Greedy Deconvolution of Point-like Objects

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***Abstract*—The orthogonal matching pursuit (OMP) is an algorithm to solve sparse approximation problems. In [1] a sufficient condition for exact recovery is derived, in [2] the authors transfer it to noisy signals. We will use OMP for reconstruction of an inverse problem, namely the deconvolution problem.**

In sparse approximation problems one often has to deal with the problem of redundancy of a dictionary, i.e. the atoms are not linearly independent. However, one expects them to be approximatively orthogonal and this is quantified by incoherence.

This idea cannot be transfered to ill-posed inverse problems since here the atoms are typically far from orthogonal: The ill-posedness of the (typically compact) operator causes that the correlation of two distinct atoms probably gets huge, i.e. that two atoms can look much alike. Therefore in [3], [4] the authors derive a recovery condition which uses the kind of structure one assumes on the signal and works without the concept of coherence.

In this paper we will transfer these results to noisy signals. For our source we assume that it consists of a superposition of point-like objects with an a-priori known distance. We will apply it exemplarily to Dirac peaks convolved with Gaussian kernel as used in mass spectrometry.

I. Introduction

Consider a signal u in a Banach space B which is sparse in some unit-normed dictionary $\mathcal{E} := \{e_i\}_{i\in\mathbb{Z}} \subset B$. With sparse we mean here that there exists a finite decomposition of u with N atoms $e_i \in \mathcal{E}$,

$$u = \sum_{i\in\mathbb{Z}} \alpha_i e_i \quad \text{with} \quad \alpha_i \in \mathbb{R}, \quad \|\alpha\|_{\ell^0} =: N < \infty.$$

In the following we denote with I the support of α. This setting corresponds with a lot of signal processing problems, e.g. with mass spectrometry [5] where the signal is modeled as a sum of Dirac peaks (so-called impulse trains):

$$u = \sum_{i\in\mathbb{Z}} \alpha_i \, \delta(\cdot - x_i).$$

Other applications for instance can be found in astronomical signal processing problems or digital holography, cf. [6], where images arise as superposition of characteristic functions of balls with different centers x_i and radii r_j,

$$u = \sum_{\substack{i\in\mathbb{Z}^2\\ j\in\mathbb{N}}} \alpha_{i,j} \, \chi_{B_{r_j}}(\cdot - x_i).$$

In this paper, to such an element $u \in B$, we consider the inverse convolution problem

$$Ku := u * k = v \tag{1}$$

with the convolution operator K with kernel k. The operator K maps from the Banach space B in a certain Hilbert space H. Typically K is compact and hence the operator equation (1) is not continuously invertible, i.e. the solution does not depend continuously on the data. This turns out to be a challenge for the case where only noisy data v^ε with noise level $\|v - v^\varepsilon\| \leq \varepsilon$ are available—as it is always the case in praxis. First a small perturbation ε can cause an arbitrarily large error in the reconstruction u of "$Ku = v^\varepsilon$" and second no solution u exists if v^ε is not in the range of K.

In the following, the solution of (1) shall be found via deriving iteratively the correlation between residual and the unit-normed atoms of the dictionary

$$\mathcal{D} := \{d_i\}_{i\in\mathbb{Z}} := \Big\{\frac{Ke_i}{\|Ke_i\|}\Big\}_{i\in\mathbb{Z}} = \Big\{\frac{e_i * k}{\|e_i * k\|}\Big\}_{i\in\mathbb{Z}}.$$

Remark that since the convolution operator K is injective we get that $e_i * k \neq 0$ for all $i \in \mathbb{Z}$ and hence the dictionary $\mathcal{D}$ is well defined. To stabilize the inversion of "$Ku = v^\varepsilon$" the iteration has be stopped early enough.

For deconvolution of noiseless data and the case only noisy data v^ε with bounded noise $\|v - v^\varepsilon\| \leq \varepsilon$ are available we will use the orthogonal matching pursuit, first proposed in the signal processing context by Davis et al. in [7] and Pati et al. in [8] independently:

Algorithm I.1 (Orthogonal Matching Pursuit)**.**
Set $k := 0$ and $I^0 := \emptyset$. Initialize $r^0 := v^\varepsilon$ resp. v and $\widehat{u}^0 := 0$.
While $\|r^k\| > \varepsilon$ (resp. $\|r^k\| \neq 0$ for $\varepsilon = 0$)

$$\begin{aligned} k &:= k + 1, \\ i_k &:= \operatorname{argsup}\big\{|\langle r^{k-1}, d_i\rangle| \,\big|\, d_i \in \mathcal{D}\big\}, \\ I^k &:= I^{k-1} \cup \{i_k\}, \end{aligned}$$

Project u onto $\operatorname{span} \mathcal{E}(I^k) := \{e_i \in \mathcal{E} \,|\, i \in I^k\}$, i.e.

$$\begin{aligned} \widehat{u}^k &:= \operatorname{argmin}\big\{\|v^\varepsilon - K\widehat{u}\|^2 \,\big|\, \widehat{u} \in \operatorname{span} \mathcal{E}(I^k)\big\}, \\ r^k &:= v^\varepsilon - K\widehat{u}^k. \end{aligned}$$

Remark that in infinite dimensional Hilbert spaces the supremum

$$\sup\{|\langle r^{k-1}, d_i\rangle| \,|\, d_i \in \mathcal{D}\} \tag{2}$$

does not have to be realized. Because of that OMP has a variant—called weak orthogonal matching pursuit (WOMP)—which does not choose the optimal atom in the sense of (2) but only one that is nearly optimal, i.e. $i_k \in \mathbb{Z}$ with

$$|\langle r^{k-1}, d_{i_k}\rangle| \geq \omega \sup\{|\langle r^{k-1}, d_i\rangle| \mid d_i \in \mathcal{D}\},$$

with a fixed weakness parameter $\omega \in (0, 1]$.

In [1] a sufficient condition for exact recovery with algorithm I.1 is derived, in [2] the authors transfer it to noisy signals with the concept of coherence, which quantifies the magnitude of redundancy. This idea cannot be transfered to ill-posed inverse problems—such as (1)—since the (typically compact) operator causes that the correlation of two distinct atoms gets huge. Therefore in [3], [4] the authors derive a recovery condition which works without the concept of coherence. For a comprehensive presentation of OMP cf. eg. [9].

In this paper in section II we will first reflect the results for OMP derived in [1] and [3], [4]. In section III we will transfer these results to noisy signals with the techniques of [2]. In section IV we will apply these results to one example from mass spectrometry. Here, the data are given as sums of Dirac peaks convolved with a Gaussian kernel. To the end of this section we will utilize the deduced condition for simulated data of an isotope pattern. For another example—characteristic functions convolved with an oscillating Fresnel kernel as appear in digital holography—see [10].

II. Exact Recovery Conditions

In [1], Tropp gives a sufficient and necessary condition for exact recovery with OMP. Next, we list this result in the language of infinite dimensional inverse problems.

The OMP chooses the right atom d_i in step $k+1$ if

$$\sup_{i\in I} |\langle r^k, d_i\rangle| > \sup_{i\notin I} |\langle r^k, d_i\rangle|.$$

Define the linear continuous synthesis operator for the dictionary $\mathcal{D} = \{d_i\} = \left\{\frac{Ke_i}{\|Ke_i\|}\right\}$ via

$$\begin{array}{llll} D: & \ell^1 & \to H, \\ & (\beta_i)_{i\in\mathbb{Z}} & \mapsto \sum\limits_{i\in\mathbb{Z}} \beta_i d_i = \sum\limits_{i\in\mathbb{Z}} \beta_i \frac{Ke_i}{\|Ke_i\|}. \end{array}$$

Since D is linear and bounded, the Banach space adjoint operator

$$D^* : H \to (\ell^1)^* = \ell^\infty$$

exists and arises as

$$D^* v = (\langle v, d_i\rangle)_{i\in\mathbb{Z}} = \left(\left\langle v, \frac{Ke_i}{\|Ke_i\|}\right\rangle\right)_{i\in\mathbb{Z}}.$$

With this definition Tropp characterizes a correct choice of the OMP in the $(k+1)$-th step via

$$\frac{\|P_{I^\complement} D^* r^k\|_{\ell^\infty}}{\|P_I D^* r^k\|_{\ell^\infty}} < 1,$$

where P_I and $P_{I^\complement}$ are the projection onto I and $I^\complement := \mathbb{Z} \setminus I$, respectively. With that he derives a condition which ensures the exact recovery with OMP:

Theorem II.1 (Tropp [1])**.** *Let $\alpha \in \ell^0$ with $\operatorname{supp}\alpha = I$, $u = \sum_{i\in\mathbb{Z}} \alpha_i e_i$ be the source and $v = Ku$ the measured signal. If for the operator $K : B \to H$ and the dictionary $\mathcal{E} = \{e_i\}_{i\in\mathbb{Z}}$ the* Exact Recovery Condition (ERC)

$$\sup_{d\in\mathcal{D}(I^\complement)} \|(DP_I)^\dagger d\|_{\ell^1} < 1 \tag{3}$$

holds, then OMP recovers α.

Theorem II.1 gives a sufficient condition for exact recovery with OMP. In [1] Tropp shows that condition (3) is even necessary, i.e. if

$$\sup_{d\in\mathcal{D}(I^\complement)} \|(DP_I)^\dagger d\|_{\ell^1} \geq 1$$

then there exists a signal with support I for which OMP does not recover α with $v = Ku = K\sum \alpha_i e_i$.

The expression in condition (3) is hard to evaluate. Therefore Dosall and Mallat [3] and Gribonval and Nielsen [4] derive a weaker sufficient but not necessary recovery condition that depends on inner products of the dictionary atoms of $\mathcal{D}(I)$ and $\mathcal{D}(I^\complement)$.

Proposition II.2 (Dosall and Mallat [3], Gribonval and Nielsen [4])**.** *Let $\alpha \in \ell^0$ with $\operatorname{supp}\alpha = I$, $u = \sum_{i\in\mathbb{Z}} \alpha_i e_i$ be the source and $v = Ku$ the measured signal. If for the operator $K : B \to H$ and the basis $\mathcal{E} = \{e_i\}_{i\in\mathbb{Z}}$ the* Neumann ERC

$$\sup_{i\in I} \sum_{\substack{j\in I\\ j\neq i}} |\langle d_i, d_j\rangle| + \sup_{i\in I^\complement} \sum_{j\in I} |\langle d_i, d_j\rangle| < 1 \tag{4}$$

holds, then OMP recovers α.

Proof: By theorem II.1, OMP recovers right, if

$$\|(P_I D^* D P_I)^{-1}\|_{\ell^1,\ell^1} \sup_{i\in I^\complement} \|P_I D^* d_i\|_{\ell^1} < 1.$$

The condition (4) in particular implies

$$\sup_{i\in I} \sum_{\substack{j\in I\\ j\neq i}} |\langle d_i, d_j\rangle| < 1,$$

hence by Neumann series we can estimate the first term via

$$\|(P_I D^* D P_I)^{-1}\|_{\ell^1,\ell^1} \leq \frac{1}{1 - \sup\limits_{i\in I} \sum\limits_{\substack{j\in I\\ j\neq i}} |\langle d_i, d_j\rangle|}. \tag{5}$$

With that and rewriting the second term,

$$\sup_{i\in I^\complement} \|P_I D^* d_i\|_{\ell^1} = \sup_{i\in I^\complement} \sum_{j\in I} |\langle d_i, d_j\rangle|,$$

we get the assumption. ∎

Remark II.3. Obviously the condition (4) is not necessary for exact recovery. Assume $I \subset \mathbb{Z}$ with

$$\sup_{i\in I^\complement} \sum_{j\in I} |\langle d_i, d_j\rangle| = 0 \quad \text{and} \quad \sup_{i\in I} \sum_{\substack{j\in I\\ j\neq i}} |\langle d_i, d_j\rangle| \geq 1.$$

Here the Neumann ERC fails but for any signal with support I OMP will recover exactly since the atoms d_i, $i \in I$, and d_j, $j \in I^{\complement}$, are uncorrelated and OMP never chooses an atom twice.

Remark II.4. The sufficient conditions for WOMP with weakness parameter $\omega \in (0, 1]$ are

$$\sup_{d \in \mathcal{D}(I^{\complement})} \|(DP_I)^{\dagger} d\|_{\ell^1} < \omega$$

and

$$\sup_{i \in I} \sum_{\substack{j \in I \\ j \neq i}} |\langle d_i, d_j \rangle| + \frac{1}{\omega} \sup_{i \in I^{\complement}} \sum_{j \in I} |\langle d_i, d_j \rangle| < 1,$$

according to theorem II.1 and for proposition II.2, respectively. They are proved analogously to the OMP case—same as all other following WOMP results.

III. Exact Recovery in Presence of Noise

In [2], Donoho, Elad and Temlyakov transfer Tropp's result [1] to noisy signals. They derive a condition in terms of incoherence of a dictionary. This condition is—just as remarked in [2]—an obvious weaker condition. As already mentioned, in particular for ill-posed problems this condition is too restrictive. In the following we will give exact recovery conditions in presence of noise which is closer to the results of theorem II.1 and proposition II.2.

Assume that instead of exact data $v = Ku \in H$ only a noisy version

$$v^{\varepsilon} = v + \eta = Ku + \eta$$

with noise level $\|v - v^{\varepsilon}\| = \|\eta\| \leq \varepsilon$ can be observed. Now, the greedy method has to stop as soon as the representation error r^k is smaller or equal to the noise level ε, i.e. if $\varepsilon \geq \|r^k\|$. With these assumptions similar estimations as in [1] lead to

Theorem III.1 (ERC in Presence of Noise)**.** *Let $\alpha \in \ell^0$ with $\operatorname{supp} \alpha = I$. Let $u = \sum_{i \in \mathbb{Z}} \alpha_i e_i$ be the source and $v^{\varepsilon} = Ku + \eta$ the noisy data with noise level $\|\eta\| \leq \varepsilon$ and noise-to-signal-ratio*

$$r_{\varepsilon/\alpha} := \frac{\sup\limits_{i \in \mathbb{Z}} |\langle \eta, d_i \rangle|}{\min\limits_{i \in I} |\alpha_i| \|Ke_i\|}.$$

If for the operator $K : B \to H$ and the basis $\mathcal{E} = \{e_i\}_{i \in \mathbb{Z}}$ the Exact Recovery Condition in Presence of Noise (εERC)

$$\sup_{d \in \mathcal{D}(I^{\complement})} \|(DP_I)^{\dagger} d\|_{\ell^1} < 1 - 2\, r_{\varepsilon/\alpha} \tag{6}$$

holds, then OMP recovers the correct support I of α.

A comprehensive proof of theorem III.1 can be found in [10].

Remark III.2. In particular, to ensure the εERC (6) one has necessarily for the noise-to-signal-ratio

$$r_{\varepsilon/\alpha} < \frac{1}{2}.$$

Remark III.3. A rough upper bound for $\sup_{i \in \mathbb{Z}} |\langle \eta, d_i \rangle|$ is ε and hence

$$r_{\varepsilon/\alpha} \leq \frac{\varepsilon}{\min\limits_{i \in I} |\alpha_i| \|Ke_i\|}.$$

Same as before for the noiseless case, the expression in condition (6) is hard to evaluate. Analogously to [3] we next give a weaker sufficient recovery condition that depends on inner products of the dictionary atoms.

Proposition III.4 (Neumann ERC in Presence of Noise)**.** *Let $\alpha \in \ell^0$ with $\operatorname{supp} \alpha = I$. Let $u = \sum_{i \in \mathbb{Z}} \alpha_i e_i$ be the source and $v^{\varepsilon} = Ku + \eta$ the noisy data with noise level $\|\eta\| \leq \varepsilon$ and noise-to-signal-ratio $r_{\varepsilon/\alpha} < 1/2$. If for the operator $K : B \to H$ and the basis $\mathcal{E} = \{e_i\}_{i \in \mathbb{N}}$ the* Neumann εERC

$$\sup_{i \in I} \sum_{\substack{j \in I \\ j \neq i}} |\langle d_i, d_j \rangle| + \frac{1}{1 - 2\, r_{\varepsilon/\alpha}} \sup_{i \in I^{\complement}} \sum_{j \in I} |\langle d_i, d_j \rangle| < 1 \tag{7}$$

holds, then OMP recovers the correct support I of α.

Remark III.5. Notice that $1/(1 - 2\, r_{\varepsilon/\alpha})$ is monotonically increasing in $r_{\varepsilon/\alpha}$ and that for a small noise-to-signal-ratio the condition (7) gives almost the same as (4), since

$$\frac{1}{1 - 2\, r_{\varepsilon/\alpha}} \to 1 \qquad \text{for} \qquad r_{\varepsilon/\alpha} \to 0.$$

Remark III.6. The according sufficient conditions for WOMP with weakness parameter $\omega \in (0, 1]$ for the case of noisy data with noise-to-signal-ratio $r_{\varepsilon/\alpha} < \omega/2$ are

$$\sup_{d \in \mathcal{D}(I^{\complement})} \|(DP_I)^{\dagger} d\|_{\ell^1} < \omega - 2\, r_{\varepsilon/\alpha},$$

and

$$\sup_{i \in I} \sum_{\substack{j \in I \\ j \neq i}} |\langle d_i, d_j \rangle| + \frac{1}{\omega - 2\, r_{\varepsilon/\alpha}} \sup_{i \in I^{\complement}} \sum_{j \in I} |\langle d_i, d_j \rangle| < 1,$$

analog to theorem III.1 and proposition III.4, respectively.

IV. Resolution Bounds for Mass Spectrometry

Granted, to apply the Neumann conditions of proposition II.2 and proposition III.4, respectively, one has to know the support I, too. For deconvolution problems, however, with certain prior knowledge the equations (4) resp. (7) are easier to evaluate than condition (3) resp. (6). In the following we will do this simplification exemplarily with impulse trains convolved with Gaussian kernel as e.g. occurs in mass spectrometry, cf. [5].

Analysis

In mass spectrometry the source u is given as sum of Dirac peaks at integer positions $i \in \mathbb{Z}$,

$$u = \sum_{i \in \mathbb{Z}} \alpha_i \, \delta(\cdot - i),$$

with $|\operatorname{supp}\alpha| = |I| = N$. Since the measuring procedure is influenced by Gaussian noise the measured data can be modeled by a convolution operator K with Gaussian kernel

$$k(x) = \frac{1}{\pi^{1/4}\sigma^{1/2}} \exp\Big(-\frac{x^2}{2\sigma^2}\Big),$$

which are also the atoms of the dictionary $\mathcal{D} = \{\delta(\cdot - i) * k\} = \{k(\cdot - i)\}$, since $\|k(\cdot - i)\|_{L_2} = 1$, $i \in \mathbb{Z}$.

To verify the conditions (4) and (7) respectively, we need the autocorrelation of two atoms $k(\cdot - i)$ and $k(\cdot - j)$. In $L_2(\mathbb{R}, \mathbb{R})$ it arises as

$$\begin{aligned}
&\langle k(\cdot - i), k(\cdot - j)\rangle_{L_2} \\
&= \int\limits_{\mathbb{R}} \frac{1}{\sqrt{\pi}\sigma} \exp\Big(-\frac{(x-i)^2}{2\sigma^2}\Big) \exp\Big(-\frac{(x-j)^2}{2\sigma^2}\Big)\, \mathrm{d}\,x \\
&= \exp\Big(-\frac{(i-j)^2}{4\sigma^2}\Big),
\end{aligned}$$

which is positive and monotonically decreasing in the distance $|i-j|$. If we additionally assume that the peaks of any source u have the minimal distance

$$\rho := \min_{i,j \in \operatorname{supp}\alpha} |i - j|,$$

then w.l.o.g. we can estimate the sums of correlations to above as follows. Here, ϑ_3 denotes the Jacobi theta function of the third kind and ι represents the imaginary unit. For $\rho \in \mathbb{N}$ we get for the correlations of support atoms

$$\begin{aligned}
\sup_{i \in I} \sum_{\substack{j \in I \\ j \neq i}} |\langle d_i, d_j\rangle| &\leq \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \langle k, k(\cdot - j\rho)\rangle \\
&= \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \exp\Big(-\frac{(j\rho)^2}{4\sigma^2}\Big) = \vartheta_3\Big(0, \exp\Big(-\frac{\rho^2}{4\sigma^2}\Big)\Big) - 1.
\end{aligned}$$

For the correlations of support atoms and non-support atoms we have to distinguish between two cases for ρ. For $\rho \geq 2$ we get

$$\begin{aligned}
\sup_{i \in I^{\complement}} \sum_{j \in I} |\langle d_i, d_j\rangle| &\leq \sup_{i \in \{1,\dots,\rho-1\}} \sum_{j \in \mathbb{Z}} \langle k(\cdot - i), k(\cdot - j\rho)\rangle \\
&= \sup_{i \in \{1,\dots,\rho-1\}} \sum_{j \in \mathbb{Z}} \exp\Big(-\frac{(i - j\rho)^2}{4\sigma^2}\Big) \\
&= \sup_{i \in \{1,\dots,\rho-1\}} \exp\Big(-\frac{i^2}{4\sigma^2}\Big)\, \vartheta_3\Big(-\frac{i\rho}{4\sigma^2}\,\iota, \exp\Big(-\frac{\rho^2}{4\sigma^2}\Big)\Big),
\end{aligned}$$

and for $\rho = 1$

$$\begin{aligned}
\sup_{i \in I^{\complement}} \sum_{j \in I} |\langle d_i, d_j\rangle| &\leq \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \langle k, k(\cdot - j)\rangle \\
&= \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \exp\Big(-\frac{j^2}{4\sigma^2}\Big) = \vartheta_3\Big(0, \exp\Big(-\frac{1}{4\sigma^2}\Big)\Big) - 1.
\end{aligned}$$

With that we can formulate the Neumann ERC and the Neumann εERC for Dirac peaks convolved with Gaussian kernel.

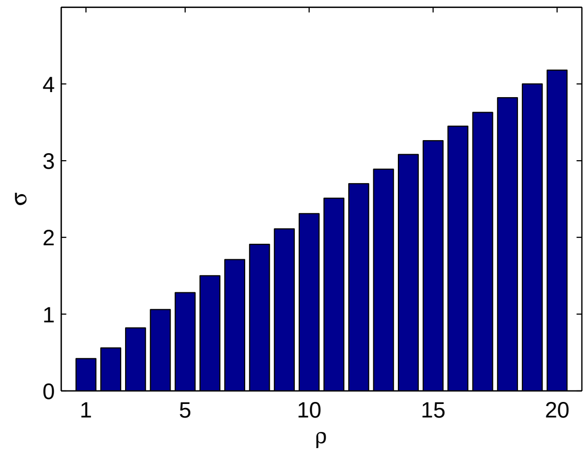

Fig. 1. ERC for different combinations of σ and ρ with $r_{\varepsilon/\alpha} = 0$.

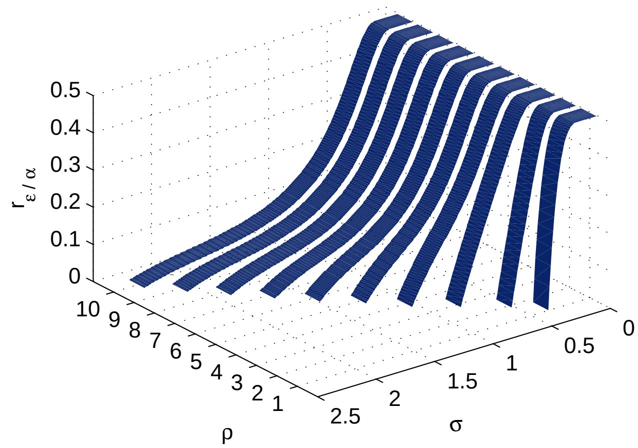

Fig. 2. εERC for combinations of σ, ρ and $r_{\varepsilon/\alpha}$.

Proposition IV.1. *An estimation to above for the ERC (i.e. $r_{\varepsilon/\alpha} = 0$) and εERC (i.e. $0 < r_{\varepsilon/\alpha} < \frac{1}{2}$) for Dirac peaks convolved with Gaussian kernel is for $\rho \geq 2$*

$$\begin{aligned}
&\sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \exp\Big(-\frac{(j\rho)^2}{4\sigma^2}\Big) \\
&\quad + \frac{1}{1 - 2\,r_{\varepsilon/\alpha}} \sup_{i \in \{1,\dots,\rho-1\}} \sum_{j \in \mathbb{Z}} \exp\Big(-\frac{(i - j\rho)^2}{4\sigma^2}\Big) < 1,
\end{aligned}$$

and for $\rho = 1$

$$\Big(1 + \frac{1}{1 - 2\,r_{\varepsilon/\alpha}}\Big) \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \exp\Big(-\frac{j^2}{4\sigma^2}\Big) < 1.$$

The condition of proposition IV.1 is plotted for some combinations of σ, ρ and $r_{\varepsilon/\alpha}$ in figure 1 and in figure 2. The area of the bars in figure 1 and the space beneath the straps in figure 2 describe the combinations where the Neumann ERC is fulfilled.

Remark IV.2. Often for deconvolution problems the autocorrelation of two atoms $|\langle d(\cdot-i), d(\cdot-j)\rangle|$ is not monotonically decreasing in the distance $|i-j|$ and it obviously depends on the kernel k. However, if the correlation of two atoms can be estimated from above via a monotonically decreasing function w.r.t. an appropriate distance then we can use a similar estimate. We do this exemplarily for an oscillating kernel in [10], namely, for Fresnel-convolved characteristic functions as appear in digital holography.

Numerical Examples

We apply the Neumann εERC of proposition IV.1 to simulated data of an isotope pattern. Here the data consist of equidistant peaks with different heights. In our example we use four peaks with a distance of $\rho = 5$ and heights of 130, 220, 180 and 90, cf. the balls at the top of figure 3.

After convolving with Gaussian kernel with $\sigma = 1.125$ we apply a Poisson noise model. This is convenient, because in mass spectrometry a finite number of particles is counted. For $r_{\varepsilon/\alpha}$ we estimate with Cauchy-Schwarz inequality

$$|\langle \eta, d_i\rangle| \approx \int_{i-4\sigma}^{i+4\sigma} \eta(x)\, k(x-i)\ \mathrm{d}\,x \leq \Big(\int_{i-4\sigma}^{i+4\sigma} \eta^2(x)\ \mathrm{d}\,x\Big)^{\frac{1}{2}}.$$

In the first example with low noise (mean and variance of 1.5 for regions without peaks) the Neumann εERC is fulfilled and hence OMP recovered the support exactly, see middle of figure 3. However, the condition is restrictive: For the second example the signal is disturbed with huge noise (mean and variance of 30 for regions without peaks) and the Neumann εERC is not fulfilled. Certainly, OMP recovered the support exactly, see bottom of figure 3.

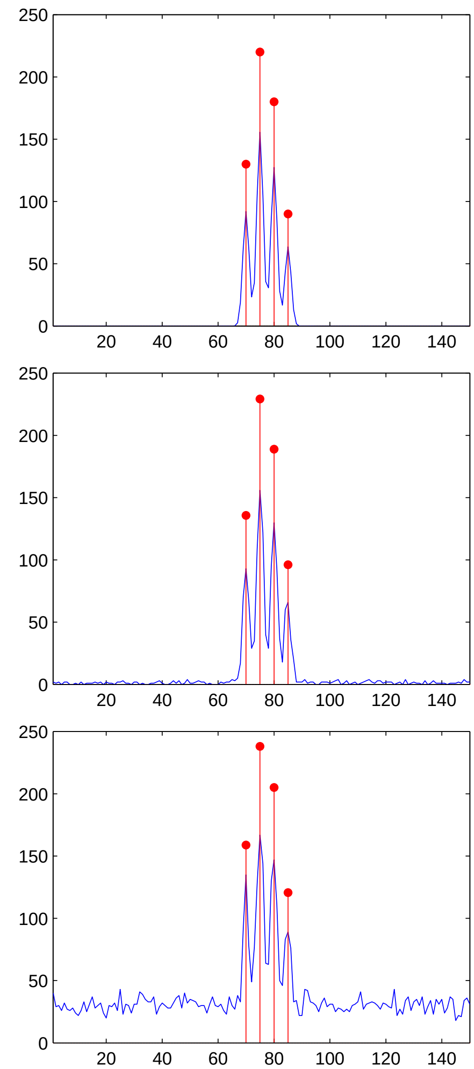

Fig. 3. Simulated isotope pattern. Top: support and Gaussian-convolved data without noise. Middle: low noise, Neumann εERC satisfied. Bottom: high noise, Neumann εERC not satisfied but still exact recovery possible.

V. Conclusion and Future Prospects

In this paper we gave an exact recovery condition for noisy signals that works without the concept of coherence. For our source we assumed that it consists of a superposition of point-like objects with an a-priori known distance. The example from mass spectrometry showed that the condition is restrictive. An idea to come to a tighter exact recovery condition is to bring in more prior knowledge, as e.g. a non-negativity constraint, cf. [11]. We postpone this idea for future work and hope for a tighter condition—especially for the mass spectrometry application.

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