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Acyclic edge-colouring of planar graphs

Nathann Cohen[*] , Frédéric Havet [*] , Tobias Müller [†]

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Abstract: A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph G, denoted $\chi'_a(G)$ is the minimum k such that G admits an *acyclic edge-colouring* with k colours. We conjecture that if G is planar and $\Delta(G)$ is large enough then $\chi'_a(G) = \Delta(G)$. We settle this conjecture for planar graphs with girth at least 5 and outerplanar graphs. We also show that $\chi'_a(G) \leq \Delta(G) + 25$ for all planar graph G, which improves a previous result by Muthu et al.

Key-words: edge-colouring, planar graphs, bounded density graph

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[†] School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel. `tobias@post.tau.ac.il`

Arête-coloration acyclique des graphes planaires.

Résumé : Une arête-coloration propre telle que tout cycle ait des arêtes de trois couleurs différentes est appelée *arête-coloration acyclique*. L'*indice chromatique acyclique* d'un graphe G, denoté $\chi'_a(G)$, est le plus petit entier k tel que G admette une arête-coloration acyclique avec k couleurs. Nous conjecturons que si G est planaire et $\Delta(G)$ suffisamment grand alors $\chi'_a(G) = \Delta(G)$. Nous montrons cette conjecture pour les graphes planaires de maille au moins 5 et les graphes planaires extérieurs. Nous prouvons également que $\chi'_a(G) \leq \Delta(G) + 25$ pour tout graphe planaire G, améliorant ainsi un résultat de Muthu et al.

Mots-clés : arête-coloration, graphe planaire, graphe de densité bornée

1 Introduction

A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph G, denoted $\chi'_a(G)$ is the minimum k such that G admits an *acyclic edge-colouring* with k colours.

Conjecture 1 (Alon, Sudakov and Zaks [2]). *For every graph G, $\chi'_a(G) \leq \Delta(G)+2$.*

This conjecture would be tight as there are cases where more than $\Delta+1$ colours are needed. However, the only graphs G for which we know that $\chi'_a(G) > \Delta(G)+1$ are the subgraphs of K_{2n} that have at least $2n^2-2n+2$ edges (because at most one colour class can contain n edges (a perfect matching), and all other colour classes can contain at most $n-1$ edges each). Therefore the following conjecture might even be true:

Conjecture 2 (Alon, Sudakov and Zaks [2]). *If G is a Δ-regular graph then $\chi'_a(G) = \Delta(G)+1$ unless $G = K_{2n}$.*

Alon, McDiarmid and Reed [1] showed an upper bound of $64\Delta(G)$ for $\chi'_a(G)$ which was later improved by to $16\Delta(G)$ by Molloy and Reed [6]. For graphs with large girth, better upper bounds are known. Muthu et al [7] showed that if G has girth at least 9 then $\chi'_a(G) \leq 6\Delta(G)$ and if it has girth at least 220 then $\chi'_a(G) \leq 4.52\Delta(G)$. Finally, Alon. Sudakov and Saks also showed that Conjecture 1 is true for graphs with girth at least $C\Delta\log(\Delta)$ for some fixed constant C.

Muthu et al [8] proved that $\chi'_a(G) \leq \Delta(G)+1$ for outerplanar graphs. The same authors [9] proved that $\chi'_a(G) \leq 2\Delta(G)+29$ if G is planar and $\chi'_a(G) \leq \Delta(G)+6$ if G is planar and triangle-free.

Sanders and Zhao [10] showed that planar graphs with maximum degree $\Delta \geq 7$ have chromatic index Δ. Vizing edge-colouring conjecture [11] asserts that planar graphs of maximum degree 6 are also 6-edge-colourable. This would be best possible as for any $\Delta \in \{2,3,4,5\}$, there are some planar graphs with maximum degree Δ with chromatic index $\Delta+1$ [11].

We propose a conjecture analogous to the above one of Vizing.

Conjecture 3. There exists Δ_0 such that every planar graph with maximum degree $\Delta \geq \Delta_0$ has an acyclic edge-colouring with Δ colours.

In this paper, we give some evidences to this conjecture. Firstly, in Section 2, we show that every planar graph has an acyclic edge-colouring with $\Delta+25$ colours thus improving the $2\Delta+29$ bound of Muthu et al [9]. In Section 3, we show that Conjecture 3 holds for planar graphs of girth at least 5 (with $\Delta_0 = 19$). and more generally for graphs with maximum average degree less than $4-\varepsilon$ for any $\varepsilon > 0$. Recall that the *girth* of a graph is the minimum length of a cycle it contains or $+\infty$ if it has no cycles. The *maximum average degree* of G is $Mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} \mid H \text{ is a subgraph of } G\}$. It is well known a planar graph of girth g has maximum average degree less than $2+\frac{4}{g-2}$. Finally, in Section 4, we show that Conjecture 3 holds for outerplanar graphs (with $\Delta_0 = 9$). Note that $\sup\{Mad(G) \mid G \text{ is outerplanar}\} = 4$.

2 Planar graphs

In this section we will prove the following result:

Theorem 4. $\chi'_a(G) \leq \Delta(G)+25$ *for all planar graphs G.*

The proof of Theorem 4 relies heavily on a structural result by Borodin et al. [4]. Before stating this structural result we need to introduce some notation and terminology. Let G be a graph.

Definition 5 (Good vertex). *A vertex $v \in V(G)$ is said to be* good *if:*

(i) $d(v) \leq 5$*, and*

(ii) there is a vertex $w \in N(v)$ *such that:*

(a) $d(u) \leq 25$ *for all* $u \in N(v) \setminus \{w\}$*, and*

(b) $\sum_{u \in N(v) \setminus \{w\}} d(u) \leq 38$.

Definition 6 (Bunch)**.** *A sequence of paths* $(P_1, \ldots, P_m)$ *in G is called a* bunch *with* poles p, q *and* length m *if:*

(i) Each P_i *is a pq-path of length 1 or 2, and;*

(ii) For each $1 \leq i \leq m-1$ *the cycle formed by* P_i *and* P_{i+1} *is not separating.*

See Figure 2 for an example of a bunch. A path $P_i = pq$ of length 1 in the bunch will be referred to as a *parental edge*. If a path P_i in the bunch has length 2, i.e. $P_i = pv_iq$, then the middle vertex v_i is called a *bunch vertex*. If $1 < i < m$ (that is P_i is neither the first nor the last path of the bunch) then its middle vertex v_i is an *internal bunch vertex*. We will say that two bunches are *internally disjoint* if they do not share any internal bunch vertices. (I.e. internally disjoint bunches may share poles and the middle vertices of their first and last paths, but no other bunch vertices).

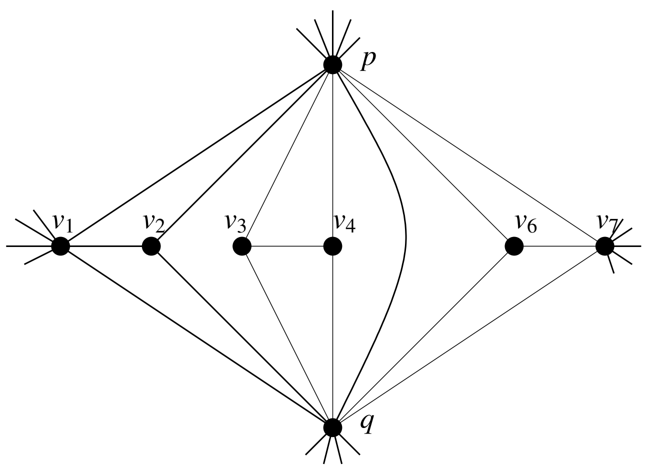


Figure 1: Example of a bunch of length 7 with a parental edge.

Theorem 7 (Borodin et al. [4])**.** *For every plane graph G at least one of the following holds:*

(i) G has a good vertex;

(ii) G has a vertex v of degree $d(v) \geq 26$ *that is a pole for a bunch of length at least* $d(v)/5$.

Here it should be remarked that Theorem 7 is a straightforward simplification of the corresponding theorem in [4] (which had more subcases). Theorem 7 has the following corollary, which is a key ingredient to our proof of Theorem 4.

Theorem 8. *For every plane graph at least one of the following holds:*

(i) G has a good vertex v;

(ii) G has a vertex v that is a pole for $1 \leq k \leq 6$ *internally disjoint bunches of length at least 6, and has at most* $24 - 4k$ *neighbours that do not belong to these bunches.*

Proof. Let G be an arbitrary plane graph. If there is a good vertex in G, then we are done, so let us assume that there is no such vertex. We shall iteratively define a sequence of graphs $G_0, G_1, G_2, \ldots, G_m$ such that in $G_0, \ldots, G_{m-1}$ has no good vertex and G_m has (at least) one. Set $G_0 = G$. Suppose that G_i has been defined. If there is a good vertex in G_i then we set $m = i$ and we stop. If not then by Theorem 7 we can find a vertex p of degree $d(p) \geq 26$ that is a pole for a bunch $(P_1, P_2, \ldots, P_k)$ of length $k \geq \lceil d(v)/5 \rceil \geq 6$. Since $k \geq 6$ and G has at most one parental edge, the paths P_3, P_4 exist and at least one of them is not a parental edge. W.l.o.g. let us assume P_3 is not a parental edge and let v be the middle vertex of P_3. We now set $G_{i+1} := G_i \setminus v$. That is, we delete v and all edges incident to it from G_i to obtain G_{i+1}. Note that by construction out of any bunch $(P_1, \ldots, P_k)$ in G at least 5 of the paths P_j must still exist in G_i. Therefore a pole p of bunch of length 6 in G never gets deleted (as the vertex that gets deleted from G_i to obtain G_{i+1} has degree at most 4 in G_i).

Eventually this process of deleting vertices will end at some iteration m, when G_m has a good vertex x. By construction, in G_{m-1} there is a vertex p that has degree at least 26 and is a pole for a bunch of length ≥ 6 that had the middle vertex of its third or fourth path deleted to obtain G_m. Let us denote the other pole of this bunch by q. The only vertices whose degrees in $G_m = G_{m-1} \setminus v$ are different from their degrees in G_{m-1} are the poles p, q and possibly up to two bunch vertices adjacent to v. Thus, x is either equal to one of p, q or adjacent to one of p, q. Therefore either p or q must have degree at most 25 in G_m. W.l.o.g. we can suppose it is p. As observed before, out of any bunch in G from which vertices have been deleted a bunch of length at least 5 remains in G_m. Hence p is a pole for at least 1 and at most $\lfloor 25/4 \rfloor = 6$ internally disjoint bunches of length at least 6 in G (note that the first and last bunch vertex may be shared by two different bunches of the pole p).

Moreover, if p was pole for only one bunch of length at least 6 in G, then it has at most $25 - 5 = 20$ neighbours not in this bunch. Observe that if p is a bunch for $2 \leq k \leq 6$ internally disjoint bunches, and each bunch shares its first and last bunch vertex with another bunch, then p cannot have any neighbour that is not part of these bunches (it is "surrounded" by them). On the other hand, if there is at least one bunch whose first bunch vertex is not the last vertex of any other bunch, then at least $5 + 4(k-1)$ vertices incident with p in G_m belong to the k internally disjoint bunches of length at least 6 in G. Hence, indeed at most $24 - 4k$ neighbours of p are not part of these bunches. □

of Theorem 4. Suppose the statement is false, and let G be a graph with maximum degree at most $\Delta := \Delta(G)$ that does not allow an acyclic edge-colouring with $\Delta + 25$ colours but for which all of its subgraphs do allow such an edge-colouring (that is, G is a minimal counterexample).

Claim 8.1. *If $(P_1, \ldots, P_m)$ is a bunch in G with $m \geq 6$ and poles p, q and if for some $2 \leq i \leq m-2$ neither of the paths P_i, P_{i+1} is a parental edge, then $v_i v_{i+1}$ is not an edge (writing $P_i = p v_i q, P_{i+1} = p v_{i+1} q$).*

of Claim 8.1. Suppose that $v_i v_{i+1}$ is an edge for some $2 \leq i \leq m-2$. Let us first suppose that $3 \leq i \leq m-3$. As G is a minimal counterexample, $G \setminus v_i v_{i+1}$ has an acyclic edge-colouring. There are at most $\Delta + 8 < \Delta + 25$ edges in $G \setminus v_i v_{i+1}$ that are incident with one of the vertices p, v_{i-1}, v_i, v_{i+1} or v_{i+2}. Observe that any cycle that through $v_i v_{i+1}$ contains one of the paths $v_i v_{i+1} p, p v_i v_{i+1}, v_{i-1} v_i v_{i+1}$ or $v_i v_{i+1} v_{i+2}$. Thus, if we colour $v_i v_{i+1}$ with a colour disctinct from those used on the edges incident with p, v_{i-1}, v_i, v_{i+1} or v_{i+2} then we get an acyclic colouring of G, a contradiction. Hence $v_i v_{i+1} \notin G$ for $i = 3, \ldots, m-3$.

Now let us consider the case when $i = 2$ (the case when $i = m-2$ is analogous). Again $G \setminus v_2 v_3$ has an acyclic edge-colouring by minimality of G. Any cycle through $v_2 v_3$ contains one of the paths $q v_2 v_3 p$, $p v_2 v_3 q$, $v_1 v_2 v_3 p$ or $v_1 v_2 v_3 q$ (recall $v_3 v_4$ is not an edge). First suppose that the colour of $v_1 v_2$ is distinct from both of the colours used on $p v_3, q v_3$. Then, colouring $v_2 v_3$ with any of the $\Delta + 25 - (d(p) + 3) \geq 1$ colours not used on either $v_1 v_2, q v_2, q v_3$ or edges incident with p, will produce an acyclic edge-colouring of G, a contradiction. Hence $v_1 v_2$ must have the same colour as either $p v_3$ or $q v_3$. By symmetry we can assume that it is $p v_3$. But in this case again colouring $v_2 v_3$ with any of the $\Delta + 25 - (d(p) + 3) \geq 1$ colours not used on either $v_1 v_2, q v_2, q v_3$ or edges incident with p, will produce an acyclic edge-colouring of G. This contradiction concludes the proof of Claim 8.1. □

Claim 8.2. *G does not have a good vertex.*

of Claim 8.2. Suppose there is a good vertex v. We shall assume that $d(v)=5$ (the case when $d(v)<5$ is similar). Let us write $N(v)=\{u_1,u_2,u_3,u_4,u_5\}$ where $d(u_1)\geq d(u_2)\geq d(u_3)\geq d(u_4)\geq d(u_5)$. By minimality, $G\setminus v$ has an acyclic edge-colouring with $\Delta+25$ colours. We shall extend this colouring of $G\setminus v$ to a colouring of G, arriving at a contradiction. Let C_i denote the set of colours present at u_i. Observe that $|C_i|=d(u_i)-1$.

First suppose that $d(u_2)\leq 13$. Because $|C_1|+|C_2|+|C_3|\leq\Delta-1+12+12<\Delta+25$, we can find a colour $c_2\notin C_1\cup C_2\cup C_3$ and colour vu_2 with this colour. Observe that this choice of c_2 assures that no cycle through u_1vu_2 or u_2vu_3 can be bicoloured. Similarly, as $|C_1|+|C_3|+|C_4|+1<\Delta+25$ we can find a colour $c_3\notin C_1\cup C_3\cup C_4\cup\{c_2\}$ and colour vu_3 with this colour. This choice of c_3 makes sure no cycle through u_1vu_3 or u_3vu_4 can be bicoloured. Since $d(u_5)\leq 9=\lfloor 38/4\rfloor$ we have $|C_2|+|C_4|+|C_5|+2<\Delta-1+12+8+2<\Delta+25$. Hence we can find $c_4\notin C_2\cup C_4\cup C_5\cup\{c_2,c_3\}$ and colour vu_4 with this colour. In this way no cycle through u_2vu_4 or u_4vu_5 can be bicoloured. Similarly, $|C_2|+|C_3|+|C_5|+3<\Delta+25$ and we can find a $c_5\notin C_2\cup C_3\cup C_5\cup\{c_2,c_3,c_4\}$ and colour vu_5 with this colour. This makes sure no cycle through u_2vu_5 or u_3vu_5 can be bicoloured. Since $d(u_4)+d(u_5)\leq 38/2=19$, we have $|C_1|+|C_4|+|C_5|+4<\Delta+25$ and we can pick $c_1\notin C_1\cup C_4\cup C_5\cup\{u_2,u_3,u_4,u_5\}$ and colour vu_1 with this colour. This choice of c_1 prevents cycles through u_1vu_4 or u_1vu_5 from being bicoloured. Since no cycle through v can be bicoloured, the c_i are distinct and $c_i\notin C_i$ for each i we have constructed an acyclic edge-colouring of G using at most $\Delta+25$ colours. But this contradicts that choice of G.

Hence, $d(u_2)\geq 14$. By definition of a good vertex we also have that $d(u_2)\leq 25$. Because $|C_1|+|C_2|\leq\Delta-1+24<\Delta+25$ we can find a $c_2\notin C_1\cup C_2$ and colour vu_2 with this colour. This ensures that no cycle through u_1vu_2 is bicoloured. As $\Delta+25\geq d(u_2)+25>38\geq|C_2|+|C_3|+|C_4|+|C_5|+4$, we can find distinct colours $c_3,c_4,c_5\notin\{c_2\}\cup C_2\cup C_3\cup C_4\cup C_5$, and colour the edges vu_3,vu_4,vu_5 with these colours. Observe that in this way no cycle through u_ivu_j can be bicoloured for all $2\leq i<j\leq 5$. Finally, since $d(u_3)+d(u_4)+d(u_5)\leq 38-14=24$, we have $|C_1|+|C_3|+|C_4|+|C_5|+4<\Delta+25$. Hence we can pick $c_1\notin C_1\cup C_3\cup C_4\cup C_5\cup\{c_2,c_3,c_4,c_5\}$ and colour vu_1 with this colour. This prevents all cylces through u_1vu_3,u_1vu_4 or u_1vu_5 from being bicoloured. Again no cycle through v is bicoloured, the c_i are distinct and $c_i\notin C_i$ for each i, so that we have constructed an acyclic edge-colouring of G with at most $\Delta+25$ colours. This again contradicticts the choice of G, and concludes the proof. □

Since there is no good vertex, by Theorem 8, there exists a vertex p that is a pole for $1\leq k\leq 6$ internally disjoint bunches of length at least 6 and has at most $24-4k$ neighbours not in those bunches. Let us denote those bunches by $B_1=(P^1_1,\dots,P^1_{m_1}),\dots,B_k=(P^k_1,\dots,P^k_{m_k})$ and let us denote the other pole of B_i by q_i. If it exists, we will denote the middle vertex of P^i_j by v^i_j. By claim 8.1 we have $d(v^i_j)=2$ for all $3\leq j\leq m_i-2$ and $1\leq i\leq k$.

One of P^1_3,P^1_4 is not a parental edge. W.l.o.g. let us assume it is P^1_3. By minimality of G, there exists an acyclic edge-colouring of $G\setminus v^1_3$ with $\Delta+25$ colours. We shall extend this colouring to an acyclic edge-colouring of G with $\Delta+25$ colours, arriving at a contradiction.

Let C_1 denote the set of colours used on edges incident with p and let C_2 denote the set of colours used on the edges $v^i_1v^i_2,v^i_{m_i-1}v^i_{m_i},i=1,\dots,k$. Then $|C_1|+|C_2|\leq\Delta-1+12<\Delta+25$. Hence we can find a colour $c'\notin C_1\cup C_2$ and colour pv^1_3 with c'.

For each $1\leq i\leq k$ there is at most one edge $v^i_jq_i$ that has colour c'. Let A_1 denote the set of colours used on the edges pv^i_j for which $v^i_jq_i$ has colour c', and observe that for each $1\leq i\leq k$ there is at most one edge $v^i_jq_i$ with colour c'. Let A_2 denote the set of colours used on the parental edges pq_i for $2\leq i\leq k$ (insofar as these parental edges exist). Let A_3 denote the set of colours used on the edges $pv^i_1,pv^i_{m_i}$ for $i=1,\dots,k$. Let A_4 denote the set of colours used on edges incident with q_1, and let A_5 denote the set of colours used on edges incident with p, that are not part of the k internally disjoint bunches of length at least 6. Then,

$$\begin{array}{rcl}|A_1|+|A_2|+|A_3|+|A_4|+|A_5|+1 & \leq & k+(k-1)+2k+\Delta-1+24-4k+1\\ & = & \Delta+23<\Delta+25.\end{array}$$

Hence, we can find $c''\notin A_1\cup A_2\cup A_3\cup A_4\cup A_5\cup\{c'\}$ and colour $v^1_3q_1$ with this colour. We now claim that this gives an acyclic edge-colouring of G. To see this, note that any cycle through v^1_3 contains one of the following paths:

(i) $q_1v_3^1pw$ with w a not in any of the k bunches, or;

(ii) $q_1v_3^1pv_1^i$ with $1 \leq i \leq k$, or;

(iii) $q_1v_3^1pv_{m_i}^i$ with $1 \leq i \leq k$, or;

(iv) $q_1v_3^1pv_2^iv_1^i$ with $1 \leq i \leq k$, or;

(v) $q_1v_3^1pv_{m_i-1}^iv_{m_i}^i$ with $1 \leq i \leq k$, or;

(vi) $q_1v_3^1pq_i$ for $2 \leq i \leq k$, or;

(vii) $q_1v_3^1pv_j^iq_i$ with $1 \leq i \leq k$ and $1 \leq j \leq m_i$.

Any cycle that satisfies (i) is not bicoloured because $c'' \notin A_5$. If it satisfies (ii) or (iii) then it is not bicoloured because $c'' \notin A_3$. If it satisfies (iv) or (v) then it is not bicoloured because $c' \notin C_1$. If a cycle satisfies (vi) then it is not bicoloured because $c'' \notin A_2$. If a cycle satisfies (vii) then it is not bicoloured because either the colour of $v_j^iq_i$ is not equal to c' or otherwise the colour of pv_j^i does not equal c'', since $c'' \notin A_1$. We have thus constructed an acyclic edge-colouring of G with at most $\Delta+25$ colours, which contradicts the choice of G and finishes the proof. □

3 Planar graphs of girth at least 5

The aim of this section is to prove Conjecture 3 for planar graphs of girth at least 5. Actually, we prove the conjecture for a more general class of graphs: the graphs of maximum average degree at most $10/3$.

The *average degree* of a graph G is $Ad(G) = \frac{1}{|V(G)|}\sum_{v\in V(G)} d(v) = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G is $Mad(G) = \max\{Ad(H) \mid H \text{ is a subgraph of } G\}$. It is well known that the girth and the maximum average degree of a planar graph are related to each other:

Proposition 9. *Let G be a planar graph of girth g.*

$$Mad(G) < 2 + \frac{4}{g-2}.$$

Theorem 10. *Let $\Delta \geq 19$ and G be a graph with maximum degree at most Δ and maximum average degree less than $\frac{10}{3}$. Then $\chi'_a(G) \leq \Delta$.*

Theorem 10 and Proposition 9 immediately yield the following.

Corollary 11. *Let $\Delta \geq 19$ and G be a planar graph with maximum degree at most Δ and girth at least 5. Then $\chi'_a(G) \leq \Delta$.*

More generally than Theorem 10, we show the following.

Theorem 12. *For any $\varepsilon > 0$, there exists an integer Δ_ε such that every graph G with maximum degree at most Δ with $\Delta \geq \Delta_\varepsilon$ and maximum average degree less than $4-\varepsilon$ is acyclically Δ-edge-colourable.*

In order to prove Theorems 10 and 12, we first establish some properties of Δ-*minimal graphs* which are graphs with maximum degree at most Δ, non acyclically Δ-edge-colourable but such that every proper subgraph is. Then, by the Discharging Method, we deduce that such a graph has maximum average degree at least $4-\varepsilon$ (resp. $10/3$) if Δ is at least Δ_ε (resp. 22). We will first prove, in Subsection 3.2, Theorem 12 for its discharging procedure is simpler because we only establish the existence of Δ_ε and make no attempt to minimize it. We then show in Subsection 3.3 Theorem 10.

3.1 Properties of Δ-minimal graphs

Proposition 13. *A Δ-minimal graph G is 2-connected. In particular, $\delta(G) \geq 2$.*

Proof. If G is not connected, it is the disjoint union of G_1 and G_2. Both G_1 and G_2 admits an acyclic Δ-edge-colouring by minimality of G. The union of these two edge-colourings is an acyclic Δ-edge-colouring of G.

Suppose now that G has a cutvertex v. Let C_i, for $1 \leq i \leq p$ be the components of $G - v$ and G_i the graph induced by $C_i \cup \{v\}$. By minimality of G, all the G_i admit an acyclic Δ-edge-colouring by minimality of G. Moreover, free to permute the colours we may assume that two edges incident to v get different colours. Hence the union of these edge-colourings is an an acyclic Δ-edge-colouring of G. because any cycle of G is entirely contained in one of the G_i. □

Proposition 14. *Let G be a Δ-minimal graph. For every vertex $v \in V(G)$, $\sum_{u \in N(v)} d(u) \geq \Delta + 1$.*

Proof. Suppose by way of contradiction that there is a vertex v such that $\sum_{u \in N(v)} d(u) \leq \Delta$. Let w be a neighbour of v. By minimality of G, $G \setminus vw$ admits an acyclic edge-colouring with Δ colours. Now colour vw with a colour distinct from the ones of the edges incident to a neighbour of v. This is possible as there are at most $\Delta - 1$ such edges distinct from vw. Doing so we clearly obtain a proper edge-colouring. Let us now show that there is no bicoloured cycle. A cycle that does not contain vw has edges of at least three colours as the edge-colouring of G was acyclic. Now a cycle containing vw must contains an edge vu and an edge tu with $u \in N(v) \setminus \{w\}$. By construction, the colours of tu, uv and vw are distinct. □

A *thread* is a path of length two whose internal vertex has degree 2.

Proposition 15. *Let $k \geq 2$ be an integer and G a Δ-minimal graph. In G, a Δ-vertex is the end of at most k threads whose other endvertex has degree at most k.*

To prove this proposition we need the following lemma.

Lemma 16. *Let $H = ((A,B),E)$ be a bipartite graph with $|A| = |B| = c$ such that for any vertex $a \in A$ $d(a) = 1$. If the number of vertices of B incident to at least one edge differs from 2 then there exists a perfect matching M such that the bipartite graph $((A,B), E \cup M)$ has girth at least 6.*

Proof. Let m be the number of vertices of B of degree at least one. Let $b_1, \dots, b_c$ be the vertices of B with $d(b_i) \geq 1$ if $i \leq m$ and $d(b_i) = 0$ otherwise. And let $a_1, \dots, a_c$ be the vertices of A with $a_i b_i \in E$ for all $1 \leq i \leq m$. If $m \geq 3$, let $M = \{a_i b_{i+1} \mid 1 \leq i < m\} \cup \{a_m b_1\} \cup \{a_i b_i \mid m < i \leq c\}$. Then the unique cycle in $((A,B), E \cup M)$ is $C = (a_1, b_2, a_2, b_3, \dots, a_{m-1}, b_m, a_1)$. It has length $2m \geq 6$. If $m = 1$, let $M = \{a_i b_{i+1} \mid 1 \leq i \leq c\} \cup \{a_c b_1\}$. Then $((A,B), E \cup M)$ has no cycle has the unique vertex of degree at least two in B is b_1. □

of Proposition 15. . Suppose for a contradiction that there is a Δ-vertex u with $c = k+1$ threads uv_iw_i, $1 \leq i \leq c$, such $d(w_i) \leq k$. Note that $c \geq 3$.

Set $A = \{v_1, \dots, v_c\}$. By Proposition 13, $w_i \notin A$ for all $1 \leq i \leq c$. By minimality of G, $G - A$ admits an acyclic Δ-edge-colouring.

Let us first extend it to the v_iw_i as follows. Let F be the set of colours assigned to the edges incident to u and no vertex of A and for $1 \leq i \leq c$ let F_i be the set of colours assigned to the edges incident to w_i (and distinct from v_iw_i). Then $|F| = \Delta - c$ and $|F_i| \leq k - 1$. For all $1 \leq i \leq c$, let S_i be the set of colours not in $F \cup F_i$. Since $|F| + |F_i| = \Delta - c + k - 1 = \Delta - 2$ then $|S_i| \geq 2$. If $|\bigcup_{i=1}^{c} S_i| \geq 3$, then one can assign to each v_iw_i a colour in S_i in such a way that at least 3 colours appear on such edges. If not, then $S_i = S_j$, and colour all the v_iw_i, $1 \leq i \leq c$ with the same colour.

We will now colour the edges uv_i for $1 \leq i \leq c$. Therefore let $H_1 = ((A,B),E_1)$ be the bipartite graph with B the set of c colours $\{b_1, \dots, b_c\}$ not in F and in which v_i is adjacent to b_j if $c(v_iw_i) = b_j$. As long as some

v_i has degree 0 then add an edge between a_i and an isolated b_j to obtain a bipartite graph $H_1 = ((A,B),E_2)$. By construction, at least three or exactly one colours appear on the v_iw_i. Thus H_2 fulfils the hypothesis of Lemma 16, so there exists a matching M such that $((A,B),E_2 \cup M)$ has girth at least 6. For $1 \le i \le c$, assign to each v_i the colour to which it is linked in M.

Let us now prove that this edge-colouring of G is acyclic. It is obvious that it is proper since v_i is not linked to $c(v_iw_i)$ in M. Let us now prove that it is acyclic. Let C be a cycle of G. If it contains no vertex of A then it contains edges of three different colours because the edge-colouring of $G-A$ is acyclic. Suppose now that C contains a unique vertex of A, say v_i. Then C contains w_iv_i, v_iu and ut with t a neighbour of u not in A. Then $c(ut) \in F$, so by construction, $c(w_iv_i) \neq c(ut)$. Hence the colours of w_iv_i, v_iu and ut are distinct. Suppose finally that C contains two vertices of A, say v_i and v_j. Then C contains w_iv_i, v_iu, w_jv_j and v_ju. Since $((A,B),E_2 \cup M)$ has girth at least 6, either $c(v_iu) \neq c(w_jv_j)$ or $c(v_ju) \neq c(w_iv_i)$. In both cases, C has edges of three different colours. □

Proposition 17. *Let k and l be two positive integers and G a Δ-minimal graph. In G, a $(\Delta - l)$-vertex is the end of at most $k-1-l$ threads whose other endvertex has degree at most k.*

To prove this proposition we need the following lemma.

Lemma 18. *Let $H = ((A,B),E)$ be a bipartite graph with $c = |A| < |B|$ such that for any vertex $a \in A$ $d(a) = 1$. Then there exists a perfect matching M such that the bipartite graph $((A,B),E \cup M$ has no cycle.*

Proof. Let $c' = |B|$. Let $b_1,\dots,b_{c'}$ be the vertices of B with $d(b_i) \ge 1$ if $i \le m$ and $d(b_i) = 0$ otherwise. And let $a_1,\dots,a_c$ be the vertices of A with $a_ib_i \in E$ for all $1 \le i \le m$. Let $M = \{a_ib_{i+1} \mid 1 \le i \le c\}$. This is well-defined since $c' > c$. Then $((A,B),E \cup M)$ has no cycle. □

of Proposition 17. . Suppose for a contradiction that there is a $(\Delta - l)$-vertex u with $c = k-l$ threads uv_iw_i, $1 \le i \le c$, such $d(w_i) \le k$.

Set $A = \{v_1,\dots,v_c\}$. By minimality of G, $G-A$ admits an acyclic Δ-edge-colouring. Let us first extend it to the v_iw_i as follows. Let F be the set of colours assigned to the edges incident to u and no vertex of A and for $1 \le i \le c$ let F_i be the set of colours assigned to the edges incident to w_i (and distinct from v_iw_i). Then $|F| = \Delta - l - c$ and $|F_i| \le k-1$.

For all $1 \le i \le c$ colour v_iw_i with a colour not in $F \cup F_i$ and distinct from the colours. This is possible since $|F| + |F_i| = \Delta - l - c + k - 1 = \Delta - 1$.

We will now colour the edges uv_i for $1 \le i \le c$. Therefore let $H_1 = ((A,B),E_1)$ be the bipartite graph with B the set of $c+j$ colours $\{b_1,\dots,b_{c+j}\}$ not in F and in which v_i is adjacent to b_j if $c(v_iw_i) = b_j$. As long as some v_i has degree 0 then add an edge between a_i and an isolated b_j to obtain a bipartite graph $H_1 = ((A,B),E_2)$. Then H_2 fulfils the hypothesis of Lemma 16 so there exists a matching M such that $((A,B),E_2 \cup M)$ has no cycle. For $1 \le i \le c$, assign to each v_i the colour to which it is linked in M.

In the same way as in the proof of Proposition 15, one shows that the obtained edge-colouring is acyclic. □

3.2 Proof of Theorem 12

Lemma 19. *Let $\varepsilon > 0$. There exists Δ_ε such that if $\Delta \ge \Delta_\varepsilon$ then any Δ-minimal graph has average degree at least $4 - \varepsilon$.*

Proof. The result for $\varepsilon = \frac{1}{2}$ implies the result for larger values of ε. Hence we assume that $\varepsilon \le \frac{1}{2}$. Let us assign an initial charge of $d(v)$ to each vertex $v \in V(G)$ Set $d_\varepsilon = \lceil \frac{8}{\varepsilon} - 2 \rceil$.

We perform the following discharging rules.

R1: for $4 \le d < d_\varepsilon$, every d-vertex sends $a(d) = 1 - \frac{4-\varepsilon}{d}$ to each neighbour.

R2: for $d_\varepsilon \le d \le \Delta+1-d_\varepsilon$ then every d-vertex sends $1-\frac{\varepsilon}{2}$ to each neighbour.

R3: for $\Delta+2-d_\varepsilon \le d \le \Delta$ then every d-vertex sends

- $1-\varepsilon$ to each 3-neighbour;
- $2-\varepsilon$ to each 2-neighbour whose second neighbour has degree 2 or 3;
- $b(d) = 2-\varepsilon-a(d)$ to each 2-neighbour whose second neighbour has degree d with $4 \le d < d_\varepsilon$;
- $1-\frac{\varepsilon}{2}$ to each 2-neighbour whose second neighbour has degree $d \ge d_\varepsilon$.

Let us now check that every vertex v has final charge $f(v)$ at least $4-\varepsilon$.

If v is a 2-vertex then let u and w be its two neighbours with $d(u) \le d(w)$. If $d(u) \le 3$ then $d(w) \ge \Delta-2$ by Proposition 14. Hence v receives $2-\varepsilon$ from w by R3, so $f(v) \ge 2+2-\varepsilon = 4-\varepsilon$. If $4 \le d(u) < d_\varepsilon$ then $d(w) > \Delta+1-d_\varepsilon$ by Proposition 14. Hence v receives $a(d)$ from u by R2 and $b(d)$ from w by R3. So $f(v) = 4-\varepsilon$. If $d(u) \ge 10$ then v receives $1-\frac{\varepsilon}{2}$ from u and $1-\frac{\varepsilon}{2}$ from w by R3. So $f(v) = 4-\varepsilon$.

Suppose that v is a 3-vertex. Then by Proposition 14 it has at least two (≥ 8)-neighbours. Hence it receives at least $2 \times 1/2$ by R1, R2 or R3 because $\varepsilon \le \frac{1}{2}$. So $f(v) \ge 4$.

Suppose $4 \le d(v) < d_\varepsilon$. Then v sends $d(v)$ times $1-\frac{4-\varepsilon}{d(v)}$ so $f(v) \ge 4-\varepsilon$.

Suppose $d_\varepsilon \le d(v) \le \Delta+1-d_\varepsilon$. Then v sends at most $d(v)$ times $1-\frac{\varepsilon}{2}$ so $f(v) \ge d(v) \times \frac{\varepsilon}{2} \ge 4-\varepsilon$.

Suppose now that $d(v) \ge \Delta+2-d_\varepsilon$. Then by Propositions 15 and 17, the most v can send is when it has three 2-neighbours with second neighbour of degree at most 3, one 2-neighbour with second neighbour of degree d for all $4 \le d \le d_\varepsilon - 1$ and $\Delta - d_\varepsilon + 1$ 2-neighbours with second neighbour of degree at least d_ε. Hence

$$\begin{aligned} f(v) &\ge \Delta+2-d_\varepsilon-3(2-\varepsilon) - \sum_{d=4}^{d_\varepsilon-1} b(d) - (\Delta-d_\varepsilon+1)(1-\frac{\varepsilon}{2}) \\ &\ge \Delta\frac{\varepsilon}{2} - S_\varepsilon \end{aligned}$$

with $S_\varepsilon = d_\varepsilon - 2 + 3(2-\varepsilon) + \sum_{d=4}^{d_\varepsilon-1} b(d) - (1-\frac{\varepsilon}{2})(d_\varepsilon-1)$. Setting $\Delta_\varepsilon = \lceil \frac{2}{\varepsilon}(S_\varepsilon+4-\varepsilon) \rceil$, if $\Delta \ge \Delta_\varepsilon$, $f(v) \ge 4-\varepsilon$. □

of Theorem 12. If Theorem 12 were false, then a minimum counterexample G would be a Δ-minimum graph. So by Lemma 19, its average degree would be at least $4-\varepsilon$, a contradiction. □

3.3 Proof of Theorem 10

Lemma 20. *Let $\Delta \ge 19$ and G be a Δ-minimal graph. Then $Mad(G) \ge Ad(G) \ge 10/3$.*

Proof. Let us assign an initial charge of $d(v)$ to each vertex $v \in V(G)$ and perform the following discharging rules.

R1: every 4-vertex sends 4/9 to each of its (≤ 3)-neighbours;

R2: every 5-vertex sends 7/12 to each 2-neighbour and $1/3$ to each 3-neighbour;

R3: for $6 \le d \le 9$, every d-vertex sends $1-10/3d$ to each neighbour.

R4: for $10 \le d \le \Delta-9$ then every d-vertex sends $2/3$ to each neighbour.

R5: for $\Delta-8 \le d \le \Delta$ then every d-vertex sends

- 2/3 to each d-neighbour with $3 \leq d \leq 5$;
- 4/3 to each 2-neighbour whose second neighbour has degree 2 or 3;
- 8/9 to each 2-neighbour whose second neighbour has degree 4;
- 9/12 to each 2-neighbour whose second neighbour has degree 5;
- $1/3 + 10/3d$ to each 2-neighbour whose second neighbour has degree d with $6 \leq d \leq 9$;
- 2/3 to each 2-neighbour whose second neighbour has degree $d \geq 10$.

Let us now check that every vertex v has final charge $f(v)$ at least $\frac{10}{3}$.

If v is a 2-vertex then let u and w be its two neighbours with $d(u) \leq d(w)$. If $d(u) \leq 3$ then $d(w) \geq \Delta - 2$ by Proposition 14. Hence v receives 4/3 from w by R5, so $f(v) \geq 2 + 4/3 = 10/3$. If $d(u) = 4$ then $d(w) \geq \Delta - 3$ by Proposition 14. Hence v receives 4/9 from u by R1 and 8/9 from w by R5. So $f(v) = 10/3$. If $d(u) = 5$ then $d(w) \geq \Delta - 4$ by Proposition 14. Hence v receives 7/12 from u by R2 and 9/12 from w by R5. So $f(v) = 10/3$. If $6 \leq d(u) \leq 9$ then $d(w) \geq \Delta - 8$ by Proposition 14. Hence v receives $1 - 10/3d$ from u by R3 and $1/3 + 10/3d$ from w by R5. So $f(v) = 10/3$. If $d(u) \geq 10$ then v receives $2/3$ from u by R4 and $2/3$ from w by R5. So $f(v) = 10/3$.

Suppose that v is a 3-vertex. Then, since $\Delta \geq 10$, by Proposition 14 it has either a (≥ 5)-neighbour or two 4-neighbours. Hence it receives either at least 1/3 by R2, R3, R4 or R5, or $2 \times 4/9 \geq 1/3$ by R1. In both cases, $f(v) \geq 3 + 1/3 = 10/3$.

Suppose that v is a 4-vertex. Then, since $\Delta \geq 18$, by Proposition 14, it has either three (≤ 3)-neighbours and one (≥ 10)-neighbour or at most two (≤ 3)-neighbours. In the first case, it sends 4/9 to each of its 3-neighbours and receives 2/3 form its (≥ 10)-neighbour. So $w(v) \geq 4 - 3 \times \frac{4}{9} + \frac{2}{3} = 10/3$. In the second case, it sends 4/9 to at most 2 neighbours. So $w(v) \geq 4 - 2 \times \frac{4}{9} > 10/3$.

Suppose that v is a 5-vertex.
Assume first that v has at most three (≤ 3)-neighbours. If it has at least one (3)-neighbour it sends at most $3/2$ so $f(v) \geq 5 - 3/2 > 10/3$. If not it has three 2-neighbours. Let u_1 and u_2 be the two (≥ 4)-neighbours of v. By Proposition 14, $d(u_1) + d(u_2) \geq 11$ since $\Delta \geq 16$. Hence one of these two vertices is a (≥ 6)-vertex and it sends at least 4/9 to u. Hence $f(v) \geq 5 + 4/9 - 7/4 > 10/3$.
Assume now that v has at least four (≤ 3)-neighbours. Let i be the number of 2-neighbours of v. Then by Proposition 14, v has exactly $4 - i$ 3-neighbour and its fifth neighbour has degree at least $6 + i$ since $\Delta \geq 17$. Hence $f(v) \geq 5 - i.\frac{7}{12} - (4 - i)\frac{1}{3} + 1 - \frac{10}{3(6+i)} > 10/3$.

Suppose $6 \leq d(v) \leq 9$. Then v sends $d(v)$ times $1 - 10/3d(v)$ so $f(v) \geq d(v) - d(v)(1 - 10/3d) = 10/3$.

Suppose $10 \leq d(v) \leq \Delta - 10$. Then v sends at most $d(v)$ times $2/3$ so $f(v) \geq d(v)(1 - 2/3) \geq 10/3$.

Suppose that $d(v) = \Delta - l$ for $1 \leq l \leq 7$. By Proposition 17, v is incident to at most $\Delta - l - 1$ threads so its has at least one (≥ 3)-neighbour to which it sends at most $2/3$. Moreover the most it can send is when it has exactly one 2-neighbour with second neighbour of degree d for each $l + 2 \leq d \leq 9$ and $\Delta - 9$ 2-neighbours with second neighbour of degree at least 10. Hence its final charge is

$$\begin{aligned} f(v) &\geq \Delta - l - \left((\Delta - 8)\frac{2}{3} + \sum_{d=l+2}^{9} s(d) \right) \\ &\geq \frac{1}{3}\Delta + \frac{16}{3} - \left(l + \sum_{d=l+2}^{9} s(d) \right) \end{aligned}$$

with $s(3) = 4/3$, $s(4) = 8/9$, $s(5) = 9/12$ and $s(d) = 1/3 + 10/3d$ for $6 \leq d \leq 9$. Since $s(3) > 1$ and $s(d) < 1$ when $d \geq 4$, then $l + \sum_{d=l+2}^{9} s(d)$ is minimum when $l = 2$. Hence

$$\begin{aligned} f(v) &\geq \frac{1}{3}\Delta + \frac{16}{3} - \left(2 + \sum_{d=4}^{9} s(d)\right) \\ &\geq \frac{1}{3}\Delta + \frac{61}{36} - \frac{10}{3}\sum_{d=6}^{9}\frac{1}{d} \\ &\geq \frac{1}{3}\Delta + \frac{61}{36} - \frac{10}{3} \times \frac{275}{504} \geq \frac{10}{3} \end{aligned}$$

because $\Delta \geq 11$.

Suppose $d(v) = \Delta$. By Proposition 15, the most it can send is when it has three 2-neighbours with second neighbour of degree at most 3, exactly one 2-neighbour with second neighbour of degree d for $4 \leq d \leq 9$ and $\Delta - 9$ 2-neighbours with second neighbour of degree at least 10. In this case it sends

$$\begin{aligned} 3 \times \frac{4}{3} + \frac{8}{9} + \frac{9}{12} + \sum_{d=6}^{9}\left(\frac{1}{3} + \frac{10}{3d}\right) + (\Delta - 9)\frac{2}{3} &= \frac{2}{3}\Delta + \frac{35}{36} + \frac{10}{3}\sum_{d=6}^{9}\frac{1}{d} \\ &= \frac{2}{3}\Delta + \frac{35}{36} + \frac{10}{3} \times \frac{275}{504} \\ &\leq \Delta - \frac{10}{3} \end{aligned}$$

because $\Delta \geq 19$. Hence $f(v) \geq \frac{10}{3}$.

Now $Ad(G) = \frac{1}{|V|}\sum_{v \in V(G)} d(v) = \frac{1}{|V|}\sum_{v \in V(G)} f(v) \geq \frac{10}{3}$. □

of Theorem 10. If Theorem 10 would be false, a minimum counterexample G would be a Δ-minimum graph. So by Lemma 20, its average degree is at least $4 - \varepsilon$, a contradiction. □

4 Outerplanar graphs

The aim of this section is to prove the following theorem:

Theorem 21. *Every outerplanar G with maximum degree at most $\Delta \geq 9$ has an acyclic Δ-edge-colouring.*

In order to prove it, we need some structural lemmas on 2-connected outerplanar graphs. The first one is a restriction to 2-connected outerplanar graphs of a result of Expired and Ochem [5].

Lemma 22 (Esperet and Ochem [5]). *Let G be a 2-connected outerplanar graph. Then G contains one of the configurations C_i, $1 \leq i \leq 3$, depicted Figure 2.*

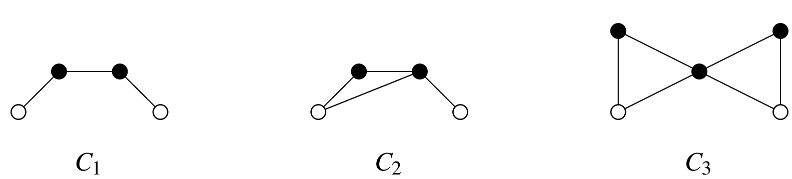


Figure 2: Unavoidable configurations in a 2-connected outerplanar graph.

Drawing and naming conventions: In all the depicted configurations the black vertices cannot be connected to non-drawn vertices in the whole graph, while the white can. The square vertices have degree at least $\Delta-2$. We call A (resp. B) a configuration C_2 in which the two white vertices have degree at least $\Delta-2$ in the whole graph. We will name some configurations by words on $\{A,\bar{A},B\}$ as they can be seen as concatenations (the rightmost white vertex of the first graph is identified with the leftmost white vertex of the second, and so on) of graphs A, $\bar{A}$ and B, where $\bar{A}$ denotes the mirror image of A. A configuration C_2 (resp. C_3) which is not a A (resp. B) is a configuration C_2' (resp. C_3').

Lemma 23. *Let G be a 2-connected outerplanar graph with maximum degree at most $\Delta \geq 9$. Then G contains a configuration C_1, or one of the configurations C_2', C_3', AA, $\bar{A}A$, $A\bar{A}$, AB, $\bar{A}B$ or BB depicted in Figure 3.*

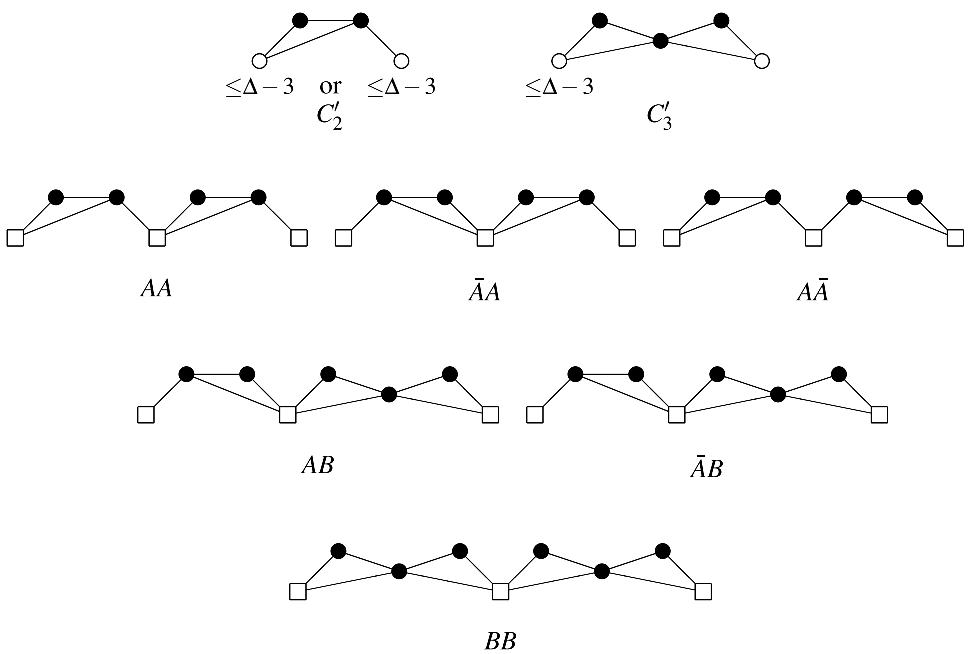
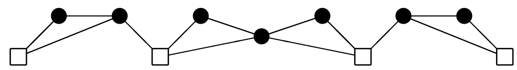


Figure 3: Unavoidable configurations of a 2-connected outerplanar graph. In C_2' and C_3', one of the two white vertices has degree at most $\Delta-3$.

Proof. By induction on the number of vertices, the result holding trivially when G has three vertices.

Suppose now that G has more than three vertices. According to Lemma 22, G contains one of the configurations C_i, $1 \leq i \leq 3$. Hence if G does not contain any configuration C_1, C_2' and C_3', it must contain a configuration H_1 isomorphic to C_2 or C_3 such that the two white vertices have degree at least $\Delta-2$. In other words, H_1 is a A or a B. Let G' be the graph obtained by removing the black vertices of H_1 and adding an edge e_1 between its two white vertices. Then G' is 2-connected. So, by the induction hypothesis, G' contains one subgraph H_2 among the configurations (C_i), $1 \leq i \leq 3$. The white vertices of H_1 have degree at least $\Delta-2$ in G, they have degree at least $\Delta-4 \geq 5$ in G'. So they may not be the black vertices of H_2. In particular, the edge e_1 is not in $E(H_2)$. Moreover, if H_2 is a configuration C_1, AA, $a\bar{A}$, $\bar{A}A$, AB, $\bar{A}B$ or BB in G' it is the same configuration in G. Assume now it is a configuration C_2' or a C_3' in G'. If one of its white vertices has degree at most $\Delta-3$ in G, it is a configuration C_2' or C_3' in G. If not, then the white vertex H_2 with degree at most $\Delta-3$ in G' has degree at least $\Delta-2$ in G. Hence it must be one of the white vertices of H_1. Then the union of H_1 and H_2 is one of the configurations AA, $A\bar{A}$, $\bar{A}A$, AB, $\bar{A}B$ or BB. $\square$

Lemma 24. *Let G be a* 2*-connected outerplanar graph with maximum degree at most* $\Delta \geq 9$*. Then G contains one of the following configurations :* C_1*,* C_2'*,* C_3'*, AA,* $\bar{A}A$*,* $\bar{A}B$*, BB, or the configuration* $AB\bar{A}$ *(see Figure 24).*



$AB\bar{A}$

Proof. By induction on the number of vertices, the result holding trivially when G has three vertices.

Suppose now that G has more than three vertices. By Lemma 23, it contains one of the configurations C_1, C_2', C_3', AA, $\bar{A}A$, $\bar{A}B$, $A\bar{A}$, AB or BB. If it a configuration C_1, C_2', C_3', AA, $\bar{A}A$, $\bar{A}B$ or BB, we have the result. If it is a configuration H_1 in $\{A\bar{A}, AB\}$. Let G' be the graph obtained by removing the black vertices of H_1 and adding an edge between its leftmost white vertex and its middle white vertex and one edge between its middle white vertex and its rightmost one vertex. Then G' is 2-connected. By the induction hypothesis, G' contains one subgraph H_2 among the configurations of Lemma 24. As the white vertices of H_1 have degree at least $\Delta - 2$ in G, they have degree at least $\Delta - 4 \geq 5$ in G'. So they may not be the black vertices of H_2. Hence, if H_2 is a configuration of Lemma 24 in G' but is not a configuration in G, then it is a subgraph isomorphic to A or B, one of whose white vertex is of degree at most $\Delta - 3$ is G' and at least $\Delta - 2$ in G. Hence this vertex must be one of the white vertices of H_1. Then the union of H_1 and H_2 contains one of the following configurations:

- $A\bar{A}\bar{A}$, and so contains (the mirror image of) AA,
- $A\bar{A}A$, and so contains $\bar{A}A$,
- $A\bar{A}B$, and so contains $\bar{A}B$,
- ABA, and so contains(the mirror image of) $\bar{A}B$,
- ABB, and so contains BB,
- BAB, and so contains (the mirror image of) $\bar{A}B$,
- AAB, and so contains AA,
- $\bar{A}AB$, and so contains $\bar{A}A$, or
- $AB\bar{A}$.

□

Lemma 25. *Let G be a 2-connected outerplanar graph of maximum degree at most* $\Delta \geq 9$*. Then G contains one of the following configurations :* C_1*,* C_2'*,* C_3'*, AA,* $\bar{A}A$*,* $\bar{A}B$ *or BB.*

Proof. The proof is very similar to the one of Theorem 24 in which Lemma 24 plays the role of Lemma 23. At the end the union of H_1 and H_2 must contain one of the following configurations:

- $AAB\bar{A}$, and so contains AA,
- $BAB\bar{A}$, and so contains (the mirror image of) $\bar{A}B$, or
- $\bar{A}AB\bar{A}$, and so contains $\bar{A}A$.

□

of Theorem 21. By induction on the number of edges of G, the result holding trivially if G has one edge.

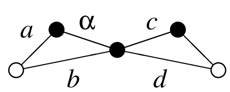
If G is not connected then, by the induction hypothesis, all its connected components admit an acyclic Δ-edge-colouring. The union of these colourings is an acyclic Δ-edge-colouring of G.

If G has a cutvertex x. Let H_i, $1 \leq i \leq p$ be the connected components of $G-x$ and for $1 \leq i \leq p$, let G_i be the graph induced by $H_i \cup \{x\}$. By the induction hypothesis, each G_i admits an acyclic Δ-edge-colouring c_i. Free to permute the colours, we may assume that the edges incidents to x get different colours. Thus the union of the c_i is an acyclic Δ-edge-colouring c_i because every cycle in G is a cycle of one of the G_i.

Suppose now that G is 2-connected. By Lemma 25, it must contain a configuration H isomorphic to C_1, C'_2, C'_3, AA, $\bar{A}A$, $\bar{A}B$, or BB.

If H is a configuration C_1, let G' be the graph obtained from G by contracting the edge e between the two black vertices of H. By the induction hypothesis, G' admits an acyclic Δ-edge-colouring which is an acyclic Δ-edge-colouring of $G \setminus e$ such that the two edges adjacent to e are assigned different colours. Hence colouring e with a colour distinct from those two, we obtained a Δ-edge-colouring of G.

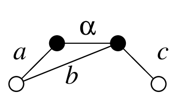
If H is a configuration C'_3. Without loss of generality, we may assume that the left white vertex v_l has degree at most $\Delta-3$. By the induction hypothesis, $G \setminus \alpha$ admits an acyclic Δ-edge-colouring. Assigning α a colour distinct from the ones of c, d and all the edges incident to v_l, we obtain an acyclic Δ-edge-colouring of G.



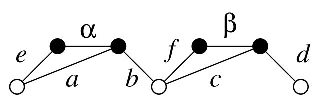
If H is one of the configurations C'_2, AA, $\bar{A}A$), $\bar{A}B$, or BB, then we consider an acyclic Δ-edge-colouring of the graph G' obtained from G by deleting the edges between two black vertices of H. We then extend this colouring into an acyclic Δ-edge-colouring of G by colouring these edges (denoted by Greek letters) possibly after some recolouring of the already coloured edges of H (denoted by roman letters). We now detail how to do that for each configuration. In the first cases, we explain why the obtained edge-colouring is acyclic. (It is always obvious it is proper). We do not do so in the other ones as the arguments are similar to the one of the first cases. We denote by c_x the (original) colour of an edge x of G'.

Configuration C'_2:

- If the left white vertex v_l is of degree at most $\Delta-3$, we give α a colour different from c_c and the colours of the edges incident to v_l.
- If the right white vertex v_r is of degree at most $\Delta-2$, we give α a colour different from c_a, c_b and the colours of the edges incident to v_r.



Configuration AA :



1. $c_e \neq c_b$ and $c_f \neq c_d$. We colour α with $c_\alpha \notin \{c_e, c_a, c_b\}$ and β with $c_\beta \notin \{c_f, c_c, c_d\}$. There is no bicoloured cycle through the edges α and β because $c_e \neq c_b$ and $c_f \neq c_d$. So the colouring is acyclic.

2. $c_e = c_b$ and $c_f \neq c_d$:

   2.1. $c_a \neq c_f$. We colour α with $c_\alpha = c_f$ and β with $c_\beta \notin \{c_f, c_c, c_d, c_b\}$. There is no bicoloured cycle through β as $c_f \neq c_d$. And there there is no bicoloured cycle through α because such a cycle must have the two colours c_f and c_b and hence go through f which has only one neighbour coloured with c_b. So the colouring is acyclic.

   2.2. $c_a = c_f$

   2.2.1 $c_d \neq c_b$. We colour α with $c_\alpha = c_c$ and β with $c_\beta \notin \{c_a, c_c, c_d\}$. There is no bicoloured cycle through β because $c_f \neq c_d$. And there there is no bicoloured cycle through α because such a cycle must have the two colours c_c and c_b and hence go through c which has only one neighbour coloured with c_b as $c_d \neq c_b$. So the colouring is acyclic.

   2.2.2 $c_d = c_b$. We recolour a and f with $c_b = c_e$ and b and e with $c_a = c_f$. It is simple matter to check that this does not create any bicoloured cycles as such cycles must contain at least one of the edges α or β, which have no colour yet. We then colour α with c_c and β with c_a. There is no bicoloured cycle through α because such a cycle must have the two colours c_c and c_a and hence go through c which has only one neighbour coloured with c_a. There is no bicoloured cycle through β because such a cycle must have the two colours c_a and c_b and hence go through e which has only one neighbour coloured with c_b.

3. $c_e = c_b$ and $c_f = c_d$.

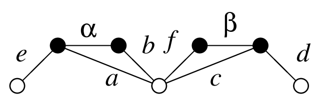
   3.1 $c_a \notin \{c_c, c_d\}$. We colour α with c_c and β with c_b.

   3.2 $c_a = c_d$. we recolour a with c_b, b with c_c, c with c_b and e with c_a. We then colour β with c_c and α with $c_\alpha \notin \{c_a, c_b, c_c\}$.

   3.3 $c_a = c_c$. We recolour b with $c_f = c_d$, c with c_b and f with $c_c = c_a$. We then colour α with $c_\alpha \notin \{c_a, c_b, c_d\}$ and β with $c_\beta \notin \{c_a, c_b, c_d\}$.

4. $c_e \neq c_b$ and $c_f = c_d$. We colour α with $c_\alpha \notin \{c_e, c_b, c_a\}$ and β with c_b.

Configuration $\bar{A}A$:

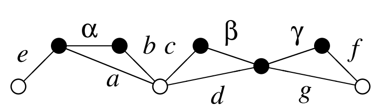


1. $c_e \neq c_b$ and $c_f \neq c_d$. We colour α with $c_\alpha \notin \{c_e, c_a, c_b\}$ and β with $c_\beta \notin \{c_f, c_c, c_d\}$.

2. $c_e = c_b$ and $c_f \neq c_d$. We colour α with c_f and β with $c_\beta \notin \{f, c, d\}$. By symmetry we have the result if $c_e = c_b$ and $c_f \neq c_d$.

3. $c_e = c_b$ and $c_f = c_d$. We colour α with c_d and β with c_a.

Configuration $\bar{A}B$:

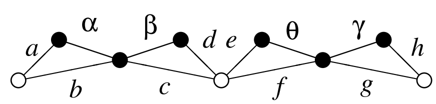


1. $c_e \neq c_b$, $c_c \neq c_g$ and $c_f \neq c_d$. We colour α with $c_\alpha \notin \{c_e, c_a, c_b\}$, β with $c_\beta \notin \{c_c, c_d, c_g\}$ and γ with $c_\gamma \notin \{c_f, c_g c_\beta, c_d\}$.
2. $c_e \neq c_b$, $c_c \neq c_g$ and $c_f = c_d$. We colour α with $c_\alpha \notin \{c_e, c_a, c_b\}$, γ with c_c and β with $c_\beta \notin \{c_c, c_d, c_g\}$.
3. $c_e \neq c_b$, $c_c = c_g$ and $c_f \neq c_d$. We colour α with $c_\alpha \notin \{c_e, c_a, c_b\}$, β with c_f, and γ with $c_\gamma \notin \{c_d, c_f, c_c\}$.
4. $c_e = c_b$, $c_c \neq c_g$ and $c_f \neq c_d$. We colour α with c_c, β $c_\beta \notin \{c_c, c_d, c_g\}$ and γ with $c_\gamma \notin \{c_d, c_f, c_g, c_\beta\}$.
5. $c_e = c_b$, $c_c = c_g$ and $c_f \neq c_d$. We colour α with c_d, β with c_b, and γ with $c_\gamma \notin \{c_d, c_f, c_g, c_b\}$.
6. $c_e = c_b$, $c_c \neq c_g$ and $c_f = c_d$. We colour α with c_c, γ with c_b, and β with $c_\beta \notin \{c_c, c_d, c_g, c_b\}$.
7. $c_e \neq c_b$, $c_c = c_g$ and $c_f = c_d$.

   7.1 $c_e \neq c_d$. We colour α with c_d, γ with c_a, and β with c_b.

   7.2. $c_e \neq c_c$. We colour α with c_c, γ with c_b, and β with c_a.
8. $c_e = c_b$, $c_c = c_g$ and $c_f = c_d$. We colour α with c_c, γ with c_b, and β with c_a.

Configuration *BB* :



1. $c_a \neq c_c$, $c_b \neq c_d$, $c_e \neq c_g$, and $c_f \neq c_h$. We colour α with $c_\alpha \notin \{c_a, c_b, c_c, c_d\}$, β with $c_\beta \notin \{c_\alpha, c_b, c_c, c_d\}$, θ with $c_\theta \notin \{c_e, c_f, c_g, c_h\}$, and γ with $c_\gamma \notin \{c_\theta, c_f, c_g, c_h\}$.
2. $c_a = c_c$, $c_b \neq c_d$, $c_e \neq c_g$, and $c_f \neq c_h$. We colour α with $c_\alpha \in \{c_d, c_e\} \backslash \{c_b\}$, β with $c_\beta \notin \{c_\alpha, c_b, c_c, c_d\}$, θ with $c_\theta \notin \{c_e, c_f, c_g, c_h\}$, and γ with $c_\gamma \notin \{c_\theta, c_f, c_g, c_h\}$.
3. $c_a \neq c_c$, $c_b \neq c_d$, $c_e \neq c_g$, and $c_f = c_h$. We have the result by symmetry with the previous case.
4. $c_a \neq c_c$, $c_b = c_d$, $c_e \neq c_g$, and $c_f \neq c_h$. We colour α with $c_\alpha \in \{c_e, c_f\} \backslash \{c_f\}$, β with c_a, θ with $c_\theta \notin \{c_e, c_f, c_g, c_h\}$, and γ with $c_\gamma \notin \{c_\theta, c_f, c_g, c_h\}$
5. $c_a \neq c_c$, $c_b \neq c_d$, $c_e = c_g$, and $c_f \neq c_h$. We have the result by symmetry with the previous case.
6. $c_a = c_c$, $c_b \neq c_d$, $c_e = c_g$, and $c_f \neq c_h$. We colour α with $c_\alpha \in \{c_d, c_f\} \backslash \{c_b\}$, β with $c_\beta \notin \{c_b, c_c, c_d, c_\alpha\}$, θ with c_a, and γ with $c_\gamma \notin \{c_a, c_f, c_g, c_h\}$.
7. $c_a \neq c_c$, $c_b = c_d$, $c_e \neq c_g$, and $c_f = c_h$. We have the result by symmetry with the previous case.
8. $c_a = c_c$, $c_b \neq c_d$, $c_e \neq c_g$, and $c_f = c_h$. We colour α with $c_\alpha \in \{c_d, c_e\} \backslash \{c_b\}$, β with $c_\beta \notin \{c_b, c_a, c_d, c_e, c_f\}$, γ with $c_\gamma \in \{c_d, c_e\} \backslash \{c_f\}$, and θ with $c_\theta \notin \{c_a, c_g, c_f, c_e, c_d\}$.
9. $c_a \neq c_c$, $c_b = c_d$, $c_e = c_g$, and $c_f \neq c_h$. We colour β with c_f, θ with c_c, γ with $c_\gamma \notin \{c_h, c_e, c_f, c_c\}$, and α with $c_\alpha \notin \{c_a, c_b, c_c, c_f\}$.
10. $c_a = c_c$, $c_b = c_d$, $c_e \neq c_g$, and $c_f \neq c_h$.

    10.1 $c_a \neq c_g$. We colour α with c_f, β with c_e, θ with $c_\theta \notin \{c_b, c_e, c_f, c_g, c_h\}$ and γ with $c_\gamma \notin \{c_\theta, c_f, c_g, c_h\}$.

10.2 $c_a = c_g$. We colour α with c_e, β with c_f, θ with $c_\theta \notin \{c_a, c_f, c_g, c_h\}$ and γ with $c_\gamma \notin \{c_\theta, c_f, c_g, c_h\}$.

11. $c_a \neq c_c$, $c_b \neq c_d$, $c_e = c_g$ and $c_f = c_h$. We have the result by symmetry with the previous case.

12. $c_a = c_c$, $c_b = c_d$, $c_e = c_g$ and $c_f \neq c_h$. We colour α with c_e, β with c_f, θ with c_h, and γ with $c_\gamma \notin \{c_e, c_f, c_h\}$.

13. $c_a \neq c_c$, $c_b = c_d$, $c_e = c_g$ and $c_f = c_h$. We have the result by symmetry with the previous case.

14. $c_a = c_c$, $c_b = c_d$, $c_e \neq c_g$ and $c_f = c_h$.

    14.1. $c_a \neq c_g$. We colour α with c_f, β with c_e, γ with c_e, and θ with $c_\theta \notin \{c_g, c_e, c_f\}$.

    14.2. $c_a = c_g$. We colour α with c_e, β with c_f, γ with c_e, and θ with $c_\theta \notin \{c_g, c_e, c_f\}$.

15. $c_a = c_c$, $c_b \neq c_d$, $c_e = c_g$ and $c_f = c_h$. We have the result by symmetry with the previous case.

16. $c_a = c_c$, $c_b = c_d$, $c_e = c_g$ and $c_f = c_h$. We colour α with c_f, β with c_e, θ with c_a, and γ with c_b.

□

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