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# Bounds on eigenvalues and singular values of interval matrices

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**Abstract:**

We study bounds on eigenvalues of interval matrices, and our aim is to develop fast computable formulae that produce as-sharp-as-possible bounds. We consider two cases: general (unsymmetric) and symmetric interval matrices.

We focus on the latter case, since on one hand these such interval matrices have many applications in mechanics and engineering, and on the other many results from classical matrix analysis could be applied to them. We also provide bounds for the singular values of (generally non-square) interval matrices.

Finally, we illustrate and compare the various approaches by a series of examples.

**Key-words:** Interval matrix, real eigenvalues, singular values, eigenvalue bounds, matrix regularity, interval analysis.

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# Calcul de bornes sur les valeurs propres et singulières des matrices intervalles

**Résumé :**

**Mots-clés :** Matrices intervalles, valeurs propres réelles, valeurs singulières, bornes sur les valeurs propres, régularit'e des matrices, analyse par intervalles.

# 1 Introduction

Many real-life problems suffer from diverse uncertainties, for example due to data measurement errors. Considering intervals instead of fixed real numbers is one possible way to tackle such uncertainties. In this paper, we study real eigenvalues of matrices, the entries of which vary simultaneously and independently inside some given intervals. The set of all possible eigenvalues forms a union of several compact real intervals, and our aim is to compute their bounds.

The problem of computing lower and upper bounds for the eigenvalue set is well-studied, e.g. [3, 8, 14, 24, 25, 26, 27, 29]. In the past and recent years some effort was made in developing and extending diverse inclusion sets for eigenvalues [6, 19] like Gerschorin discs or Cassini ovals. Even though such inclusion sets are more or less easy to compute and can be extended to interval matrices, the intervals that they produce are big over-estimations of the true ones.

The interval eigenvalue problem has a lot of applications in the field of mechanics and engineering. Let us mention for instance automobile suspension system [24], mass structures [23], vibrating systems [9], principal component analysis [10], and robotics [5]. In many cases, the properties of a system is given by the eigenvalues (or singular values) of a Jacobian matrix. A modern approach is to consider that the parameters of this matrix vary in a set of continuous states. The propagation of an interval representation of the parameters in the matrix allows us to bound the properties of the system over all its states. This is useful for designing a system, as well as to certify its performance.

Our goal is to revise and possibly improve the existing formulae for bounding eigenvalues of interval matrices. We focus on algorithms that are useful from a practical point of view; meaning that sometimes we sacrifice the accuracy of the results for speed. Nevertheless, the bounds that we derive are sharp enough for almost all practical purposes and are excellent candidates for initial estimate for various iterative algorithms [14].

Let us introduce some notation. An interval matrix is defined as

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n};\ \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}$, $\overline{A} \in \mathbb{R}^{m\times n}$, $\underline{A} \le \overline{A}$, are given matrices. By

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A})$$

we denote the midpoint and the radius of $\boldsymbol{A}$, respectively. We assume that the reader is familiar with a basic interval arithmetic, otherwise see e.g. [2, 12, 21].

By an *inner approximation* of a set $\mathcal{S}$ we mean any subset of $\mathcal{S}$, and by an *outer approximation* of $\mathcal{S}$ we mean a set containing $\mathcal{S}$ as a subset. Our aim is to develop formulae for calculating an outer approximation of the eigenvalue set of an (unsymmetric or symmetric) interval matrix.

The paper consists of two parts: the first is devoted to an unsymmetric interval matrix, while the second one devoted to a symmetric interval matrix.

Symmetry causes dependency between interval quantities, but—on the other hand—stronger theorems are applicable. Moreover, bounds of singular values of interval matrices are obtained as a simple consequence.

Moreover, the following notation will used through the paper:

| | |
|---|---|
| $\lvert\boldsymbol{v}\rvert = \max\{-\underline{v}, \overline{v}\}$ | magnitude (absolute value) of an interval $\boldsymbol{v}$; |
| $\lvert\boldsymbol{A}\rvert$ | magnitude (absolute value) of an interval matrix $\boldsymbol{A}$, i.e., $\lvert\boldsymbol{A}\rvert_{ij} = \lvert\boldsymbol{A}_{ij}\rvert$; |
| $\lVert A\rVert_p = \max_{x\neq 0} \frac{\lVert Ax\rVert_p}{\lVert x\rVert_p}$ | matrix p-norm; |
| $\kappa_p(A) = \lVert A\rVert_p \lVert A^{-1}\rVert_p$ | condition number (in p-norm); |
| $\sigma_{\max}(A)$ | maximal singular value of a matrix $A$; |
| $\rho(A)$ | spectral radius of a matrix $A$; |
| $\lambda^{\mathrm{Re}}(A)$ | real part of an eigenvalue of a matrix $A$; |
| $\lambda^{\mathrm{Im}}(A)$ | imaginary part of an eigenvalue of a matrix $A$; |

The rest of the paper is structured as follows: In the next section we present bounds for non-symmetric interval matrices. In Section 3 we present several algorithms or improvements for computing bounds for the eigenvalues of real symmetric interval matrices, we extend them to compute the bounds for singular values of general matrices, and we present a comparison of the various methods. The last section summarizes our results.

## 2 Unsymmetric interval matrix

Let

$$\Lambda := \{\lambda \in \mathbb{R};\ Ax = \lambda x,\ x \neq 0,\ A \in \boldsymbol{A}\}$$

be the set of all real eigenvalues of $\boldsymbol{A}$. It is always formed by a union of compact real intervals. The computation of this set is considered a very difficult task; even checking whether $0 \in \Lambda$ is an NP-hard problem, since it is equivalent to checking regularity of the interval matrix $\boldsymbol{A}$, which is NP-hard [22]. Therefore, we focus on a fast computation of initial (hopefully sharp enough) outer approximation of $\Lambda$.

For other approaches that estimate $\Lambda$, we refer the reader to [8, 24, 29]. Let us recall a method proposed in [27, Theorem 2] that we will improve in the sequel:

**Theorem 1** (Rohn, 1998)**.** *Let*

$$S_c := \frac{1}{2}\left(A_c + A_c^T\right),$$
$$S_\Delta := \frac{1}{2}\left(A_\Delta + A_\Delta^T\right).$$

*Then* $\Lambda \subseteq \boldsymbol{\lambda}^0 := [\underline{\lambda}^0, \overline{\lambda}^0]$*, where*

$$\underline{\lambda}^0 = \lambda_{\min}(S_c) - \rho(S_\Delta),$$
$$\overline{\lambda}^0 = \lambda_{\max}(S_c) + \rho(S_\Delta),$$

*and* $\lambda_{\min}(S_c)$*,* $\lambda_{\max}(S_c)$ *denotes the minimal and maximal eigenvalue of* $S_c$*, respectively.*

In most of the cases, the previous theorem provides a good estimation of the eigenvalue set $\Lambda$ (cf. [14]). However, its main disadvantage is the fact that it produces non-empty estimations, even in the case where the eigenvalue set is empty (see Example 1). To overcome this drawback we propose an alternative approach that utilizes Bauer–Fike theorem [11, 15, 30]:

**Theorem 2** (Bauer–Fike, 1960)**.** *Let* $A, B \in \mathbb{R}^{n\times n}$ *and suppose that* $A$ *is diagonalizable, that is,* $V^{-1}AV = \operatorname{diag}(\mu_1, \ldots, \mu_n)$ *for some* $V \in \mathbb{C}^{n\times n}$ *and* $\mu_1, \ldots, \mu_n \in \mathbb{C}$*. For every (complex) eigenvalue* $\lambda$ *of* $A + B$*, there exists an index* $i \in \{1, \ldots, n\}$ *such that*

$$|\lambda - \mu_i| \leq \kappa_p(V) \cdot \|B\|_p.$$

For almost all practical cases, the 2-norm, seems to be the most suitable choice. In what follows we will use the previous theorem with $p = 2$.

**Proposition 1.** *Let an interval matrix* $\boldsymbol{A}$ *be such that* $A_c$ *is diagonalizable, that is* $V^{-1}A_cV = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ *for* $V \in \mathbb{C}^{n\times n}$ *and* $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$*. Denote by* $\lambda^{\mathrm{Re}}_{\min}(A_c)$ *and* $\lambda^{\mathrm{Re}}_{\max}(A_c)$ *the minimal and the maximal real part of eigenvalues of* $A_c$*, respectively. Then* $\Lambda \subseteq \boldsymbol{\lambda} := [\underline{\lambda}, \overline{\lambda}]$*, where*

$$\underline{\lambda}^1 = \lambda^{\mathrm{Re}}_{\min}(A_c) - \kappa_2(V) \cdot \sigma_{\max}(A_\Delta),$$
$$\overline{\lambda}^1 = \lambda^{\mathrm{Re}}_{\max}(A_c) + \kappa_2(V) \cdot \sigma_{\max}(A_\Delta).$$

*Proof.* Every $A \in \boldsymbol{A}$ can be written as $A = A_c + A'$, where $|A'| \leq A_\Delta$ (where the inequality applies element-wise). By Bauer–Fike theorem with 2-norm we have for each real eigenvalue $\lambda(A)$ of $A$ that

$$\lambda(A) \leq \lambda^{\mathrm{Re}}_{\max}(A_c + A') \leq \lambda^{\mathrm{Re}}_{\max}(A_c) + \kappa_2(V) \cdot \|A'\|_2 = \lambda^{\mathrm{Re}}_{\max}(A_c) + \kappa_2(V) \cdot \sigma_{\max}(A').$$

As $|A'| \leq A_\Delta$, we have $\sigma_{\max}(A') \leq \sigma_{\max}(A_\Delta)$. Hence

$$\lambda(A) \leq \lambda^{\mathrm{Re}}_{\max}(A_c) + \kappa_2(V) \cdot \sigma_{\max}(A_\Delta).$$

In the same manner we get the lower bound on real eigenvalues

$$\lambda(A) \geq \lambda^{\mathrm{Re}}_{\min}(A_c + A') \geq \lambda^{\mathrm{Re}}_{\min}(A_c) - \kappa_2(V) \cdot \|A'\|_2 \geq \lambda^{\mathrm{Re}}_{\min}(A_c) - \kappa_2(V) \cdot \sigma_{\max}(A_\Delta).$$

□

Theorem 1 provides a simple formula for bounding real parts of eigenvalues of all matrices in $\boldsymbol{A}$. For real eigenvalues itself, better bounds can be derived.

**Proposition 2.** *Let $A_c$ be diagonalizable, i.e., $V^{-1}A_cV$ is diagonal for some $V \in \mathbb{C}^{n\times n}$. Then $\Lambda \subseteq (\bigcup_{i=1}^{n} \boldsymbol{\lambda}_i)$, where*

$$\underline{\lambda}_i = \lambda_i^{\mathrm{Re}}(A_c) - \sqrt{\left(\kappa_2(V)\cdot\sigma_{\max}(A_\Delta)\right)^2 - \lambda_i^{\mathrm{Im}}(A_c)^2}, \tag{1}$$

$$\overline{\lambda}_i = \lambda_i^{\mathrm{Re}}(A_c) + \sqrt{\left(\kappa_2(V)\cdot\sigma_{\max}(A_\Delta)\right)^2 - \lambda_i^{\mathrm{Im}}(A_c)^2}, \tag{2}$$

*provided that $\left(\kappa_2(V)\cdot\sigma_{\max}(A_\Delta)\right)^2 \geq \lambda_i^{\mathrm{Im}}(A_c)^2$; otherwise $\boldsymbol{\lambda}_i = \emptyset$.*

*Proof.* Similarly as in the proof of Proposition 1 we take any $A \in \boldsymbol{A}$ and write it in the form $A = A_c + A'$, where $|A'| \leq A_\Delta$. Bauer–Fike theorem with 2-norm implies that for each complex eigenvalue $\lambda(A)$ there is some complex eigenvalue $\lambda_i(A_c)$ such that

$$|\lambda(A) - \lambda_i(A_c)| \leq \kappa_2(V)\cdot\|A'\|_2 \leq \kappa_2(V)\cdot\sigma_{\max}(A_\Delta).$$

Thus all complex eigenvalues of all matrices $A \in \boldsymbol{A}$ lie in the circles having the centers in $\lambda_i(A_c)$-s with corresponding radii $\kappa_2(V)\cdot\sigma_{\max}(A_\Delta)$. The formulae (1)–(2) represent an intersection of these circles with the real axis. □

Notice that both a pair of complex conjugate eigenvalues $\lambda_i(A_c)$ and $\lambda_j(A_c)$ yields the same interval $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_j$, so it suffices to consider only one of them.

Proposition 2 is a very useful tool for estimating $\Lambda$ in the case where the "large" complex eigenvalues of $A_c$, have also large imaginary parts. In general is equivalent to Rohn's theorem, since it is not neither provably better, nor provably worse. Therefore, it is advisable, in practice, to use both of them.

We present one more improvement for computing bounds of $\Lambda$, that is based on a theorem by Horn & Johnson [16]:

**Theorem 3.** *Let $A \in \mathbb{R}^{n\times n}$. Then*

$$\lambda_{\min}\left(\frac{A+A^T}{2}\right) \leq \lambda^{\mathrm{Re}}(A) \leq \lambda_{\max}\left(\frac{A+A^T}{2}\right)$$

*for every (complex) eigenvalue $\lambda(A)$ of the matrix $A$.*

The theorem says that any upper or lower bound of the eigenvalue set of the symmetric interval matrix $\frac{A+A^T}{2}$ is also a bound of $\Lambda$. Symmetric interval matrices are in details studied in Section 3 and the results obtained there can be used here to bound $\Lambda$ via Theorem 3. Note that in this way, the Rohn's bounds from Theorem 4 yield the same bounds as that from Theorem 1.

**Example 1.** Let

$$\boldsymbol{A} = \begin{pmatrix} [1,2] & [1,2] \\ [-2,-1] & [1,2] \end{pmatrix}.$$

The Rohn's theorem leads to the outer estimation $\Lambda \subseteq [0.5, 2.5]$, and the same interval is produced by Proposition 1.

The matrix $A_c = \left(\begin{smallmatrix} 1.5 & 1.5 \\ -1.5 & 1.5 \end{smallmatrix}\right)$ has two complex eigenvalues, $1.5 - 1.5\,i$ and $1.5 + 1.5\,i$, and the radius is $\kappa_2(V)\cdot\sigma_{\max}(A_\Delta) = 1$. Thus, by Proposition 2, $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_2 = \emptyset$, and no matrix $A \in \boldsymbol{A}$ has a real eigenvalue. That is, $\Lambda = \emptyset$.

**Example 2.** Let

$$\boldsymbol{A} = \begin{pmatrix} [-5,-4] & [-9,-8] & [14,15] & [4.6,5] & [-1.2,-1] \\ [17,18] & [17,18] & [1,2] & [4,5] & [10,11] \\ [17,17.2] & [-3.5,-2.7] & [1.9,2.1] & [-13,-12] & [6,6.4] \\ [18,19] & [2,3] & [18,19] & [5,6] & [6,7] \\ [13,14] & [18,19] & [9,10] & [-18,-17] & [10,11] \end{pmatrix}.$$

The Rohn's outer estimation $[-22.104, 35.4999]$ and by using Proposition 1 we obtain $[-24.486, 29.3101]$.

Proposition 2 produces union of two intervals. Eigenvalues of $A_c$ are:

$$-15.8973,\ -4.0671,\ 15.1215 + 15.9556\,i,\ 15.1215 - 15.9556\,i,\ \text{and } 20.7214.$$

The radius $\kappa_2(V) \cdot \sigma_{\max}(A_\Delta) = 8.5887$. Hence

$$\boldsymbol{\lambda}_1 = [-24.486, -7.30853],\ \boldsymbol{\lambda}_2 = [-12.6559, 4.5216],\ \boldsymbol{\lambda}_3 = \boldsymbol{\lambda}_4 = \emptyset,\ \boldsymbol{\lambda}_5 = [12.1327, 29.3101].$$

The resulting outer approximation of $\Lambda$ is $[-24.486, 4.5216] \cup [12.1327, 29.3101]$.

Taking into account all the methods and intersecting the intervals we obtain better result $[-22.104, 4.5216] \cup [12.1327, 29.3101]$.

To have a comparison on the sharpness of these estimates we note that the exact description of $\Lambda$ produced by the Hladík & Daney algorithm [14] is as follows:

$$\Lambda = [-17.5116, -13.7578] \cup [-6.7033, -1.4582] \cup [16.7804, 23.6143].$$

# 3 Symmetric interval matrix

Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix. It has $n$ real eigenvalues, which are in decreasing order:

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

Let $\boldsymbol{A}$ be an interval matrix such that $\underline{A}$ and $\overline{A}$ are symmetric. It is important to note that not every matrix in $\boldsymbol{A}$ is symmetric. Let the *symmetric interval matrix* be

$$\boldsymbol{A}^S := \{A \in \boldsymbol{A} \mid A = A^T\}.$$

We denote by

$$\boldsymbol{\lambda}_i(\boldsymbol{A}^S) := \left\{\lambda_i(A) \mid A \in \boldsymbol{A}^S\right\}$$

the set of $i$-th eigenvalues. Each of these sets is a compact real interval; this is a consequence of the continuity of the eigenvalue function and the compactness of $\boldsymbol{A}^S$. It can happen that the sets $\boldsymbol{\lambda}_i(\boldsymbol{A}^S)$ and $\boldsymbol{\lambda}_j(\boldsymbol{A}^S)$, where $i \neq j$, overlap.

Our aim is to derive as-sharp-as-possible bounds of the eigenvalue sets. The upper bound $\lambda_i^u(\boldsymbol{A}^S)$, $i \in \{1, \ldots, n\}$, is any real number satisfying $\lambda_i^u(\boldsymbol{A}^S) \geq \overline{\lambda_i}(\boldsymbol{A}^S)$. We do not pay special attention to lower bounds $\lambda_i^l(\boldsymbol{A}^S)$, $i \in \{1, \ldots, n\}$, as they are computable as upper bounds of $-\boldsymbol{A}^S$.

The symmetric case is very important for the real-life applications as symmetric matrices appear very often in engineering problems. Under the concept of interval computations, symmetry induces dependencies between the matrix elements, which are hard to deal with, in general. The straightforward approach is to "forget" the dependencies and apply the methods from the previous section, to get rigorous bounds on eigenvalues. Unfortunately, these bounds are far from being sharp, since the loss of dependency implies a big overestimation on the computed intervals.

We should mention that there are very few theoretical results concerning symmetric interval matrices. Let us only mention that it is not known how to compute all the exact boundary points of the eigenvalues set. Such a result could be of extremely practical importance since it can be used for testing the accuracy of existing approximation algorithms. In this line of research, let us mention the work of Deif [8] and Hertz [13]. The former provides an exact description of eigenvalues set, but it works only under some not-easy-to-verify assumptions on sign pattern invariance of eigenvectors; the latter, see also [28], proposes a formula for computing the exact extremal values $\overline{\lambda}_1(\boldsymbol{A}^S)$ and $\underline{\lambda}_n(\boldsymbol{A}^S)$, which consists of $2^{n-1}$ iterations. Theoretical results could also be found in the work of Qiu & Wang [25]. However, some of them turned out to be incorrect [31].

Since the exact problem of computing the eigenvalues set(s) is a difficult one, several approximation algorithms were developed in the recent years. An evolution strategy method by Yuan et al. [31] yields inner approximation of the eigenvalues set. By means of matrix perturbation theory, Qiu et al. [23] proposed an algorithm for approximate bounds, and Leng & He [18] for outer estimation. Outer estimation was also given by Beaumont [4]; he used a polyhedral approximation of eigenpairs and an iterative improvement. Kolev [17] developed an outer estimation algorithm for general case with non-linear dependencies.

## 3.1 Basic bounds

The following theorem is due to Rohn [28]; to make the paper self-contained, we include its proof.

**Theorem 4.** *It holds that*

$$\boldsymbol{\lambda}_i(\boldsymbol{A}^S) \subseteq [\lambda_i(A_c) - \rho(A_\Delta), \lambda_i(A_c) + \rho(A_\Delta)].$$

*Proof.* By Weyl's theorem [11, 15, 20, 30], for any symmetric matrices $B, C \in \mathbb{R}^{n\times n}$ one has

$$\lambda_i(B) + \lambda_n(C) \leq \lambda_i(B + C) \leq \lambda_i(B) + \lambda_1(C) \quad \forall i = 1, \ldots, n.$$

Particularly, for every $A \in \boldsymbol{A}$ in the form of $A = A_c + A'$, $A' \in [-A_\Delta, A_\Delta]$, we have

$$\lambda_i(A) = \lambda_i(A_c + A') \leq \lambda_i(A_c) + \lambda_1(A') \leq \lambda_i(A_c) + \rho(A') \quad \forall i = 1, \ldots, n.$$

As $|A'| \leq A_\Delta$, we get $\rho(A') \leq \rho(A_\Delta)$, whence

$$\lambda_i(A) \leq \lambda_i(A_c) + \rho(A_\Delta).$$

Working similarly, we can prove that $\lambda_i(A) \geq \lambda_i(A_c) - \rho(A_\Delta)$. □

The bounds obtained by this theorem are usually quite sharp. The main shortcoming of this formula is that all the intervals $\boldsymbol{\lambda}_i(\boldsymbol{A}^S)$, $1 \leq i \leq n$, have the same width.

The following proposition provides an upper bound for the largest eigenvalue of $\boldsymbol{A}^S$, that is a upper bound for the right endpoint of $\boldsymbol{\lambda}_1(\boldsymbol{A}^S)$. Even though the formula is very simple and the bound is not very sharp, there are cases that it yields better bound than the one by Rohn's theorem. In particular it provides better bounds for non-negative interval matrices, and for interval matrices like that ones we consider in Section 3.3 and have the form $[-A_\Delta, A_\Delta]$.

**Proposition 3.** *It holds*

$$\overline{\lambda}_1(\boldsymbol{A}^S) \leq \lambda_1(|\boldsymbol{A}|).$$

*Proof.* Using the well-known Courant–Fischer theorem [11, 15, 20, 30], we have for every $A \in \boldsymbol{A}$

$$\begin{aligned}\lambda_1(A) &= \max_{x^Tx=1} x^TAx \leq \max_{x^Tx=1} |x^TAx| \\ &\leq \max_{x^Tx=1} |x|^T|A||x| \leq \max_{x^Tx=1} |x|^T|\boldsymbol{A}||x| \\ &= \max_{x^Tx=1} x^T|\boldsymbol{A}|x = \lambda_1(|\boldsymbol{A}|).\end{aligned}$$

□

In the same way we can compute a lower bound for the eigenvalues of $\boldsymbol{A}$: $\underline{\lambda}_n(\boldsymbol{A}^S) \geq -\overline{\lambda}_1(|\boldsymbol{A}|)$. However, this inequality is not so useful.

## 3.2 Interlacing approach, direct version

The approach that we propose in this section is based on Cauchy's interlacing property for eigenvalues of the symmetric matrix [11, 15, 20, 30].

**Theorem 5** (Interlacing property, Cauchy, 1829)**.** *Let $A \in \mathbb{R}^n$ be a symmetric matrix and let $A_i$ be a matrix obtained from $A$ by removing the $i$-th row and column. Then*

$$\lambda_1(A) \geq \lambda_1(A_i) \geq \lambda_2(A) \geq \lambda_2(A_i) \geq \cdots \geq \lambda_{n-1}(A_i) \geq \lambda_n(A).$$

We develop two methods based on the interlacing property; the direct and the indirect one. These methods are useful as long as the intervals $\boldsymbol{\lambda}_i(\boldsymbol{A}^S)$, $i = 1, \ldots, n$, do overlap, or as long as there is a narrow gap between them. Overlapping happens, for example, when there are multiple eigenvalues in $\boldsymbol{A}^S$.

If none of the previous cases occur, then the bounds are not so sharp; see Example 3.

The first method uses the interlacing property directly. Bounds on the eigenvalues of the principal minor $\boldsymbol{A}_i^S$ are also bounds on the eigenvalues of matrices in $\boldsymbol{A}^S$ (except the upper bound of $\overline{\lambda}_1(\boldsymbol{A}^S)$ and the lower bound of $\underline{\lambda}_n(\boldsymbol{A}^S)$). The basic idea is to compute the bounds recursively. However, such a recursive algorithm would be of exponential complexity. Therefore, we propose a simple local search that requires only a linear number of iterations and the results of which are quite satisfactory. It consists of selecting the most promising principal minor $\boldsymbol{A}_i$ and recursively using only this. To obtain as good results as possible we call this procedure in the reverse order, as well. That is we begin with some diagonal element $\boldsymbol{a}_{ii}$ of $\boldsymbol{A}^S$, which is a matrix one-by-one, and iteratively increase its dimension until we obtain $\boldsymbol{A}^S$.

The algorithmic scheme is presented in Algorithm 1. We often need to compute an upper bound $\lambda_1^u(\boldsymbol{B}^S)$ for the maximal eigenvalue of any matrix in $\boldsymbol{B}^S$ (steps 3 and 12). For this purpose we can call Theorem 4, Proposition 3, or, to obtain the best results, we choose the minimum of the two. Notice that the algorithm computes only upper bounds for $\overline{\lambda}_i(\boldsymbol{A}^S)$, $i = 1, \ldots, n$. Lower bounds for $\underline{\lambda}_i(\boldsymbol{A}^S)$, $i = 1, \ldots, n$, can be obtained by calling the algorithm using $-\boldsymbol{A}^S$ as input matrix.

**Algorithm 1** (Interlacing approach, direct version)

1: $\boldsymbol{B}^S := \boldsymbol{A}^S$;
2: **for** $k = 1, \ldots, n$ **do**
3: compute $\lambda_1^u(\boldsymbol{B}^S)$;
4: $\lambda_k^u(\boldsymbol{A}^S) := \lambda_1^u(\boldsymbol{B}^S)$;
5: select the most promising index $i \in \{1, \ldots, n-k+1\}$;
6: remove the $i$-th row and the $i$-th column from $\boldsymbol{B}^S$;
7: **end for**
8: put $I = \emptyset$;
9: **for** $k = 1, \ldots, n$ **do**
10: select the most promising index $i \in \{1, \ldots, n\} \setminus I$, and put $I := I \cup \{i\}$;
11: let $\boldsymbol{B}^S$ be a sub-matrix of $\boldsymbol{A}^S$ restricted to the rows and columns indexed by $I$;
12: compute $\lambda_1^u(\boldsymbol{B}^S)$;
13: $\lambda_{n-k+1}^u(\boldsymbol{A}^S) := \min\left\{\lambda_{n-k+1}^u(\boldsymbol{A}^S),\ \lambda_1^u(\boldsymbol{B}^S)\right\}$;
14: **end for**
15: **return** $\lambda_k^u(\boldsymbol{A}^S),\ k = 1, \ldots, n$.

An important ingredient of the algorithm is the selection of the index $i$, in steps 5 and 10. We describe the selection for step 5; for step 10 we work similarly. In the essence, there are two basic choices:

$$i := \arg \min_{j=1,\ldots,n-k+1} \lambda_1^u(\boldsymbol{B}_j^S), \tag{3}$$

and

$$i := \arg \min_{j=1,\ldots,n-k+1} \sum_{r,s \neq j} |\boldsymbol{B}_{r,s}|^2. \tag{4}$$

In both cases we select an index $i$ so that to possibly minimize $\overline{\lambda}_1(\boldsymbol{B}_i^S)$.

The first formula requires more computations and yields the optimal index more probably than the second one. The latter is based on the well-known result [15, 30] that the Frobenius norm of a matrix (i.e., the sum of squares of its entries) equals the sum of squares of its eigenvalues. Therefore, the most promising index is the one that maximizes the sum of squares of the absolute values (magnitudes) of the removed components.

The selection rule (3) causes a quadratic time complexity of Algorithm 1 with respect to the number of calculations of spectral radii or eigenvalues. Using the selection rule (4) results only a linear number of such calculations.

### 3.3 Interlacing approach, indirect version

The second method uses also the interlacing property, and is based on the following idea. Every matrix $A \in \boldsymbol{A}^S$ can be written as $A = A_c + A_\delta$ with $A_\delta \in [-A_\Delta, A_\Delta]^S$. We compute the eigenvalues of the real matrix $A_c$, and bounds on eigenvalues of matrices in $[-A_\Delta, A_\Delta]^S$, and we "merge" them to obtain bounds on eigenvalues of matrices in $\boldsymbol{A}^S$. For the "merging" step we use a theorem for perturbed eigenvalues.

The algorithm is presented in Algorithm 2. It returns only upper bounds $\lambda_i^u(\boldsymbol{A}^S)$, $i = 1, \ldots, n$ for $\overline{\lambda}_i(\boldsymbol{A}^S)$, $i = 1, \ldots, n$, since lower bounds are computable likewise. The bounds required in step 2 are computed using Algorithm 1.

The following theorem due to Weyl [15, 30] gives very nice formulae for the eigenvalues of a matrix sum.

**Theorem 6** (Weyl, 1912)**.** *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices with eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \cdots \geq \lambda_n(B)$, respectively. Then one has*

$$\begin{aligned} \lambda_{r+s-1}(A+B) &\leq \lambda_r(A) + \lambda_s(B) \quad \forall r, s \in \{1, \ldots, n\},\ r+s \leq n+1, \\ \lambda_{r+s-n}(A+B) &\geq \lambda_r(A) + \lambda_s(B) \quad \forall r, s \in \{1, \ldots, n\},\ r+s \geq n+1. \end{aligned}$$

### 3.4 Diagonal maximization

In this subsection we show that the largest eigenvalues are achieved when the diagonal entries of $A \in \boldsymbol{A}^S$ are the maximum ones. Therefore, we can fix them and consider only a subset of $A \in \boldsymbol{A}^S$. Similar results can be obtained for the smallest eigenvalues.

**Lemma 1.** *Let $i \in \{1, \ldots, n\}$. Then there is some matrix $A \in \boldsymbol{A}^S$ with diagonal entries $A_{j,j} = \overline{A}_{j,j}$ such that $\lambda_i(A) = \overline{\lambda}_i(\boldsymbol{A}^S)$.*

**Algorithm 2** (Interlacing approach, indirect version)

1: Compute eigenvalues $\lambda_1(A_c) \geq \cdots \geq \lambda_n(A_c)$;
2: compute bounds $\lambda_1^u\big([-A_\Delta, A_\Delta]^S\big), \ldots, \lambda_n^u\big([-A_\Delta, A_\Delta]^S\big)$;
3: **for** $k = 1, \ldots, n$ **do**
4: $\quad \lambda_k^u(\boldsymbol{A}^S) := \min\limits_{i=1,\ldots,k} \big\{\lambda_i(A_c) + \lambda_{k-i+1}^u\big([-A_\Delta, A_\Delta]^S\big)\big\}$;
5: **end for**
6: **return** $\lambda_k^u(\boldsymbol{A}^S),\ k = 1, \ldots, n$.

*Proof.* Let $A' \in \boldsymbol{A}^S$ be such that $\lambda_i(A') = \overline{\lambda}_i(\boldsymbol{A}^S)$. Such a matrix always exists, since $\overline{\lambda}_i(\boldsymbol{A}^S)$ is defined as the maximum of a continuous function on a compact set. We define $A \in \boldsymbol{A}^S$ as follows: $A_{ij} := A'_{ij}$ if $i \neq j$, and $A_{ij} := \overline{A}_{ij}$ if $i = j$. By the Courant–Fischer theorem [11, 15, 20, 30], we have

$$\begin{aligned}\lambda_i(A') &= \max_{V \subseteq \mathbb{R}^n;\, \dim V = i} \ \min_{x \in V;\, x^T x = 1} x^T A' x \\ &\leq \max_{V \subseteq \mathbb{R}^n;\, \dim V = i} \ \min_{x \in V;\, x^T x = 1} x^T A x \\ &= \lambda_i(A).\end{aligned}$$

Hence $\lambda_i(A) = \lambda_i(A)' = \overline{\lambda}_i(\boldsymbol{A}^S)$. □

This lemma implies that for computing upper bounds $\lambda_i^u(\boldsymbol{A}^S)$ of $\overline{\lambda}_i(\boldsymbol{A}^S)$, $i = 1, \ldots, n$, it suffices to consider only the symmetric interval matrix $\boldsymbol{A}_r^S \subseteq \boldsymbol{A}^S$ defined as

$$\boldsymbol{A}_r^S := \{A \in \boldsymbol{A}^S \mid A_{j,j} = \overline{A}_{j,j} \forall j = 1, \ldots, n\}.$$

To this matrix we can apply all the algorithms developed in the previous subsections. The resulting bound are sometimes sharper and sometimes not so sharp; see Examples 3–4. So the best possible results are obtained by using all the methods together.

## 3.5 Singular values

Let $A \in \mathbb{R}^{m \times n}$ and denote $q := \min\{m, n\}$. By $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ we denote the singular values of $A$. It is well known [11, 15, 20] that the singular values of $A$ are identical with the $q$ largest eigenvalues of the Jordan–Wielandt matrix

$$\begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix},$$

which is symmetric. Consider an interval matrix $\boldsymbol{A} \subset \mathbb{R}^{m \times n}$. By

$$\boldsymbol{\sigma}_i(\boldsymbol{A}) := \{\sigma_i(A) \mid A \in \boldsymbol{A}\}, \quad i = 1, \ldots, q,$$

we denote the singular value sets of $\boldsymbol{A}$. The problem of approximating the singular value sets was considered e.g. in [1, 7]. Deif's method [7] produces exact

singular value sets, but only under some assumption that are generally difficult to verify. Ahn & Chen [1] presented a method for calculating the largest possible singular value $\overline{\sigma}_1(\boldsymbol{A})$. It is a slight modification of [13] and time complexity is exponential ($2^{m+n-1}$ iterations). They also proposed a lower bound for the smallest possible singular value $\underline{\sigma}_n(\boldsymbol{A})$ by means of interval matrix inversion.

To get an outer approximation of the singular value set of $\boldsymbol{A}$ we can exhibit the methods proposed in the previous subsections and apply them on eigenvalue set of the symmetric interval matrix

$$\begin{pmatrix} 0 & \boldsymbol{A}^T \\ \boldsymbol{A} & 0 \end{pmatrix}^S. \tag{5}$$

Diagonal maximization (Subsection 3.4) does not work, as the diagonal of the symmetric interval matrix (5) consists of zeros only. The other methods work well. Even though they run very fast, they can be accelerated a bit, as some of them can be slightly modified and used directly on $\boldsymbol{A}$ instead of (5). Particularly, Proposition 3 and interlacing property are easy to modify for singular values. We discuss it very briefly.

In the first case we get that the largest singular value over $A \in \boldsymbol{A}$ is bounded from above by $\sigma_1(|\boldsymbol{A}|)$, the largest singular value of the real matrix $|\boldsymbol{A}|$.

In the second case we base on the interlacing property of singular values [11, 15, 16]. Let $A \in \mathbb{R}^{m\times n}$. Remove any row or column from $A$ and the result denote by $A'$. Then

$$\sigma_1(A) \geq \sigma_1(A') \geq \sigma_2(A) \geq \cdots \geq \sigma_q(A') \geq \sigma_q(A) \geq \sigma_q(A'),$$

where $\sigma_q(A')$ is defined to be zero provided that $q$ is less than both the dimensions of $A'$. Using the singular values interlacing property, Algorithm 1 and 2 can be adapted accordingly. We leave this simple consequence to the reader.

### 3.6 Case of study

The aim of the following examples is to show that no presented method is better than the other one. In different situations, different variants are the best.

**Example 3.** Consider the example given by Qiu et al. [23] (see also [31]):

$$\boldsymbol{A}^S = \begin{pmatrix} [2975,3025] & [-2015,-1985] & 0 & 0 \\ [-2015,-1985] & [4965,5035] & [-3020,-2980] & 0 \\ 0 & [-3020,-2980] & [6955,7045] & [-4025,-3975] \\ 0 & 0 & [-4025,-3975] & [8945,9055] \end{pmatrix}^S.$$

Proposition 3 yields the upper bound $\lambda_1^u(\boldsymbol{A}^S) = 12720.2273$, which is—by chance—the optimal value. The other outer approximations of the eigenvalues sets $\boldsymbol{\lambda}_i(\boldsymbol{A}^S)$, $i = 1,\ldots,n$, are listed in the table below. The corresponding items are as follows:

(R) bounds computed by the Rohn's theorem (Theorem 4);

(D1) bounds computed by Algorithm 1 with the index selection rule (3);

(D2) bounds computed by Algorithm 1 with the index selection rule (4);

(I1) bounds computed by Algorithm 2 with the index selection rule (3);

(I2) bounds computed by Algorithm 2 with the index selection rule (4);

(DD1) bounds computed by diagonal maximization by using Algorithm 1 and the index selection rule (3);

(DI1) bounds computed by diagonal maximization by using Algorithm 2 and the index selection rule (3);

(B) bounds obtained by using Theorem 4, Algorithm 1 and 2, and then choosing the best ones; the index selection rule is (3);

(O) optimal bounds; they are known provided an inner and outer approximation (calculated or known from references) coincide.

| | $[\lambda_1^l(\boldsymbol{A}^S),\ \lambda_1^u(\boldsymbol{A}^S)]$ | $[\lambda_2^l(\boldsymbol{A}^S),\ \lambda_2^u(\boldsymbol{A}^S)]$ | $[\lambda_3^l(\boldsymbol{A}^S),\ \lambda_3^u(\boldsymbol{A}^S)]$ | $[\lambda_4^l(\boldsymbol{A}^S),\ \lambda_4^u(\boldsymbol{A}^S)]$ |
|---|---|---|---|---|
| (R) | [12560.6296, 12720.4331] | [6984.5571, 7144.3606] | [3309.9466, 3469.7501] | [825.2597, 985.0632] |
| (D1) | [8945.0000, 12720.2273] | [4945.00000, 9055.0000] | [2924.5049, 6281.7216] | [825.2597, 3025.0000] |
| (D2) | [8945.0000, 12720.2273] | [2945.0000, 9453.4449] | [1708.9320, 6281.7216] | [825.2597, 3025.0000] |
| (I1) | [12560.6296, 12720.4331] | [6984.5571, 7144.3606] | [3309.9466, 3469.7501] | [825.2597, 985.0632] |
| (I2) | [12560.6296, 12720.4331] | [6984.5571, 7144.3606] | [3309.9466, 3469.7501] | [825.2597, 985.0632] |
| (DD1) | [8945.0000, 12720.2273] | [4965.0000, 9055.0000] | [2950.0000, 6281.7216] | [837.0637, 3025.0000] |
| (DI1) | [12557.7243, 12723.3526] | [6990.7616, 7138.1800] | [3320.2863, 3459.4322] | [837.0637, 973.1993] |
| (B) | [12560.6296, 12720.2273] | [6990.7616, 7138.1800] | [3320.2863, 3459.4322] | [837.0637, 973.1993] |
| (O) | [12560.8377, 12720.2273] | [7002.2828, 7126.8283] | [3337.0785, 3443.3127] | [842.9251, 967.1082] |

The table shows that the direct interlacing methods (D1), (D2) and (DD1) are not efficient; gaps between the eigenvalues sets $\boldsymbol{\lambda}_i(\boldsymbol{A}^S)$, $i = 1, \ldots, n$, are too wide. The indirect interlacing methods (I1) and (I2) yield the same intervals as the Rohn method (R). The indirect interlacing method using diagonal maximization is several times better (e.g. for $\lambda_4^l(\boldsymbol{A}^S)$, $\lambda_4^u(\boldsymbol{A}^S)$) and several times worse (e.g. for $\lambda_1^l(\boldsymbol{A}^S)$, $\lambda_1^u(\boldsymbol{A}^S)$) than (R). The combination (B) of all the methods produces good outer estimation of the eigenvalue set, particularly that of $\boldsymbol{\lambda}_1(\boldsymbol{A}^S)$.

For this example, Qiu et al. [23] obtained the approximate values

$$\underline{\lambda}_1(\boldsymbol{A}^S) \approx 12588.29,\ \overline{\lambda}_1(\boldsymbol{A}^S) \approx 12692.77,\ \underline{\lambda}_2(\boldsymbol{A}^S) \approx 7000.195,\ \overline{\lambda}_2(\boldsymbol{A}^S) \approx 7128.723,$$
$$\underline{\lambda}_3(\boldsymbol{A}^S) \approx 3331.162,\ \overline{\lambda}_3(\boldsymbol{A}^S) \approx 3448.535,\ \underline{\lambda}_4(\boldsymbol{A}^S) \approx 826.7372,\ \overline{\lambda}_4(\boldsymbol{A}^S) \approx 983.5858.$$

However, these values form neither inner nor outer estimation of the eigenvalue set. The method of Leng & He [18] based on matrix perturbation theory results in bounds

$$\lambda_1^l(\boldsymbol{A}^S) = 12550.53,\ \lambda_1^u(\boldsymbol{A}^S) = 12730.53,\ \lambda_2^l(\boldsymbol{A}^S) = 6974.459,\ \lambda_2^u(\boldsymbol{A}^S) = 7154.459,$$
$$\lambda_3^l(\boldsymbol{A}^S) = 3299.848,\ \lambda_3^u(\boldsymbol{A}^S) = 3479.848,\ \lambda_4^l(\boldsymbol{A}^S) = 815.1615,\ \lambda_4^u(\boldsymbol{A}^S) = 995.1615.$$

In comparison to (B), they are not so sharp. The evolution strategy method proposed by Yuan et al. [31] returns an inner estimation of the eigenvalues set, which is equal to the optimal result (see (O) in the table) in this example.

**Example 4.** Consider the symmetric interval matrix

$$\boldsymbol{A}^S = \begin{pmatrix} [0,2] & [-7,3] & [-2,2] \\ [-7,3] & [4,8] & [-3,5] \\ [-2,2] & [-3,5] & [1,5] \end{pmatrix}^S .$$

Following the notation used in Example 3 we display in the table below results obtained by the presented methods.

| | $[\lambda_1^l(\boldsymbol{A}^S),\ \lambda_1^u(\boldsymbol{A}^S)]$ | $[\lambda_2^l(\boldsymbol{A}^S),\ \lambda_2^u(\boldsymbol{A}^S)]$ | $[\lambda_3^l(\boldsymbol{A}^S),\ \lambda_3^u(\boldsymbol{A}^S)]$ |
|---|---|---|---|
| (R) | [−2.2298, 16.0881] | [−6.3445, 11.9734] | [−8.9026, 9.4154] |
| (D1) | [4.0000, 15.3275] | [−2.5616, 6.0000] | [−8.9026, 2.0000] |
| (D2) | [4.0000, 15.3275] | [−2.5616, 6.0000] | [−8.9026, 2.0000] |
| (I1) | [−0.7436, 16.0881] | [−3.3052, 10.4907] | [−8.9026, 6.3760] |
| (I2) | [−0.7436, 16.0881] | [−3.3052, 10.4907] | [−8.9026, 6.3760] |
| (DD1) | [4.0000, 15.3275] | [−2.0000, 6.0000] | [−8.3759, 2.0000] |
| (DI1) | [−0.9115, 16.3089] | [−2.9115, 10.8445] | [−8.3759, 6.7850] |
| (B) | [4.0000, 15.3275] | [−2.0000, 6.0000] | [−8.3759, 2.0000] |
| (O) | [?, 15.3275] | [?, ?] | [−7.8184, ?] |

This example illustrates the case when direct interlacing methods (D1)–(D2) yields better results than the indirect ones (I1)–(I2). The same is true for the diagonal maximization variants (DD1) and (DI1). The Rohn method (R) is not very convenient here. Optimal bounds are known only for $\lambda_1^u(\boldsymbol{A}^S)$ and $\lambda_3^l(\boldsymbol{A}^S)$.

**Example 5.** Herein, we show several examples concerning bounding singular value sets of interval matrices.

(1) The first example comes from the paper by Deif [7]. Given an interval matrix

$$\boldsymbol{A} = \begin{pmatrix} [2,3] & [1,1] \\ [0,2] & [0,1] \\ [0,1] & [2,3] \end{pmatrix}$$

Deif's method yields the following estimation of the singular value sets

$$\boldsymbol{\sigma}_1(\boldsymbol{A}) \approx [2.5616,\ 4.5431], \quad \boldsymbol{\sigma}_2(\boldsymbol{A}) \approx [1.3134,\ 2.8541].$$

Ahn & Chen [1] confirmed that $\overline{\sigma}_1(\boldsymbol{A}) = 4.5431$, but the real value of $\underline{\sigma}_2(\boldsymbol{A})$ must be smaller. Namely, it is less or equal to one since $\underline{\sigma}_2(A) = 1$ for $A^T = \left(\begin{smallmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{smallmatrix}\right)$. Our approach using combination of all presented method together results in an outer estimation

$$\boldsymbol{\sigma}_1(\boldsymbol{A}) \subseteq [2.0489,\ 4.5431], \quad \boldsymbol{\sigma}_2(\boldsymbol{A}) \subseteq [0.4239,\ 3.1817].$$

(2) Next, consider an example by Ahn & Chen [1]

$$\boldsymbol{A} = \begin{pmatrix} [0.75,\ 2.25] & [-0.015,\ -0.005] & [1.7,\ 5.1] \\ [3.55,\ 10.65] & [-5.1,\ -1.7] & [-1.95,\ -0.65] \\ [1.05,\ 3.15] & [0.005,\ 0.015] & [-10.5,\ -3.5] \end{pmatrix}.$$

Our method gives on outer estimation for singular value sets

$$\boldsymbol{\sigma}_1(\boldsymbol{A}) \subseteq [4.3308,\ 14.0115],\quad \boldsymbol{\sigma}_2(\boldsymbol{A}) \subseteq [1.9305,\ 11.6111],\quad \boldsymbol{\sigma}_3(\boldsymbol{A}) \subseteq [0.0000,\ 5.1000].$$

Ahn & Chen calculated that $\overline{\sigma}_1(\boldsymbol{A}) = 13.9371$ and $\underline{\sigma}_3(\boldsymbol{A}) \geq 0.1147$.

# 4 Conclusion and future work

In this paper we considered outer approximations of the eigenvalue sets of unsymmetric and symmetric interval matrices. For both cases, we presented several improvements. Computing sharp outer approximations of the eigenvalue set of an unsymmetric interval matrix is a difficult problem. The proposed methods provide quite satisfactory results, as indicated by Examples 1–2. Examples 3–5 demonstrate that we are able to bound quite sharply the eigenvalues of symmetric interval matrices and the singular values of interval matrices. Our bounds are quite close to the optimal ones.

At the current state, there is no algorithm that computes better bounds in all the cases. Since the computational cost of the presented algorithms is rather low, it is advisable use all of them in practice and select the best one depending on the particular instance.

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