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► **To cite this version:**

Stéphane Chrétien. The Two Stage l1 Approach to the Compressed Sensing Problem. Rémi Gribonval. SPARS'09 - Signal Processing with Adaptive Sparse Structured Representations, Apr 2009, Saint Malo, France. 2009. <inria-00369534>

HAL Id: inria-00369534

<https://hal.inria.fr/inria-00369534>

Submitted on 20 Mar 2009

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The Two Stage l_1 Approach to the Compressed Sensing Problem

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Résumé—This paper gives new results on the recovery of sparse signals using l_1 -norm minimization. We introduce a two-stage l_1 algorithm equivalent to the first two iterations of the alternating l_1 relaxation introduced in [1].

I. INTRODUCTION

The Compressed sensing problem is currently the focus of an extensive research activity and can be stated as follows : Given a sparse vector $x^* \in \mathbb{R}^n$ and an observation matrix $A \in \mathbb{R}^{m \times n}$ with $m \ll n$, try to recover the vector x from the small measurement vector $y = Ax^*$. Although the problem consists of solving an overdetermined system of linear equations, enough sparsity will allow to succeed as shown by the following lemma (where Σ_s will denote the set of all s -sparse vectors, i.e. vectors whose components are all zero except for at most s of them),

Lemma I.1 [2] *If A is any $m \times n$ matrix and $2s \leq m$, then the following properties are equivalent :*

i. The decoder $\Delta_0(y)$ given by

$$\Delta_0(y) = \operatorname{argmin}_{x \in \mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad Ax = y. \quad (1)$$

k satisfies $\Delta_0(Ax) = x$, for all $x \in \Sigma_s$,

ii. For any set of indices T with $\#T = 2k$, the matrix A_T has rank $2s$ where A_T stands for the submatrix of A composed of the columns indexed by T only.

A. The l_1 and the Reweighted l_1 relaxations

The main problem with decoder Δ_0 is that the optimization problem (1) is in general NP-hard. For this reason, the now standard l_1 relaxation strategy is adopted, i.e. the decoder $\Delta_1(y)$ is obtained as

$$\Delta_1(y) = \operatorname{argmin}_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad Ax = y. \quad (2)$$

Now, solving (2) can be done in polynomial time and thus $\Delta_1(y)$ can be efficiently computed. The second problem is to give robust conditions under which exact recovery holds. One such condition was given by Candes Romberg and Tao [3] and is now known as the Uniform Uncertainty Principle (UUP) or as the Restricted Isometry Property (RIP).

One of the main remaining challenges is to reduce the number of observations m needed to recover a given sparse

signal x . One idea is the use of l_p , $p < 0 < 1$ decoders $\Delta_p(y)$. The main drawback of the approach using l_p , $p < 0 < 1$ norm minimization is that the resulting decoding scheme is again NP-Hard. Another idea is to use a reweighted l_1 approach as proposed in [4].

Algorithm 1 Reweighted l_1 algorithm (Rew- l_1)

Input $u > 0$ and $L \in \mathbb{N}_*$
 $z_u^{(0)} = e$
 $x_u^{(0)} \in \min_{x \in \mathbb{R}^n, Ax=y} \|x\|_1$
 $l = 1$
while $l \leq N$ **do**
 $z_u^{(l)} = \frac{1}{|x^{(l)}|+u}$ componentwise
 $x_u^{(l)} \in \operatorname{argmax}_{x \in \mathbb{R}^n, Ax=y} \sum_{i=1}^n w_i^{(l-1)} |x_i|$
 $l \leftarrow l + 1$
end while
Output $z_u^{(L)}$ and $x_u^{(L)}$.

The main intuition behind this reweighted l_1 relaxation is the following. The greater the component x_i becomes, the smaller weight it should receive since it can be considered that this component should not be set to zero.

The main drawback of the reweighted l_1 approach is that an unknown parameter is to be tuned whose order of magnitude is hard to know ahead of time.

B. The Alternating l_1 algorithm

Another approach was proposed in [1] and uses Lagrange duality. Let us write down problem (1), to which Δ_0 is the solution map, as the following equivalent problem

$$\max_{z \in \{0,1\}^n, x \in \mathbb{R}^n} e^t z \quad \text{s.t.} \quad z_i x_i = 0, \quad i = 1, \dots, n, \quad Ax = y$$

where e denotes the vector of all ones. Here since the sum of the z_i 's is maximized, the variable z plays the role of an indicator function for the event that $x_i = 0$. This problem is clearly nonconvex due to the quadratic equality constraints $z_i x_i = 0$, $i = 1, \dots, n$. However, these constraints can be merged into the unique constraint $\|D(z)x\|_1 = 0$, leading to the following equivalent problem

$$\max_{z \in \{0,1\}^n, x \in \mathbb{R}^n} e^t z \quad \text{s.t.} \quad \|D(z)x\|_1 = 0, \quad Ax = y. \quad (3)$$

The Alternating l_1 algorithm consists of a suboptimal alternating minimization procedure to approximate the dual function at u . The algorithm is as follows.

Algorithm 2 Alternating l_1 algorithm (Alt- l_1)

Input $u > 0$ and $L \in \mathbb{N}_*$
 $z_u^{(0)} = e$
 $x_u^{(0)} \in \max_{x \in \mathbb{R}^n, Ax=y} \mathcal{L}(x, z^{(0)}, u)$
 $l = 1$
while $l \leq N$ **do**
 $z_u^{(l)} \in \operatorname{argmax}_{z \in \{0,1\}^n} \mathcal{L}(x_u^{(l)}, z, u)$
 $x_u^{(l)} \in \operatorname{argmax}_{x \in \mathbb{R}^n, Ax=y} \mathcal{L}(x, z_u^{(l)}, u)$
 $l \leftarrow l + 1$
end while
Output $z_u^{(L)}$ and $x_u^{(L)}$.

Notice that, similarly to the reweighted l_1 algorithm, the Alternating l_1 method also requires the tuning of an unknown parameter u . However, the main motivation for this proposal is that this parameter u has a clear meaning : it is a dual variable which, in the case where the dual function $\theta(u)$ is well approximated by the sequence $\mathcal{L}(x^{(l)}, z^{(l)}, u)$, can be efficiently optimized without additional prior information, due to the convexity of the dual function.

II. THE TWO STAGE l_1 METHOD

The main remark about the alternating l_1 method is the following (see [1]) : for a given dual variable u , the alternating l_1 algorithm can be seen a sequence $(x_u^{(l)})_{l \in \mathbb{N}}$ of truncated l_1 -norm minimizers of the type

$$x_u^l = \operatorname{argmin}_{x \in \mathbb{R}^n} \|x_{T_u^l}\|_1 \quad \text{s.t.} \quad Ax = y. \quad (4)$$

where T_u^l is the set of indices for which $|x_i^{l-1}| < \frac{1}{u}$. Therefore, the Alternating l_1 algorithm can be seen as an iterative thresholding scheme with threshold value equal to $\frac{1}{u}$. Now assume for instance that a fraction ρm of the non zero components is well identified by the plain l_1 step with solution $x^{(l)}$. Then, the practitioner might ask if the appropriate value for u is the one which imposes an l_1 penalty on the index set corresponding to the $n - \rho m$ smallest components of $x^{(l)}$. Moreover, the large scale simulation experiments which have been performed on the plain l_1 relaxation seemed to agree on the fact that the breakdown point occurs near $\frac{m}{4}$. Thus, a practitioner could be tempted to wonder whether $\rho = \frac{1}{4}$ is a sensible value. Motivated by the previous practical considerations, the two stage l_1 algorithm is defined as follows (the parameter u is now replaced by the parameter $\rho = \frac{1}{u}$).

Notice that we restrict ρ to lie in $(0, \frac{1}{2})$. The reason should be obvious since, due to Lemma I.1, even decoder $\Delta_0(y)$ is unable to identify more than $\frac{m}{2}$ -sparse vectors. Another remark is that the procedure could be continued for more than 2 steps but simulation experiments of the Alternating l_1 method seem to confirm that in most cases two steps suffice to converge.

Algorithm 3 Two stage l_1 algorithm (2Stage- l_1)

Input $\rho \in (0, \frac{1}{2})$
Step 0 : $x^{(0)} \in \operatorname{argmax}_{x \in \mathbb{R}^n, Ax=y} \|x\|_1$ and $T = \text{index set of the } \rho m \text{ largest components of } x^{(0)}$
Step 1 : $x^{(1)} \in \operatorname{argmax}_{x \in \mathbb{R}^n, Ax=y} \|x_{T^c}\|_1$
Output $x_\rho^{(1)}$.

III. MAIN RESULTS

At Step 1 of the method, a subset T is selected with cardinal ρm and optimization is then performed with objective function $\|x_{T^c}\|_1$. In this section, we will adopt the following notations : S will denote the support of x^* , T will denote the index set of the ρm largest components of $x^{(0)}$ as defined in the two-stage l_1 algorithm. T_g^c will be an abbreviation of $(T^c)_g$, the "good" subset of T^c or, in mathematical terms, the subset of indices of S which also belong to T^c . On the other hand, T_b^c will denote the complement of T_g^c in T^c .

Lemma III.1 *Assume the cardinal of T_g^c is less than $\gamma/2$ and that A satisfies $RIP(\delta, \gamma s)$. Let $h^{(1)} = x^{(1)} - x^*$. Then, there exists a positive number C^* depending on x^* such that $\|h_T^{(1)}\|_1 \leq C^* \|x_{T_b^c}^*\|_1$. Moreover, if $\|h_T^{(1)}\|_1 = 0$, then $\|x_{T_b^c}^*\|_1 = 0$.*

Proof. Let $N(h_T)$ denote the optimal value of the problem

$$\min_{h_{T_b^c} \in \mathbb{R}^{n-s}} \|x_{T_b^c}^* + h_{T_b^c}\|_1 \quad (5)$$

subject to

$$A_{T_b^c}(x_{T_b^c}^* + h_{T_b^c}) + A_{T_g^c}x_g^{(1)} = y - A_T(x_T^* + h_T).$$

Assume that $x_{T_b^c}^* = 0$. Then, $N(h_T)$ plays the role of a norm for h_T although it does not satisfy the triangle inequality. In particular, $N(h_T)$ is nonnegative, convex and $N(h_T) = 0$ implies that $h_T = 0$.

Nonnegativity and convexity are straightforward. Assume that $N(h_T) = 0$, i.e. the solution \tilde{h} of (5) is null. This implies that $A_{T_g^c}x_{T_g^c}^{(1)} = y - A_Tx_T^* = A_{T_b^c}x_{T_b^c}^*$, which implies that $x_{T_g^c}^{(1)} - x_{T_b^c}^*$ is in the kernel of $A_{T_g^c}$. Using the fact that T_g^c has cardinal less than $\gamma s/2$ and the $RIP(\delta, \gamma s)$ assumption, we conclude that $x_{T_g^c}^{(1)} = x_{T_b^c}^*$. In order to finish the proof of the lemma, it remains to recall that $N(h_T)$ is convex and that, by Theorem 1.1 in [5], $\|h^{(1)}\|_1$ (and thus $\|h_T^{(1)}\|_1$) is bounded from above by $C \inf_{\#U \leq \gamma/2} \|x_{U^c}\|_1$ in order to obtain existence of a sufficiently small positive constant C^* depending on x^* such that $N(h_T) \geq C^* \|h_T\|_1$ for all h_T in the ball $B(0, C \|x^*\|_1)$. The desired result then follows.

To prove that $N(h_T) = 0$ if $h_T = 0$ is a bit harder. Thus, assume that $h_T = 0$. Then, the solution \tilde{h} of (5) is just the solution of

$$\min_{h_{T_b^c} \in \mathbb{R}^{n-s}} \|h_{T_b^c}\|_1 \quad A_{T_b^c}h_{T_b^c} = y - A_T - x_T^* - A_{T_g^c}x_g^{(1)}.$$

Now since $y = Ax^*$, we obtain that $y - A_T x_T^* - A_{T_g^c} x_g^{(1)} = A_{T_g^c} (x_{T_g^c}^* - x_g^{(1)})$ and thus, the right hand side term is nothing but the image of a $\gamma s/2$ -sparse vector. Now, recalling that we assumed $RIP(\delta, \gamma s)$, Theorem 1.1 in [5] implies that \tilde{h} must be the sparsest solution of the system $A_{T_g^c} h_{T_g^c} = y - A_T x_T^* - A_{T_g^c} x_g^{(1)}$ from which we deduce that \tilde{h} is $\gamma s/2$ -sparse. Therefore the vector $(h_{T_g^c}, x_{T_g^c}^{(1)})$ is γs which solves $A_{T_g^c} x_{T_g^c} = y - A_T x_T^*$. On the other hand, $x_{T_g^c}^*$ also solves $A_{T_g^c} x_{T_g^c} = y - A_T x_T^*$ and its support is included in the support of $(h_{T_g^c}, x_{T_g^c}^{(1)})$. Therefore, $(h_{T_g^c}, x_{T_g^c}^{(1)}) - x_{T_g^c}^*$ is a $\gamma s/2$ sparse vector which lies in the kernel of A . Using again the fact that $RIP(\delta, \gamma s)$ holds, we conclude that $(h_{T_g^c}, x_{T_g^c}^{(1)}) - x_{T_g^c}^* = 0$. Thus, $h_{T_g^c} = 0$ and $x_{T_g^c}^{(1)} = x_{T_g^c}^*$. \square

Using this lemma, we deduce the following theorem.

Theorem III.1 Assume that $RIP(\delta, \gamma s)$ holds and that an index set T_g of cardinal greater than or equal to $(1 - \gamma/2)s$ has been recovered at Step 0 after thresholding, then $x^{(1)}$ satisfies

$$\|x^{(1)} - x^*\|_1 \leq C^{**} \|x_{T_g^c}^*\|_1.$$

for some constant C^{**} depending on x^* .

Proof. The vector $x^{(1)}$ satisfies

$$\|x_{T_g^c}^{(1)}\|_1 \leq \|x_{T_g^c}^*\|_1. \quad (6)$$

Let us write $h^{(1)} = x^{(1)} - x^*$. Using (6), a now standard decomposition gives

$$\|x_{T_g^c}^*\|_1 - \|h_{T_g^c}\|_1 + \|h_{T_g^c}\|_1 - \|x_{T_g^c}^*\|_1 \leq \|x_{T_g^c}^*\|_1 + \|x_{T_g^c}^*\|_1.$$

We thus obtain

$$\|h_{T_g^c}\|_1 \leq \|h_{T_g^c}\|_1 + 2\|x_{T_g^c}^*\|_1. \quad (7)$$

However, since $RIP(\delta, \gamma s)$ holds, $NSP(C, \gamma/2s)$ holds too, with $C < 1$. Therefore, we obtain that

$$\|h_{T_g^c}\|_1 \leq C(\|h_{T_g^c}\|_1 + \|h_T\|_1). \quad (8)$$

Combining (7) and (8), we obtain

$$\|h_{T_g^c}\|_1 \leq \frac{C}{1-C} \|h_T\|_1 + \frac{2}{1-C} \|x_{T_g^c}^*\|_1. \quad (9)$$

As a consequence, we obtain that

$$\begin{aligned} \|h\|_1 &\leq C(2\|x_{T_g^c}^*\|_1 + C'\|h_T\|_1) + \|h_T\|_1 + 2\|x_{T_g^c}^*\|_1 \\ &\quad + C\|h_T\|_1 + \|h_T\|_1 \\ &= (1 + C + CC')\|h_T\|_1 + 2(1 + C)\|x_{T_g^c}^*\|_1. \end{aligned}$$

which, using Lemma III.1, implies

$$\|h\|_1 \leq ((1 + C + CC')C^* + 2(1 + C))\|x_{T_g^c}^*\|_1. \quad (10)$$

which is the desired bound. \square

The following corollary is a straightforward consequence of the previous theorem.

Corollary III.1 Assume that the assumptions of Theorem III.1 are satisfied. Then, exact reconstruction is obtained if $x_{T_g^c}^* = 0$, i.e. x^* is s -sparse.

IV. MONTE CARLO EXPERIMENTS

The following Monte Carlo experiments show that the performance of the two-stage l_1 algorithm which drops the penalty over the index set of the $m/4$ largest components of the solution of plain l_1 are almost as good as the performance of the reweighted l_1 with the best parameter which is usually unknown in practice. A Python program is available at <http://stephane.g.chretien.googlepages.com/alternatingl1> and can be used to perform these experiments and other involving the Alternating l_1 algorithm.

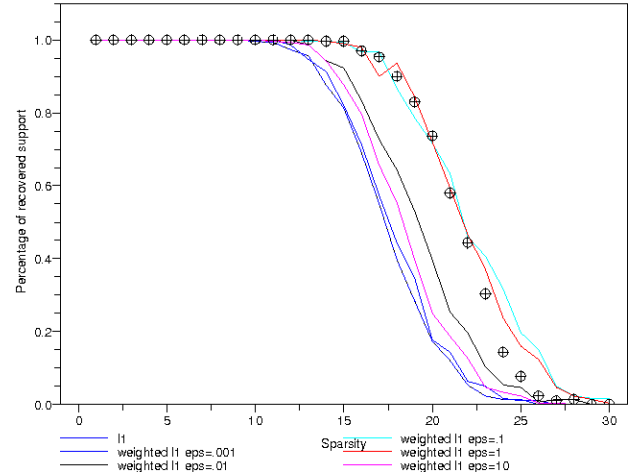


FIG. 1. Rate of success over 300 Monte Carlo experiments in recovering the support of the signal vs. signal sparsity k for $n = 128$, $m = 50$, $L = 4$, $u = 3$. A and nonnull components of x were drawn from the gaussian $\mathcal{N}(0, 1)$ distribution. The results for the two-stage l_1 method are represented by the "+" in a circle" sign.

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