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Abstract

This paper is aimed at studying the formation of patches in a cross-diffusion system without reaction terms when the diffusion matrix can be negative but with positive self-diffusion. We prove existence results for small data and global a priori bounds in space-time Lebesgue spaces for a large class of 'diffusion' matrices. This result indicates that blow-up should occur on the gradient. One can tackle this issue using a relaxation system with global solutions and prove uniform a priori estimates. Our proofs are based on a duality argument à la M. Pierre which we extend to treat degeneracy and growth of the diffusion matrix.

We also analyze the linearized instability of the relaxation system and a Turing type mechanism can occur. This gives the range of parameters and data for which instability may occur. Numerical simulations show that patterns arise indeed in this range and the solutions tend to exhibit patches with stiff gradients on bounded solutions, in accordance with the theory.

1 Introduction

The dynamics of interacting population with cross-diffusion have been widely investigated by several researchers. The concept of this phenomena was studied by Levin [18], Levin and Segel, [17], Okubo [27], Mimura and Murray [24], Mimura and Kawasaki [23], Mimura and Yamaguti [25], and many other authors. All these papers base the pattern formation on a reaction term as prey-predator interactions.

Spatial patterns can however emerge from pure diffusions without reaction terms nor oriented drift at the individual level. This is the case of N populations described microscopically by a brownian process which intensity depends upon the macroscopic density $U = (U_1, ..., U_N)$ of the populations

$$dX_k(t) = \sigma_k\big(U(X,t)\big)dW_k(t), \qquad 1 \leq k \leq N.$$

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When set on a bounded domain with reflexion on the boundary, the corresponding models for the population density are cross-diffusions

$$\begin{cases} \frac{\partial}{\partial t}U - \Delta A(U) = 0, \quad \text{in } \Omega, \\ \frac{\partial}{\partial n}A(U) = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $U = U(x,t) \in \mathbb{R}^N$, $x \in \Omega$ a smooth bounded domain of $\mathbb{R}^d$, n denotes the outward normal to Ω. Finally $A : \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinearity related to the intensity of the interactions by the relation

$$A_k(U) = U_k a_k(U), \tag{2}$$

and $a_k(U) = \frac{1}{2}\sigma_k(U).\sigma_k^t(U)$. We also complete the system with an initial data

$$U(t=0) = U^0 = (U_1^0, ..., U_N^0) \quad \text{with } U_k^0 \geq 0.$$

The properties, and pattern formation capacity of such systems are better described by introducing the more general form

$$\frac{\partial}{\partial t}U_k - \sum_{l=1}^{N} \text{div}[D_{kl}(U)\nabla U_l] = 0, \tag{3}$$

where $D_{kl}(U)$ are the components of a $N \times N$ matrix, the derivative of A in the case (1). Boundary conditions have to be imposed and we consider here the case of Neumann conditions

$$D(U).\nabla U.n = 0, \qquad \text{on } \partial\Omega.$$

For such boundary conditions, mass conservation yields naturally

$$\langle U(t) \rangle = \langle U^0 \rangle, \qquad \forall t \geq 0,$$

where $\langle U^0 \rangle$ denotes the average

$$\langle U \rangle = \frac{1}{|\Omega|}\int_\Omega U(x)dx.$$

The Lotka-Volterra competition with cross-diffusion has recently received great attention. They are many established results concerning the global existence of classical solutions (see [33, 19] and the references therein) where most of the proofs rely on Amann's theorem [1, 2]. We point out that standard parabolic theory is not directly applicable to our model due to the presence of cross-diffusion terms.

In opposition with pattern formations, an important issue has been widely studied which is to know in which circumstances the solutions exist globally and behave like in the case a single heat equation, i.e., relax to a constant state as $t \to \infty$. Typically three kinds of special methods have been helpful in this scope. The first method is to rely on the maximum principle. It can occur on certain combinations of the U_i as in [1, 2, 3]. Entropy methods also applies to particular systems and has also been a useful tool because of the related symmetrization of the system following [15, 8]. It provides a natural method both for existence and relaxation to steady state as in the recent studies in [6, 7] of the Shigezada-Kawasaki prey-predator system [31], or for tumous models [12]. This method typically applies in the special case of the square entropy when D is definite positive meaning that there is $\nu > 0$ such that

$$\sum_{k,l=1}^{N} \xi_k D_{kl}(U)\xi_l \geq \nu|\xi|^2, \quad \forall \xi \in \mathbb{R}^N. \tag{4}$$

This strong positivity property gives the energy inequality

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|U(t)|^2dx=\sum_{k,l=1}^{N}\nabla U_k D_{kl}(U)\nabla U_l\leq -\nu\int_{\Omega}|\nabla U(t)|^2dx.$$

The third method, by duality, has been used in [5] on a particular upper-diagonal diffusion with Dirichlet boundary condition; we show here that the method can be extended to very general systems with Neuman conditions.

Concerning instabilities, the interplay between diffusion and reaction terms has raised surprising results in the spirit of the Turing instability mechanism [32]. The question to know if cross-diffusion or self-diffusion gives an advantage to competing species is studied in [20, 21].

Our interest in this paper concerns instability mechanisms that may appear only from the diffusion intensity and the lost of positivity in the second order matrix. We study in which circumstances the increase of this intensity with higher density of the other species can lead to a segregation phenomena. Of course such instability is incompatible with any entropy inequality, and thus (4) cannot hold. This rises several mathematical questions which seem to be new in the domain of cross-diffusions. Can it still be that small solutions exist globally even though the maximum principle does not hold in general? For large data, what kind of regularity or 'blow-up' can we expect? Finally, how do regularized systems behave in the 'instability' regime. We will study these questions with a model problem in mind related to Shigezada-Kawasaki's system and that represents two species with stronger interactions

$$\begin{cases} \frac{\partial}{\partial t}U_1-\Delta[U_1(1+a_{11}U_1^p+a_{12}U_2^p)]=0, \qquad x\in\Omega,\\ \frac{\partial}{\partial t}U_2-\Delta[U_2(1+a_{21}U_1^p+a_{22}U_2^p)]=0, \end{cases}\tag{5}$$

still with Neumann boundary conditions. One can check that as soon as $p>1$, the matrix D_{kl} is negative for U large.

We approach these questions both theoretically and numerically. In particular we prove existence for small initial data (section 2) and we give a priori bounds in $L^p_{t,x}$ for possible solutions to (5) thus showing that the break-down should come from the blow-up of gradient estimates rather than usual L^p norms (section 3). Our main tool here is a general estimate due to M. Pierre [29, 28] in the context of semilinear parabolic systems (arising in population dynamics or more generally reaction-diffusion systems) (see also [9]). In section 4, these bounds are extended to a relaxation system that takes into account a local measurement of densities; we show that the method is well adapted to general (even not parabolic) cross-diffusions and prove global existence for the relaxation system. For non-parabolic cases, we show that Turing instability occurs in a certain range of data and for small relaxation parameters. Numerical simulations of this relaxation system are performed in section 5. They show that the oscillatory initial regime reorganizes to create patches where one species density dominates the other and interfaces are generated which width is related to the relaxation length. The technical and general extension of M. Pierre's estimate to bounded domains for Neumann boundary conditions is kept for an Appendix as well as another remarkable energy estimate which holds for particular cross-diffusion coefficients in (5).

2 Global solutions for small data

The lack of maximum principle for general diffusion systems is a major difficulty that arises for systems as (3). Using stronger H^1 estimates, we can show in one dimension that for small initial data there is global existence. Such solutions decay to the constant state for large time and this is incompatible with the patterns formation we are interested in. This indicates that large initial data are necessary for pattern formation as expected in general.

We consider the system under the form (1)–(2) and assume that

$$a_k \in C^1(\mathbb{R}^N), \tag{6}$$

$$a_k(0) \geq \nu > 0. \tag{7}$$

Theorem 2.1 (Global small solutions in 1 dimension) *In one dimension, under assumptions (6),(7), and for an initial data satisfying, with α small enough,*

$$\|U^0\|_{L^\infty(\Omega)} + \|\nabla U^0\|_{L^2(\Omega)} \leq \alpha,$$

there is a global solution $U(t,x)$ to the cross-diffusion system (1)–(2). It satisfies for all $t > 0$

$$\|U(t)\|_{L^\infty(\Omega)} \leq C\alpha, \qquad \|\nabla U(t)\|_{L^2(\Omega)} \leq \alpha,$$

with C independent of t and

$$U(t) \underset{t\to\infty}{\longrightarrow} \langle U^0 \rangle.$$

Proof. Firstly, since $\langle U(t)\rangle$ is a priori conserved, we notice that

$$\|U(t)\|_{L^\infty(\Omega)} \leq \|\langle U^0\rangle\|_{L^\infty(\Omega)} + \sqrt{|\Omega|}\|\nabla U\|_{L^2(\Omega)} \leq \alpha + \sqrt{|\Omega|}\|\nabla U\|_{L^2(\Omega)}.$$

Secondly, we multiply (1) by the vector $A'(U)$ and differentiate. We obtain

$$\partial_t \nabla A(U) = \nabla\big(A'(U)\partial_t U\big) = \nabla\big(A'(U)\Delta A(U)\big),$$

we multiply by $\nabla A(U)$ and integrate

$$\frac{d}{dt}\int_\Omega \frac{|\nabla A(U)|^2}{2} = \int_\Omega \nabla A(U)\nabla\big(A'(U)\Delta A(U)\big) = -\int_\Omega \Delta A(U)\ A'(U)\ \Delta A(U). \tag{8}$$

But from (2) and (7), we have $A'(0)_{kl} = a_{kk}(0)\delta_{kl}$, and thus for $\|U\|_{L^\infty(\Omega)} \leq \varepsilon$ small enough, we have

$$XA'(U)X \geq \frac{\nu}{2}\|X\|^2, \qquad \forall X \in \mathbb{R}^N, \tag{9}$$

$$|A'(U)X|^2 \geq \frac{\nu^2}{2}\|X\|^2, \qquad \forall X \in \mathbb{R}^N. \tag{10}$$

We now consider T^* defined by

$$T^* := \sup\{t \geq 0, \|U(t)\| \leq \varepsilon\}.$$

For $\alpha < \varepsilon$, then $T^* > 0$. Suppose $T^* < \infty$, then for $0 \leq t \leq T^*$, we have, from (8) and (9),

$$\int_\Omega |\nabla A(U(t))|^2 dx \leq \int_\Omega |\nabla A(U^0)|^2 dx.$$

Therefore,

$$\int_\Omega |A'\big(U(t)\big)\nabla U(t)|^2 \leq \int_\Omega |\nabla A(U^0)|^2 dx.$$

Using (10), this leads to

$$\int_\Omega |\nabla U(t)|^2 dx \le \frac{2}{\nu^2}\int_\Omega |\nabla A(U^0)|^2 dx \le \frac{2C}{\nu^2}\int_\Omega |\nabla U^0|^2 dx.$$

Now, we choose α such that $\alpha \le \frac{\varepsilon}{3}$ and $\frac{\sqrt{2C|\Omega|}}{\nu}\alpha \le \frac{\varepsilon}{3}$, this ensures, $\|U(t)\|_{L^\infty(\Omega)} \le \frac{2\varepsilon}{3}$ and thus, T^* is not maximal. Therefore $T^* = \infty$.

Finally, the existence of a solution for small times follows from standard parabolic theory and the a priori bound above shows that these are global solutions.

We now prove the time convergence to $\langle U^0\rangle$. Because $\int_\Omega (U - \langle U^0\rangle)^2 = \int_\Omega U^2 - \langle U\rangle^2$, we compute

$$\frac{d}{dt}\int_\Omega (U - \langle U\rangle)^2 = \frac{d}{dt}\int_\Omega U^2 = \int_\Omega \Delta A(U) U = -\int_\Omega A'(U)\nabla U.\nabla U.$$

But as the solution stays in the domain $\|U(t)\|_{L^\infty(\Omega)} \le \varepsilon$, we have for some constant C_0,

$$\frac{d}{dt}\int_\Omega (U - \langle U^0\rangle)^2 \le -\frac{\nu}{2}\int_\Omega |\nabla U|^2 \le -2C_0\int_\Omega (U - \langle U^0\rangle)^2,$$

thanks to the Poincaré Wirtinger inequality. We conclude from Gronwall lemma that

$$\|U - \langle U^0\rangle\|_{L^2(\Omega)} \le e^{-C_0 t}\|U^0 - \langle U^0\rangle\|_{L^2(\Omega)}.$$

□

3 A priori bounds for large data

For large initial data and when the condition (4) does not hold, we cannot expect in general the existence of solutions for the cross-diffusion system (3)–(2). For a single equation, the corresponding situation is when $A'(u)$ can be negative on some interval $I \subset \mathbb{R}_+$.

$$\frac{\partial}{\partial t}u - \Delta A(u) = 0.$$

The situation is analyzed in [30, 26, 11] (see also the survey in [10]) and it is better analyzed in term of relaxation systems, an approach which we will follow later. We expect that oscillations or jumps occur at positive times, but a first issue is a priori control in L^∞ for possible solutions. This follows from the maximum principle for a single equation (and possibly from entropy constructions for relaxation systems, see [30]). For systems this is an open question and we give here a first a priori bound

Theorem 3.1 *Smooth solutions to (3)–(2) satisfy the a priori bounds*

$$\Big(\int_0^T\int_\Omega \sum_{k=1}^N A_k(U)\,\sum_{k=1}^N U_k\,dxdt\Big)^{\frac12} \le C_1(\Omega)\|U^0\|_{L^2(\Omega)} + C_2\big(\Omega, \sum_{k=1}^N \langle U_k^0\rangle\big)\sqrt{T}. \tag{11}$$

Therefore if we assume in (2) that

$$a_k(U) \ge \nu > 0 \qquad \forall k = 1, ..., N, \tag{12}$$

then we also have, with the notation $Q_T = \Omega \times [0, T]$,

$$\nu\|U\|_{L^2(Q_T)} \le C_1(\Omega)\|U^0\|_{L^2(\Omega)} + C_2(\Omega, \langle U^0\rangle)\sqrt{T}. \tag{13}$$

In the particular case of model (5), we observe that the larger is p, the best is the bound in (11). In particular, $A(U)$ is always integrable. The L^2 estimate in (13) is much weaker.

Proof. Our proof is based on a variant of a general duality argument due to [28, 9], that is presented in Appendix A. We denote $w = \sum_{k=1}^{N} U_k$. We sum up the equations and we find

$$\partial_t w - \Delta \sum_{k=1}^{N} a_k(U) U_k = 0,$$

which we write

$$\partial_t w - \Delta \alpha(t,x) w = 0, \qquad \alpha(t,x) := \frac{\sum_{k=1}^{N} a_k(U(t,x)) U_k(t,x)}{w(t,x)} = \alpha(U(t,x)).$$

We can use now (27) in Appendix A and obtain,

$$\|\sqrt{\alpha}\, w\|_{L^2(Q_T)} \leq C(\Omega) \|w^0\|_{L^2(\Omega)} + 2\langle w^0 \rangle \|\sqrt{\alpha}\|_{L^2(Q_T)}. \tag{14}$$

In order to control the right hand side by $\|\sqrt{\alpha}\, w\|_{L^2(Q_T)}$, we use a truncation method. Since the coefficients $a_k(U)$ are continuous, and U_k are nonnegative, we may define for any $R > 0$,

$$\sup_{w \leq R} \alpha(U) := M(R) < +\infty.$$

Furthermore, we may truncate w away from values less than R, a parameter to be fixed later on, with the indicator function $\mathbb{I}_{\{w \geq R\}}$ and rewrite (14) as

$$\|\sqrt{\alpha}\, w\, \mathbb{I}_{\{w \geq R\}}\|_{L^2(Q_T)} \leq C(\Omega) \|w^0\|_{L^2(\Omega)} + 2\langle w^0 \rangle \|\sqrt{\alpha} \mathbb{I}_{\{w \geq R\}}\|_{L^2(Q_T)} + 2\langle w^0 \rangle \|\sqrt{\alpha}\, \mathbb{I}_{\{w \leq R\}}\|_{L^2(Q_T)},$$

$$\|\sqrt{\alpha}\, w\, \mathbb{I}_{\{w \geq R\}}\|_{L^2(Q_T)} \leq C(\Omega) \|w^0\|_{L^2(\Omega)} + 2\langle w^0 \rangle \|\sqrt{\alpha}\, \frac{w}{R}\, \mathbb{I}_{\{w \geq R\}}\|_{L^2(Q_T)} + 2\langle w^0 \rangle \|\sqrt{M(R)}\|_{L^2(Q_T)}.$$

We choose $R = 4\langle w^0 \rangle$ and obtain

$$\|\sqrt{\alpha}\, w\, \mathbb{I}_{\{w \geq R\}}\|_{L^2(Q_T)} \leq 2C(\Omega) \|w^0\|_{L^2(\Omega)} + 4\langle w^0 \rangle \sqrt{|\Omega|\; T\; M(R)}.$$

Since we also know that

$$\|\sqrt{\alpha} w \mathbb{I}_{\{w \leq R\}}\|_{L^2(Q_T)} \leq R\sqrt{M(R)|\Omega| T} = 4\langle w^0 \rangle \sqrt{M(R)|\Omega| T},$$

we conclude

$$\|\sqrt{\alpha}\, w\|_{L^2(Q_T)} \leq 2C(\Omega) \|w^0\|_{L^2(\Omega)} + C_2(\langle w^0 \rangle) \sqrt{|\Omega|\; T},$$

with $C_2(\langle w^0 \rangle) = 8\langle w^0 \rangle \sqrt{M(4\langle w^0 \rangle)}$. This is exactly the a priori estimate (11).

The other statement is a simple and direct consequence. $\square$

4 A relaxation system

If we assume that the intensity of the brownian motion depends on the density of the populations measured with a space scale $\delta > 0$ and not at the exact location x, then the system (5) can be replaced by a cross-diffusion relaxation system

$$\begin{cases} \frac{\partial}{\partial t} u_k - \Delta[a_k(\tilde{u}) u_k] = 0, \qquad x \in \Omega, \quad k = 1, ..., N, \\ -\delta^2 \Delta \tilde{u}_k + \tilde{u}_k = u_k, \end{cases} \tag{15}$$

together with Neumann boundary conditions both on u_k and $\tilde{u}_k$. Relaxation procedures are usual and several other examples for cross-diffusions can be found in [14, 4] and for phase transitions see [13, 10, 30]. In terms of the ecological interpretation, it is also more realistic than the initial system (5), because individuals are unlikely to be able to access a pointwise density, but might estimate their environment from sensing at a smaller scale.

We can expect that the system (15) is well-posed, and we first study this question. Then we prove uniform bounds independent of δ which indicates that instability should arise from the blow-up of gradients. To tackle the question of instabilities, we show that the system exhibits Turing patterns for δ small, this is our second goal in this section.

We keep in mind the example (5) and assume that for some $p > 0$ one has

$$0 < \nu \leq a_k(U) \leq C_0(1 + |U|^p), \qquad \forall k \in \{1, ..., N\}. \tag{16}$$

For later purpose, we also introduce the assumption that for some constant $K > 0$ and some $\eta > 0$, we have

$$\left| \frac{\nabla a_k(U)}{a_k^\eta(U)} \right| \leq K |\nabla U|. \tag{17}$$

We have in mind coefficients of the form $(1 + \tilde{U}_j^p)$ and then we can take $\eta = \frac{p-1}{p}$.

4.1 Uniform estimates for $p < 2$

We first extend the a priori estimate of section 3 to this relaxation system. The coupling induces a limitation on the possible growth of the nonlinearities $a_k(U)$ and we have the

Theorem 4.1 *Assume that (16) holds for some $0 < p < 2$, then, the a priori bound holds for a constant C independent of δ*

$$\|u\|_{L^2(Q_T)} \leq C(\|u^0\|_{L^2(\Omega)}, T), \qquad \forall T > 0.$$

This is weaker than the a priori bound in (11). The difficulty in the case at hand comes from the dependency of $a_k(\tilde{u})$ which we cannot lower bound from u itself.

Proof. We denote by $\tilde{a}_k$ the quantity $a_k(\tilde{u})$. The estimate (27) of Appendix A gives, for all $k \in \{1, ..., N\}$

$$\|\sqrt{\tilde{a}_k}\, u_k\|_{L^2(Q_T)} \leq C(\Omega) \|u_k^0\|_{L^2(\Omega)} + 2\langle u_k^0 \rangle \|\sqrt{\tilde{a}_k}\|_{L^2(Q_T)}.$$

The last term may be estimated as

$$\|\sqrt{\tilde a_k}\|^2_{L^2(Q_T)} = \int_0^T \int_\Omega \tilde a_k \le C_0 \int_0^T \int_\Omega (1 + \sum_l \tilde u_l^p).$$

Thanks to Holder inequality and direct estimate on the solution to the elliptic equation on $\tilde u_l$, we have,

$$\int_0^T \int_\Omega \tilde u_l^p \le (|\Omega|\, T)^{\frac{2-p}{2}} \|\tilde u_l\|^{\frac{p}{2}}_{L^2(Q_T)} \le C(\Omega) T^{\frac{2-p}{2}} \|u_l\|^p_{L^2(Q_T)}.$$

Finally, back to the original inequality we arrive at

$$\sqrt{\nu}\, \|u_k\|_{L^2(Q_T)} \le C(\Omega)\|u_k^0\|_{L^2(\Omega)} + 2\sqrt{C_0\, T\, |\Omega|}\langle u_k^0\rangle + 2\langle u_k^0\rangle \sqrt{C_0}(|\Omega| T)^{\frac{2-p}{4}} \|u\|^{\frac{p}{2}}_{L^2(Q_T)},$$

which leads to

$$\sqrt{\nu}\|u\|_{L^2(Q_T)} \le C(\Omega, \|u^0\|_{L^2(\Omega)}, T) + C(\Omega, \langle u^0\rangle) T^{\frac{2-p}{2}} \|u\|^{\frac{p}{2}}_{L^2(Q_T)}.$$

As $p/2 < 1$, this proves that $\|u\|_{L^2(Q_T)}$ is a priori bounded as by a constant depending only on Ω, T, $\|u^0\|_{L^2(\Omega)}$ and the two constants in (16). □

4.2 Existence of solutions

We now show stronger estimates from which strong compactness of solutions follows. They use fundamentally the regularity on $\tilde u$ in (15) by elliptic regularizing effects. Existence of global solutions follow and the details are carried out in [16].

The main result of this section is the following

Proposition 4.2 *Assume that (16) holds with $p > 1$, and $p < \frac{2d}{d-2}$ when $d > 2$. Then, the a priori estimate holds*

$$\|\sqrt{\tilde a_i} u_i\|_{L^2(Q_T)} \le C(\delta, \|u^0\|_{L^1\cap L^2(\Omega)}, T). \tag{18}$$

Furthermore, if we assume (17) in dimension 1 with any $\eta > 0$, and in dimension 2 with $0 < \eta < 1$, then we have for all $1 \le q < \infty$,

$$\|u(t)\|_{L^q(\Omega)} \le C(q, \delta, \|u^0\|_{L^1\cap L^q(\Omega)}, T), \qquad 0 \le t \le T, \tag{19}$$

$$\int_0^T \int_\Omega |\nabla u^{q/2}|^2 dxdt \le C(q, \delta, \|u^0\|_{L^1\cap L^q(\Omega)}, T). \tag{20}$$

Proof. We begin with the proof of (18) which improves that of the theorem 4.1. We use again the estimate (27) applied to u_i which yields

$$\sqrt{\nu}\|u_i\|_{L^2(Q_T)} \le \|\sqrt{\tilde a_i} u_i\|_{L^2(Q_T)} \le C(\Omega)\|u_i^0\|_{L^2(\Omega)} + 2\langle u_i^0\rangle \|\sqrt{\tilde a_i}\|_{L^2(Q_T)}. \tag{21}$$

We use the hypothesis (16) to get

$$\|\sqrt{\tilde a_i}\|_{L^2(Q_T)} \le \sqrt{C_0|\Omega|\, T + C_0 \int_0^T \int_\Omega |\tilde u|^p} \le C'(\Omega)\sqrt{T} + \sqrt{\int_0^T \int_\Omega |\tilde u|^p} = \sqrt{C_0|}(\sqrt{\Omega|\, T} + \sqrt{\int_0^T \|\tilde u\|_p^p}).$$

Thanks to elliptic regularity we also have

$$\|\tilde{u}\|_p \leq C(\delta, r)\|u\|_r,$$

for any $r > 1$ satisfying also $\frac{1}{p} \geq \frac{1}{r} - \frac{2}{d}$ (particularly it is true for any r if $d = 1, 2$). Then, using interpolation inequality and choosing $r < 2$, we find successively

$$\|\tilde{u}\|_p \leq C(\delta, r)\|u\|_1^{1-\theta(r)}\|u\|_{L^2(\Omega)}^{\theta(r)} = C(\delta, r)\|u^0\|_1^{1-\theta(r)}\|u\|_{L^2(\Omega)}^{\theta(r)} \qquad \theta(r) = \frac{1-\frac{1}{r}}{1-\frac{1}{2}} = 2(1-\frac{1}{r}),$$

$$\|\sqrt{\tilde{a}_i}\|_{L^2(Q_T)} \leq \sqrt{C_0}\big(\sqrt{\Omega T} + \sqrt{\int_0^T C(\delta, r)^p \|u^0\|_1^{(1-\theta(r))p}\|u\|_{L^2(\Omega)}^{p\theta(r)}}\big),$$

$$\|\sqrt{\tilde{a}_i}\|_{L^2(Q_T)} \leq \sqrt{C_0}\big(\sqrt{\Omega T} + C(\delta, r)^{p/2}\|u^0\|_1^{(1-\theta(r))p/2}\sqrt{\int_0^T \|u(t)\|_{L^2(\Omega)}^{p\theta(r)}dt}\big). \tag{22}$$

Now, if we may choose r such that $\theta(r)p < 2$, we get, thanks to Jensen's inequality

$$\int_0^T \|u(t)\|_{L^2(\Omega)}^{p\theta(r)}dt = \int_0^T \|u(t)\|_{L^2(\Omega)}^{2(p\theta(r)/2)}dt \leq T^{1-p\theta(r)/2}\Big(\int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt\Big)^{p\theta(r)/2},$$

which we rewrite as

$$\int_0^T \|u(t)\|_{L^2(\Omega)}^{p\theta(r)}dt \leq T^{1-p\theta(r)/2}\|u\|_{L^2(Q_T)}^{p\theta(r)}.$$

Replacing in (22), we have

$$\|\sqrt{\tilde{a}_i}\|_{L^2(Q_T)} \leq \sqrt{C_0}\sqrt{\Omega T} + C(\delta, r, p, \|u^0\|_1)T^{\frac{2-p\theta(r)}{4}}\|u\|_{L^2(Q_T)}^{\frac{p\theta(r)}{2}}$$

And replacing in (21), we obtain

$$\nu\|u\|_{L^2(Q_T)} \leq C(\Omega)\|u^0\|_{L^2(\Omega)} + C'(\Omega)\sqrt{T} + C(\delta, r, p, \|u^0\|_1)T^{\frac{2-p\theta(r)}{4}}\|u\|_{L^2(Q_T)}^{\frac{p\theta(r)}{2}} \tag{23}$$

This concludes the first inequality when $p\theta(r)/2 < 1$, and it remains to find the range of p in order to fulfill the constraints. These can be obtained choosing r close enough to 1 for $d = 1, 2$. For $d > 2$, we need the conditions

$$\begin{cases} \frac{1}{p} \geq \frac{1}{r} - \frac{2}{d}, \qquad 1 < r < 2, \\ \frac{p\theta(r)}{2} = p(1-\frac{1}{r}) < 1. \end{cases}$$

We choose to satisfy the second line $\frac{1}{r} > \frac{p-1}{p}$, but close to equality (which gives $1 < r < 2$ as we check it a posteriori). This leads, in the first line, to the condition $p < \frac{2d}{d-2}$, but close to equality (which imposes $r < \frac{2d}{d+2}$). The bound on $\|u\|_{L^2(Q_T)}$ gives then a bound on $\|\sqrt{\tilde{a}_i}u_i\|_{L^2(Q_T)}$ thus concluding the proof of (18).

This estimate leads to the stronger a priori bounds (19) that we prove now. We go back to the equation and write

$$\frac{d}{dt}\int u^q + C_q\int \tilde{a}|\nabla u^{q/2}|^2 = -C_q^1\int u^{q/2}\nabla u^{q/2}\nabla\tilde{a}.$$

From this equality, and writing $\tilde{a}|\nabla u^{q/2}| = |\nabla(\tilde{a}^{1/2}u^{q/2}) - u^{q/2}\nabla\tilde{a}^{1/2}|$, we derive directly the inequality

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int u^q + \tfrac{C_q}{3}\int \tilde{a}|\nabla u^{q/2}|^2 + \tfrac{C_q}{3}\int |\nabla(\tilde{a}^{1/2}u^{q/2})|^2 &\leq C_q^2 \textstyle\int u^q \frac{|\nabla\tilde{a}|^2}{\tilde{a}} \\ &\leq C_q \textstyle\int u^q|\nabla\tilde{u}|^2\tilde{a}^{2\eta-1}. \end{aligned} \tag{24}$$

We show separately how in dimensions 1 and 2 this allows us to control any L^q norm for $q < +\infty$.
The case $d = 1$. The proof is easier for $d = 1$ because there exists a constant C (depending only on δ and Ω) such that

$$\|\tilde{u}\|_{L^\infty(\Omega)} + \|\nabla\tilde{u}\|_{L^\infty(\Omega)} \leq C\int |u| \leq C(\|u^0\|_1).$$

Therefore from (24) we deduce

$$\frac{d}{dt}\int u^q \leq C.C_q \int u^q.$$

We conclude thanks to Gronwall lemma.
The case $d = 2$. In dimension 2, we divide the proof into two steps.
Step 1, $1 < q < 2$**.** We first focus on small values of the exponent q namely $1 < q < 2$ and $2\eta < 1 + \frac{q}{2}$ (the limitation $\eta < 1$ comes from choosing q close to 2). From (24) and using successively (17) and Hölder inequality, we obtain

$$\begin{aligned} \tfrac{d}{dt}\int u^q &\leq C_q^3 \int u^q \tilde{a}^{q/2}|\nabla\tilde{u}|^2. \\ &\leq C_q^3 \big\|u\sqrt{\tilde{a}}\big\|_{qr}^q \|\nabla\tilde{u}\|_{2r'}^2 \\ &\leq C(q,\delta,r)\big\|u\sqrt{\tilde{a}}\big\|_{qr}^q \|u\|_m^2 \end{aligned}$$

thanks to elliptic regularity, with

$$\frac{1}{m} = \frac{1}{2r'} + \frac{1}{2} = 1 - \frac{1}{2r}.$$

Choosing $r = \frac{2}{q}$ then $\frac{1}{m} = 1 - \frac{q}{4}$ and we arrive at

$$\begin{aligned} \tfrac{d}{dt}\int u^q &\leq C(\delta,r)\|u\sqrt{\tilde{a}}\|_2^q \|u\|_m^2 \\ &\leq C(\delta,r)\|u\sqrt{\tilde{a}}\|_2^q \|u\|_q^{2\theta}\,\|u\|_{L^2(\Omega)}^{2(1-\theta)} \\ &\leq C(\delta,r)\|u\sqrt{\tilde{a}}\|_2^q \|u\|_q^q \|u\|_{L^2(\Omega)}^{2-q?} \end{aligned}$$

because, as we have $\frac{1}{2} < \frac{1}{m} < \frac{1}{q}$, we may interpolate m between q and 2 with

$$\theta = \frac{\frac{1}{m} - \frac{1}{2}}{\frac{1}{q} - \frac{1}{2}} = \frac{\frac{2-q}{4}}{\frac{2-q}{2q}} = \frac{q}{2}.$$

We finally obtain by Young's inequality,

$$\frac{d}{dt}\int u^q \leq C(q,\delta,r)\Big[\|u\sqrt{\tilde{a}}\|_2^2 + \|u\|_{L^2(\Omega)}^2\Big]\,\|u\|_q^q,$$

and we may then conclude with Gronwall lemma using the estimates on $\int_0^T \|u\sqrt{\tilde{a}}\|_2^2$ and $\int_0^T \|u\|_2^2$ in (18). By interpolation, this also gives a priori bound for any L^q norm for $q \in [1, 2[$. This ends step 1.

Step 2 We now focus on L^q norms for $q \geq 2$. We notice that now, controlling any L^q norm for $q < 2$, we control by elliptic regularity any L^q norm of $\nabla\tilde{u}$ except the L^∞ norm. We also control the L^∞ norm of $\tilde{u}$ and therefore of $\tilde{a}$. We go back to (24) and conclude

$$\begin{aligned}
\frac{d}{dt}\int u^q + C\int |\nabla u^{q/2}|^2 &\leq C\int u^q \frac{|\nabla\tilde{a}|^2}{\tilde{a}} \leq C(\delta, q, T)\int u^q|\nabla\tilde{u}|^2 \\
&\leq C(\delta, q, T)\|u\|_{qr}^q \|\nabla\tilde{u}\|_{2r'}^2 \\
&\leq C(\delta, q, r, T)\|u\|_{qr}^q.
\end{aligned}$$

We use now interpolation: for any $s > r > 1$, we have

$$\|u\|_{qr} \leq \|u\|_q^{1-\theta}\|u\|_{qs}^\theta \qquad \theta = \frac{\frac{1}{qr} - \frac{1}{qs}}{\frac{1}{q} - \frac{1}{qs}} = \frac{\frac{1}{r} - \frac{1}{s}}{1 - \frac{1}{s}}.$$

Using Young's inequality, we obtain for ε small (to be chosen later)

$$\frac{d}{dt}\int u^q + C\int |\nabla u^{q/2}|^2 \leq C(\delta, q, r, T, \varepsilon)\|u\|_q^q + \varepsilon\|u\|_{qs}^q.$$

We have by Poincaré Wirtinger inequality

$$\begin{aligned}
\|u\|_{qs}^q = \|u^{q/2}\|_{2s}^2 &\leq 2\|u^{q/2} - \langle u^{q/2}\rangle\|_{2s}^2 + 2\|\langle u^{q/2}\rangle\|_{2s}^2 \\
&\leq C(\Omega, s, \nu)\int_\Omega |\nabla u^{q/2}|^2 + C(\Omega, s)\|u\|_{q/2}^q \\
&\leq C(\Omega, s, \nu)\int_\Omega |\nabla u^{q/2}|^2 + C(\Omega, s)\big[\|u\|_1^q + \|u\|_q^q\big],
\end{aligned}$$

from interpolation and Young's inequality. We fix s as above, choose ε small enough and we obtain

$$\frac{d}{dt}\int u^q \leq C(\Omega, q)\big[\|u\|_1^q + \|u\|_q^q\big].$$

And we conclude (19) by Gronwall lemma. The last estimate (20) also follows from (24). $\square$

4.3 Turing patterns

In order to go further and study the instability occurring in the regularized model, we consider the following particular system:

$$\begin{cases}
\partial_t u - \Delta\big(u(1 + \tilde{v}^2)\big) = 0, \\
\partial_t v - \Delta\big(v(1 + \tilde{u}^2)\big) = 0, \\
-\delta^2\Delta\tilde{u} + \tilde{u} = u, \\
-\delta^2\Delta\tilde{v} + \tilde{v} = v,
\end{cases} \tag{25}$$

still with Neumann boundary conditions and initial data u^0, v^0.

It is rather intuitive that for δ large, diffusion is dominant; this is also the case for small initial data thanks to the argument in section 2. Therefore, the appearance of patterns depends upon a relation between the average densities of populations u and v and the parameter δ. In order to study this in details, we begin with some notations

- the only possible constant steady state of the system is given by $u = \tilde{u} = \langle u^0 \rangle$ and $v = \tilde{v} = \langle v^0 \rangle$,
- we denote by $(\lambda > 0,\ w)$ the non-zero solutions to the Neumann eigenproblem

$$-\Delta w = \lambda w, \qquad \partial_n w = 0 \quad \text{on } \partial\Omega$$

  we also denote by λ_1 the first eigenvalue for the Laplacian.

In order to investigate when the (in)stability of the constant steady state occurs, we study the linearized system:

$$\begin{cases} \partial_t u - (1 + \langle v^0 \rangle^2)\Delta u - 2\langle u^0 \rangle \langle v^0 \rangle \Delta \tilde{v} = 0, \\ \partial_t v - (1 + \langle u^0 \rangle^2)\Delta v - 2\langle u^0 \rangle \langle v^0 \rangle \Delta \tilde{u} = 0, \\ -\delta^2 \Delta \tilde{u} + \tilde{u} = u, \\ -\delta^2 \Delta \tilde{v} + \tilde{v} = v. \end{cases}$$

As usual, we look for solutions of type $e^{\mu t}(a, b, c, d)w$. Such solutions should satisfy

$$\begin{cases} c = \frac{a}{1+\delta^2\lambda}, \qquad d = \frac{b}{1+\delta^2\lambda}, \\ \mu a + \lambda a(1 + \langle v^0 \rangle^2) + \lambda 2\langle u^0 \rangle \langle v^0 \rangle \frac{b}{1+\delta^2\lambda} = 0, \\ \mu b + \lambda b(1 + \langle u^0 \rangle^2) + \lambda 2\langle u^0 \rangle \langle v^0 \rangle \frac{a}{1+\delta^2\lambda} = 0, \end{cases}$$

which may be written under the matrix form

$$\begin{pmatrix} 1 + \langle v^0 \rangle^2 & \frac{2\langle u^0 \rangle \langle v^0 \rangle}{1+\delta^2\lambda} \\ \frac{2\langle u^0 \rangle \langle v^0 \rangle}{1+\delta^2\lambda} & 1 + \langle u^0 \rangle^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\mu}{\lambda} \begin{pmatrix} a \\ b \end{pmatrix}.$$

We denote by $M = M(\langle u^0 \rangle, \langle v^0 \rangle, \delta, \lambda)$ this symmetric matrix.

The question of the stability of the constant steady state can now be formulated in terms of eigenvalues of the matrix M. It is unstable if $\mu > 0$ and thus if M has negative eigenvalues. In this case, local behavior around the equilibrium should lead to segregation since the associated eigenvector to $-\mu/\lambda$ should satisfy $a.b < 0$. We have the following

Lemma 4.3 *If the initial populations $\langle u^0 \rangle$ and $\langle v^0 \rangle$ are large enough and the relaxation parameter δ is small enough, then the constant steady state is linearly unstable. More precisely, it occurs under the conditions*

$$\gamma := 4\langle u^0 \rangle^2 \langle v^0 \rangle^2 - \big(1 + \langle u^0 \rangle^2\big)\big(1 + \langle v^0 \rangle^2\big) > 0,$$

$$\delta^2 < \frac{2\langle u^0 \rangle \langle v^0 \rangle - \sqrt{\big(1 + \langle u^0 \rangle^2\big)\big(1 + \langle v^0 \rangle^2\big)}}{\lambda_1 \sqrt{\big(1 + \langle u^0 \rangle^2\big)\big(1 + \langle v^0 \rangle^2\big)}}. \tag{26}$$

The domain influences instability only through the smallness condition on $\delta^2\lambda_1(\Omega)$ when the initial data are such that $\gamma > 0$.

Notice that the first condition ensures that the limiting system ($\delta = 0$) has a negative 'diffusion' matrix D in the setting (3).

Proof. As mentioned earlier, the constant steady state is unstable if the symmetric matrix M admits negative eigenvalues, i.e., if $det(M) < 0$. We calculate

$$\det(M) = -\frac{4\langle u^0\rangle^2\langle v^0\rangle^2}{(1+\delta^2\lambda)^2} + (1+\langle u^0\rangle^2)(1+\langle v^0\rangle^2) \geq -\gamma.$$

As $det(M)$ is a non-decreasing function of δ, with limit γ as $\delta \to 0$ we first need $\gamma > 0$ that is our first condition. The second condition gives the upper bound on δ to satisfy this inequality. $\square$

5 Numerical results

The theoretical results indicate that solutions of the relaxation system (15) remain bounded in L^2. Therefore, we expect that the instability obtained for large initial data or a small δ (through Turing mechanism) should lead to stiff gradients.

We present several numerical tests for the particular cubic system (25). They aim at showing that (i) the conditions of Lemma 4.3 are accurate and describe the numerical transition to instability, (ii) stationary patterns are indeed obtained in this range of data with stiff gradients. These numerical results also show the variety of possible steady state, an interesting phenomena widely studied theoretically ([21] and the references therein). We have performed both 1D and 2D simulations in the following domains

- In interval $\Omega = [0, 1]$ (1D simulation)
- In rectangle $\Omega = [0, 2] \times [0, 0.5]$ (in both cases $|\Omega| = 1$)
- In unit square $\Omega =]0, 1[^2$

In 2D, the computations use an unstructured grid and a mixed finite element method for space and backward Euler scheme for time. The method is already presented in [22].

We recall the eigenvectors of Laplace operator with Neumann boundary condition: for $\Omega = [0, 1]$, $e_n(x) = \cos(n\pi x)$ and particularly, the first nonzero eigenvalue is π^2, associated to the eigenvector $\cos(\pi * x)$. For $\Omega = [0, 2] \times [0, 0.5]$, the eigenvectors are given by $e_{n,m}(x, y) = \cos(\frac{n\pi x}{2})\cos(2m\pi y)$, the first nonzero eigenvalue is $\pi^2/4$.

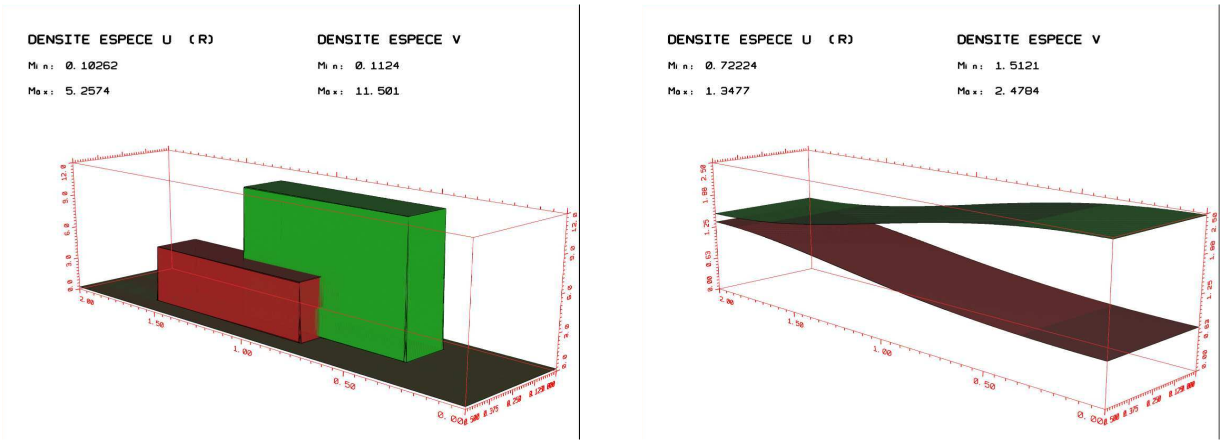
We compare the theoretical formula of Lemma 4.3 and the numerical stability of the steady state. In all simulations we take $\langle u^0\rangle = 2, \langle v^0\rangle = 1$. In this case, instability might occur, since

$$\gamma = 16 - 10 = 6 > 0$$

and the limiting values of δ_0 are given in the table 1. Therefore, in a first series of numerical tests, we choose the parameters $\delta^2 = 0.26 < \delta_0^2$ and $\delta^2 = 0.27 > \delta_0^2$ in 1D and for $\delta^2 = 0.1 < \delta_0^2$, and $\delta^2 = 0.11 > \delta_0^2$ in 2D. In both cases, we have obtained relaxation to constant equilibrium when δ is taken larger than the critical value (for all the initial data we have tested), and instability of the constant equilibrium when δ is smaller than the critical value.

Domain	$\Omega =]0,1[$	$\Omega =]0,2[\times]0,0.5[$	$\Omega =]0,1[^2$
Critical value δ_0^2	0.2649111	0.1073644	0.02684

Table 1: Critical value of the parameter δ_0 for Turing instability, computed from formula (26).



(a) Intital condition

(b) Steady state

Figure 1: Initial condition (left) and steady state (right) in 2D simulations. The relaxation parameter $\delta^2 = .1$ is small enough to fulfill condition (26). The scales for the solutions are not the same in the two figures.

We illustrate the instability case with steady states in figure 3 for 1D simulations and in figure 1 for 2D simulations. For the $1D$ simulation, we took $v^0 \equiv 1$ and $u^0 = 1.9 + 0.2\mathbb{1}_{\{]0.1,0.6[\}}$.

Next we study the singularity that occurs on the transients for small relaxation parameter δ. Numerical solutions show that strong oscillations occur. In figure 2 we depict, for the same initial data, the effect of δ on the solution at a given time.

A Appendix: Michel Pierre's estimate

Consider the problem

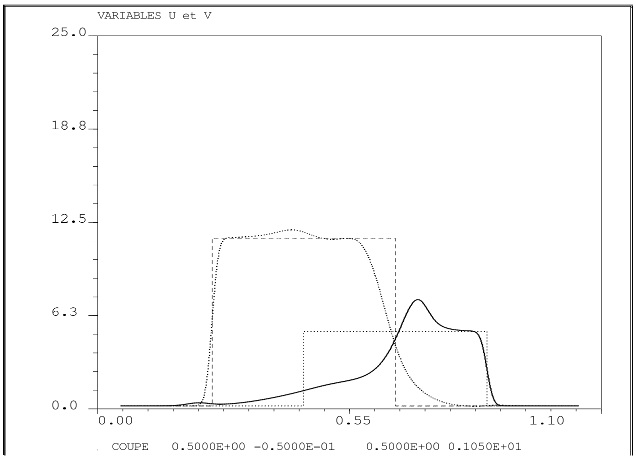
$$\begin{cases} \partial_t u - \Delta[a(t,x)u] = 0, \\ u(t=0) = u^0, \end{cases}$$

together with Neumann boundary condition in a bounded domain Ω. We denote $Q_T = (0,T)\times\Omega$. We assume that $a(t,x) > 0$ is smooth and u is a weak solution. We can also assume withour lack of generality that $\langle u^0\rangle \geq 0$. Then we have the a priori estimate

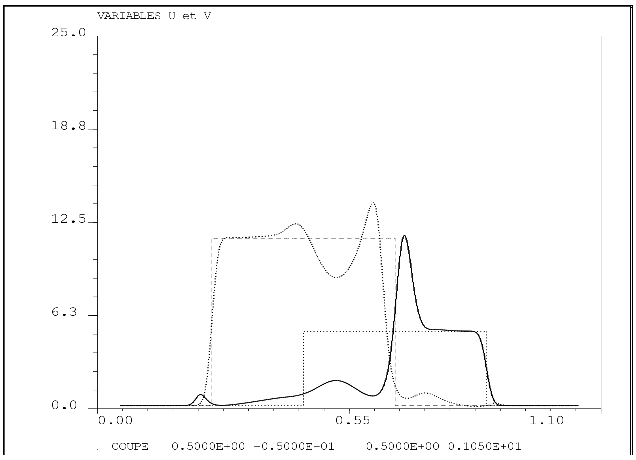
Lemma A.1 *For any $T > 0$, we have*

$$\|\sqrt{a}\, u\|_{L^2(Q_T)} \leq C(\Omega)\|u^0\|_{L^2(\Omega)} + 2\langle u^0\rangle \|\sqrt{a}\|_{L^2(Q_T)}, \tag{27}$$

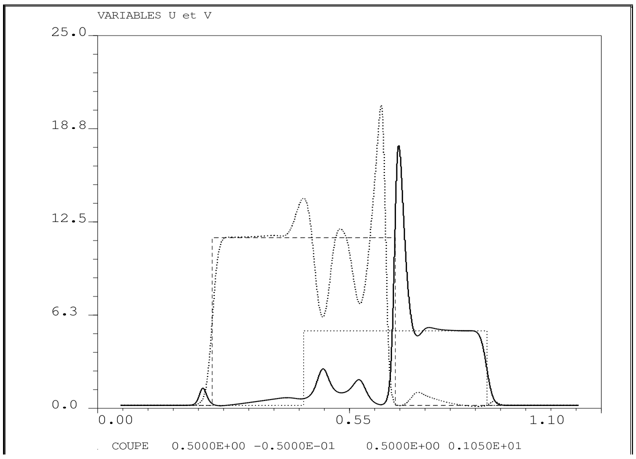
where $C(\Omega)$ is the constant of Poincaré Wirtinger's inequality.



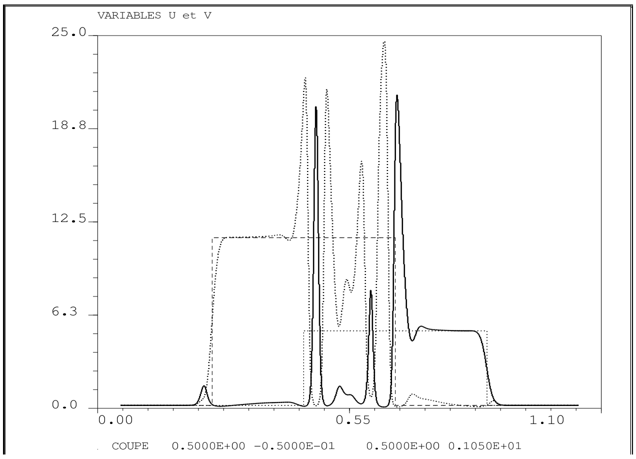
(a) $\delta^2 = 2.\ 10^{-3}$



(b) $\delta^2 = 2.\ 10^{-4}$

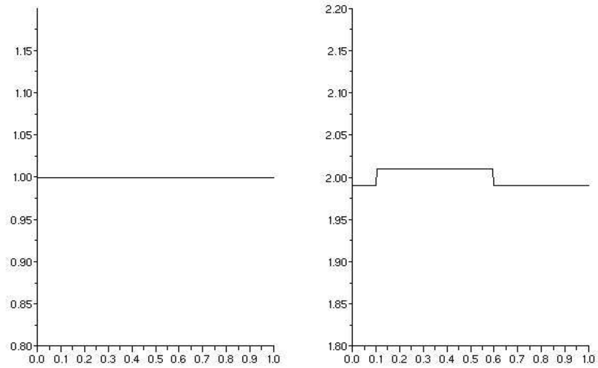


(c) $\delta^2 = 5.\ 10^{-5}$

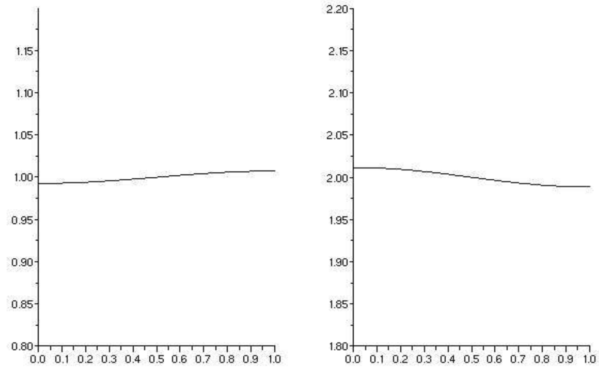


(d) $\delta^2 = 2.\ 10^{-5}$

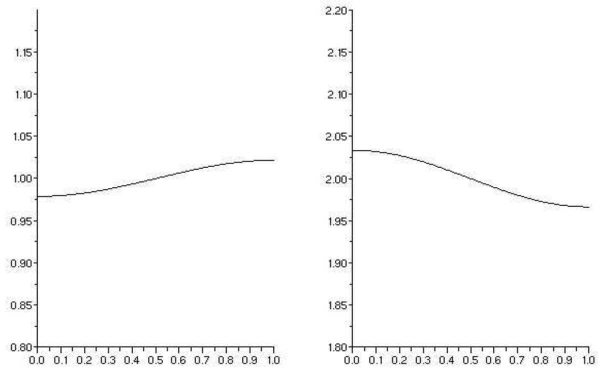
Figure 2: Cuts, at a given time, in the y direction and in the middle of the domain $\Omega = (0,1)^2$ in 2D. The piecewise initial condition is also represented in dashed line. As expected strong oscillations occur with species seggregation. These oscillations are stronger when δ is smaller.



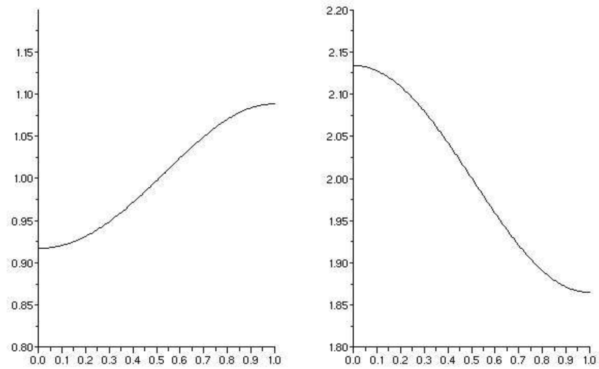
(a) time $t = 0$



(b) time $t = 2$



(c) time $t = 5$



(d) time $t = 9$

Figure 3: Time evolution for a $1D$ simulation for $\delta^2 = 0.25 < \delta_0^2$. This figure shows how a small perturbation is amplified and creates a steady pattern. Because δ is large (close to δ_0) there are not strong oscillations as in the case of smaller values.

Proof. Consider smooth functions $F(t,x)$ and the solutions to the adjoint problem

$$\begin{cases} \partial_t v + a(t,x)\Delta v = F(t,x), \\ v(t=T) = 0, \end{cases} \tag{28}$$

still with Neumann conditions. We have

$$\frac{d}{dt}\int_\Omega uv = \int_\Omega Fu,$$

and thanks to the final condition for the adjoint problem,

$$-\int_\Omega u^0 v^0 = \int_0^T \int_\Omega Fu. \tag{29}$$

Multiplying (28) by Δv, we get

$$\int_\Omega \partial_t v \Delta v + \int_\Omega a|\Delta v|^2 = \int_\Omega F\Delta v,$$

integrating by parts on Ω , we obtain,

$$-\frac{d}{dt}\int_\Omega \frac{|\nabla v|^2}{2} + \int_\Omega a|\Delta v|^2 \leq \int_\Omega (\frac{F^2}{2a} + \frac{a}{2}|\Delta v|^2),$$

which gives after integration in time, using again $v(T) = 0$,

$$\int_\Omega |\nabla v^0|^2 + \int_0^T \int_\Omega a|\Delta v|^2 \leq \int_0^T \int_\Omega \frac{F^2}{a},$$

and by consequence,

$$\|\nabla v^0\|_{L^2(\Omega)} \leq \|\frac{F}{\sqrt{a}}\|_{L^2(Q_T)}, \tag{30}$$

$$\|\sqrt{a}\Delta v\|_{L^2(Q_T)} \leq \|\frac{F}{\sqrt{a}}\|_{L^2(Q_T)}. \tag{31}$$

We need additionally a bound on $\int v^0$ that we derive as follows. We use again (28) to find

$$|\int_\Omega v^0| = |\int_0^T \int_\Omega a\Delta v - F| \leq \int_0^T \int_\Omega \sqrt{a}\Big(\sqrt{a}|\Delta v| + \frac{F}{\sqrt{a}}\Big),$$

which gives, thanks to the Cauchy Schwarz inequality and (31),

$$\left|\int_\Omega v^0\right| \leq 2\|\sqrt{a}\|_{L^2(Q_T)}\|\frac{F}{\sqrt{a}}\|_{L^2(Q_T)}. \tag{32}$$

Finally, we get using Poincaré-Wirtinger ineqality, (32) and then (30),

$$\begin{aligned} \left|\int_\Omega u^0 v^0\right| &\leq \left|\int_\Omega u^0(v^0 - \langle v^0\rangle)\right| + |\int_\Omega \langle u^0\rangle v^0| \\ &\leq C(\Omega)\|u^0\|_{L^2(\Omega)}\|\nabla v^0\|_{L^2(\Omega)} + 2\langle u^0\rangle\|\sqrt{a}\|_{L^2(Q_T)}\|\frac{F}{\sqrt{a}}\|_{L^2(Q_T)} \\ &\leq C(\Omega)\|u^0\|_{L^2(\Omega)}\|\frac{F}{\sqrt{a}}\|_{L^2(Q_T)} + 2\langle u^0\rangle\|\sqrt{a}\|_{L^2(Q_T)}\|\frac{F}{\sqrt{a}}\|_{L^2(Q_T)}. \end{aligned}$$

Back to (29), we conclude that

$$\Big|\int_0^T\int_\Omega Fu\Big| = \Big|\int_0^T\int_\Omega \frac{F}{\sqrt{a}}\sqrt{a}u\Big| \le \Big(C(\Omega)\|u^0\|_{L^2(\Omega)} + 2\langle u^0\rangle\|\sqrt{a}\|_{L^2(Q_T)}\Big)\|\frac{F}{\sqrt{a}}\|_{L^2(Q_T)},$$

which is equivalent to (27). □

B Energy for a particular cross-diffusion system

A particular choice of cross-diffusion terms in (5) permits for an energy inequality even for negative second order matrices. This is the case of the system

$$\begin{cases} \partial_t U - \Delta(U(1+V^2)) = 0, \\ \partial_t V - \Delta(V(1+U^2)) = 0, \end{cases} \tag{33}$$

still with Neumann boundary conditions and initial data U^0, V^0.

For this system, the energy is given by

$$E(x,t) := (1+U^2)(1+V^2).$$

One can easily check that it holds

$$\frac{\partial}{\partial t}E(x,t) = 2U(1+V^2)\Delta(U(1+V^2)) + 2(V(1+U^2))\Delta(V(1+U^2)),$$

which leads immediately to

$$\frac{d}{dt}\int_\Omega (1+U^2)(1+V^2) = -2\int |\nabla(U(1+V^2))|^2 - 2\int |\nabla(V(1+U^2))|^2 \le 0.$$

It follows an a priori estimate in the space $L^\infty_t(L^2_x)$ that completes the L^p_{tx} bound proved in section 3.

The system (33) is not always elliptic. This is related to the non-convexity of this energy (still for large data), an important difference with the Shigezada-Kawasaki prey-predator system which comes with a convex entropy functional ([6]).

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