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# Stability of $l_1$ minimisation in compressed sensing

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## Abstract

We discuss known results (c.f. [16, 6]) about stability of  $l_1$  minimisation (denoted  $\Delta_1$ ) with respect to the measurement error and how those results depend on the measurement matrix  $\Phi$ . Then we produce a large class of measurement matrices  $\Phi$  for which we can apply results from [16] so we have estimate

$$\|\Delta_1(\Phi(x) + r) - x\|_2 \leq C(\|r\|_2 + k^{-1/2}\sigma_k^1(x)). \quad (1)$$

We conclude with a modification of  $l_1$  minimisation which gives (1) for most random measurement matrices considered in compressed sensing literature. We also discuss stability of instance optimality in probability.

## 1 General description of compressed sensing

Let us start by explaining the general setup of compressed sensing. Given  $N \gg n$  we look for  $n \times N$  a matrix  $\Phi$  such that vector  $y = \Phi x \in \mathbb{R}^n$  preserves information about  $x \in \mathbb{R}^N$ . We need a decoder (generally nonlinear)  $\Delta : \mathbb{R}^n \rightarrow \mathbb{R}^N$  such that  $\Delta(\Phi x)$  looks like  $x$ . We require a  $k$  so  $\Delta(\Phi x) = x$  for  $x$  any  $k$ -sparse vector. We want  $\Delta$  to be numerically friendly and  $k$  big. This leads to requiring that  $\Phi$  has RIP( $k, \delta$ ).

**Definition 1.1** ([3]) *Matrix  $\Phi$  has RIP( $k, \delta$ ),  $0 < \delta < 1$  if*

$$(1 - \delta)\|x\|_2 \leq \|\Phi x\|_2 \leq (1 + \delta)\|x\|_2$$

*for every  $k$  sparse vector  $x \in \mathbb{R}^N$ .*

The largest possible  $k$  is  $\sim n/\log(N/n)$ .

All matrices which perform well for this maximal range of  $k$  are random. In this paper we consider only random matrices  $\Phi(\omega) = (\phi_{i,j}(\omega))$  where  $\phi_{i,j}$ 's are independent, i.i.d. subgaussian random variables e.g. Gaussian, Bernoulli. We normalize  $\mathbb{E}|\phi_{i,j}|^2 = 1/n$  so columns  $\Phi_j$  of  $\Phi$  have typically norm one. Let us call them standard matrices.

Given  $0 < \delta < 1$  standard  $\Phi(\omega)$  satisfies RIP( $k, \delta$ ) for  $k = \lfloor c_1(\delta)n/\log(N/n) \rfloor$  with probability  $\geq 1 - e^{-c_2(\delta)n}$  where  $c_1, c_2 > 0$ . This is well known, see e.g. [7, 3, 1, 11]. The important point is that RIP is not practically verifiable even for moderately large  $N$  and  $k$ .

There are two main approaches to finding  $\Delta$  for the above matrices. One approach, introduced by (E. Candes, D. Donoho et. al. see e.g. [3, 7]) is  $l_1$  minimization  $\Delta_1$  i.e.

$$\Delta_1(y) = \text{Argmin}\{\|z\|_1 : \Phi(z) = y\}. \quad (2)$$

Another uses greedy algorithm. We start with an algorithm AL which for  $y \in \mathbb{R}^n$  and vectors  $(\Phi_j)_{j=1}^N$  gives a subset  $\Lambda \subset \{1, \dots, N\}$  with  $\#\Lambda \leq k$  and

$\sum_{j \in \Lambda} a_j \Phi_j$ . We set  $\Delta_{AL}(y) = \sum_{j=1}^N a_j e_j$ . This approach was proposed with AL=OMP by A.Gilbert, J. Tropp et al. see e.g. [15]. Some variants of OMP were used by D.Needell, J.Tropp, R.Vershynin et al. see [13, 14, 5].

Instance optimality originate in the work of A.Gilbert, M.Strauss and coauthors and was formally introduces in [4]. Define  $\sigma_k^p(x) = \inf\{\|x - z\|_p : z \text{ is } k \text{ sparse}\}$ . We would like to have

$$\|x - \Delta(\Phi(x))\|_2 \leq C\sigma_k^2(x) \text{ for all } x \in \mathbb{R}^N. \quad (3)$$

Cohen Dahmen DeVore [4] showed this is impossible. So a pair random measurement matrix  $\Phi(\omega)$  and decoder  $\Delta(\omega)$  is *instance optimal in probability* (for  $k$  with constant  $C$ ) if for every  $x \in \mathbb{R}^N$  there exists a set  $\Omega(x)$  of probability very close to 1 such that for  $\omega \in \Omega(x)$  we have  $\|x - \Delta(\Phi(x))\|_2 \leq C\sigma_k^2(x)$ . This is possible – it was shown [4] that there exists such a decoder but it was totally impractical. In [16], [6] and [5] instance optimality in probability was shown for more practical decoders.

From practical point of view it is unrealistic to expect that we can get  $\Phi(x)$  exactly; generally we should expect some measurement error. So we apply our decoder to  $\Phi(x) + r$  for some  $r \in \mathbb{R}^n$  and expect the result to be close to  $x$ . For  $\ell_1$  minimisation results of this type were proved first in [16] for Gaussian measurement matrix and later in [6] for standard matrices.

## 2 Stability of $\ell_1$ minimization

Our arguments are largely geometric. On  $\mathbb{R}^n$  and  $\mathbb{R}^N$  we will consider the following well known norms  $\|\cdot\|_2$ ,  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$ . The unit balls in those norms will be denoted  $\mathcal{B}_2$ ,  $\mathcal{B}_1$  and  $\mathcal{B}_\infty$  respectively with the dimension of the space added as superscript if needed. We will also use on  $\mathbb{R}^n$  the following family of norms:  $\|x\|_{J(\alpha)} = \max(\|x\|_2, \alpha\|x\|_\infty)$  for  $\alpha > 1$ . The unit ball in this norm will be denoted  $\mathcal{B}_{J(\alpha)}$ . If  $\alpha = \sqrt{N}$  this norm will be denoted by  $\|\cdot\|_J$  and the ball  $\mathcal{B}_J$ . By  $\mathcal{S}$  we will denote the euclidean unit sphere.

In [16] and in [6] we introduced the following new geometric properties of the measurement matrix

**Definition 2.1** A matrix  $\Phi$  has  $LQ(\mu, k)$  property if for every vector  $y \in \mathbb{R}^n$  there exists  $x \in \mathbb{R}^N$  such that  $\Phi x = y$  and  $\|x\|_1 \leq \mu^{-1}\sqrt{k}\|y\|_2$ . Equivalently  $\Phi(\mathcal{B}_1^N)$  (which is the convex hull of  $(\pm\Phi_j)_{j=1}^N$ ) contains  $(\mu/\sqrt{k})\mathcal{B}_2$ .

In [16] it was proved that for some  $\mu > 0$  and  $c > 0$  and  $\delta > 0$  the Gaussian random measurement matrix satisfies  $LQ(\mu, k)$  and  $RIP(k, \delta)$  with  $k = \lfloor cn/\log(N/n) \rfloor$  with overwhelming probability.

**Definition 2.2** A matrix  $\Phi$  has  $J(\mu, k)$  property if for every vector  $y \in \mathbb{R}^n$  there is  $x \in \mathbb{R}^N$  such that  $\Phi x = y$  and  $\|x\|_1 \leq \mu^{-1}\sqrt{k} \max(\|y\|_2, \sqrt{n/k}\|y\|_\infty)$ . Equivalently  $\Phi(\mathcal{B}_1^N)$  (which is the convex hull of  $(\pm\Phi_j)_{j=1}^N$ ) contains  $(\mu/\sqrt{k})\mathcal{B}_{J(\sqrt{n/k})}$ .

In [9] (see also [6]) it was proved that any standard measurement matrix for some  $c, \mu, \delta > 0$  has  $J(\mu, k)$  and  $RIP(k, \delta)$  with  $k = \lfloor cn/\log(N/n) \rfloor$  with overwhelming probability.

Using those concepts we proved

**Theorem 2.3 ([16])** If  $\Phi$  satisfies  $RIP(k, \delta)$  and  $LQ(\mu, k)$  then there exists  $C$  such that for every  $x \in \mathbb{R}^N$  and  $r \in \mathbb{R}^n$

$$\|\Delta_1(\Phi(x) + r) - x\|_2 \leq C(\|r\|_2 + \frac{\sigma_k^1(x)}{\sqrt{k}}) \quad (4)$$

and

$$\|\Delta_1(\Phi(x) + r) - x\|_2 \leq C(\|r\|_2 + \sigma_k^2(x) + \|\Phi(x|S^c)\|_2) \quad (5)$$

where  $S$  is a  $k$  element set for which  $\sigma_k^2(x) = \|x|S^c\|_2$ .

**Theorem 2.4 ([6])** If  $\Phi$  satisfies  $RIP(k, \delta)$  and  $J(\mu, k)$  then there exists  $C$  such that for every  $x \in \mathbb{R}^N$  and  $r \in \mathbb{R}^n$

$$\|\Delta(\Phi(x) + r) - x\|_2 \leq C(\|r\|_J + \frac{\sigma_k^1(x)}{\sqrt{k}}) \quad (6)$$

and

$$\|\Delta(\Phi(x) + r) - x\|_2 \leq C(\|r\|_J + \sigma_k^2(x) + \|\Phi(x|S^c)\|_J) \quad (7)$$

where  $S$  is a  $k$  element set for which  $\sigma_k^2(x) = \|x|S^c\|_2$ .

This in particular implies that  $\Delta_1$  is instance optimal in probability for all standard matrices. But it also shows stability of this decoder with respect to the measurement error  $r$ .

Let us note that (4) and (6) build upon and improve results from [2].

Actually Theorem 2.4 was not stated in [6]. We were interested in instance optimality and did not want to get into abstract formulations. However all the arguments needed to the proof of Theorem 2.4 are in [6] (see also [16]). If we compare Theorems 2.3 and 2.4 we see that when  $r = 0$  they have the same conclusions. The only difference is in how the measurement error influences the estimate. Since  $\|x\|_2 \leq \|x\|_J$  the estimates in Theorem 2.3 are better. Later on we will comment on this in more detail. Unfortunately LQ in the maximal range of  $k$  was shown in [16] only for Gaussian standard measurement matrix and for matrix whose columns are independently drawn from  $\mathcal{S}$  (but this is quite close to Gaussian). LQ clearly fails for Bernoulli random matrix. This is the main drawback of Theorem 2.3.

### 3 Random matrices satisfying LQ

Now we want to produce a large family of measurement matrices which satisfy  $\text{LQ}(\mu, k)$  for  $k = \lfloor cn \log(N/n) \rfloor$ . First we need a unitary matrix in  $\mathbb{R}^n$ . In his fundamental paper [8] B.S. Kashin formulated the following

**Theorem 3.1** *There exists a constant  $c > 0$  such that with overwhelming probability a unitary matrix  $U$  satisfies*

$$c\|x\|_2 \leq \frac{1}{2\sqrt{n}}(\|U(x)\|_1 + \|x\|_1) \leq \|x\|_2 \quad (8)$$

for all  $x \in \mathbb{R}^n$ .

He provided only a sketch of the proof, and stated only the existence of a matrix satisfying (8) but his proof easily gives that it holds with overwhelming probability on  $U$ . By duality we get that if a unitary matrix  $U$  satisfies (8) then  $\text{conv}(U^*(\frac{1}{\sqrt{n}}B_\infty) \cup \frac{1}{\sqrt{n}}B_\infty) \supset cB_2$ . To see this note that if this is not

the case then there exists  $x$  with  $\|x\|_2 = 1$  such that  $|\langle x, U^*z \rangle| < c\sqrt{n}$  and  $|\langle x, z \rangle| < c\sqrt{n}$  for all  $z \in B_\infty$ . Thus we obtain  $\|x\|_1 < c\sqrt{n}$  and  $\|Ux\|_1 < c\sqrt{n}$  what contradicts (8). Since  $B_J \supset \frac{1}{\sqrt{n}}B_\infty$  we get

**Corollary 3.2** *With overwhelming probability on  $U$  we have  $\text{conv} U(B_J) \cup B_J \supset cB_2$ .*

We fix one such unitary matrix. Now we define a random matrix  $\Phi(\omega)$  as follows. We take two symmetric, subgaussian random variables  $\eta$  and  $\tau$  such that  $\mathbb{E}|\eta|^2 = \mathbb{E}|\tau|^2 = 1$ . Let  $\eta_{j,i}$  and  $\tau_{j,i}$  denote independent copies of  $\eta$  (resp.  $\tau$ ) and also  $\eta_{j,i}$ 's are independent from  $\tau_{i,j}$ 's. Our matrix  $\Phi$  has columns  $\phi_j$  where  $\phi_j = n^{-1/2}(\eta_{j,1}, \dots, \eta_{j,n})$  for  $j < N/2$  and  $\phi_j = n^{-1/2}U(\tau_{j,1}, \dots, \tau_{j,n})$  for  $j \geq N/2$ . From [6] we know that with overwhelming probability  $\text{conv}(\phi_j)_{j < N/2} \supset cB_J$  and  $\text{conv}(\phi_j)_{j \geq N/2} \supset cU(B_J)$  so  $\text{conv}(\phi_j)_{j=1}^N \supset c'B_2$ . So we need to show that the matrix  $\Phi$  satisfies RIP. Actually we will show the appropriate concentration inequality. Let us denote  $\Phi_1 = (\phi_j)_{j=1}^{N/2}$  and  $\Phi_2 = (\phi_j)_{j \geq N/2}$ . We know (see e.g. [6]) that each of those matrices satisfies the concentration estimate: for each  $\epsilon > 0$  there exists a constant  $c(\epsilon) > 0$  such that for each  $x$

$$\mathbb{P}(|\|\Phi_s(x)\|^2 - \|x\|^2| > \epsilon\|x\|^2) \leq 2e^{-nc(\epsilon)}. \quad (9)$$

Thus (see [1]) matrices  $\Phi_1$  and  $\Phi_2$  satisfy  $\text{RIP}(k, \delta)$ . Now given  $x \in \mathbb{R}^N$  we write  $x = x_1 + x_2$  where  $x_1$  equals  $x$  on coordinates  $< N/2$  and  $x_2$  equals  $x$  on coordinates  $\geq N/2$ . Note that

$$\begin{aligned} \|\Phi(x)\|^2 &= \|\Phi_1(x_1) + \Phi_2(x_2)\|^2 \\ &= \|\Phi_1(x_1)\|^2 + \|\Phi_2(x_2)\|^2 + 2\langle \Phi_1(x_1), \Phi_2(x_2) \rangle. \end{aligned} \quad (10)$$

Since  $\Phi_1$  and  $\Phi_2$  are independent for a fixed  $b = \Phi_2(x_2)$  we have

$$\begin{aligned} \mathbb{P}(|\langle \Phi_1(x_1), b \rangle| > \frac{\epsilon}{3}\|b\| \cdot \|x_1\|) \\ = \mathbb{P}\left(\left|\sum_i \sum_j^{N/2} x_j b_i \eta_{i,j}\right| > \frac{\epsilon}{3}\|b\| \cdot \|x_1\|\right). \end{aligned}$$

Since  $\eta_{i,j}$  are independent subgaussian variables and  $\sum_i \sum_j^{N/2} |x_j|^2 |b_i|^2 = \|b\|^2 \|x_1\|^2$  we can continue as

$$\leq 2e^{-nc(\epsilon/3)} \quad (11)$$

From (9) for  $s = 2$  we see that  $\|b\| \sim \|x_2\|$  with big probability so from (11), (11) and (9) we get

$$\mathbb{P}\left(\left|\|\Phi(x)\|^2 - \|x\|^2\right| > \epsilon \|x\|^2\right) \leq 6e^{-nc(\epsilon/3)}.$$

So we proved a concentration inequality and thus we infer (see [1]) that  $\Phi$  satisfies RIP( $k, \delta$ ) for  $\delta > 0$  with  $k = \lfloor c(\delta)n/\log(n/N) \rfloor$ .

## 4 Norm $\|\cdot\|_J$

In Theorem 2.4 the norm  $\|\cdot\|_J$  appears as a measure which estimates the influence of the measurement error on the accuracy of the recovery. Now we want to make some remarks on this norm. First we observe that

$$\|x\|_J \leq \sqrt{\log N} \|x\|_2 \quad (12)$$

so Theorem 2.4 implies for any standard measurement matrix the error estimate

$$\|\Delta(\Phi(x)+r)-x\|_2 \leq C(\sqrt{\log N}\|r\|_2 + \frac{\sigma_k^1(x)}{\sqrt{k}}) \quad (13)$$

and also shows that for each  $x \in \mathbb{R}^N$  with big probability on the draw of  $\Phi$  we have

$$\|\Delta(\Phi(x)+r)-x\|_2 \leq C(\sqrt{\log N}\|r\|_2 + \sigma_k^2(x)). \quad (14)$$

To see (14) we must use that with the big probability on the draw of  $\Phi$  we have  $\|\Phi(x|S^c)\|_J \leq \sigma_k^2(x)$ . But this was already proved in [6] Lemma 5.5. The point is that for a random vector  $z = \sum_j z_j \eta_j$  where  $\eta_j$ 's are independent symmetric random variables with the same subgaussian distribution it is very unlikely that  $\|z\|_2 < \|z\|_J$ .

But we can make the same observation about the error. If the error is as above e.g. random Gaussian it is exponentially unlikely that  $\|z\|_2 < \|z\|_J$ . The same is true if our error is uniformly drawn from  $\mathcal{S}$ ,  $\sigma$  is the normalized surface measure on  $\mathcal{S}$ . The following is well known (see e.g. [12, p.5])

**Lemma 4.1** For  $0 \leq \alpha \leq 1$

$$\sigma(\{x \in \mathcal{S} : x_1 \geq \alpha\}) \leq \sqrt{\frac{\pi}{2}} e^{-\frac{\alpha^2 n}{2}}. \quad (15)$$

Note that  $\{x \in \mathcal{S} : x_1 \geq \alpha\} \subset \{x \in \mathcal{S} : x_1 \geq \sin \alpha\}$ .

Let us denote  $I_n = \int_{-\pi/2}^{\pi/2} \cos^n \theta d\theta$ . From classical formulas for  $\sigma$  we get

$$\sigma(\{x \in \mathcal{S} : x_1 \geq \sin \alpha\}) = I_n^{-1} \int_{\alpha}^{\pi/2} \cos^n \theta d\theta.$$

Since  $\cos \theta \leq e^{-\theta^2/2}$  for  $0 \leq \theta \leq \pi/2$  we get

$$\begin{aligned} \int_{\alpha}^{\pi/2} \cos^n \theta d\theta &= \frac{1}{\sqrt{n}} \int_{\alpha\sqrt{n}}^{\sqrt{n}\pi/2} \cos^n\left(\frac{t}{\sqrt{n}}\right) dt \\ &\leq \frac{1}{\sqrt{n}} \int_{\alpha\sqrt{n}}^{\sqrt{n}\pi/2} e^{-t^2/2} dt \end{aligned}$$

and substituting  $u = t - \alpha\sqrt{n}$  we can continue

$$\leq \frac{1}{\sqrt{n}} e^{-\alpha^2 n/2} \int_0^{\infty} e^{-u^2/2} du = \sqrt{\frac{\pi}{2n}} e^{-\alpha^2 n/2}.$$

To estimate  $I_n$  from below we integrate by parts to get  $I_n = \frac{n-2}{n-1} I_{n-2} \geq \sqrt{\frac{n-3}{n-1}} I_{n-2}$ . From this we get  $\sqrt{n} I_n \geq 1$ .

Now if  $\|z\|_J > \|z\|_2$  then for some coordinate  $s$  we have  $|x_s| > \frac{1}{\sqrt{\log N}}$ . The probability that this happens is at most

$$2n\sigma\left\{x \in \mathcal{S} : x_1 > \frac{1}{\sqrt{\log N}}\right\} \leq 2n\sqrt{\frac{\pi}{2}} e^{-\frac{n}{2\log N}}.$$

## 5 New algorithm

Now we want to suggest a modified  $\ell_1$  minimization algorithm which will have stability properties in  $\ell_2$  norm. The main point in our improvement is that we do not change the measurement matrix as it may be prescribed by other considerations. So suppose we have a fixed measurement matrix  $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}^n$  satisfying RIP( $k, \delta$ ) for  $k = \lfloor cn/\log n/N \rfloor$  and some  $\delta > 0$ . Given a signal  $x \in \mathbb{R}^N$  we observe  $y = \Phi(x)$  or  $y = \Phi(x) + r$ . Our algorithm works as follows:

1. We fix at random  $n \times N$  Gaussian matrix  $\Psi = \left(\frac{1}{\sqrt{n}} \gamma_{i,j}\right)_{i=1, j=1}^{n, N}$ .

- We consider  $n \times 2N$  matrix  $\Gamma = (\Phi, \Psi)$  and solve the  $\ell_1$  minimization problem

$$\bar{z} = \text{Argmin } \{z \in \mathbb{R}^{2N} : \Gamma(z) = y\}$$

- We define the decoder as  $\Delta(y) = \bar{z}|_{\{1, \dots, N\}}$

If the matrix  $\Phi$  is not fixed but is a random standard measurement matrix we proceed exactly the same but we must make sure that  $\Psi$  and  $\Phi$  are independent.

The main observation is that with big probability on the draw of  $\Psi$  (or with big probability on the joint draw of  $\Phi$  and  $\Psi$ ) the matrix  $\Gamma$  satisfies RIP and LQ for some constants and  $k = \lfloor cn/\log N/n \rfloor$ . Since  $\|x - \Delta(y)\| \leq \|x - \bar{z}\|$  for every norm we are considering we get:

- $\|x - \Delta(y)\|_2 \leq C(\|r\|_2 + \frac{\sigma_k^1(x)}{\sqrt{k}})$
- If  $\Phi$  is random then for every  $x \in \mathbb{R}^N$  with big probability we get

$$\|x - \Delta(x)\|_2 \leq C(\|r\|_2 + \sigma_k^2(x)).$$

This is an immediate consequence of Theorem 2.3 once we show that the matrix  $\Gamma$  with big probability satisfies RIP and LQ for some constants and  $k = \lfloor cn/\log N/n \rfloor$ . But  $\Gamma$  has LQ because  $\Psi$  has. We will show that  $\Gamma$  has RIP( $k, \frac{4}{5}\delta$ ). The argument for this is standard so we will provide only a sketch.

- We take  $k$  columns from  $\Gamma$ . If all are columns of  $\Phi$  or  $\Psi$  then we are done, so assume we have  $k > s > 0$  such columns  $(\phi_j)_{j \in A}$  from matrix  $\Phi$  and  $k - s$  columns  $(\psi_j)_{j \in B}$  from  $\Psi$ . There are  $\sum_{s=1}^{k-1} \binom{N}{s} \binom{N}{k-s} < N^{k-1}$  such possibilities.

- Let  $X = \text{span}(\phi_j)_{j \in A}$  and  $Z = \text{span}(\psi_j)_{j \in B}$ . In order to show RIP it suffices to show that  $|\langle x, z \rangle| \leq \frac{\delta}{10} \|x\| \|z\|$  for  $x \in X$  and  $z \in Z$ .

- It is sufficient to show this for  $x$  and  $z$  from  $\eta$  nets in the unit balls (with appropriate  $\eta$ . Since (see e.g. [10, Ch. 15 Prop. 1.3.]) the unit ball in  $d$  dimensional space has an  $\eta$  net of cardinality not exceeding  $(6/\eta)^d$  applying this to  $X$  and  $Z$  we see that we must to consider  $(6/\eta)^k$  pairs  $x, z$ .

- A standard estimate shows that  $\mathbb{P}(|\langle x, z \rangle| \geq \lambda \|x\| \|z\|) \leq 4e^{-\lambda^2 n/2}$ . This is true both when  $\Phi$  is fixed and when  $\Phi$  is random.

- We put those estimates together (see e.g. [1]) and infer that we can find  $k$  of the desired magnitude.

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