2}(\Gamma)} + \|\mu\|_{H^1(\Gamma)}\|u\|_{H^2(\Omega_+)}\right)\\ &\le C\left(1 + C(k,n^2,\mu)\right)\left(\|h\|_{H^{1/2}(\Gamma)} + \|\mu\|_{C^1(\Gamma)}\|g\|_{L^2(\Omega_+)}\right).\end{aligned}$$

An application of the triangle inequality to $v = u - w$ shows a $H^1(\Omega_+)$ a-priori bound for u and finishes the proof, again by [9, Theorem 2.15]. □

Corollary 7. *(a) The solution $u^{\varepsilon,0}$ of (19) exits in $H^1(\Omega_+)$, it is independent of $\varepsilon > 0$ and satisfies the bound $\|u^{\varepsilon,0}\|_{H^1(\Omega_+)} \le C\|g\|_{L^2(\Omega)}$.*

(b) The solution $u^{\varepsilon,2}$ of (19) exists in $H^1(\Omega_+)$ for all $\varepsilon > 0$ and satisfies $\|u^{\varepsilon,1}\|_{H^1(\Omega_+)} \le C\|g\|_{L^2(\Omega)}$ with C independent of $\varepsilon \in (0,\varepsilon_0]$ for arbitrary ε_0.

(c) There is $\varepsilon_0 > 0$ such that the solution $u^{\varepsilon,2}$ of (19) exists in $H^1(\Omega_+)$ for all $\varepsilon \in (0,\varepsilon_0]$ and satisfies the bound $\|u^{\varepsilon,1}\|_{H^1(\Omega_+)} \le C\|g\|_{L^2(\Omega)}$ with C independent of $\varepsilon \in (0,\varepsilon_0]$.

Proof. (a) This follows from Theorem 11.

(b) The impedance boundary condition for $p = 1$ is $u^{\varepsilon,1} + (\varepsilon/\alpha)(\partial u^{\varepsilon,1}/\partial\nu) = 0$ on Γ. Therefore, Theorem 6 holds with $|q|^2/\operatorname{Im}(q) = \sqrt{2}\varepsilon$ and $\left\||\mu|^2/\operatorname{Im}(\mu)\right\|_{L^\infty(\Gamma)}\|1/\mu\|_{L^\infty(\Gamma)} = \sqrt{2}$.

(c) For $p = 2$, the impedance boundary condition in (19) is $u^{\varepsilon,2} + (\varepsilon/\alpha - \mathrm{i}H\varepsilon^2)(\partial u^{\varepsilon,2}/\partial\nu) = 0$ on Γ. Choose ε_0 such that $|H\varepsilon| < 1/(2\sqrt{2})$ for $\varepsilon \in (0,\varepsilon_0]$. Then, Theorem 6 holds with $|q|^2/\operatorname{Im}(q) \le \sqrt{2}\varepsilon$, $\left\||\mu|^2/\operatorname{Im}(\mu)\right\|_{L^\infty(\Gamma)}\|1/\mu\|_{L^\infty(\Gamma)} \le 2\sqrt{2}$ and $\|\mu\|_{C^1(\Gamma)} \le 2\varepsilon + \varepsilon^2\|\nabla H\|_{L^\infty(\Gamma)}$. Note here that we assumed Γ to be of class C^2, such that $\|\nabla H\|_{L^\infty(\Gamma)}$ is well defined. □

Now we finish this section with the proof of the main theorem, Theorem 2, showing the convergence $u^{\varepsilon,p} \to u^\varepsilon$ in $H^1(\Omega_+)$ of optimal order $p+1$.

Proof of Theorem 2. In view of (22) we only need to show that $\|\tilde{u}^{\varepsilon,p} - u^{\varepsilon,p}\|_{H^1(\Omega_+)} \le C\varepsilon^{p+1}$. By (17), the difference $e^{\varepsilon,p} = \tilde{u}^{\varepsilon,p} - u^{\varepsilon,p}$ solves

$$\Delta e^{\varepsilon,p} + k^2 n^2 e^{\varepsilon,p} = 0 \quad \text{in } \Omega_+, \qquad \frac{\partial e^{\varepsilon,p}}{\partial\nu} = T^+_{n_+^2}(e^{\varepsilon,p}) \quad \text{on } \Gamma_a,$$
$$\text{and} \qquad e^{\varepsilon,p} + D_{\varepsilon,p}\left(\frac{\partial e^{\varepsilon,p}}{\partial\nu}\right) = \varepsilon^{p+1} r^{\varepsilon,p} \quad \text{on } \Gamma.$$

where $r^{\varepsilon,p}$ is given by (18). By Lemma 1 we observe that $\|r^{\varepsilon,p}\|_{H^{1/2}(\Omega)} \le C(p)$ for $p \in \{0,1,2,3\}$. Now, Theorem 7 yields the claim of Theorem 2. □

5 GIBC of Order Three

Let us now start to investigate the Neumann-to-Dirichlet impedance boundary condition of order three, with the aim to prove an analogous convergence result as in Theorem 2 for $p = 3$. As we saw in Section 3, formal expansion yields the following candidate for an Neumann-to-Dirichlet impedance condition of order three,

$$u^{\varepsilon,3} + \frac{\varepsilon\sqrt{2}}{2}\left((1+\mathrm{i}) - \sqrt{2}\mathrm{i}\varepsilon H - (1-\mathrm{i})\frac{\varepsilon^2}{2}(3H^2 - G + k^2 + \Delta_\Gamma)\right)\frac{\partial u^{\varepsilon,3}}{\partial\nu} = 0 \qquad \text{on } \Gamma. \tag{33}$$

The difficulty with this boundary condition is that the operator applied to the normal derivative is a tangential differential operator of order 2. In contrast, the exact Neumann-to-Dirichlet operator is a pseudo differential operator of order -1, mapping $H^{-1/2}(\Gamma)$ continuously into $H^{1/2}(\Gamma)$. Also, we remark that for the boundary condition in (33), the term $\mathrm{Im}\int_\Gamma \overline{u^{\varepsilon,3}}\,(\partial u^{\varepsilon,3}/\partial\nu)\,\mathrm{d}s$ is indefinite, a fact which would cause trouble in our later analysis. For these two reasons we modify the above condition in the following way. First considering all real terms of the boundary operator in (33), we formally compute that

$$\frac{\varepsilon\sqrt{2}}{2}\left(1-\frac{\varepsilon^2}{2}(3H^2-G+k^2+\Delta_\Gamma)\right) = \frac{\varepsilon\sqrt{2}}{2}\left(1-\frac{\varepsilon^2}{2}(3H^2-G+k^2)-\frac{\varepsilon^2}{2}(1-\varepsilon^2\Delta_\Gamma)^{-1}\Delta_\Gamma)\right)+\mathcal{O}(\varepsilon^4),$$

since $\Delta_\Gamma-(I-\varepsilon^2\Delta_\Gamma)^{-1}1\Delta_\Gamma=-\varepsilon^2\Delta_\Gamma^2(I-\varepsilon^2\Delta_\Gamma)^{-1}$. We note that $1-\varepsilon^2\Delta_\Gamma$ is bounded and coercive on $H^1(\Gamma)$, hence invertible due to Lax-Milgram's theorem, and that the term on the right constitutes a tangential differential operator on Γ of order zero. Next, we use a Padé approximation to change the sign of the surface Laplacian appearing in the complex terms of (33). Using that $(1-\varepsilon^2\Delta_\Gamma)(1+\varepsilon^2\Delta_\Gamma)=1-\varepsilon^4\Delta_\Gamma^2$, we find

$$\begin{aligned}\frac{\varepsilon\sqrt{2}}{2}&\left(1-\varepsilon\sqrt{2}H+\frac{\varepsilon^2}{2}(3H^2-G+k^2+\Delta_\Gamma)\right)\\ &=\frac{\varepsilon\sqrt{2}}{2}\left(\frac{1}{2}-\varepsilon\sqrt{2}H+\frac{\varepsilon^2}{2}(3H^2-G+k^2)+\frac{1}{2}+\frac{\varepsilon^2}{2}\Delta_\Gamma\right)\\ &=\frac{\varepsilon\sqrt{2}}{2}\left(\frac{1}{2}-\varepsilon\sqrt{2}H+\frac{\varepsilon^2}{2}(3H^2-G+k^2)+\frac{1}{2}\left(1-\varepsilon^2\Delta_\Gamma\right)^{-1}\right)+\mathcal{O}(\varepsilon^4).\end{aligned}$$

Therefore we define a modified boundary operator $\hat{D}_{\varepsilon,3}$ for a generalized impedance boundary condition of order three as follows,

$$\begin{aligned}\hat{D}_{\varepsilon,3}\phi=\frac{\varepsilon\sqrt{2}}{2}&\left(1-\frac{\varepsilon^2}{2}(3H^2-G+k^2)-\frac{\varepsilon^2}{2}(1-\varepsilon^2\Delta_\Gamma)^{-1}\Delta_\Gamma)\right)\phi\\ &+\frac{\mathrm{i}\varepsilon\sqrt{2}}{2}\left(\frac{1}{2}-\varepsilon\sqrt{2}H+\frac{\varepsilon^2}{2}(3H^2-G+k^2)+\frac{1}{2}\left(I-\varepsilon^2\Delta_\Gamma\right)^{-1}\right)\phi.\end{aligned}$$

Using this operator, we define $u^{\varepsilon,3}$ via the following boundary value problem

$$\Delta u^{\varepsilon,3}+k^2n^2u^{\varepsilon,3}=g \quad \text{in } \Omega_+, \qquad \frac{\partial u^{\varepsilon,3}}{\partial\nu}=T^+_{n_+^2}(u^{\varepsilon,3}) \quad \text{on } \Gamma_a,$$
$$u^{\varepsilon,3}+\hat{D}_{\varepsilon,3}(\partial u^{\varepsilon,3}/\partial\nu)=0 \quad \text{on } \Gamma. \tag{34}$$

Three important properties of $\hat{D}_{\varepsilon,3}$ are collected in the next lemma, namely boundedness and coercivity on $L^2(\Gamma)$, which results of course in invertibility of $\hat{D}_{\varepsilon,3}$ on $L^2(\Gamma)$.

Lemma 8. *There are constants $\varepsilon_0>0$ and $C_{1,2,3,4}$ independent of $\varepsilon>0$ such that*

$$\|\hat{D}_{\varepsilon,3}\phi\|_{L^2(\Gamma)}\le C_1\varepsilon\|\phi\|_{L^2(\Gamma)}, \qquad \mathrm{Re}\langle\hat{D}_{\varepsilon,3}\phi,\phi\rangle\ge C_2\varepsilon\|\phi\|^2_{L^2(\Gamma)}, \qquad \textit{and}$$
$$\mathrm{Im}\langle\hat{D}_{\varepsilon,3}\phi,\phi\rangle\ge C_3\varepsilon\|\phi\|^2_{L^2(\Gamma)} \qquad \textit{for all } \phi\in L^2(\Gamma) \textit{ and } \varepsilon\in(0,\varepsilon_0]. \tag{35}$$

Proof. We start with some preparations. First, the identity $(I-\varepsilon^2\Delta_\Gamma)(1-\varepsilon^2\Delta_\Gamma)^{-1}\phi = \phi$ for $\phi \in L^2(\Gamma)$ implies by one partial integration that

$$\int_\Gamma \left(\varepsilon^2 \nabla_\Gamma((1-\varepsilon^2\Delta_\Gamma)^{-1}\phi)\cdot\nabla_\Gamma\overline{\psi} + ((1-\varepsilon^2\Delta_\Gamma)^{-1}\phi)\overline{\psi}\right)\,\mathrm{d}s = \int_\Gamma (1-\varepsilon^2\Delta_\Gamma)^{-1}\phi\overline{\psi}\,\mathrm{d}s \tag{36}$$

for all $\psi \in H^1(\Gamma)$. Setting $\psi = (1-\varepsilon^2\Delta_\Gamma)^{-1}\phi$ and taking the complex conjugate of (36) we obtain

$$\langle (I-\varepsilon^2\Delta_\Gamma)^{-1}\phi,\phi\rangle = \|(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi\|^2_{L^2(\Gamma)} + \varepsilon^2\|\nabla_\Gamma(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi\|^2_{L^2(\Gamma)} \tag{37}$$

as well as $\psi = \Delta_\Gamma(1-\varepsilon^2\Delta_\Gamma)^{-1}$ results by further integrations by parts in

$$-\langle \Delta_\Gamma(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi,\phi\rangle = \|\nabla_\Gamma(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi\|^2_{L^2(\Gamma)} + \varepsilon^2\|\Delta_\Gamma(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi\|^2_{L^2(\Gamma)}. \tag{38}$$

The Cauchy-Schwarz inequality applied to (36) shows that $\|(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi\|^2_{L^2(\Gamma)} \le \|\phi\|^2_{L^2(\Gamma)}$. Therefore (37) implies

$$\varepsilon^2\|\nabla_\Gamma(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi\|^2_{L^2(\Gamma)} \le \|\phi\|^2_{L^2(\Gamma)}. \tag{39}$$

The identity $-\varepsilon^2\Delta_\Gamma(I-\varepsilon^2\Delta_\Gamma)^{-1} = I-(I-\varepsilon^2\Delta_\Gamma)^{-1}$ and (37) finally shows that $-\varepsilon^2\langle\Delta_\Gamma(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi,\phi\rangle \le \|\phi\|^2_{L^2(\Gamma)}$ as well, and hence (38) yields $\varepsilon^4\|\Delta_\Gamma(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi\|^2_{L^2(\Gamma)} \le \|\phi\|^2_{L^2(\Gamma)}$.

A brief look at the definition of $\hat{D}_{\varepsilon,3}$ shows that we can write

$$\hat{D}_{\varepsilon,3} = \varepsilon\left(a_\varepsilon - \sqrt{2}\varepsilon^2/4\,(1-\varepsilon^2\Delta_\Gamma)^{-1}\Delta_\Gamma)\right) + \mathrm{i}\varepsilon\left(b_\varepsilon + \frac{1}{2}\left(I-\varepsilon^2\Delta_\Gamma\right)^{-1}\right)$$

and that we observe that there is $\varepsilon_0 > 0$ such that the coefficients a_ε and b_ε on Γ are bounded between two constants $0 < c_0 < C$ uniformly in $\varepsilon \in (0,\varepsilon_0]$. Consequently,

$$\langle \hat{D}_{\varepsilon,3}\phi,\psi\rangle = \varepsilon\langle a_\varepsilon\phi,\psi\rangle - \frac{\sqrt{2}\varepsilon^3}{4}\langle(1-\varepsilon^2\Delta_\Gamma)^{-1}\Delta_\Gamma\phi,\psi\rangle + \mathrm{i}\varepsilon\langle b_\varepsilon\phi,\psi\rangle - \frac{\mathrm{i}\varepsilon}{2}\langle\left(I+\varepsilon^2\Delta_\Gamma\right)^{-1}\phi,\psi\rangle$$

for all $\psi \in L^2(\Gamma)$ and for $\psi = \phi$ we can estimate real and imaginary part of this expression from below due to our above computations, $c_0\|\phi\|^2_{L^2(\Gamma)} \le \mathrm{Re}\langle\hat{D}_{\varepsilon,3}\phi,\phi\rangle$ and $c_0\|\phi\|^2_{L^2(\Gamma)} \le \mathrm{Im}\langle\hat{D}_{\varepsilon,3}\phi,\phi\rangle$. For an upper bound on $L^2(\Gamma)$, we set $\psi = \hat{D}_{\varepsilon,3}\phi$, which yields by the Cauchy-Schwarz inequality

$$\begin{aligned}\|\hat{D}_{\varepsilon,3}\phi\|_{L^2(\Gamma)} &\le C\varepsilon\|\phi\|_{L^2(\Gamma)} \\ &\quad + C\varepsilon^3\|(I-\varepsilon^2\Delta_\Gamma)^{-1}\Delta_\Gamma\phi\|_{L^2(\Gamma)} + C\varepsilon\|(I-\varepsilon^2\Delta_\Gamma)^{-1}\phi\|_{L^2(\Gamma)} \le C\varepsilon\|\phi\|_{L^2(\Gamma)}.\end{aligned}$$

□

In the next step, we show that the approximated scattering problem involving the Neumann-to-Dirichlet impedance operator $\hat{D}_{\varepsilon,3}$ is uniquely solvable. Recall that this scattering problem is to find $u^{\varepsilon,3} \in H^1(\Omega_+)$ such that

$$\Delta u + k^2 u = g \quad \text{in } \Omega_+, \qquad \frac{\partial u}{\partial\nu} = T^+_{n_+^2}(u) \quad \text{on } \Gamma_a, \qquad u + \hat{D}_{\varepsilon,3}\left(\frac{\partial u}{\partial\nu}\right) = 0 \quad \text{on } \Gamma. \tag{40}$$

Actually, we are going to tackle this problem in more generality, which will be helpful in the later error analysis, and solve for an inhomogeneous boundary condition $u + \hat{D}_{\varepsilon,3}(\partial u/\partial\nu) = h$.

Before stating our result on existence and uniqueness of solution, we note that the variational form of the problem then reads

$$\int_{\Omega_+} \left(\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v}\right) \,\mathrm{d}s - \int_{\Gamma_a} \overline{v}\, T^+_{n_+^2}(u) \,\mathrm{d}s + \int_\Gamma \overline{v} \hat{D}^{-1}_{\varepsilon,3}(u) \,\mathrm{d}s = - \int_{\Omega_+} g \overline{v} \,\mathrm{d}x + \int_\Gamma \overline{v} \hat{D}^{-1}_{\varepsilon,3} h \,\mathrm{d}s \quad (41)$$

for all $v \in H^1(\Omega_+)$. Note that $\hat{D}^{-1}_{\varepsilon,3}$ is indeed boundedly invertible on $L^2(\Gamma)$ for ε in some interval $(0, \varepsilon_0]$ due to its coercivity.

Theorem 9. *For $g \in L^2(\Omega_+)$ and $h \in H^{1/2}(\Gamma)$ there is a unique solution of the variational problem* (41) *which satisfies*

$$\|u^{\varepsilon,3}\|_{H^1(\Omega_+)} + C_1 \sqrt{\varepsilon} \left\| \frac{\partial u^{\varepsilon,3}}{\partial \nu} \right\|_{L^2(\Gamma)} \leq C_2 \left(\|g\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\Gamma)} \right)$$

for $C_{1,2}$ independent of $\varepsilon \in (0, \varepsilon_0]$.

Proof. As in the proof of Theorem 6 we consider first the case $g = 0$; the general case will again follow from this special situation. By an integration by parts in Ω_+, we find

$$\int_{\Omega_+} \left(\left|\nabla u^{\varepsilon,3}\right|^2 - k^2 n^2 |u^{\varepsilon,3}|^2 \right) \,\mathrm{d}x - \int_{\Gamma_a} \overline{u^{\varepsilon,3}}\, T^+_{n_+^2}(u^{\varepsilon,3}) \,\mathrm{d}s - \int_\Gamma \overline{u^{\varepsilon,3}} \frac{\partial u^{\varepsilon,3}}{\partial \nu} \,\mathrm{d}s = 0. \quad (42)$$

Taking the imaginary part of this equation, we obtain

$$\operatorname{Im} \int_{\Gamma_a} \overline{u^{\varepsilon,3}}\, T^+_{n_+^2}(u^{\varepsilon,3}) \,\mathrm{d}s + \operatorname{Im} \int_\Gamma \overline{u^{\varepsilon,3}} \frac{\partial u^{\varepsilon,3}}{\partial \nu} \,\mathrm{d}s = 0.$$

Moreover, exploiting the coercivity of $\operatorname{Im} \hat{D}_{\varepsilon,3}$ proven in Lemma 8

$$\begin{aligned} \operatorname{Im} \int_\Gamma \overline{u^{\varepsilon,3}} \frac{\partial u^{\varepsilon,3}}{\partial \nu} \,\mathrm{d}s &= \operatorname{Im} \int_\Gamma \hat{D}_{\varepsilon,3} \left(\frac{\partial u^{\varepsilon,3}}{\partial \nu} \right) \overline{\frac{\partial u^{\varepsilon,3}}{\partial \nu}} \,\mathrm{d}s + \operatorname{Im} \int_\Gamma \overline{h} \frac{\partial u^{\varepsilon,3}}{\partial \nu} \,\mathrm{d}s \\ &\geq c\varepsilon \left\| \frac{\partial u^{\varepsilon,3}}{\partial \nu} \right\|^2_{L^2(\Gamma)} + \operatorname{Im} \int_\Gamma \overline{h} \frac{\partial u^{\varepsilon,3}}{\partial \nu} \,\mathrm{d}s. \quad (43) \end{aligned}$$

Therefore, by Lemma 12 and the fact that $\operatorname{Im} \int_{\Gamma_a} \overline{u^{\varepsilon,3}}\, T^+_{n_+^2}(u^{\varepsilon,3}) \,\mathrm{d}s \geq 0$,

$$\varepsilon \left\| \frac{\partial u^{\varepsilon,3}}{\partial \nu} \right\|^2_{L^2(\Gamma)} \leq \|h\|_{H^{1/2}(\Gamma)} \left\| \frac{\partial u^{\varepsilon,3}}{\partial \nu} \right\|_{H^{-1/2}(\Gamma)}. \quad (44)$$

Due to Lemma 8, $D^{\varepsilon,3}$ is bounded on $L^2(\Gamma)$ with bound $C_1\sqrt{\varepsilon}$ for $\varepsilon \in (\varepsilon, \varepsilon_0]$. Exploiting the impedance boundary condition in (41), we obtain

$$\begin{aligned} \|u^{\varepsilon,3}\|^2_{L^2(\Gamma)} &\leq 2 \|D^{\varepsilon,3}(\partial u^{\varepsilon,3}/\partial \nu)\|^2_{L^2(\Gamma)} + 2\|h\|^2_{L^2(\Gamma)} \leq C\varepsilon^2 \|\partial u^{\varepsilon,3}/\partial \nu\|^2_{L^2(\Gamma)} + 2\|h\|^2_{L^2(\Gamma)} \\ &\leq C\varepsilon \|h\|_{H^{1/2}(\Gamma)} \left\| \frac{\partial u^{\varepsilon,3}}{\partial \nu} \right\|_{H^{-1/2}(\Gamma)} + 2\|h\|^2_{L^2(\Gamma)} \leq C \|h\|_{H^{1/2}(\Gamma)} \|u^{\varepsilon,3}\|_{H^1(\Omega_+)} + 2\|h\|^2_{L^2(\Gamma)}, \end{aligned}$$

for a different constant C. We take the real part of (42) and exploit that $-\operatorname{Re}\int_{\Gamma_a} \overline{u^{\varepsilon,3}}\, T^+_{n_+^2}(u^{\varepsilon,3})\,\mathrm{d}s \geq 0$ by the representation of $T^+_{n_+^2}$ in (3), and that

$$-\operatorname{Re}\int_\Gamma \overline{u^{\varepsilon,3}}\,\frac{\partial u^{\varepsilon,3}}{\partial\nu}\,\mathrm{d}s = \operatorname{Re}\int_\Gamma \hat{D}_{\varepsilon,3}\left(\frac{\partial u^{\varepsilon,3}}{\partial\nu}\right)\frac{\partial\overline{u^{\varepsilon,3}}}{\partial\nu}\,\mathrm{d}s - \operatorname{Re}\int_\Gamma \overline{h}\frac{\partial u^{\varepsilon,3}}{\partial\nu}\,\mathrm{d}s$$
$$\geq c\varepsilon\left\|\frac{\partial u^{\varepsilon,3}}{\partial\nu}\right\|^2_{L^2(\Gamma)} - \operatorname{Re}\int_\Gamma \overline{h}\frac{\partial u^{\varepsilon,3}}{\partial\nu}\,\mathrm{d}s.$$

With the help of Lemma 12, (42) therefore implies

$$\begin{aligned}\|u^{\varepsilon,3}\|^2_{H^1(\Omega_+)} + c\varepsilon\left\|\frac{\partial u^{\varepsilon,3}}{\partial\nu}\right\|^2_{L^2(\Gamma)} &\leq \|h\|_{H^{1/2}(\Gamma)}\|\partial u^{\varepsilon,3}/\partial\nu\|_{H^{-1/2}(\Gamma)} + (1+k^2\|n^2\|_\infty)\|u^{\varepsilon,3}\|^2_{L^2(\Omega_+)}\\ &\leq C_1\|h\|_{H^{1/2}(\Gamma)}\|u^{\varepsilon,3}\|_{H^1(\Omega_+)} + C_2\|u^{\varepsilon,3}\|^2_{L^2(\Gamma)}\\ &\leq \tilde{C}_1\|h\|_{H^{1/2}(\Gamma)}\|u^{\varepsilon,3}\|_{H^1(\Omega_+)} + 2C_2\|h\|^2_{L^2(\Gamma)},\end{aligned}$$

where C_1, C_2 and $\tilde{C}_1$ are constants independent of ε. Hence, we conclude that

$$\|u^{\varepsilon,3}\|_{H^1(\Omega_+)} + c\sqrt{\varepsilon}\left\|\frac{\partial u^{\varepsilon,3}}{\partial\nu}\right\|_{L^2(\Gamma)} \leq C\|h\|_{H^{1/2}(\Gamma)} \tag{45}$$

for some constant C independent of ε.

Consider now the general case $0 \neq g \in L^2(\Omega_+)$. Again, let $w \in H^1(\Omega_+)$ be the variational solution of $\Delta w + k^2n^2w = 0$ in Ω_+ subject to the homogeneous Dirichlet boundary condition $u|_\Gamma = 0$ and the radiation condition $\partial w/\partial\nu = T^+_{n_+^2}(w)$ on Γ_a. Recall that Theorem 11 states $\|w\|_{H^1(\Omega_+)} + \|\partial u/\partial\nu\|_{L^2(\Gamma)} \leq C\|g\|_{L^2(\Omega_+)}$ and by the regularity result contained in the same theorem we also note the bound $\|w\|_{H^2(\Omega_+)} \leq C\|g\|_{L^2(\Omega_+)}$. The difference $v = u - w$ solves $\Delta v + k^2n^2v = 0$ in Ω_+, $v + \hat{D}_{\varepsilon,3}(\partial v/\partial\nu) = h - \hat{D}_{\varepsilon,3}(\partial w)/(\partial\nu)$ and $\partial v/\partial\nu = T^+_{n_+^2}(w)$ on Γ_a, again in a variational sense. The boundedness of $\hat{D}_{\varepsilon,3}$ on $H^{1/2}(\Gamma)$ shown in Lemma (8) implies

$$\begin{aligned}\|v\|_{H^1(\Omega_+)} + c\sqrt{\varepsilon}\left\|\frac{\partial u^{\varepsilon,3}}{\partial\nu}\right\|_{L^2(\Gamma)} &\leq C(1+k^2\|n^2\|_\infty)\left(\|h\|_{H^{1/2}(\Gamma)} + \|\hat{D}_{\varepsilon,3}(\partial w)/(\partial\nu)\|_{H^{1/2}(\Gamma)}\right)\\ &\leq C(1+k^2\|n^2\|_\infty)\left(\|h\|_{H^{1/2}(\Gamma)} + C\|\partial w/\partial\nu\|_{H^{1/2}(\Gamma)}\right)\\ &\leq C(1+k^2\|n^2\|_\infty)\left(\|h\|_{H^{1/2}(\Gamma)} + C\|u\|_{H^2(\Omega_+)}\right)\end{aligned}$$

We conclude that

$$\begin{aligned}\|u\|_{H^1(\Omega_+)} + c\sqrt{\varepsilon}\left\|\frac{\partial u^{\varepsilon,3}}{\partial\nu}\right\|_{L^2(\Gamma)} &\leq \|v\|_{H^1(\Omega_+)} + \|w\|_{H^1(\Omega_+)}\\ &\leq C(1+k^2\|n^2\|_\infty)\left(\|h\|_{H^{1/2}(\Gamma)} + C\|g\|_{L^2(\Omega_+)}\right),\end{aligned}$$

which is $H^1(\Omega_+)$ a-priori bound for u and finishes the proof, since this a-priori bound establishes existence and uniqueness of solution, again by [9, Chapter 2]. □

The latter existence and stability results provides the mean to prove an optimal error estimate for $\|u^\varepsilon - u^{\varepsilon,3}\|_{H^1(\Omega_+)}$. Of course, this estimate makes use of several of lemmas contained in the previous Section 4. The error estimate given here requires smoothness assumptions on the data g, n^2 and Γ of our scattering problem. We give those assumptions explicitly, noting that we could of course also simply require that all data be smooth enough.

Theorem 10. *Assume that $g \in H^6(\Omega_+)$, $n^2 \in C^{6,1}(\overline{\Omega})$ and that Γ is of class $C^{7,1}$. Then there are constants $\varepsilon_0 > 0$ and $C > 0$, independent of $\varepsilon \in (0, \varepsilon_0]$, such that*

$$\|u^\varepsilon - u^{\varepsilon,3}\|_{H^1(\Omega_+)} \leq C\varepsilon^4. \tag{46}$$

Proof. As in (22) we split the quantity we need to estimate in two parts,

$$\|u^\varepsilon - u^{\varepsilon,3}\|_{H^1(\Omega_+)} \leq \|u^\varepsilon - \tilde{u}^{\varepsilon,3}\|_{H^1(\Omega_+)} + \|\tilde{u}^{\varepsilon,3} - u^{\varepsilon,3}\|_{H^1(\Omega_+)}.$$

The first part has been already treated in Lemma 5. For the second part, we note that $e^{\varepsilon,3} := \tilde{u}^{\varepsilon,3} - u^{\varepsilon,3}$ solves

$$\Delta e^{\varepsilon,3} + k^2 n^2 e^{\varepsilon,3} = 0 \quad \text{in } \Omega_+, \qquad \frac{\partial e^{\varepsilon,3}}{\partial \nu} = T^+_{n_+^2}(e^{\varepsilon,3}) \quad \text{on } \Gamma_a,$$
$$\text{and} \qquad e^{\varepsilon,3} + \hat{D}_{\varepsilon,3}\left(\frac{\partial e^{\varepsilon,k}}{\partial \nu}\right) = \varepsilon^4 \hat{r}^{\varepsilon,3} \quad \text{on } \Gamma.$$

The right hand side $\hat{r}^{\varepsilon,3}$ is different from $r^{\varepsilon,3}$ given in (18), since $\hat{r}^{\varepsilon,3}$ relies on the modified boundary operator $\hat{D}_{\varepsilon,3}$ instead of $D_{\varepsilon,3}$. More precisely,

$$\varepsilon^4 \hat{r}^{\varepsilon,3} = \varepsilon^4 r^{\varepsilon,3} + \left(\hat{D}_{\varepsilon,3} - D_{\varepsilon,3}\right)\left(\frac{\partial \tilde{u}^{\varepsilon,3}}{\partial \nu}\right).$$

From Lemma 1 it follows that $\|r^{\varepsilon,3}\|_{H^{1/2}(\Gamma)}$ is bounded by some constant independent of ε. However, the difference $\hat{D}_{\varepsilon,3} - D_{\varepsilon,3}$ is, modulo constants independent of ε, sum of the two terms

$$\varepsilon^2 \Delta_\Gamma - \varepsilon^2 (I - \varepsilon^2 \Delta_\Gamma)^{-1} \Delta_\Gamma = -\varepsilon^4 \Delta_\Gamma^2 (I - \varepsilon^2 \Delta_\Gamma)^{-1}$$

and

$$1 + \varepsilon^2 \Delta_\Gamma - (1 - \varepsilon^2 \Delta_\Gamma)^{-1} = \varepsilon^4 \Delta_\Gamma^2 (1 - \varepsilon^2 \Delta_\Gamma)^{-1}.$$

Thus, to show that $\|(\hat{D}_{\varepsilon,3} - D_{\varepsilon,3})(\partial \tilde{u}^{\varepsilon,3}/\partial \nu)\|_{H^{1/2}(\Gamma)} \leq C$ for C independent of ε, we need to prove that $\|\Delta_\Gamma^2 (1 - \varepsilon^2 \Delta_\Gamma)^{-1} (\partial \tilde{u}^{\varepsilon,3}/\partial \nu)\|_{H^{1/2}(\Gamma)}$ is bounded independently of ε. Therefore it is sufficient to show that this property holds for $\|\Delta_\Gamma^2 (\partial \tilde{u}^{\varepsilon,3}/\partial \nu)\|_{H^{1/2}(\Gamma)}$, that is, we need that $\|\partial \tilde{u}^{\varepsilon,3}/\partial \nu\|_{H^{9/2}(\Gamma)}$ is bounded. However, Lemma 1 shows that such a bound is guaranteed if only g, n^2 and Γ are smooth enough; for our purpose, we need for instance $g \in H^6(\Omega_+)$. If all data is smooth enough, an application of Theorem 9 finishes the proof. □

6 Preliminary Numerical Results

This section is dedicated to the presentation of some preliminary numerical results that test the accuracy of derived GIBCs. These tests are restricted to the 2-D case. We opted for the use of finite element methods to compute the numerical solution using FreeFem++ library (`http://www.freefem.org/ff++`). The major difficulty linked to the use of volumic methods is indeed the truncation of the computational domain. One possible approach is the use of Dirichlet-to-Neumann maps introduced in the second section of this paper. The second possibility, adopted here, is the use of so-called perfectly matched absorbing layers (PML) (see Fig. 2). We refer to [4] for description of this method in the context of rough surface scattering problems.

The numerical results presented in the following intends to only give first numerical validations and also hints on possible numerical difficulties that need to be addressed in future works.

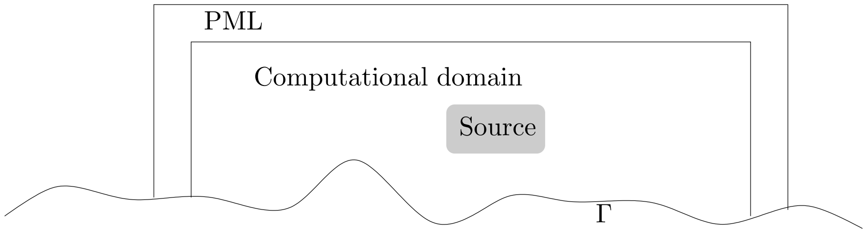
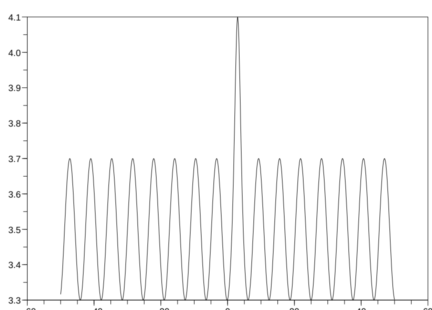


Figure 2: The numerical experiments settings

The two examples, shown in Fig. 3 and 4 correspond to a computational domain which is 100 λ large and 5 λ thick, where $\lambda = 2\pi/k$ (= 1 in our cases). The PML width is λ and we used an (optimized) linear profile for the absorption coefficient going from 1 at the inner boundary of the PML to $\sigma_0 = 5.75 + 3i$ at the outer boundary (see [4]). The numerical examples are computed using $P2$ finite elements. A reference solution is computed using a fine mesh of the two layered medium.

We observe in both experiments how the use of higher order GIBC significantly improves the accuracy of the approximate model. For all GIBCs, the error decreases with respect to ε. However, we failed in obtaining confirmation of convergence rates as predicted by our theory. We believe that this is mainly due to the numerical error induced by lateral bounds of the computational domain (the PML method can only be justified if an infinite horizontal layer is used above the rough surface). We observed that the residual error due to this truncation cannot be made smaller than 1%, which would also explain why the three curves (for GIBCs) meet for small ε. We therefore think that the design of efficient method to bound the computational domain constitutes one important step before going deeper in the numerical validation of these approximate boundary conditions. Addressing this issue is far beyond the scope of the present work.

We end us this discussion by noticing that when comparing the two examples, one observes that better results are obtained for rough surfaces with less sharp variations, which is somehow a predictable behavior since more regularity is needed for higher order GIBCs.



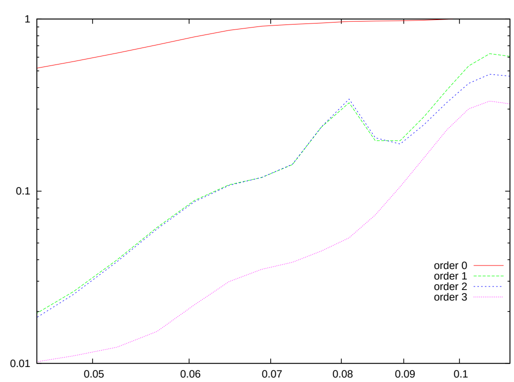


Figure 3: Left: surfcae profile: $f(t) = 3.5 + 0.2\cos(t-3) + 0.4\exp(-(t-3)^2)$. Right: L^2 error versus ε (log-log scale) for GIBCs of order 0, 1, 2 and 3.

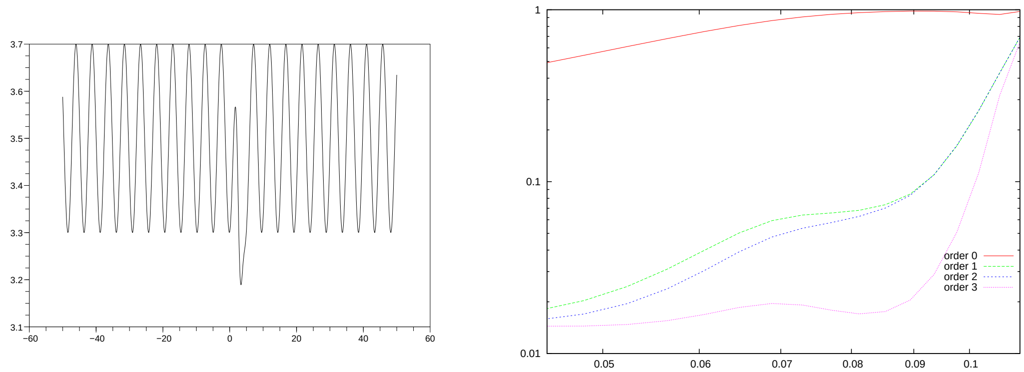


Figure 4: Left: surfcae profile: $f(t) = 3.5 + 0.2\cos(1.3t - 3) - 0.4\exp(-(t-3)^2)$. Right: L^2 error versus ε (log-log scale) for GIBCs of order 0, 1, 2 and 3.

A Auxiliary Results on Rough Surface Scattering

In this section, we provide some results on existence and uniqueness for Dirichlet scattering problems in Ω_+. The following Rellich identity, which is valid for any solution v in $H^2(\Omega_+)$ of the Helmholtz equation $\Delta v + k^2 n^2 v = g$, subject to the radiation condition $\partial v/\partial \nu = T^+_{n_+^2}(v)$ on Γ_a, irrespectively of the actual boundary condition on Γ, has been shown in [10],

$$\int_{\Omega_+} \left(2\left|\frac{\partial v}{\partial x_m}\right|^2 + k^2(x_m + a)\nu_m \frac{\partial n^2}{\partial x_m}|v^2|^2 \right) \mathrm{d}x + 2h\int_{\Gamma_a} \left(|\nabla v|^2 - 2\left|\frac{\partial v}{\partial \nu}\right|^2 - k^2 n^2 |v|^2 \right) \mathrm{d}s$$
$$- \int_{\Gamma_a} \overline{v} T^+_{n_+^2}(v)\,\mathrm{d}s + \int_\Gamma (x_m + a)\left(\nu_m |\nabla v|^2 - 2\,\mathrm{Re}\left(\frac{\partial v}{\partial x_m}\frac{\partial v}{\partial \nu} \right) - \nu_m k^2 |v|^2 \right) \mathrm{d}s$$
$$- \int_\Gamma \overline{v}\frac{\partial v}{\partial \nu}\,\mathrm{d}s = -2\int_\Gamma (x_m + a)\,\mathrm{Re}\left(\frac{\partial v}{\partial x_m}\overline{g} \right) \mathrm{d}x - \int_{\Omega_+} g\overline{v}\,\mathrm{d}x. \quad (47)$$

Theorem 11. *Assume that $n^2 \in C^{0,1}(\Omega_+)$, $\partial n^2/\partial x_m \le 0$ in Ω_+ and that Γ is of class C^2. Then there is unique variational solution of $\Delta u + k^2 n^2 u = g$ in Ω_+ subject to the Dirichlet boundary condition $u = h$ on Γ, that is, u satisfies*

$$\int_{\Omega_+} \left(\nabla u \cdot \nabla \overline{v} - k^2 n^2 u \overline{v} \right) \mathrm{d}x - \int_{\Gamma_a} \overline{v} T^+_{n_+^2}(u)\,\mathrm{d}s = -\int_{\Omega_+} g\overline{v}\,\mathrm{d}x$$

for all $v \in H^1_0(\Omega_+) := \{v \in H^1(\Omega_+),\ v = 0 \text{ on } \Gamma\}$ and the boundary condition $u|_\Gamma = h$ in the trace sense. If $h = 0$, the solution satisfies

$$\|u\|_{H^1(\Omega)} + \|\partial u/\partial \nu\|_{L^2(\Gamma)} \le C\|g\|_{L^2(\Omega)}, \quad (48)$$

and elliptic regularity implies $\|u\|_{H^2(\Omega)} \le C\|g\|_{L^2(\Omega)}$. In the general case,

$$\|u\|_{H^1(\Omega)} \le C\left(\|g\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\Gamma)} \right).$$

Generally, elliptic regularity yields that

$$\|u\|_{H^{p+2}(\Omega)} \le C\left(\|g\|_{H^p(\Omega)} + \|h\|_{H^{p+3/2}(\Gamma)} \right),$$

under the smoothness assumption $n^2 \in C^{p,1}(\overline{\Omega_+})$ and Γ of class $C^{p+1,1}$.

Proof. The existence statement for $h = 0$ in $H_0^1(\Omega_+)$ is shown as in [3], where a Rellich identity similar to (47) is used to prove existence of solution in $H^1(\Omega_+)$. For $h = 0$, the regularity statement concerning $\|u\|_{L^2(\Gamma)}$ also stems from the identity (47); in fact, this is shown explicitly in the proof of Lemma 12 below. The existence result for arbitrary h is shown as in the standard proof for inhomogeneous elliptic boundary value problems [11, Theorem 4.10], noting that (48) even implies $\|u\|_{H^1(\Omega)} \leq C\|g\|_{(H_0^1(\Omega_+))^*}$, where $(H_0^1(\Omega_+))^*$ denotes the space of bounded linear functionals on $(H_0^1(\Omega_+))^*$ with obvious norm. The regularity statement in the general case follows from the corresponding regularity result for bounded domains [11, Theorem 4.19] and a technique already introduced in [10]. For shake of completness we shall hereafter sketch some details of the proof. By abuse of notation we do not distinguish between a solution $u \in H^1(\Omega_+)$ of the Helmholtz equation satisfying $\partial u/\partial\nu = T^+_{n_+^2}(u)$ on Γ_a and its unique radiating extension to $\Omega_+ \cup \Gamma_a \cup U_a^+$.

Consider the open cube $Q = (-2, 2)^{m-1} \times (-2h, 2h)$ and set $Q_j := j + Q$ for $j \in (3\mathbb{Z})^{m-1}$. The cubes Q_j cover Ω:

$$\Omega \subset \bigcup_{j\in(3\mathbb{Z})^{m-1}} Q_j.$$

By $Q_j^2 := j + 2Q$ we denote an even larger cube containing Q_j. From the boundary regularity result [11, Theorem 4.28] we obtain the following estimate in each cube Q_j,

$$\|u\|_{H^{p+2}(Q_j\cap\Omega_+)} \leq C_j\|u\|_{H^{p+1}(Q_j^2\cap\Omega_+)} + C_j\|\partial u/\partial\nu\|_{H^{p+1/2}(Q_j^2\cap\Gamma)} + C_j\|g\|_{H^p(Q_j^2\cap\Omega_+)}.$$

The constants C_j depend on p, the local smoothness of the coefficient n^2 and the local regularity of Γ. Hence, the lemma's assumptions that Γ be of class $C^{p+1,1}$ and n^2 be $C^{0,1}$ imply a uniform bound C for the numbers C_j. Now we can exploit that the Q_j cover Ω,

$$\begin{aligned} \|u\|_{H^{p+2}(\Omega_+)} &\leq C \sum_{j\in(3\mathbb{Z})^{m-1}} \|u\|_{H^{p+2}(Q_j\cup\Omega_+)} \\ &\leq C \sum_{j\in(3\mathbb{Z})^{m-1}} \|u\|_{H^{p+1}(Q_j^2\cap\Omega_+)} + C\|\partial u/\partial\nu\|_{H^{p+1/2}(Q_j^2\cap\Gamma)} + C\|g\|_{H^p(Q_j^2\cap\Omega_+)} \\ &\leq C\|u\|_{H^p(\{f(\tilde{x})<x_m<2h)\})} + C\|\partial u/\partial\nu\|_{H^{p+1/2}(\Gamma)} + C\|g\|_{H^p(\Omega_+)} \end{aligned} \tag{49}$$

The regularity result can then easily derived from (48) and (49) by a simple induction. ☐

Several times in this text, for instance in the proof of Theorem 9, we rely on the proof of the following auxiliary result.

Lemma 12 (L^2 solution operator)**.** *For a weak solution $u \in H^1(\Omega_+)$ of the Helmholtz equation $\Delta u + k^2n^2u = 0$ in Ω_+ which takes boundary values $\phi \in H^{1/2}(\Gamma)$ and satisfies the radiation condition $\partial u/\partial\nu = T^+_{n_+^2}(u)$ on Γ_a, it holds that $\|u\|_{L^2(\Omega_+)} \leq C\|u\|_{L^2(\Gamma)}$ for a constant C independent of u. Hence, the solution operator $\phi \mapsto u$ has a bounded extension from $L^2(\Gamma)$ to $L^2(\Omega_+)$.*

For bounded domains, this is a well known result [11, Theorem 4.25]. One possibility how to prove Lemma 12 is indeed to mimic the proof of the result for bounded domains, exploiting results from variational theory of rough surface scattering. The other option is to use results from integral equation theory for rough surface scattering problems, namely invertibility in $L^2(\Gamma)$ of the boundary integral operator arising in a Brackhage-Werner ansatz for the rough surface Dirichlet scatter problem. This approach is of course restricted to a constant index of refraction in Ω_+, and therefore we prefer the first option in the sequel.

Proof of Lemma 12. Due to Theorem 11 there is a unique weak solution $u \in H^1(\Omega_+)$ of the problem

$$\Delta u + k^2 n^2 u = 0 \quad \text{in } \Omega_+, \qquad \frac{\partial u}{\partial \nu} = T^+_{n_+^2}(u) \quad \text{on } \Gamma_a, \qquad u = h \quad \text{on } \Gamma,$$

for all $h \in H^{1/2}(\Gamma)$, and $\|u\|_{H^1(\Omega_+)} \leq C\|h\|_{H^{1/2}(\Gamma)}$ for C independent of u. Moreover, Theorem 11 states that for arbitrary $g \in L^2(\Omega_+)$ there is a unique solution of the problem

$$\Delta w + k^2 n^2 w = g \quad \text{in } \Omega_+, \qquad \frac{\partial w}{\partial \nu} = T^+_{n_+^2}(u) \quad \text{on } \Gamma_a, \qquad w = 0 \quad \text{on } \Gamma,$$

with norm bound $\|w\|_{H^2(\Omega_+)} \leq C\|g\|_{L^2(\Omega_+)}$, again for C independent of u. Moreover, by the Rellich identity (47) and the Dirichlet boundary condition $w = 0$ on Γ which implies that $\partial w/\partial x_m = e_m \cdot (\nu \nabla w) = \nu_m(\partial w/\partial \nu)$,

$$\int_{\Omega_+} \left(2\left|\frac{\partial w}{\partial x_m}\right|^2 + k^2(x_m + a)\nu_m \frac{\partial n^2}{\partial x_m}|w^2|^2 \right) \mathrm{d}x - \int_{\Gamma_a} \overline{w} T^+_{n_+^2}(w)\,\mathrm{d}s - \int_\Gamma (x_m + a)\nu_m \left|\frac{\partial w}{\partial \nu}\right|^2 \mathrm{d}s$$
$$+ 2h \int_{\Gamma_a} \left(|\nabla w|^2 - 2\left|\frac{\partial w}{\partial \nu}\right|^2 - k^2 n^2 |w|^2 \right) \mathrm{d}s = -2\int_\Gamma (x_m + a)\,\mathrm{Re}\left(\frac{\partial w}{\partial x_m}\overline{g}\right) \mathrm{d}x - \mathrm{Re}\int_{\Omega_+} g\overline{w}\,\mathrm{d}x.$$

Due to [3, Lemma 2.2] and since $n^2\big|_{\Gamma_a} = n_+^2$ it holds

$$2h \int_{\Gamma_a} \left(-|\nabla w|^2 + 2\left|\frac{\partial w}{\partial \nu}\right|^2 + k^2 n_+^2 |w|^2 \right) \mathrm{d}s \leq 2k\,\mathrm{Im}\int_{\Gamma_a} \overline{w} T^+_{n_+^2}(w)\,\mathrm{d}s$$
$$= 2k\,\mathrm{Im}\int_{\Omega_+} g\overline{w}\,\mathrm{d}x \leq 2k\|g\|_{L^2(\Omega_+)}\|w\|_{L^2(\Omega_+)}.$$

The inequality after the line break follows from taking the imaginary part of

$$\int_{\Omega_+} \left(|\nabla w|^2 - k^2 n^2 |w|^2\right) \mathrm{d}x - \int_{\Gamma_a} \overline{w} T^+_{n_+^2}(w)\,\mathrm{d}s = -\int_\Omega g\overline{w}\,\mathrm{d}x.$$

Moreover, since $(x_m + a)\nu_m < 0$ and $-\,\mathrm{Re}\int_{\Gamma_a} \overline{v} T^+_{n_+^2}(v)\,\mathrm{d}s \geq 0$ we obtain that

$$\|\partial w/\partial \nu\|^2_{L^2(\Gamma)} \leq C\|g\|_{L^2(\Omega_+)}\|w\|_{H^1(\Omega_+)} \leq C\|g\|^2_{L^2(\Omega_+)}. \tag{50}$$

Let us denote $\Omega_R^+ := \Omega_+ \cap \Omega_R$. Twice applying Green's first identity yields

$$0 = \int_{\Omega_R^+} (\Delta u + k^2 n^2 u)\overline{w}\,\mathrm{d}x = \int_{\Omega_R^+} \left(\nabla u \cdot \nabla \overline{w} - k^2 n^2\right) \mathrm{d}s - \int_{\partial\Omega_R^+} \frac{\partial u}{\partial \nu}\overline{w}\,\mathrm{d}s$$
$$= -\int_{\Omega_R^+} u\left(\Delta\overline{w} - k^2 n^2 \overline{w}\right) \mathrm{d}s + \int_{\partial\Omega_R^+} u\frac{\partial \overline{w}}{\partial \nu}\,\mathrm{d}s - \int_{\partial\Omega_R^+} \frac{\partial u}{\partial \nu}\overline{w}\,\mathrm{d}s.$$

Since $u \in H^1(\Omega_+)$ and $w \in H^2(\Omega_+)$, both volume integrals converge, as $\mathbb{R} \to \infty$. Moreover,

$$\int_{\partial\Omega_R^+} u\frac{\partial \overline{w}}{\partial \nu}\,\mathrm{d}s = \int_{\Gamma_a \cap \partial\Omega_R^+} u\frac{\partial \overline{w}}{\partial \nu}\,\mathrm{d}s + \int_{\Gamma \cup \Omega_R} u\frac{\partial \overline{w}}{\partial \nu}\,\mathrm{d}s - \int_{\Omega \setminus \Omega_R^+} \left(\nabla u \cdot \nabla\overline{w} - k^2 n^2 u\overline{w}\right) \mathrm{d}s - \int_{\Omega \setminus \Omega_R^+} u\overline{g}\,\mathrm{d}s.$$

The two last terms tend to zero as $R \to \infty$ and the limit for the remaining two terms on the right exist. Hence

$$\int_{\partial\Omega_R^+} u\frac{\partial \overline{w}}{\partial\nu}\,\mathrm{d}s \to \int_\Gamma u\frac{\partial \overline{w}}{\partial\nu}\,\mathrm{d}s + \int_{\Gamma_a} u\frac{\partial \overline{w}}{\partial\nu}\,\mathrm{d}s.$$

The analogous convergence result holds for $\int_{\partial\Omega_R^+} \partial u/\partial\nu\,\overline{w}\,\mathrm{d}s$. Therefore we conclude that

$$\begin{aligned} 0 = -\int_{\Omega_+}(\Delta u + k^2n^2u)\overline{w}\,\mathrm{d}x + \int_\Gamma \frac{\partial u}{\partial\nu}\overline{w}\,\mathrm{d}s = \int_{\Omega_+}\left(\nabla u\cdot\nabla\overline{w} - k^2n^2u\overline{w}\right)\,\mathrm{d}s - \int_{\Gamma_a}\frac{\partial u}{\partial\nu}\overline{w}\,\mathrm{d}s \\ = -\int_{\Omega_+} u\left(\Delta\overline{w} - k^2n^2\overline{w}\right)\,\mathrm{d}s + \int_\Gamma u\frac{\partial\overline{w}}{\partial\nu}\,\mathrm{d}s - \int_{\Gamma_a}\frac{\partial u}{\partial\nu}\overline{w}\,\mathrm{d}s + \int_{\Gamma_a} u\frac{\partial\overline{w}}{\partial\nu}\,\mathrm{d}s \\ = -\int_{\Omega_+} u\overline{g}\,\mathrm{d}s + \int_\Gamma h\frac{\partial\overline{w}}{\partial\nu}\,\mathrm{d}s + 2\mathrm{i}\,\mathrm{Im}\int_{\Gamma_a} u\frac{\partial\overline{w}}{\partial\nu}\,\mathrm{d}s. \end{aligned}$$

We set $g = u$, take the real part of the latter equation and obtain with the help of (50)

$$\|u\|^2_{L^2(\Omega_+)} \le \|h\|_{L^2(\Gamma)}\|\partial w/\partial\nu\|_{L^2(\Gamma)} \le C\|h\|_{L^2(\Gamma)}\|u\|_{L^2(\Omega_+)}$$

which is precisely the claim of the lemma. □

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