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Christophe Hazard

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Christophe Hazard*

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Abstract: The generalized Fourier transform associated with a selfadjoint Sturm–Liouville operator is a unitary transformation which converts the action of this operator into a simple product by a spectral variable. For a particular operator defined on the half-line and which involves a step function, we show how to extend such a transformation to generalized functions, or distributions, with a suitable definition of such distributions. This extension is based essentially on the fact that, as the usual Fourier transform, this transformation has the property to exchange regularity and decay between the physical and spectral variables.

Key-words: distributions, generalized Fourier transform

* ENSTA / POEMS, 32 Boulevard Victor, 75015 Paris, France. Email: christophe.hazard@ensta.fr

Centre de recherche INRIA Paris – Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Un espace de distributions pour la transformation de Fourier généralisée associée à un opérateur de Sturm–Liouville

Résumé : La transformation de Fourier généralisée associée à un opérateur de Sturm–Liouville auto-adjoint est une transformation unitaire qui convertit l'action de cet opérateur en une simple multiplication par une variable spectrale. Dans le cas particulier d'un opérateur défini sur la demi-droite réelle dont les coefficients prennent deux valeurs constantes de part et d'autre d'un point, on montre comment une telle transformation peut être prolongée aux distributions, moyennant une définition convenable de ces distributions. Ce prolongement repose essentiellement sur le fait que, comme pour la transformation de Fourier habituelle, cette transformation possède la propriété d'échanger régularité et décroissance entre les variables physiques et spectrales.

Mots-clés : distributions, transformation de Fourier généralisée

1 Introduction

One of the fundamental properties of the usual Fourier transform of square integrable functions lies in the possibility to be extended to generalized functions, or distributions, which makes it a power tool for the solution of partial differential equations with constant coefficients. The aim of the present paper is to show how such a property holds for the generalized Fourier transform associated with a Sturm–Liouville operator. For the sake of simplicity, we consider a very simple operator defined on the half-line, for which most calculations are explicit. We denote by A the unbounded selfadjoint operator in $L^2(\mathbb{R}^+)$ given by

$$A\varphi := -\varphi'' - k^2\varphi \quad \forall\varphi \in \mathrm{D}(A) := \{\psi \in H^2(\mathbb{R}^+);\ \psi(0) = 0\}, \tag{1}$$

where $k = k(x)$ is the function defined on $\mathbb{R}^+$ by

$$k(x) := \begin{cases} \dot{k} & \text{if } 0 \leq x < h, \\ \ddot{k} & \text{if } x \geq h, \end{cases} \qquad \text{with} \quad 0 < \ddot{k} < \dot{k}. \tag{2}$$

As we will see, the latter assumption makes possible the presence of eigenvalues in the spectrum of A, denoted by Λ.

A generalized Fourier transform $\mathcal{F}$ associated with A is a *unitary* transformation from the 'physical space' $L^2(\mathbb{R}^+)$ to a spectral space $\widehat{\mathcal{H}}$ (which contains functions of the spectral variable $\lambda \in \Lambda$) which *diagonalizes* A in the sense that it converts the action of A into a simple product by λ in $\widehat{\mathcal{H}}$, that is,

$$A = \mathcal{F}^*\ \lambda\ \mathcal{F}.$$

The usual Fourier transform plays this role in the case of constant coefficients (here: $\ddot{k} = \dot{k}$). For Sturm–Liouville operators with variable coefficients, such a transformation has been introduced in the first half of the last century in the original works of H. Weyl and E.C. Titchmarsh [6]. More generally, for every selfadjoint operator in a Hilbert space, one can construct a generalized Fourier transform, which can be interpreted as a decomposition on a family of generalized eigenfunctions [1]. It has numerous applications, for instance in scattering theory [4] or quantum mechanics.

It is well-known that the decay properties for large λ of the usual Fourier transform $\widehat{\varphi}(\lambda)$ of a function $\varphi(x)$ is related to the regularity of $\varphi(x)$, and conversely, the regularity of $\widehat{\varphi}(\lambda)$ is related to the behaviour of $\varphi(x)$ for large x. The main purpose of this paper is to study how this exchange of regularity and decay between the physical and spectral spaces holds for the generalized Fourier transform $\mathcal{F}$. This will allow us to extend $\mathcal{F}$ to a space larger than $L^2(\mathbb{R}^+)$ similar to the usual Schwartz space of distributions [5].

A possible application of this extension concerns the propagation of time-harmonic waves in a slab waveguide [2], more precisely the solution of Helmholtz equation in a half-plane:

$$\begin{aligned} &-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - k^2\, u = f \quad \text{in } \mathbb{R}^+ \times \mathbb{R}, \\ &u(0, y) = 0 \quad \forall y \in \mathbb{R}, \end{aligned}$$

where k is given by (2): it only depends on x. To solve such a problem, one can use the usual Fourier transform in the variable y, which diagonalizes the y-part of the laplacian. This leads to a differential equation in x with a variable coefficient. Here we are interested in using instead the generalized Fourier transform in the x-direction. At least formally, the above system can be rewritten as

$$-\frac{\partial^2 u}{\partial y^2} + A\,u = f,$$

and applying $\mathcal{F}$ then yields a differential equation in y with constant coefficients:

$$-\frac{\partial^2 \widehat{u}}{\partial y^2} + \lambda\,\widehat{u} = \widehat{f} \quad \text{in } \mathbb{R},$$

where $\widehat{u} := \mathcal{F}u$ and $\widehat{f} := \mathcal{F}f$. But for a fixed $y \in \mathbb{R}$, function $u(\cdot, y)$ does not belong to $L^2(\mathbb{R}^+)$ in general. In order to justify the use of $\mathcal{F}$, we have to make clear in what sense we apply this transformation. We show here that it can be interpreted in the sense of distributions by constructing a space of distributions adapted to operator A.

The paper is organized as follows. In §2, we introduce the functional material which is needed in the rest of the paper. In §3, we recall the construction of the generalized Fourier transform $\mathcal{F}$ associated with A. Section 4 then investigates the exchange of decay and regularity under the action of $\mathcal{F}$. This provides us the basic tool for the extension of $\mathcal{F}$ to a space of distributions, which is the object of §5.

The following notations concerning some usual functional spaces are used hereafter. If X is a subset of $\mathbb{R}$ and μ is a measure on X, we denote by $L^2(X; \mathrm{d}\mu)$ the space of square integrable functions on X for the measure μ, equipped with the following inner product and associated norm:

$$(\varphi, \psi)_{X;\mathrm{d}\mu} := \int_X \varphi(x)\,\overline{\psi(x)}\,\mathrm{d}\mu(x) \quad \text{and} \quad \|\varphi\|_X := (\varphi, \varphi)^{1/2}_{X;\mathrm{d}\mu}.$$

If μ is the Lebesgue measure on X, we shall omit $\mathrm{d}\mu$ in these notations. For $r \in \mathbb{R}$, we denote by $H^r(X)$ the usual Sobolev spaces. If X is unbounded, $L^2_{\mathrm{comp}}(X)$ is the subspace of $L^2(X)$ composed of functions with compact support, $L^2_{\mathrm{loc}}(X)$ is the set of functions φ such that $\varphi|_K \in L^2(K)$ for all bounded sets $K \subset X$, and

$$\langle \varphi, \psi \rangle_X := \int_X \varphi(x)\,\overline{\psi(x)}\,\mathrm{d}x \quad \forall \varphi \in L^2_{\mathrm{comp}}(X),\ \forall \psi \in L^2_{\mathrm{loc}}(X).$$

In the whole paper, $\sqrt{\cdot}$ denotes the principal branch of the complex square root defined for all $z \in \mathbb{C} \setminus \mathbb{R}^-$ by

$$\sqrt{z} := |z|^{1/2}\,\mathrm{e}^{\mathrm{i}(\arg z)/2} \quad \text{with} \quad |\arg z| < \pi,$$

which has one-sided limits near every negative real number:

$$\sqrt{z \pm \mathrm{i}0} := \lim_{\varepsilon \searrow 0} \sqrt{z \pm \mathrm{i}\varepsilon} = \pm \mathrm{i}\,|z|^{1/2} \quad \forall z \in \mathbb{R}^-.$$

2 Preliminary information

We introduce some particular solutions to the following differential equation:

$$-\varphi'' - (k^2 + \zeta)\varphi = 0 \quad \text{on } \mathbb{R}^+, \tag{3}$$

where $k = k(x)$ is defined by (2) and $\zeta \in \mathbb{C}$. We denote $r_\zeta(x) := \sqrt{-k^2(x) - \zeta}$, that is,

$$r_\zeta(x) = \begin{cases} \dot{r}_\zeta := \sqrt{-\dot{k}^2 - \zeta} & \text{if } 0 \leq x < h \text{ and } \zeta \in \mathbb{C} \setminus [-\dot{k}^2, +\infty), \\ \ddot{r}_\zeta := \sqrt{-\ddot{k}^2 - \zeta} & \text{if } x \geq h \text{ and } \zeta \in \mathbb{C} \setminus [-\ddot{k}^2, +\infty). \end{cases}$$

Both functions $\dot{r}_\zeta$ and $\ddot{r}_\zeta$ have one-sided limits on their respective branch cuts:

$$\begin{aligned} \dot{r}_{\lambda \pm \mathrm{i}0} &= \mp \mathrm{i}\, \dot{\beta}_\lambda \ \text{ where } \ \dot{\beta}_\lambda := \left(\dot{k}^2 + \lambda\right)^{1/2} \quad \text{for} \ \ \lambda \in (-\dot{k}^2, +\infty), \\ \ddot{r}_{\lambda \pm \mathrm{i}0} &= \mp \mathrm{i}\, \ddot{\beta}_\lambda \ \text{ where } \ \ddot{\beta}_\lambda := \left(\ddot{k}^2 + \lambda\right)^{1/2} \quad \text{for} \ \ \lambda \in (-\ddot{k}^2, +\infty). \end{aligned}$$

We consider a canonical basis of solutions to (3), denoted by $c_\zeta(x)$ and $s_\zeta(x)$, associated with point $x = h$ in the sense that they satisfy

$$\begin{aligned} c_\zeta(h) = 1 \quad &\text{et} \quad c'_\zeta(h) = 0, \\ s_\zeta(h) = 0 \quad &\text{et} \quad s'_\zeta(h) = 1. \end{aligned}$$

When $\zeta \in \mathbb{C} \setminus [-\dot{k}^2, +\infty)$, these solutions write explicitly as

$$c_\zeta(x) = \cosh\left\{r_\zeta(x)\ (x-h)\right\} \quad \text{and} \quad s_\zeta(x) = \frac{\sinh\left\{r_\zeta(x)\ (x-h)\right\}}{r_\zeta(x)},$$

but these expressions actually do not depend of the choice of a branch of the complex square root, since they both are even functions of $r_\zeta(x)$. Hence c_ζ et s_ζ are *entire* functions of ζ.

Every solution to (3) is a linear combination of these functions. We consider two particular solutions. On one hand,

$$\Phi_\zeta(x) := c_\zeta(0)\, s_\zeta(x) - s_\zeta(0)\, c_\zeta(x), \tag{4}$$

is clearly an entire function of ζ, and satisfies the boundary conditions

$$\Phi_\zeta(0) = 0 \quad \text{and} \quad \Phi'_\zeta(0) = 1.$$

On the other hand,

$$\Theta_\zeta(x) := c_\zeta(x) - \ddot{r}_\zeta\, s_\zeta(x) \tag{5}$$

is analytic in $\mathbb{C} \setminus [-\ddot{k}^2, +\infty)$ but no more entire because of the coefficient $\ddot{r}_\zeta$. This is the *evanescent* solution to (3): it decreases exponentially when $x \to +\infty$ since

$$\Theta_\zeta(x) = \exp\{-\ddot{r}_\zeta\ (x-h)\} \quad \text{if} \ \ x \geq h, \tag{6}$$

where $\operatorname{Re} \ddot{r}_\zeta > 0$ because of our choice of the principal branch of the complex square root. On the branch cut $[-\ddot{k}^2, +\infty)$, this function has one-sided limits

$$\Theta_{\lambda \pm \mathrm{i}0}(x) := \lim_{\varepsilon \searrow 0} \Theta_{\lambda \pm \mathrm{i}\varepsilon}(x) = c_\lambda(x) \pm \mathrm{i}\ddot{\beta}_\lambda\, s_\lambda(x) \quad \forall \lambda \in [-\ddot{k}^2, +\infty). \tag{7}$$

Function Φ_λ may be seen as a linear combination of these limits:

$$\Phi_\lambda = \frac{\Theta_{\lambda - \mathrm{i}0}(0)\, \Theta_{\lambda + \mathrm{i}0} - \Theta_{\lambda + \mathrm{i}0}(0)\, \Theta_{\lambda - \mathrm{i}0}}{2\mathrm{i}\ddot{\beta}_\lambda} = -\operatorname{Im} \frac{\Theta_{\lambda + \mathrm{i}0}(0)\, \overline{\Theta_{\lambda + \mathrm{i}0}}}{\ddot{\beta}_\lambda}. \tag{8}$$

We will need the derivative of $\Theta_\zeta(x)$ with respect to ζ, which can be expressed as follows:

$$\frac{\partial}{\partial\zeta}\Theta_\zeta(x) = \frac{1}{2\ddot{r}_\zeta}\,\{(x-h)\,\Theta_\zeta(x) + \tilde{\Theta}_\zeta(x)\}, \tag{9}$$

where $\tilde{\Theta}_\zeta(x)$ is given by

$$\tilde{\Theta}_\zeta(x) := \begin{cases} \dfrac{\dot{k}^2 - \ddot{k}^2}{\dot{k}^2 + \zeta}\,(s_\zeta(x) - (x-h)\,c_\zeta(x)) & \text{if } 0 \leq x < h, \\ 0 & \text{if } x \geq h. \end{cases} \tag{10}$$

3 The generalized Fourier transform associated with A

The results given in this section are classical (see, *e.g.*, [3, 6, 7]), but we recall the proofs for the sake of clarity. The construction of a generalized Fourier transform associated with our operator A is based on the spectral theorem [1] which ensures the existence of a spectral measure E (also called resolution of the identity) such that for every measurable function $f : \mathbb{R} \to \mathbb{C}$, the operator $f(A)$ can be expressed as

$$f(A)\varphi = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_\lambda\varphi \quad \forall\varphi \in \mathrm{D}(f(A)), \tag{11}$$

where the domain of $f(A)$ is characterized by

$$\mathrm{D}(f(A)) = \left\{\psi \in L^2(\mathbb{R}^+);\ \int_{\mathbb{R}} |f(\lambda)|^2\,\mathrm{d}\|E_\lambda\psi\|^2_{\mathbb{R}^+} < \infty\right\}.$$

Stone's formula offers a convenient expression of E by means of the resolvent $R_\zeta := (A-\zeta)^{-1}$ of A : for every closed interval $I = [\lambda_1, \lambda_2] \subset \mathbb{R}$ and every $\varphi \in L^2(\mathbb{R}^+)$, we have

$$\|E_I\varphi\|^2_{\mathbb{R}^+} = \frac{1}{2\mathrm{i}\pi}\lim_{\eta\searrow 0}\lim_{\varepsilon\searrow 0}\int_{\lambda_1-\eta}^{\lambda_2+\eta}((R_{\lambda+\mathrm{i}\varepsilon} - R_{\lambda-\mathrm{i}\varepsilon})\varphi,\varphi)_{\mathbb{R}^+}\,\mathrm{d}\lambda. \tag{12}$$

To apply this formula, we shall use an explicit integral representation of $R_\zeta\,\varphi$ for $\zeta \in \mathbb{C}\setminus\mathbb{R}$:

$$(R_\zeta\,\varphi)(x) = \int_{\mathbb{R}^+}\gamma_\zeta(x,x')\,\varphi(x')\,\mathrm{d}x' \quad \forall x \in \mathbb{R}^+, \tag{13}$$

where the kernel γ_ζ is the Green's function of A, that is, the bounded solution to

$$\begin{aligned} &-\gamma''_\zeta(\cdot,x') - (k^2+\zeta)\,\gamma_\zeta(\cdot,x') = \delta_{x'} \quad \text{in } \mathcal{D}'(\mathbb{R}^+), \\ &\gamma_\zeta(0,x') = 0, \end{aligned}$$

where $\delta_{x'}$ denotes the Dirac measure at $x = x'$. Using the function Φ_ζ and Θ_ζ introduced in (4) and (5), this yields

$$\gamma_\zeta(x,x') = \frac{\Phi_\zeta(\min\{x,x'\})\,\Theta_\zeta(\max\{x,x'\})}{\Theta_\zeta(0)}, \tag{14}$$

Formula (13) is classically obtained by integrations by parts using the equations satisfied by $R_\zeta\,\varphi$ and the above equations for γ_ζ.

Proposition 1 *The spectrum Λ of A consists of, on one hand, a finite point spectrum $\Lambda_{\rm p}$ composed of the zeros of $\Theta_\zeta(0)$ located in $(-\dot{k}^2, -\ddot{k}^2)$, that is, the roots of the dispersion equation*

$$\tan\left(\sqrt{\dot{k}^2+\lambda}\; h\right) = -\,\frac{\sqrt{\dot{k}^2+\lambda}}{\sqrt{-\ddot{k}^2-\lambda}} \quad \textit{with} \;\; \lambda \in (-\dot{k}^2, -\ddot{k}^2), \tag{15}$$

and on the other hand, an absolutely continuous spectrum $\Lambda_{\rm c} := [-\ddot{k}^2, +\infty)$.

For every $\varphi \in L^2_{\rm comp}(\mathbb{R}^+)$, the spectral measure can be expressed as

$$\mathrm{d}\|E_\lambda \varphi\|^2_{\mathbb{R}^+} = |\langle \varphi, \Phi_\lambda\rangle_{\mathbb{R}^+}|^2\, \rho_\lambda\, \mathrm{d}\lambda|_{\Lambda_{\rm c}} \; + \sum_{\lambda\in\Lambda_{\rm p}} |(\varphi, \Phi_\lambda)_{\mathbb{R}^+}|^2\, \rho_\lambda\, \delta_\lambda, \tag{16}$$

where $\mathrm{d}\lambda|_{\Lambda_{\rm c}}$ is the Lebesgue measure restricted to $\Lambda_{\rm c}$, δ_λ is the Dirac measure at $\lambda \in \Lambda_{\rm p}$, and

$$\rho_\lambda := \begin{cases} \|\Phi_\lambda\|^{-2}_{\mathbb{R}^+} = 2\dot{\beta}^2_\lambda \left(h + \ddot{r}_\lambda^{-1}\right)^{-1} & \textit{if } \lambda \in \Lambda_{\rm p}, \\ \dfrac{\ddot{\beta}_\lambda}{\pi|\Theta_{\lambda+{\rm i}0}(0)|^2} & \textit{if } \lambda \in \Lambda_{\rm c}. \end{cases} \tag{17}$$

Proof. Formula (14) shows that γ_ζ is a meromorphic function of ζ in $\mathbb{C} \setminus [-\ddot{k}^2, +\infty)$. Its poles are the zeros of $\Theta_\zeta(0)$, which are easily seen to be located in $(-\dot{k}^2, -\ddot{k}^2]$. For such a pole λ, functions Φ_λ and Θ_λ are proportional: $\Phi_\lambda = -s_\lambda(0)\;\Theta_\lambda$. These are eigenfunctions associated with the eigenvalue λ, apart from the case when $\lambda = -\ddot{k}^2$ is a zero of $\Theta_\zeta(0)$, that is, when

$$\dot{k}^2 - \ddot{k}^2 = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{h^2} \quad \text{for some } n \in \mathbb{N}. \tag{18}$$

Indeed in this case $\Phi_\lambda \notin L^2(\mathbb{R}^+)$, so $\lambda = -\ddot{k}^2$ cannot be an eigenvalue of A. That is why this bound is excluded in the dispersion equation (15) which is equivalent to the equation $\Theta_\lambda(0) = 0$.

As a consequence, E vanishes on $(-\infty, -\dot{k}^2)$ and is a pure point measure on the interval $(-\dot{k}^2, -\ddot{k}^2)$. If $I \subset (-\dot{k}^2, -\ddot{k}^2)$ contains only one eigenvalue λ, then E_I is the spectral projection $E_{\{\lambda\}}$ associated with λ, *i.e.*,

$$E_{\{\lambda\}}\varphi = \frac{(\varphi, \Phi_\lambda)_{\mathbb{R}^+}\; \Phi_\lambda}{\|\Phi_\lambda\|^2_{\mathbb{R}^+}},$$

where one can check that that $\|\Phi_\lambda\|^2_{\mathbb{R}^+} = (h + \ddot{r}_\lambda^{-1})/(2\dot{\beta}^2_\lambda)$.

On the other hand, at every point $\lambda \in (-\ddot{k}^2, +\infty)$, the Green's function has one-sided limits

$$\gamma_{\lambda\pm{\rm i}0}(x,x') := \lim_{\varepsilon\searrow 0} \gamma_{\lambda\pm{\rm i}\varepsilon}(x,x') = \frac{\Phi_\lambda(\min\{x,x'\})\,\Theta_{\lambda\pm{\rm i}0}(\max\{x,x'\})}{\Theta_{\lambda\pm{\rm i}0}(0)}.$$

From (8), we see that

$$\frac{\Theta_{\lambda+{\rm i}0}(x)}{\Theta_{\lambda+{\rm i}0}(0)} - \frac{\Theta_{\lambda-{\rm i}0}(x)}{\Theta_{\lambda-{\rm i}0}(0)} = \frac{2{\rm i}\ddot{\beta}_\lambda}{|\Theta_{\lambda+{\rm i}0}(0)|^2}\,\Phi_\lambda(x),$$

which yields the gap of the Green's function across $(-\ddot{k}^2, +\infty)$:

$$\gamma_{\lambda+\mathrm{i}0}(x, x') - \gamma_{\lambda-\mathrm{i}0}(x, x') = 2\mathrm{i}\pi\, \rho_\lambda\, \Phi_\lambda(x)\, \Phi_\lambda(x'), \tag{19}$$

where ρ_λ is given by (17). We are now ready to use Stone's formula (12) together with the integral representation (13). If we choose $\varphi \in L^2_{\mathrm{comp}}(\mathbb{R}^+)$, we can pass to the limits $\eta \searrow 0$ and $\varepsilon \searrow 0$ thanks to the Lebesgue's dominated convergence theorem, which yields

$$\|E_I \varphi\|^2_{\mathbb{R}^+} = \frac{1}{2\mathrm{i}\pi} \int_{\lambda_1}^{\lambda_2} \int_{\mathbb{R}^+ \times \mathbb{R}^+} \{\gamma_{\lambda+\mathrm{i}0}(x, x') - \gamma_{\lambda-\mathrm{i}0}(x, x')\}\, \varphi(x')\, \overline{\varphi(x)}\, \mathrm{d}x\, \mathrm{d}x'\, \mathrm{d}\lambda.$$

for every interval $I = [\lambda_1, \lambda_2] \subset (-\ddot{k}^2, +\infty)$. Therefore, using (19) and Fubini's theorem, we have

$$\|E_I \varphi\|^2_{\mathbb{R}^+} = \int_{\lambda_1}^{\lambda_2} |\langle \varphi, \Phi_\lambda \rangle_{\mathbb{R}^+}|^2\, \rho_\lambda\, \mathrm{d}\lambda.$$

The expression (16) of the spectral measure on Λ_{c} follows. $\square$

Remark 2 *The values of the gap $\dot{k}^2 - \ddot{k}^2$ given by* (18) *define the thresholds of the problem: when the gap increases and meets one one these values, a new eigenvalue appears in Λ_{p} from the lower bound $-\ddot{k}^2$ of the continuous spectrum.*

We can now introduce the generalized Fourier transform associated with A, that is, the operator of 'decomposition' on the familly $\{\Phi_\lambda;\ \lambda \in \Lambda\}$. We denote

$$\mathcal{F}\varphi(\lambda) := \langle \varphi, \Phi_\lambda \rangle_{\mathbb{R}^+} \quad \forall \lambda \in \Lambda,\ \forall \varphi \in L^2_{\mathrm{comp}}(\mathbb{R}^+). \tag{20}$$

Let $\mathrm{d}\mu$ be the measure on $\mathbb{R}$ define by

$$\mathrm{d}\mu := \sum_{\lambda \in \Lambda_{\mathrm{p}}} \rho_\lambda\, \delta_\lambda \ + \ \rho_\lambda\, \mathrm{d}\lambda|_{\Lambda_{\mathrm{c}}}.$$

Theorem 3 *The transformation $\mathcal{F}$ defined by* (20) *extends by density to a unitary operator from $L^2(\mathbb{R}^+)$ to $L^2(\Lambda; \mathrm{d}\mu)$* (*still denoted by $\mathcal{F}$*), *which diagonalizes A in the sense that for every measurable function $f : \mathbb{R} \to \mathbb{C}$,*

$$f(A)\varphi = \mathcal{F}^* f(\lambda)\, \mathcal{F}\, \varphi, \quad \forall \varphi \in \mathrm{D}(f(A)) = \mathcal{F}^*(L^2(\Lambda; (1 + |f(\cdot)|^2)\mathrm{d}\mu)), \tag{21}$$

where $f(\lambda)$ stands for the operator of multiplication by $f(\lambda)$ in $L^2(\Lambda; \mathrm{d}\mu)$. Its adjoint $\mathcal{F}^* = \mathcal{F}^{-1}$ appears as the operator of 'recomposition' on the family $\{\Phi_\lambda;\ \lambda \in \Lambda\}$:

$$\mathcal{F}^* \widehat{\varphi} = \sum_{\lambda \in \Lambda_{\mathrm{p}}} \rho_\lambda\, \widehat{\varphi}(\lambda)\, \Phi_\lambda \ + \ \lim_{M \to +\infty} \int_{-\ddot{k}^2 + M^{-1}}^{M} \widehat{\varphi}(\lambda)\, \Phi_\lambda\, \rho_\lambda\, \mathrm{d}\lambda, \tag{22}$$

for all $\widehat{\varphi} \in L^2(\Lambda; \mathrm{d}\mu)$.

Proof. The expression of the measure $\mathrm{d}(E_\lambda\varphi,\psi)$ for $\varphi,\ \psi \in L^2_{\mathrm{comp}}(\mathbb{R}^+)$ follows from (16) by the polarization principle. Hence, using (11) with a bounded function f, the inner product $(f(A)\varphi,\psi)_{\mathbb{R}^+}$ can be written as

$$\sum_{\lambda\in\Lambda_{\mathrm{p}}} f(\lambda)\,\rho_\lambda\,(\varphi,\Phi_\lambda)_{\mathbb{R}^+}\,\overline{(\psi,\Phi_\lambda)_{\mathbb{R}^+}} \;+\; \int_{\Lambda_{\mathrm{c}}} f(\lambda)\,\langle\varphi,\Phi_\lambda\rangle_{\mathbb{R}^+}\,\overline{\langle\psi,\Phi_\lambda\rangle_{\mathbb{R}^+}}\,\rho_\lambda\,\mathrm{d}\lambda,$$

which amounts to the concise form

$$(f(A)\varphi,\psi)_{\mathbb{R}^+} = (f(\lambda)\,\mathcal{F}\varphi,\mathcal{F}\psi)_{\Lambda;\mathrm{d}\mu}. \tag{23}$$

Choosing $f(\lambda)\equiv 1$, that is, $f(A)=\mathrm{I}$, we see that $\mathcal{F}$ is isometric from $L^2_{\mathrm{comp}}(\mathbb{R}^+)$ to $L^2(\Lambda;\mathrm{d}\mu)$. Therefore it extends by density to an isometry from $L^2(\mathbb{R}^+)$ to $L^2(\Lambda;\mathrm{d}\mu)$, and the above formula holds for every measurable function f and all $\varphi\in\mathrm{D}(f(A))$, which yields (21). The expression of the adjoint of $\mathcal{F}$ is easily derived by noticing that

$$(\mathcal{F}^*\widehat{\varphi},\psi)_{\mathbb{R}^+} = (\widehat{\varphi},\mathcal{F}\psi)_{\Lambda;\mathrm{d}\mu} = \int_\Lambda \widehat{\varphi}(\lambda)\int_{\mathbb{R}^+}\overline{\psi(x)}\,\Phi_\lambda(x)\,\mathrm{d}x\,\mathrm{d}\mu(\lambda),$$

which yields

$$(\mathcal{F}^*\widehat{\varphi},\psi)_{\mathbb{R}^+} = \int_{\mathbb{R}^+}\int_\Lambda \widehat{\varphi}(\lambda)\,\Phi_\lambda(x)\,\mathrm{d}\mu(\lambda)\,\overline{\psi(x)}\,\mathrm{d}x,$$

by permuting both integrals. This permutation is justified if $\widehat{\varphi}$ has a compact support which does not contain $-\ddot{k}^2$ (note that ρ_λ becomes singular near $\lambda = -\ddot{k}^2$ for the thresholds defined by (18)): formula (22) then follows by density.

To see that $\mathcal{F}$ is unitary, it remains to prove that the range $\mathrm{R}(\mathcal{F})$ of $\mathcal{F}$ is the whole space $L^2(\Lambda;\mathrm{d}\mu)$. First notice that (21) implies $\mathcal{F}f(A)\varphi = P\,f(\lambda)\,\mathcal{F}\varphi$ where $P:=\mathcal{F}\mathcal{F}^*$ denotes the orthogonal projection on $\mathrm{R}(\mathcal{F})$. Moreover we have $\|\mathcal{F}f(A)\varphi\|_{\Lambda;\mathrm{d}\mu} = \|f(A)\varphi\|_{\mathbb{R}^+} = \|f(\lambda)\,\mathcal{F}\varphi\|_{\Lambda;\mathrm{d}\mu}$ by (23), which shows that we can remove P in the previous relation: $\mathcal{F}f(A)\varphi = f(\lambda)\,\mathcal{F}\varphi$. As a consequence, if $\widehat{\varphi}\in L^2(\Lambda;\mathrm{d}\mu)$ is orthogonal to $\mathrm{R}(\mathcal{F})$, the same holds for $f(\lambda)\widehat{\varphi}$ for every bounded function f since

$$(f(\lambda)\widehat{\varphi}\,,\,\mathcal{F}\psi)_{\Lambda;\mathrm{d}\mu} = (\widehat{\varphi}\,,\,\overline{f(\lambda)}\,\mathcal{F}\psi)_{\Lambda;\mathrm{d}\mu} = (\widehat{\varphi}\,,\,\mathcal{F}\,\overline{f(A)}\psi)_{\Lambda;\mathrm{d}\mu} = 0.$$

Hence $\widehat{\varphi}=0$, thus $\mathrm{R}(\mathcal{F}) = L^2(\Lambda;\mathrm{d}\mu)$. □

4 Relation between physical decay and spectral regularity

The spectral characterization (21) of $\mathrm{D}(f(A))$ shows how the physical regularity of a given $\varphi\in L^2(\mathbb{R}^+)$ is related to the asymptotic behaviour for large λ of $\mathcal{F}\varphi(\lambda)$. Indeed choosing for instance $f(A)=A^n$ with $n\in\mathbb{N}$ gives

$$\varphi\in\mathrm{D}(A^n) \iff (1+|\lambda|^n)\,\mathcal{F}\varphi\in L^2(\Lambda\,;\mathrm{d}\mu), \tag{24}$$

and it is easy to see from the definition (1) of $\mathrm{D}(A)$ that a function $\varphi\in\mathrm{D}(A^n)$ is characterized by the following conditions:

$$\varphi|_{(0,h)}\in H^{2n}(0,h) \quad\text{and}\quad \varphi|_{(h,\infty)}\in H^{2n}(h,\infty), \tag{25}$$

$$\mathrm{d}_x^{2\ell}\varphi(0)=0 \quad\text{for}\quad \ell=0,\ldots,n-1, \tag{26}$$

$$[(\mathrm{d}_x^2+k^2)^\ell\varphi]_h = [\mathrm{d}_x(\mathrm{d}_x^2+k^2)^\ell\varphi]_h = 0 \quad\text{for}\quad \ell=0,\ldots,n-1, \tag{27}$$

where we use the simplified notation d_x instead of $\mathrm{d}/\mathrm{d}x$, and $[\psi]_h$ stands for the gap of ψ at $x = h$, that is, $[\psi]_h := \lim_{\varepsilon \searrow 0}\{\psi(h+\varepsilon) - \psi(h-\varepsilon)\}$.

In this section, we address the complementary issue, that is, the relation between physical decay and spectral regularity. For the usual Fourier transform, both issues coalesce, since the inverse transform amounts to a conjugation. The question is not so obvious for the generalized Fourier transform $\mathcal{F}$. A first element of answer is given by the following result which is a straightforward consequence of the fact that Φ_ζ is an entire function of ζ.

Proposition 4 *If $\varphi \in L^2_{\mathrm{comp}}(\mathbb{R}^+)$, then $\mathcal{F}\varphi(\lambda)$ extends to an entire function.*

Theorem 5 below provides us a more precise answer. First notice that speaking of spectral regularity actually makes sense only on the continuous spectrum, which leads us to split $\mathcal{F}$ into two parts $\mathcal{F}_{\mathrm{c}}$ and $\mathcal{F}_{\mathrm{p}}$ which simply denote the restrictions of $\mathcal{F}$ to Λ_{c} and Λ_{p} :

$$\mathcal{F}_{\mathrm{c}}\,\varphi := (\mathcal{F}\varphi)|_{\Lambda_{\mathrm{c}}} \quad \text{and} \quad \mathcal{F}_{\mathrm{p}}\,\varphi := (\mathcal{F}\varphi)|_{\Lambda_{\mathrm{p}}}.$$

So $\mathcal{F}_{\mathrm{c}}$ is defined from $L^2(\mathbb{R}^+)$ to $L^2(\Lambda_{\mathrm{c}}\,;\rho_\lambda \mathrm{d}\lambda)$, and its adjoint corresponds to the continuous part of the expression (22) of $\mathcal{F}^*$:

$$\mathcal{F}_{\mathrm{c}}^*\widehat{\varphi} = \lim_{M\to+\infty}\int_{-\ddot{k}^2+M^{-1}}^{M} \widehat{\varphi}(\lambda)\,\Phi_\lambda\,\rho_\lambda\,\mathrm{d}\lambda \quad \forall\widehat{\varphi} \in L^2(\Lambda_{\mathrm{c}}\,;\rho_\lambda\mathrm{d}\lambda). \tag{28}$$

Similarly, $\mathcal{F}_{\mathrm{p}}^*$ corresponds to the point spectrum contribution. The diagonalization formula (21) then turns into the orthogonal decomposition

$$f(A)\varphi = \mathcal{F}_{\mathrm{c}}^* f(\lambda)\,\mathcal{F}_{\mathrm{c}}\,\varphi \;+\; \mathcal{F}_{\mathrm{p}}^* f(\lambda)\,\mathcal{F}_{\mathrm{p}}\,\varphi. \tag{29}$$

Theorem 5 *Suppose that $\dot{k}^2 - \ddot{k}^2$ is not one of the thresholds given by* (18). *Then, for every $n \in \mathbb{N}$ and $\varphi \in L^2(\mathbb{R}^+)$, we have*

$$x^n\,\varphi(x) \in L^2(\mathbb{R}^+) \iff D_\lambda^n\,\mathcal{F}_{\mathrm{c}}\,\varphi \in L^2(\Lambda_{\mathrm{c}}\,;\rho_\lambda\mathrm{d}\lambda), \tag{30}$$

where D_λ is the spectral derivation operator given by $D_\lambda\widehat{\varphi} := \mathrm{d}(\ddot{\beta}_\lambda\,\widehat{\varphi})/\mathrm{d}\lambda$.

In order to prove this result, we have to understand the link between $\mathcal{F}_{\mathrm{c}}(x\varphi)$ and $D_\lambda\,\mathcal{F}_{\mathrm{c}}\,\varphi$, which amounts to the link between $x\,\Phi_\lambda$ and $D_\lambda\Phi_\lambda$. But there is no simple relation between the latter quantities. Actually, on both intervals $(0,h)$ and $(h,+\infty)$, function Φ_λ is a linear combination of exponential functions of the form $\exp(\pm\mathrm{i}\beta_\lambda(x)\,x)$. Such a relation exists for each of these functions, but not for their combination. That is why the proof is based on the decomposition (8) of Φ_λ and the expression (9) of the spectral derivative of Θ_ζ. The former allows us to rewrite the generalized Fourier transform in a more convenient form. Indeed, since Φ_λ is real, for all real $\varphi \in L^2_{\mathrm{comp}}(\mathbb{R}^+)$, we have

$$\mathcal{F}_{\mathrm{c}}\,\varphi = -\operatorname{Im}\{\mathcal{T}\varphi\} \quad \text{where} \quad \mathcal{T}\varphi(\lambda) := \frac{\Theta_{\lambda+\mathrm{i}0}(0)}{\ddot{\beta}_\lambda}\int_{\mathbb{R}^+}\varphi(x)\,\overline{\Theta_{\lambda+\mathrm{i}0}(x)}\,\mathrm{d}x. \tag{31}$$

Lemma 6 *The operator $\mathcal{T}$ defined in* (31) *for $\varphi \in L^2_{\mathrm{comp}}(\mathbb{R}^+)$ extends by density to a bounded operator from $L^2(\mathbb{R}^+)$ to $L^2(\Lambda_{\mathrm{c}}\,;\rho_\lambda\mathrm{d}\lambda)$.*

Proof. We prove below that there exists a constant $C > 0$ such that

$$\|\mathcal{T}\varphi\|_{\Lambda_c;\rho_\lambda \mathrm{d}\lambda} \;\le\; C\;\|\varphi\|_{\mathbb{R}^+}, \tag{32}$$

for all $\varphi \in L^2_{\mathrm{comp}}(\mathbb{R}^+)$, which can be seen as a perturbed Plancherel identity. The conclusion will then follow from the density of $L^2_{\mathrm{comp}}(\mathbb{R}^+)$ in $L^2(\mathbb{R}^+)$.

First assume that the support of $\varphi \in L^2_{\mathrm{comp}}(\mathbb{R}^+)$ is contained in $[h, +\infty)$. From (6), the expression of $\mathcal{T}\varphi$ simplifies in this case as

$$\mathcal{T}\varphi(\lambda) = \frac{\Theta_{\lambda+\mathrm{i}0}(0)\,\mathrm{e}^{\mathrm{i}\ddot\beta_\lambda h}}{\ddot\beta_\lambda}\int_h^{+\infty}\varphi(x)\;\mathrm{e}^{-\mathrm{i}\ddot\beta_\lambda x}\;\mathrm{d}x,$$

where we recognize the usual Fourier transform. Hence using the change of variable $\beta = \ddot\beta_\lambda$ and Plancherel equality, we infer that

$$\int_{-\ddot k^2}^{+\infty}|\mathcal{T}\varphi(\lambda)|^2\,\rho_\lambda\,\mathrm{d}\lambda = \frac{2}{\pi}\int_0^{+\infty}\left|\int_{\mathbb{R}^+}\varphi(x)\;\mathrm{e}^{-\mathrm{i}\beta x}\;\mathrm{d}x\right|^2\mathrm{d}\beta = 2\int_{\mathbb{R}^+}|\varphi(x)|^2\,\mathrm{d}x,$$

which shows that for functions whose support is contained in $[h, +\infty)$, (32) holds with $C = \sqrt{2}$.

Assume now that the support of φ is contained in $[0, h]$. Choosing some fixed $\lambda^* > -\ddot k^2$, we split $\|\mathcal{T}\varphi\|^2_{\Lambda_c;\rho_\lambda\mathrm{d}\lambda}$ as the sum of two integrals respectively on $(-\ddot k^2, \lambda^*)$ and $(\lambda^*, +\infty)$. On one hand, we have by Cauchy–Schwarz inequality

$$\int_{-\ddot k^2}^{\lambda^*}|\mathcal{T}\varphi(\lambda)|^2\,\rho_\lambda\,\mathrm{d}\lambda = \int_{-\ddot k^2}^{\lambda^*}\left|\int_0^h\varphi(x)\;\overline{\Theta_{\lambda+\mathrm{i}0}(x)}\;\mathrm{d}x\right|^2\frac{\mathrm{d}\lambda}{\pi\ddot\beta_\lambda} \le C\;\|\varphi\|^2_{(0,h)}, \tag{33}$$

where

$$C = \frac{2\ddot\beta_{\lambda^*}}{\pi}\sup_{\lambda\in[-\ddot k^2,\,\lambda^*]}\int_0^h|\Theta_{\lambda+\mathrm{i}0}(x)|^2\,\mathrm{d}x.$$

On the other hand, for the complementary integral on $(\lambda^*, +\infty)$, we rewrite the expression (7) of $\Theta_{\lambda+\mathrm{i}0}(x)$ as

$$\Theta_{\lambda+\mathrm{i}0}(x) = \mathrm{e}^{\mathrm{i}\dot\beta_\lambda(x-h)} - \mathrm{i}\,\frac{\dot k^2 - \ddot k^2}{\dot\beta_\lambda + \ddot\beta_\lambda}\;s_\lambda(x)\quad\forall x\in(0,h).$$

Hence

$$\mathcal{T}\varphi = \mathcal{T}_1\varphi + \mathcal{T}_2\varphi\quad\text{where}\quad\left\{\begin{aligned}&\mathcal{T}_1\varphi(\lambda) := \frac{\Theta_{\lambda+\mathrm{i}0}(0)\,\mathrm{e}^{\mathrm{i}\dot\beta_\lambda h}}{\ddot\beta_\lambda}\int_0^h\varphi(x)\;\mathrm{e}^{-\mathrm{i}\dot\beta_\lambda x}\;\mathrm{d}x,\\&\mathcal{T}_2\varphi(\lambda) := \mathrm{i}\,\frac{(\dot k^2 - \ddot k^2)\,\Theta_{\lambda+\mathrm{i}0}(0)}{\ddot\beta_\lambda\,(\dot\beta_\lambda + \ddot\beta_\lambda)}\int_0^h\varphi(x)\;s_\lambda(x)\;\mathrm{d}x.\end{aligned}\right.$$

For the former contribution we proceed as above using the change of variable $\beta = \dot\beta_\lambda$ and Plancherel equality, which yields

$$\begin{aligned}\int_{\lambda^*}^{+\infty}|\mathcal{T}_1\varphi(\lambda)|^2\,\rho_\lambda\,\mathrm{d}\lambda &\le \frac{2\dot\beta_{\lambda^*}}{\pi\ddot\beta_{\lambda^*}}\int_{\dot\beta_{\lambda^*}}^{+\infty}\left|\int_0^h\varphi(x)\;\mathrm{e}^{-\mathrm{i}\beta x}\;\mathrm{d}x\right|^2\mathrm{d}\beta\\&\le \frac{2\dot\beta_{\lambda^*}}{\ddot\beta_{\lambda^*}}\int_0^h|\varphi(x)|^2\,\mathrm{d}x.\end{aligned}$$

For the latter, noticing that $\|s_\lambda\|^2_{(0,h)} \le h^3/3$, we deduce from Cauchy–Schwarz inequality that

$$\int_{\lambda^*}^{+\infty} |\mathcal{T}_2\varphi(\lambda)|^2\,\rho_\lambda\,\mathrm{d}\lambda \;\le\; \frac{h^3(\dot{k}^2-\ddot{k}^2)^2}{3\pi}\int_{\lambda^*}^{+\infty}\frac{\mathrm{d}\lambda}{\ddot{\beta}_\lambda\,(\dot{\beta}_\lambda+\ddot{\beta}_\lambda)^2}\;\int_0^h|\varphi(x)|^2\,\mathrm{d}x.$$

Both these estimates together with (33) finally show that (32) is also valid if the support of φ is contained in $[0,h]$, hence it holds for all $\varphi\in L^2_{\mathrm{comp}}(\mathbb{R}^+)$. □

Proof of Theorem 5. It is sufficient to verify (30) for $n=1$ and real φ : the general case ($n>1$ and complex φ) follows immediately. For every real $\varphi\in L^2_{\mathrm{comp}}(\mathbb{R}^+)$, we deduce from (31) and Lebesgue theorem that

$$D_\lambda\,\mathcal{F}_\mathrm{c}\,\varphi(\lambda) = -\int_{\mathbb{R}^+}\varphi(x)\,\frac{\partial}{\partial\lambda}\,\mathrm{Im}\left\{\Theta_{\lambda+\mathrm{i}0}(0)\overline{\Theta_{\lambda+\mathrm{i}0}(x)}\right\}\,\mathrm{d}x,$$

We see from (9) that

$$\frac{\partial}{\partial\lambda}\Theta_{\lambda+\mathrm{i}0}(x) = \frac{\mathrm{i}}{2\ddot{\beta}_\lambda}\left\{(x-h)\,\Theta_{\lambda+\mathrm{i}0}(x)+\tilde{\Theta}_\lambda(x)\right\}. \tag{34}$$

Hence

$$\begin{aligned}&\frac{\partial}{\partial\lambda}\,\mathrm{Im}\left\{\Theta_{\lambda+\mathrm{i}0}(0)\overline{\Theta_{\lambda+\mathrm{i}0}(x)}\right\}\\ &\qquad\qquad = \frac{-1}{2\ddot{\beta}_\lambda}\,\mathrm{Re}\left\{\left(x\Theta_{\lambda+\mathrm{i}0}(0)-\tilde{\Theta}_\lambda(0)\right)\overline{\Theta_{\lambda+\mathrm{i}0}(x)}+\Theta_{\lambda+\mathrm{i}0}(0)\tilde{\Theta}_\lambda(x)\right\}.\end{aligned}$$

As a consequence

$$D_\lambda\,\mathcal{F}_\mathrm{c}\,\varphi(\lambda) = \frac{1}{2}\,\mathrm{Re}\left\{\mathcal{T}\{x\varphi(x)\}(\lambda)-\frac{\tilde{\Theta}_\lambda(0)}{\Theta_{\lambda+\mathrm{i}0}(0)}\,\mathcal{T}\varphi(\lambda)+\tilde{\mathcal{T}}\varphi(\lambda)\right\}, \tag{35}$$

where we have denoted

$$\tilde{\mathcal{T}}\varphi(\lambda) \;:=\; \frac{\Theta_{\lambda+\mathrm{i}0}(0)}{\ddot{\beta}_\lambda}\int_0^h\varphi(x)\,\tilde{\Theta}_\lambda(x)\,\mathrm{d}x, \tag{36}$$

which defines a bounded operator from $L^2(\mathbb{R}^+)$ to $L^2(\Lambda_\mathrm{c}\,;\rho_\lambda\mathrm{d}\lambda)$. Indeed using the expression (10) of $\tilde{\Theta}_\lambda$, we notice that $\|\tilde{\Theta}_\lambda\|_{(0,h)}\le C/(\dot{k}^2+\lambda)$, and consequently, by Cauchy–Schwarz inequality,

$$\|\tilde{\mathcal{T}}\varphi\|_{\Lambda_\mathrm{c};\rho_\lambda\mathrm{d}\lambda}\;\le\;C\;\|\varphi\|_{(0,h)}.$$

Thanks to the assumption that $\dot{k}^2-\ddot{k}^2$ is not one of the thresholds given by (18), it is readily seen that $\tilde{\Theta}_\lambda(0)/\Theta_{\lambda+\mathrm{i}0}(0)$ is bounded on Λ_c. Hence, in view of Lemma 6, it follows from (35) that

$$\|D_\lambda\,\mathcal{F}_\mathrm{c}\,\varphi\|_{\Lambda_\mathrm{c};\rho_\lambda\mathrm{d}\lambda}\;\le\;C_1\;\|x\varphi\|_{\mathbb{R}^+}+C_2\;\|\varphi\|_{\mathbb{R}^+}.$$

We have proved this estimate for $\varphi\in L^2_{\mathrm{comp}}(\mathbb{R}^+)$. By density, it holds for all $\varphi\in L^2(\mathbb{R}^+)$ such that $x\varphi\in L^2(\mathbb{R}^+)$. Thus, for such a φ, we know that $D_\lambda\,\mathcal{F}_\mathrm{c}\,\varphi\in L^2(\Lambda_\mathrm{c}\,;\rho_\lambda\mathrm{d}\lambda)$.

Let us now prove the converse. By virtue of the identity $\mathcal{F}_\mathrm{c}^*\mathcal{F}_\mathrm{c}+\mathcal{F}_\mathrm{p}^*\mathcal{F}_\mathrm{p}=\mathrm{I}$ (see (29)) and the fact that $x\mathcal{F}_\mathrm{p}^*\mathcal{F}_\mathrm{p}\varphi\in L^2(\mathbb{R}^+)$ for all $\varphi\in L^2(\mathbb{R}^+)$ (since the eigenfunctions Φ_λ, $\lambda \in \Lambda_{\rm p}$, decrease exponentially at infinity), it is enough to prove that if $\widehat{\varphi} \in L^2(\Lambda_{\rm c}\,;\rho_\lambda {\rm d}\lambda)$ is such that $D_\lambda \widehat{\varphi} \in L^2(\Lambda_{\rm c}\,;\rho_\lambda {\rm d}\lambda)$, then $x\,\mathcal{F}_{\rm c}^*\,\widehat{\varphi} \in L^2(\mathbb{R}^+)$.

As in (31), we can use the decomposition (8) of Φ_λ to rewrite the expression (28) of $\mathcal{F}_{\rm c}^*$ in the following form, assuming that $\widehat{\varphi}$ is real and has a compact support:

$$\mathcal{F}_{\rm c}^*\widehat{\varphi} = {\rm Im}\{\mathcal{T}^*\widehat{\varphi}\} \quad \text{where} \quad (\mathcal{T}^*\widehat{\varphi})(x) = \frac{1}{\pi}\int_{\Lambda_{\rm c}} \widehat{\varphi}(\lambda)\,\frac{\Theta_{\lambda+{\rm i}0}(x)}{\Theta_{\lambda+{\rm i}0}(0)}\,{\rm d}\lambda. \tag{37}$$

By an integration by parts, we have

$$\begin{aligned}(\mathcal{T}^*D_\lambda\widehat{\varphi})(x) &= \frac{1}{\pi}\int_{\Lambda_{\rm c}} \frac{\rm d}{{\rm d}\lambda}(\ddot{\beta}_\lambda\,\widehat{\varphi}(\lambda))\,\frac{\Theta_{\lambda+{\rm i}0}(x)}{\Theta_{\lambda+{\rm i}0}(0)}\,{\rm d}\lambda \\ &= \frac{-1}{\pi}\int_{\Lambda_{\rm c}} \widehat{\varphi}(\lambda)\,\ddot{\beta}_\lambda\,\frac{\partial}{\partial\lambda}\left(\frac{\Theta_{\lambda+{\rm i}0}(x)}{\Theta_{\lambda+{\rm i}0}(0)}\right){\rm d}\lambda,\end{aligned}$$

where we deduce from (34) that

$$\ddot{\beta}_\lambda\,\frac{\partial}{\partial\lambda}\left(\frac{\Theta_{\lambda+{\rm i}0}(x)}{\Theta_{\lambda+{\rm i}0}(0)}\right) = \frac{\rm i}{2}\left\{\frac{x\,\Theta_{\lambda+{\rm i}0}(x)}{\Theta_{\lambda+{\rm i}0}(0)} - \frac{\tilde{\Theta}_\lambda(0)\,\Theta_{\lambda+{\rm i}0}(x)}{\Theta_{\lambda+{\rm i}0}(0)^2} + \frac{\tilde{\Theta}_\lambda(x)}{\Theta_{\lambda+{\rm i}0}(0)}\right\}.$$

Hence

$$\mathcal{T}^*D_\lambda\widehat{\varphi} = \frac{-{\rm i}}{2}\left\{x\,\mathcal{T}^*\widehat{\varphi} - \mathcal{T}^*\left(\frac{\tilde{\Theta}_\lambda(0)}{\Theta_{\lambda+{\rm i}0}(0)}\,\widehat{\varphi}\right) + \tilde{\mathcal{T}}^*\widehat{\varphi}\right\},$$

where the adjoint $\tilde{\mathcal{T}}^*$ of $\tilde{\mathcal{T}}$ (see (36)) is given by

$$(\tilde{\mathcal{T}}^*\widehat{\varphi})(x) = \frac{1}{\pi}\int_{\Lambda_{\rm c}} \widehat{\varphi}(\lambda)\,\frac{\tilde{\Theta}_\lambda(x)}{\Theta_{\lambda+{\rm i}0}(0)}\,{\rm d}\lambda.$$

Going back to (37), we finally obtain that for real $\widehat{\varphi} \in L^2_{\rm comp}(\Lambda_{\rm c}\,;\rho_\lambda{\rm d}\lambda)$,

$$x\,\mathcal{F}_{\rm c}^*\widehat{\varphi} = {\rm Im}\left\{2{\rm i}\,\mathcal{T}^*D_\lambda\widehat{\varphi} + \mathcal{T}^*\left(\frac{\tilde{\Theta}_\lambda(0)}{\Theta_{\lambda+{\rm i}0}(0)}\,\widehat{\varphi}\right) - \tilde{\mathcal{T}}^*\widehat{\varphi}\right\}.$$

Using again the fact that $\tilde{\Theta}_\lambda(0)/\Theta_{\lambda+{\rm i}0}(0)$ is bounded on $\Lambda_{\rm c}$, we infer that this relation holds by density if $\widehat{\varphi}$ and $D_\lambda\widehat{\varphi}$ belong to $L^2(\Lambda_{\rm c}\,;\rho_\lambda{\rm d}\lambda)$, which completes the proof of Theorem 5. □

5 Extension of $\mathcal{F}$ to a space of distributions

One of the basic properties which makes the usual Fourier transform a very powerful tool for solving partial differential equations is the possibility to interpret it in the sense of distributions, that is, to extend it to the Schwartz space $\mathcal{S}'(\mathbb{R}^N)$ of tempered distributions [5]. This property rests essentially on the exchange of regularity and decay between the physical and spectral spaces. In the previous section, we have studied how such an exchange occurs for the generalized Fourier transform. We have now all the ingredients to construct a space of distributions similar to the Schwartz space to which $\mathcal{F}$ can be extended.

5.1 Formal construction

In order to generalize the notion of distribution, we use the notion of rigging of a Hilbert space by linear topological spaces, more precisely by nuclear spaces [1]. Suppose that we can define a space $\mathcal{S}_A(\mathbb{R}^+)$ such that

$$\mathcal{S}_A(\mathbb{R}^+) \text{ is nuclear,} \tag{38}$$

$$\mathcal{S}_A(\mathbb{R}^+) \subset L^2(\mathbb{R}^+) \quad \text{and} \quad \mathcal{S}_A(\mathbb{R}^+) \text{ is } \textit{dense} \text{ in } L^2(\mathbb{R}^+), \tag{39}$$

$$\mathcal{S}_A(\mathbb{R}^+) \subset \mathrm{D}(A) \quad \text{and} \quad A(\mathcal{S}_A(\mathbb{R}^+)) \subset \mathcal{S}_A(\mathbb{R}^+), \tag{40}$$

where the symbol $\subset$ stands for continuous embeddings. We choose to identify $L^2(\mathbb{R}^+)$ with its dual space (here we use the term 'dual' to denote the collection of *antilinear* continuous functionals). Therefore assumption (39) shows that $L^2(\mathbb{R}^+)$ can be interpreted as a subspace of the dual space $\mathcal{S}'_A(\mathbb{R}^+)$ of $\mathcal{S}_A(\mathbb{R}^+)$, which yields the following chain of spaces:

$$\mathcal{S}_A(\mathbb{R}^+) \;\subset\; L^2(\mathbb{R}^+) = L^2(\mathbb{R}^+)' \;\subset\; \mathcal{S}'_A(\mathbb{R}^+), \tag{41}$$

where the duality product $\langle \cdot\,, \cdot \rangle_{\mathbb{R}^+}$ between $\mathcal{S}_A(\mathbb{R}^+)$ and $\mathcal{S}'_A(\mathbb{R}^+)$ appears as an extension of the inner product of $L^2(\mathbb{R}^+)$:

$$\langle \varphi, \psi \rangle_{\mathbb{R}^+} = (\varphi, \psi)_{\mathbb{R}^+} \quad \forall \varphi \in L^2(\mathbb{R}^+),\ \forall \psi \in \mathcal{S}_A(\mathbb{R}^+),$$

(which is compatible with the initial integral definition of $\langle \cdot\,, \cdot \rangle_{\mathbb{R}^+}$, see §1). In the scheme (41), $\mathcal{S}_A(\mathbb{R}^+)$ represents the space of *test functions*, whereas $\mathcal{S}'_A(\mathbb{R}^+)$ plays the role of space of *distributions*, or generalized functions.

Assumption (40) means that A is a continuous operator in $\mathcal{S}_A(\mathbb{R}^+)$. It allows us to interpret A in the sense of distributions, that is, to extend it to $\mathcal{S}'_A(\mathbb{R}^+)$ by setting

$$\langle A\varphi, \psi \rangle_{\mathbb{R}^+} := \langle \varphi, A\psi \rangle_{\mathbb{R}^+} \quad \forall \varphi \in \mathcal{S}'_A(\mathbb{R}^+),\ \forall \psi \in \mathcal{S}_A(\mathbb{R}^+).$$

The topological assumption (38) essentially assures that the kernel theorem applies in the functional scheme (41). This theorem, which was initially proved by Schwartz for $\mathcal{S}'(\mathbb{R}^N)$, states that every continuous sesquilinear form $a(\varphi, \psi)$ on $\mathcal{S}_A(\mathbb{R}^+) \times \mathcal{S}_A(\mathbb{R}^+)$ appears as an 'integral operator' with a distribution kernel in the sense that there exists κ_a in the (completed) tensor product $\mathcal{S}'_A(\mathbb{R}^+) \otimes \mathcal{S}'_A(\mathbb{R}^+)$, that is, a distribution of two variables, such that

$$a(\varphi, \psi) = \langle\!\langle \kappa_a, \overline{\varphi} \otimes \psi \rangle\!\rangle_{\mathbb{R}^+ \times \mathbb{R}^+} \quad \forall (\varphi, \psi) \in \mathcal{S}_A(\mathbb{R}^+) \times \mathcal{S}_A(\mathbb{R}^+).$$

The rigging (41) of the physical space $L^2(\mathbb{R}^+)$ is naturally converted into a rigging of the spectral space $L^2(\Lambda; \mathrm{d}\mu)$ by the generalized Fourier transform $\mathcal{F}$. Indeed setting $\widehat{\mathcal{S}}_A(\Lambda) := \mathcal{F}(\mathcal{S}_A(\mathbb{R}^+))$, we can extend $\mathcal{F}$ to distributions of $\mathcal{S}'_A(\mathbb{R}^+)$ by the formula

$$\langle \mathcal{F}\varphi, \widehat{\psi} \rangle_\Lambda \;:=\; \langle \varphi, \mathcal{F}^{-1}\widehat{\psi} \rangle_{\mathbb{R}^+} \quad \forall \varphi \in \mathcal{S}'_A(\mathbb{R}^+),\ \forall \widehat{\psi} \in \widehat{\mathcal{S}}_A(\Lambda),$$

where $\langle \cdot\,, \cdot \rangle_\Lambda$ denotes the duality product between $\widehat{\mathcal{S}}_A(\Lambda)$ and its dual space $\widehat{\mathcal{S}}'_A(\Lambda)$ which defines the space of *spectral distributions*, and $\mathcal{F}^{-1}$ is simply the restriction of $\mathcal{F}^*$ to $\widehat{\mathcal{S}}_A(\Lambda)$. This can be summarized in the following scheme:

$$\begin{array}{ccccc l} \mathcal{S}_A(\mathbb{R}^+) & \subset & L^2(\mathbb{R}^+) & \subset & \mathcal{S}'_A(\mathbb{R}^+) & \\ \downarrow & & \downarrow & & \downarrow & \mathcal{F} \\ \widehat{\mathcal{S}}_A(\Lambda) & \subset & L^2(\Lambda; \mathrm{d}\mu) & \subset & \widehat{\mathcal{S}}'_A(\Lambda) & \end{array}$$

where $\mathcal{F}$ appears as an isomorphism between each pair of corresponding spaces (vertical arrows). We show below how to construct such spaces. Note that the spectral spaces are derived from the physical spaces via the generalized Fourier transform. Such a construction has a practical interest if we are able to identify $\widehat{\mathcal{S}}_A(\Lambda)$ in an intrinsic way, not only as $\mathcal{F}(\mathcal{S}_A(\mathbb{R}^+))$. That is where we shall need the results of §4.

5.2 Physical distributions

How can we choose a space $\mathcal{S}_A(\mathbb{R}^+)$ which satisfies (38)–(40)? One could be tempted to consider the restrictions to $\mathbb{R}^+$ of functions of the usual Schwartz space $\mathcal{S}(\mathbb{R})$. But condition (40) would not be satisfied since for a regular φ, the function $A\varphi$ is in general discontinuous at $x = h$. Actually condition (40) implies that $\mathcal{S}_A(\mathbb{R}^+) \subset \mathrm{D}(A^n)$ for all $n \in \mathbb{N}$. Hence a natural way to adapt the definition of $\mathcal{S}(\mathbb{R})$ to our operator A is to choose

$$\mathcal{S}_A(\mathbb{R}^+) := \bigcap_{n,m\in\mathbb{N}} \mathcal{S}_{n,m} \tag{42}$$

where

$$\mathcal{S}_{n,m} := \{\varphi \in L^2(\mathbb{R}^+);\ \varphi \in \mathrm{D}(A^n) \text{ and } x^m A^n \varphi \in L^2(\mathbb{R}^+)\}.$$

Thus $\mathcal{S}_A(\mathbb{R}^+)$ appears as a projective limit of Hilbert spaces, equipped for instance with the countable family of norms $\|\varphi\|_{n,m}$ defined by

$$\|\varphi\|^2_{n,m} := \|\varphi\|^2_{\mathbb{R}^+} + \|(1 + x^m)A^n\varphi\|^2_{\mathbb{R}^+}.$$

According to (25)–(27) and the classical embedding theorems of weighted Sobolev spaces [1], we have the following characterization of $\mathcal{S}_A(\mathbb{R}^+)$.

Proposition 7 *A function φ belongs to $\mathcal{S}_A(\mathbb{R}^+)$ if and only if it is infinitely differentiable on both intervals $(0, h)$ and $(h, +\infty)$, rapidly decaying at infinity as well as its derivatives (in the sense that $\lim_{x\to+\infty} \mathrm{d}_x^n(x^m\varphi(x)) = 0$ for all $n, m \in \mathbb{N}$) , and satisfies*

$$\mathrm{d}_x^{2n}\varphi(0) = 0 \quad \text{and} \quad [(\mathrm{d}_x^2 + k^2)^n\varphi]_h = [\mathrm{d}_x(\mathrm{d}_x^2 + k^2)^n\varphi]_h = 0 \quad \forall n \in \mathbb{N}. \tag{43}$$

Proposition 8 *The space $\mathcal{S}_A(\mathbb{R}^+)$ defined by* (42) *fulfils assumptions* (38)–(40).

Proof. Let us first prove that $\mathcal{S}_A(\mathbb{R}^+)$ is nuclear. Following [1], we have to verify that for any $(n, m) \in \mathbb{N}^2$, one can find $(n', m') \in \mathbb{N}^2$ such that $\mathcal{S}_{n',m'} \subset \mathcal{S}_{n,m}$ and the embedding operator $\mathcal{S}_{n',m'} \to \mathcal{S}_{n,m}$ is quasinuclear, *i.e.*, a Hilbert–Schmidt operator. Each $\mathcal{S}_{n,m}$ can be seen as a closed subspace of $H^{2n}(0, h) \oplus H^{2n}_m(h, +\infty)$ (characterized by conditions (26) and (27)), where $H^{2n}_m(h, +\infty)$ is defined as the usual Sobolev space $H^{2n}(h, +\infty)$ by replacing the Lebesgue measure $\mathrm{d}x$ by the weighted measure $(1 + x^m)^2\, \mathrm{d}x$. Therefore it is enough to verify that one can find $(n', m') \in \mathbb{N}^2$ such that both embedding operators $H^{2n'}(0, h) \to H^{2n}(0, h)$ and $H^{2n'}_{m'}(h, +\infty) \to H^{2n}_m(h, +\infty)$ are quasinuclear. For the former, this holds if $n' > n$ [1, Theorem 3.2, p.137]. For the latter, we must have $n' > n$ and $m' > m$ [1, Theorem 4.4, p.147]. This completes the proof.

The continuous embedding $\mathcal{S}_A(\mathbb{R}^+) \subset L^2(\mathbb{R}^+)$ is obvious, and the density of $\mathcal{S}_A(\mathbb{R}^+)$ in $L^2(\mathbb{R}^+)$ follows from the density of $\mathcal{D}(0,h)$ and $\mathcal{D}(h,+\infty)$ respectively in $L^2(0,h)$ and $L^2(h,+\infty)$ (recall that $\mathcal{D}(X)$ denotes the space of infinitely differentiable functions in X with compact support [5]). Hence (39) is satisfied. Finally noticing that $\|A\varphi\|_{n,m} \leq C\,\|\varphi\|_{n+1,m}$ yields (40). $\square$

Proposition 7 shows that the dual space $\mathcal{S}'_A(\mathbb{R}^+)$ of $\mathcal{S}_A(\mathbb{R}^+)$ is similar to the Schwartz space $\mathcal{S}'(\mathbb{R})$. It actually contains all restrictions to $\mathbb{R}^+$ of distributions of $\mathcal{S}'(\mathbb{R})$ whose order is less than 2 in a vicinity of $x = h$. For instance, the space of locally integrable tempered functions

$$L^1_{\rm temp}(\mathbb{R}^+) := \left\{ f \in L^1_{\rm loc}(\mathbb{R}^+);\ \exists m \in \mathbb{N},\ \lim_{x\to+\infty} x^{-m} f(x) = 0 \right\} \tag{44}$$

is obviously contained in $\mathcal{S}'_A(\mathbb{R}^+)$. Owing to the boundary condition $\varphi(0) = 0$ in (43), $\mathcal{S}'_A(\mathbb{R}^+)$ also contains functions which are not integrable near $x = 0$, such as x^{-s} for $1 \leq s < 2$. The Dirac distribution δ_x at a given point $x > 0$, defined by $\langle \delta_x, \varphi\rangle_{\mathbb{R}^+} = \overline{\varphi(x)}$, belongs to $\mathcal{S}'_A(\mathbb{R}^+)$, as well as its derivative δ'_x. Further derivatives of δ_x are also in $\mathcal{S}'_A(\mathbb{R}^+)$ if $x \neq h$, but not if $x = h$ since the derivatives of order greater than 1 of functions of $\mathcal{S}_A(\mathbb{R}^+)$ are in general discontinuous at this point. On the other hand, $A^n\delta_h$ belongs to $\mathcal{S}'_A(\mathbb{R}^+)$ for every $n \in \mathbb{N}$: this is the functional $\langle A^n\delta_h, \varphi\rangle_{\mathbb{R}^+} = \overline{(-\mathrm{d}_x^2 - k^2)^n\varphi(h)}$, which is well defined by virtue of (43).

5.3 Spectral distributions

Consider now the space $\widehat{\mathcal{S}}_A(\Lambda) := \mathcal{F}(\mathcal{S}_A(\mathbb{R}^+))$ of spectral test functions, that is,

$$\widehat{\mathcal{S}}_A(\Lambda) = \bigcap_{n,m\in\mathbb{N}} \widehat{\mathcal{S}}_{n,m}, \quad \text{where } \ \widehat{\mathcal{S}}_{n,m} := \mathcal{F}(\mathcal{S}_{n,m}).$$

Theorem 3 (more precisely its consequence (24)) and Theorem 5 provide us a characterization of $\widehat{\mathcal{S}}_{n,m}$:

Corollary 9 *A function $\widehat{\varphi} \in L^2(\Lambda; \mathrm{d}\mu)$ belongs to $\widehat{\mathcal{S}}_{n,m}$ if and only if*

$$\lambda^n\widehat{\varphi} \in L^2(\Lambda; \mathrm{d}\mu) \quad \textit{and} \quad D^m_\lambda(\lambda^n\widehat{\varphi}|_{\Lambda_{\rm c}}) \in L^2(\Lambda_{\rm c}\,; \rho_\lambda \mathrm{d}\lambda).$$

The following result makes precise what kind of functions $\widehat{\mathcal{S}}_A(\Lambda)$ contains.

Proposition 10 *A function $\widehat{\varphi}$ belongs to $\widehat{\mathcal{S}}_A(\Lambda)$ if and only if there exists $\tilde{\varphi} \in \mathcal{S}(\mathbb{R})$ such that*

$$\widehat{\varphi}(\lambda) = \frac{1}{\ddot{\beta}_\lambda}\,\tilde{\varphi}(\ddot{\beta}_\lambda) \quad \forall\lambda \in \Lambda_{\rm c}.$$

Proof. Using the change of variable $\beta = \ddot{\beta}_\lambda$, that is, $\lambda = \lambda(\beta) := \beta^2 - \ddot{k}^2$, we see that

$$\int_{-\ddot{k}^2}^{+\infty} |\widehat{\varphi}(\lambda)|^2\,\rho_\lambda\,\mathrm{d}\lambda = \int_0^{+\infty} |\beta\,\widehat{\varphi}(\lambda(\beta))|^2\,\tilde{\rho}_\beta\,\mathrm{d}\beta \quad \text{where } \ \tilde{\rho}_\beta := \frac{2}{\pi|\Theta_{\lambda(\beta)+\mathrm{i}0}(0)|^2},$$

which means that the transformation C defined by

$$(C\widehat{\varphi})(\beta) := \beta\,\widehat{\varphi}(\lambda(\beta)) \quad \forall \beta \geq 0$$

is unitary from $L^2(\Lambda_{\rm c}\,;\rho_\lambda \mathrm{d}\lambda)$ to $L^2(\mathbb{R}^+;\tilde{\rho}_\beta\,\mathrm{d}\beta)$, hence an isomorphism from $L^2(\Lambda_{\rm c}\,;\rho_\lambda \mathrm{d}\lambda)$ to $L^2(\mathbb{R}^+)$, since $M_1 \leq \tilde{\rho}_\beta \leq M_2$ for some positive constants M_1 and M_2. Moreover, noticing that

$$C(\lambda^n\widehat{\varphi}) = \lambda(\beta)^n\,C\widehat{\varphi} \quad \text{and} \quad C(D_\lambda^m\widehat{\varphi}) = \frac{1}{2^m}\,\frac{\mathrm{d}^m(C\widehat{\varphi})}{\mathrm{d}\beta^m},$$

we infer that $\widehat{\varphi} \in \widehat{\mathcal{S}}_A(\Lambda)$ if and only if $C\widehat{\varphi}$ is the restriction to $\mathbb{R}^+$ of a function of $\mathcal{S}(\mathbb{R})$. The conclusion follows. $\square$

Hence a function $\widehat{\varphi}$ of $\widehat{\mathcal{S}}_A(\Lambda)$ is infinitely differentiable on $(-\ddot{k}^2,+\infty)$ with rapidly decaying derivatives. Near $\lambda = -\ddot{k}^2$, it may have a singular behaviour: the above proposition actually tells us that $(\mathrm{d}^m\widehat{\varphi}/\mathrm{d}\lambda^m)(\lambda) = O(\ddot{\beta}_\lambda^{-1-2m})$ as λ tends to $-\ddot{k}^2$.

Spectral distributions of $\widehat{\mathcal{S}}'_A(\Lambda)$ are then very similar to tempered distributions on $(-\ddot{k}^2,+\infty)$. In particular, every distribution of $\mathcal{S}'(\mathbb{R})$ whose support is contained in $(-\ddot{k}^2,+\infty)$ (completed by any scalar values at points $\lambda \in \Lambda_{\rm p}$) belongs to $\widehat{\mathcal{S}}'_A(\Lambda)$. On the other hand, we see from the proof of Proposition 10 that a function $\widehat{f} : \Lambda \mapsto \mathbb{C}$ such that $C\widehat{f} \in L^1_{\rm temp}(\mathbb{R}^+)$ (see (44)) belongs to $\widehat{\mathcal{S}}'_A(\Lambda)$. This shows that the space

$$L^1_{\rm temp}(\Lambda) := \left\{ \widehat{f} : \Lambda \mapsto \mathbb{C};\ \widehat{f}|_{\Lambda_{\rm c}} \in L^1_{\rm loc}(\Lambda_{\rm c}) \text{ and } \exists m \in \mathbb{N},\ \lim_{\lambda\to+\infty} \lambda^{-m}\widehat{f}(\lambda) = 0 \right\}$$

is contained in $\widehat{\mathcal{S}}'_A(\Lambda)$.

We end this section by a simple application of the above results. For a given $\lambda_0 \in \Lambda$, function Φ_{λ_0} clearly belongs to $L^1_{\rm temp}(\mathbb{R}^+)$. Its generalized Fourier transform is then defined in the sense of $\mathcal{S}'_A(\mathbb{R}^+)$ by

$$\langle \mathcal{F}\Phi_{\lambda_0}, \widehat{\psi}\rangle_\Lambda = \langle \Phi_{\lambda_0}, \mathcal{F}^{-1}\widehat{\psi}\rangle_{\mathbb{R}^+} = \overline{\mathcal{F}\,\mathcal{F}^{-1}\widehat{\psi}(\lambda_0)} = \overline{\widehat{\psi}(\lambda_0)},$$

for all $\widehat{\psi} \in \widehat{\mathcal{S}}_A(\Lambda)$. This means that $\mathcal{F}\Phi_{\lambda_0}$ is the Dirac measure at λ_0 :

$$\mathcal{F}\Phi_{\lambda_0} = \widehat{\delta}_{\lambda_0}.$$

On the other hand, for a given $x_0 \in \mathbb{R}^+$, the Dirac measure δ_{x_0} at x_0 belongs to $\mathcal{S}'_A(\mathbb{R}^+)$ and

$$\langle \mathcal{F}\delta_{x_0}, \widehat{\psi}\rangle_\Lambda = \langle \delta_{x_0}, \mathcal{F}^{-1}\widehat{\psi}\rangle_{\mathbb{R}^+} = \overline{\mathcal{F}^{-1}\widehat{\psi}(x_0)} = \langle \Phi_\cdot(x_0), \widehat{\psi}\rangle_\Lambda,$$

for all $\widehat{\psi} \in \widehat{\mathcal{S}}_A(\Lambda)$. As a consequence,

$$\mathcal{F}\delta_{x_0} = \Phi_\cdot(x_0).$$

These relations could have been written formally from the begining. We have now a proper functional context in which they make sense.

Conclusion

We have shown how the generalized Fourier transform associated with the Sturm–Liouville operator A defined in (1) can be interpreted in the sense of distributions by constructing suitable spaces of tempered physical and spectral distributions. This construction is based on the fact that the generalized Fourier transform exchanges regularity and decay between the physical and spectral variables. Thanks to an intensive use of the explicit form of the generalized eigenfunctions of A, the proof of this property requires only very simple tools. On the other hand, it cannot be adapted to more complicated Sturm–Liouville operators for which the generalized eigenfunctions are not known in closed form. The extension of the present results to such situations seems to be an open question.

References

[1] Y.M. Berezansky, Z.G. Sheftel, G.F. Us, Functional Analysis II, Birkhäuser, Basel, 1996.

[2] A.-S. Bonnet-Ben Dhia, G. Dakhia, C. Hazard, L. Chorfi, Diffraction by a defect in an open waveguide: a mathematical analysis based on a modal radiation condition, to appear in SIAM J. Appl. Math..

[3] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, Krieger Publishing Company, Malabar (Florida), 1955.

[4] C. Hazard, F. Loret, Generalized eigenfunction expansions for conservative scattering problems with an application to water waves, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007) 995–1035.

[5] L. Schwartz, Théorie des Distributions, Hermann, Paris, 1966.

[6] E.C. Titchmarsh, Eigenfunction expansions associated with second order differential equations, Oxford University Press, 1958.

[7] J. Weidmann, Spectral theory of ordinary differential operators, Springer-Verlag, Berlin, 1987.

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