Percolation and Connectivity in AB Random Geometric Graphs

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Abstract

Given two independent Poisson point processes $\Phi^{(1)}, \Phi^{(2)}$ in $\mathbb{R}^d$, the AB Poisson Boolean model is the graph with points of $\Phi^{(1)}$ as vertices and with edges between any pair of points for which the intersection of balls of radius $2r$ centred at these points contains at least one point of $\Phi^{(2)}$. This is a generalization of the AB percolation model on discrete lattices. We show the existence of percolation for all $d \geq 2$ and derive bounds for a critical intensity. We also provide a characterization for this critical intensity when $d = 2$. To study the connectivity problem, we consider independent Poisson point processes of intensities n and cn in the unit cube. The AB random geometric graph is defined as above but with balls of radius r. We derive a weak law result for the largest nearest neighbour distance and almost sure asymptotic bounds for the connectivity threshold.

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1 Introduction

A variant of the usual independent percolation model that has been of interest is the AB percolation model ([5, 15]). Given a graph L, each vertex is given a mark A or B independent of other vertices. Edges between vertices with similar marks (A or B) are removed. The resulting random subgraph is the AB graph model. Percolation is said to happen in this model if there exists, with positive probability, an infinite path of vertices with marks alternating between A and B. This model has been studied on lattices and some related graphs. The AB percolation model behaves quite differently as compared to the usual percolation model. For example, it is known that AB percolation does not occur in $\mathbb{Z}^2$ ([1]), but occurs on the planar triangular lattice ([14]), some periodic two-dimensional graphs ([12]) and the half close-packed graph of $\mathbb{Z}^2$ ([15]).

The following generalization of the discrete AB percolation model has been studied on various graphs by Kesten *et. al.* (see [2, 8, 9]). Mark each vertex or site of a graph L independently as 0 or 1 with probability p and $1-p$ respectively. Given any infinite sequence (referred to as a word) $w \in \{0,1\}^\infty$, the question is whether w occurs in the graph L or not. The sentences $(1,0,1,0...), (0,1,0,1..)$ correspond to AB percolation and the sequence $(1,1,1...)$ corresponds to usual percolation. More generally Kesten *et. al.* answer whether all (or almost all) infinite sequences (words) are seen in L or not. The graphs for which the answer is known in affirmative are $\mathbb{Z}^d$ for d large, triangular lattice and $\mathbb{Z}^2_{cp}$, the close-packed graph of $\mathbb{Z}^2$. Our results provide partial answers to these questions in the continuum.

Our aim is to study a generalization of the discrete AB percolation model to the continuum. We study the problem of percolation and connectivity in such models. For the percolation problem the vertex set of the graph will be a homogenous Poisson point process in $\mathbb{R}^d$. For the connectivity problem we will consider a sequence of graphs whose vertex sets will be homogenous Poisson point processes of intensity n in $[0,1]^d$. We consider different models while studying percolation and connectivity so as to be consistent with the literature. This allows for easy comparison with, as well as the use of existing results from the literature.

Our motivation for the study of AB random geometric graphs comes from applications to wireless communication. In models of ad-hoc wireless networks, the nodes are assumed to be communicating entities that are distributed randomly in space. Edges between any two nodes in the graph represents the ability of the two nodes to communicate effectively with each other. A pair of nodes share an edge if the distance between the nodes is less than a certain cutoff radius $r > 0$ that is determined by the transmission power. Percolation and connectivity thresholds for such a model have been used to derive, for example, the capacity of wireless networks ([4, 6]). Consider a transmission scheme called the frequency division half duplex, where each node transmits at a frequency f_1 and receives at frequency f_2 or vice-versa ([13]). Thus nodes with transmission-reception frequency pair (f_1, f_2) can communicate only with nodes that have transmission-reception frequency pair (f_2, f_1) that are located within the cutoff distance r. Another example where such a model would be applicable is in communication between communicating units deployed at two different levels, for example surface (or underwater) and in air. Units in a level can communicate only with those at the other level that are within a certain range. A third example is in secure communication in wireless sensor networks with two types of nodes, tagged and normal. Upon deployment, each tagged node broadcasts a key over a predetermined secure channel, which is received by all normal nodes that are within transmission range. Two normal nodes can then communicate provided there is a tagged node from which both these normal nodes have received a key, that is, the tagged node is within transmission range of both the normal nodes.

The rest of the paper is organized as follows. Sections 2, 3 define and state our main theorems on percolation and connectivity respectively. Sections 4, 5 contain the proofs of these results. We will refer to our graphs, in the percolation context as the AB Poisson Boolean model, and as the AB random geometric graph while investigating the connectivity problem. Poisson Boolean model and random geometric graphs where the nodes are of the same type are the topics of the monographs [10] and [11] respectively.

2 Percolation in the AB Poisson Boolean Model

2.1 Model Definition

We first describe the AB Poisson Boolean model. Let $\Phi^{(1)} = \{X_i\}_{i\geq 1}$ and $\Phi^{(2)} = \{Y_i\}_{i\geq 1}$ be independent Poisson point processes in $\mathbb{R}^d$, $d \geq 2$, with intensities λ and μ respectively. Let the metric on $\mathbb{R}^d$ be given by the usual Euclidean norm denoted by $|\cdot|$.

The usual continuum percolation model is defined as follows.

Definition 2.1. *Define the graph* $\tilde{G}(\lambda, r) := (\Phi^{(1)}, \tilde{E}(\lambda, r))$ *to be the graph with vertex set* $\Phi^{(1)}$ *and edge set*

$$\tilde{E}(\lambda, r) = \{\langle X_i, X_j\rangle : X_i, X_j \in \Phi^{(1)}, |X_i - X_j| \leq 2r\}.$$

The edges in all the graphs that we consider are undirected, that is, $\langle X_i, X_j\rangle \equiv \langle X_j, X_i\rangle$. We will use the notation $X_i \sim X_j$ to denote existence of an edge between X_i, X_j when the underlying graph is unambiguous. By percolation, we mean the existence of an infinite connected component in the graph. For fixed $r > 0$, define

$$\lambda_c(r) := \inf\left\{\lambda > 0 : \mathsf{P}\left(\tilde{G}(\lambda, r) \text{ percolates }\right) > 0\right\}. \tag{2.1}$$

In this usual continuum percolation model ([10]), it is known that $0 < \lambda_c(r) < \infty$.

A natural analog of this model to the AB set-up would be to consider a graph with vertex set $\Phi^{(1)}$ where each vertex is independently marked A or B. We will consider a more general model from which results for the above model will follow as a corollary.

Definition 2.2. *The AB Poisson Boolean model* $G(\lambda, \mu, r) := (\Phi^{(1)}, E(\lambda, \mu, r))$ *is the graph with vertex set* $\Phi^{(1)}$ *and edge set*

$$E(\lambda, \mu, r) := \{\langle X_i, X_j\rangle : X_i, X_j \in \Phi^{(1)}, |X_i - Y| \leq 2r, |X_j - Y| \leq 2r, \textit{ for some } Y \in \Phi^{(2)}\}.$$

Let $\theta(\lambda, \mu, r) = \mathsf{P}\left(G(\lambda, \mu, r) \text{ percolates}\right)$. It follows from the zero-one law that $\theta(\lambda, \mu, r) \in \{0, 1\}$. We are interested in characterizing the region formed by (λ, μ, r) for which $\theta(\lambda, \mu, r) = 1$.

Definition 2.3. *For fixed $\lambda, r > 0$, define the critical intensities $\mu_c(\lambda, r)$ by*

$$\mu_c(\lambda, r) := \sup\{\mu : \theta(\lambda, \mu, r) = 0\}.$$

2.2 Main Results

We start with some simple lower bounds for the critical intensity $\mu_c(\lambda, r)$.

Proposition 2.1. *Fix $\lambda, r > 0$. Let $\lambda_c(r)$, $\mu_c(\lambda, r)$ be the critical intensities as in (2.1) and Definition 2.3, respectively. Then*

1. $\mu_c(\lambda, r) \geq \lambda_c(r) - \lambda, \quad$ *if* $\quad \lambda_c(2r) < \lambda < \lambda_c(r), \quad$ *and*
2. $\mu_c(\lambda, r) = \infty, \quad$ *if* $\quad \lambda \leq \lambda_c(2r)$.

However, it is not clear that $\mu_c(\lambda, r) < \infty$ for $\lambda > \lambda_c(2r)$. We answer this in affirmative for $d = 2$.

Theorem 2.1. *Let $d = 2$ and $r > 0$ be fixed. Then for any $\lambda > \lambda_c(2r)$, we have $\mu_c(\lambda, r) < \infty$.*

Thus the AB Boolean model exhibits a *phase transition* in the plane. However, the above theorem does not tell us how to choose a μ for a given $\lambda > \lambda_c(2r)$ for $d = 2$ such that AB percolation happens, or if indeed there is a phase transition for $d \geq 3$. We obtain an upper bound for $\mu_c(\lambda, r)$ as a special case of a more general result which is the continuum analog of word percolation on discrete lattices described in Section 1. In order to state this result, we need some notation.

Definition 2.4. *For each $d \geq 2$, define the critical probabilities $p_c(d)$, and the functions $a(d, r)$ as follows.*

1. *For $d = 2$, consider the triangular site percolation model (see Figure 1) with edge length $r/2$. Around each vertex place a "flower" formed by circular arcs (see Figure 1). These arcs*

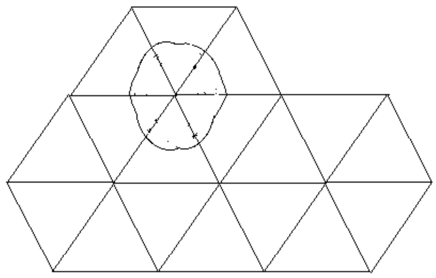

Figure 1: The triangular lattice and flower in $\mathbb{R}^2$ with area $a(2, r)$

are formed by circumferences of circles of radius $\frac{r}{2}$ drawn from the mid-points of the edges. Let $a(2, r)$ be the area of a flower. Let $p_c(2)$ be the critical probability for independent site percolation on this lattice.

2. *For $d \geq 3$, let $p_c(d)$ be the critical probability for independent site percolation on $\mathbb{Z}^d$, and define $a(d, r) = (r/\sqrt{3+d})^d$.*

It is known that $p_c(2) = \frac{1}{2}$, and $p_c(d) < 1$, for $d \geq 3$ (see [5]).

Proposition 2.2. *For any $d \geq 2$, let $p_c(d)$, $a(d, r)$ be as in Definition 2.4. Fix $k \in \mathbb{N}$ and let $(r_1, \ldots, r_k) \in \mathbb{R}_+^k$. Set $r_0 = \inf_{1 \leq i,j \leq k}\{r_i + r_j\}$. For $i = 1, \ldots, k$, let $\Phi^{(i)}$ be independent Poisson point processes of intensity $\lambda_i > 0$. A word $\omega = \{w(i)\}_{i \geq 1} \in \{1, 2, \ldots, k\}^\infty$ is said to occur if there exists a sequence of distinct elements $\{X_i\}_{i \geq 1} \subset \mathbb{R}^d$, such that $X_i \in \Phi^{(w(i))}$, and $|X_i - X_{i+1}| \leq r_{w(i)} + r_{w(i+1)}$, for $i \geq 1$. If $\prod_{i=1}^k (1 - e^{-\lambda_i a(d, r_0)}) > p_c(d)$, then almost surely, every word occurs.*

The following corollary gives an upper bound for $\mu_c(\lambda, r)$ for large λ.

Corollary 2.1. *Suppose that $d \geq 2$, $r > 0$, and $\lambda > 0$ satisfies*

$$\lambda > -\frac{\log(1 - p_c(d))}{a(d, 2r)},$$

where $p_c(d)$, $a(d, r)$ are as in Definition 2.4. Let $\mu_c(\lambda, r)$ be the critical intensity as in Definition 2.3.

Then

$$\mu_c(\lambda, r) \leq -\frac{1}{a(d,2r)} \log\left[1 - \left(\frac{p_c(d)}{1-e^{-\lambda a(d,2r)}}\right)\right]. \tag{2.2}$$

Remark 2.1. *A simple calculation (see [10], pg.88) gives* $a(2,2) \simeq 0.8227$*, and*

$$-(a(2,2))^{-1}\log(1-p_c(2)) \simeq 0.843.$$

Using these we obtain from Corollary 2.1 that $\mu_c(0.85, 1) < 6.2001$.

Remark 2.2. *It can be shown that the number of infinite components in the AB Boolean model is atmost one, almost surely. The proof of this fact follows along the same lines as the proof in Poisson Boolean model (see [10, Proposition 3.3, Proposition 3.6]), since it relies on the ergodic theorem and the topology of infinite components, and not on the specific nature of the infinite components.*

The above proposition can be used to show existence of AB percolation in the natural analog of the discrete AB percolation model (refer to the two sentences above Definition 2.2). Recall that $\Phi^{(1)}$ is a Poisson point process in $\mathbb{R}^d$ of intensity $\lambda > 0$. Let $\{m_i\}_{i\geq 1}$ be a sequence of i.i.d. marks distributed as $m \in \{A, B\}$, with $P(m = A) = p = 1 - P(m = B)$. Define the point processes Φ^A, Φ^B as

$$\Phi^A := \{X_i \in \Phi^{(1)} : m_i = A\}, \qquad \Phi^B := \Phi^{(1)} \setminus \Phi^A.$$

Definition 2.5. *For any* $\lambda, r > 0$*, and* $p \in (0,1)$*, let* Φ^A *and* Φ^B *be as defined above. Let* $\widehat{G}(\lambda, p, r) := (\Phi^A, \widehat{E}(\lambda, p, r))$ *be the graph with vertex-set* Φ^A *and edge-set*

$$\widehat{E}(\lambda, p, r) := \{< X_i, X_j >: X_i, X_j \in \Phi^A,\ |X_i - Y| \leq 2r, |X_j - Y| \leq 2r, \ \textit{for some } Y \in \Phi^B\}.$$

Corollary 2.2. *Let* $\widehat{\theta}(\lambda, p, r) := P(\widehat{G}(\lambda, p, r)$ *percolates). Then for any* λ *satisfying*

$$\lambda > -\frac{2\log\left(1 - \sqrt{p_c(d)}\right)}{a(d,2r)},$$

there exists a $p(\lambda) < \frac{1}{2}$*, such that* $\widehat{\theta}(\lambda, p, r) = 1$*, for all* $p \in (p(\lambda), 1 - p(\lambda))$.

3 Connectivity in AB Random Geometric Graphs

3.1 Model Definition

The set up for the study of connectivity in AB random geometric graphs is as follows. For each $n \geq 1$, let $\mathcal{P}_n^{(1)}$ and $\mathcal{P}_n^{(2)}$ be independent homogenous Poisson point processes in $U = [0,1]^d$, $d \geq 2$, of intensity n. We also nullify some of the technical complications arising out of boundary effects by choosing to work with the toroidal metric on the unit cube, defined as

$$d(x,y) := \inf\{|x - y + z| : z \in \mathbb{Z}^d\}, \qquad x, y \in U. \tag{3.1}$$

Definition 3.1. *For any $m, n \geq 1$, the AB random geometric graph $G_n(m,r)$ is the graph with vertex set $\mathcal{P}_n^{(1)}$ and edge set*

$$E_n(m,r) := \{\langle X_i, X_j\rangle : X_i, X_j \in \mathcal{P}_n^{(1)}, d(X_i, Y) \leq r, d(X_j, Y) \leq r, \text{ for some } Y \in \mathcal{P}_m^{(2)}\}.$$

Our goal in this section is to study the *connectivity threshold* in the sequence of graphs $G_n(cn, r)$ as $n \to \infty$ for $c > 0$. The constant c can be thought of as a measure of the relative denseness or sparseness of $\mathcal{P}_n^{(1)}$ with respect to $\mathcal{P}_{cn}^{(2)}$ (see Remark 3.1 below). It is easier to first consider the critical radius required to eleminate isolated nodes.

Definition 3.2. *For each $n \geq 1$, let $W_n(r)$ be the number of isolated nodes, that is, vertices with degree zero in $G_n(cn,r)$, and define the largest nearest neighbor radius as*

$$M_n := \sup\{r \geq 0 : W_n(r) > 0\}.$$

3.2 Main Results

Let $\theta_d := \|B_O(1)\|$ be the volume of the d-dimensional unit closed ball, where $\|.\|$ denotes the

Lebesgue measure on $\mathbb{R}^d$. For any $\beta > 0$, and $n \geq 1$, define the sequence of cut-off functions,

$$r_n(c, \beta) = \left(\frac{\log(n/\beta)}{cn\theta_d} \right)^{\frac{1}{d}}, \tag{3.2}$$

and let

$$r_n(c) = r_n(c, 1). \tag{3.3}$$

Let $\phi(a) = \arccos(a)$. For $d = 2$, define

$$A(c) = \pi^{-1} \left[2\phi\left(\frac{c^{\frac{1}{2}}}{2} \right) - \sin\left(2\phi\left(\frac{c^{\frac{1}{2}}}{2} \right) \right) \right]. \tag{3.4}$$

Define the constant c_0 to be

$$c_0 := \begin{cases} \sup\{c : A(c) + \frac{1}{c} > 1\} & \text{if } d = 2 \\ 1 & \text{if } d \geq 3. \end{cases} \tag{3.5}$$

The function $A(c) + \frac{1}{c}$ is decreasing and hence $1 < c_0 \leq 4$ for $d = 2$. The first part of the following Lemma shows that for $c < c_0$, the above choice of radius stabilizes the expected number of isolated nodes in $G_n(cn, r_n(c, \beta))$ as $n \to \infty$. The second part shows that the assumption $c < c_0$ is not merely technical. The Lemma also suggests a *phase transition* at some $\tilde{c} \in [1, 2^d]$, in the sense that, for $c < \tilde{c}$ the expected number of isolated nodes in $G_n(cn, r_n(c, \beta))$ converges to a finite limit and diverges for $c > \tilde{c}$.

Lemma 3.1. *For any $\beta, c > 0$, let $r_n(c, \beta)$ be as defined in (3.2), and $W_n(r_n(c, \beta))$ be the number of isolated nodes in $G_n(cn, r_n(c, \beta))$. Let c_0 be as defined in (3.5). Then as $n \to \infty$,*

1. $\mathsf{E}(W_n(r_n(c, \beta))) \to \beta$ *for* $c < c_0$, *and*
2. $\mathsf{E}(W_n(r_n(c, \beta))) \to \infty$ *for* $c > 2^d$.

For $c < c_0$, having found the radius that stabilizes the mean number of isolated nodes, the next theorem shows that the number of isolated nodes and the largest nearest neighbour radius in

$G_n(cn, r_n(c,\beta))$ converge in distribution as $n \to \infty$. Let $\stackrel{d}{\to}$ denote convergence in distribution and $Po(\beta)$ denote a Poisson random variable with mean β.

Theorem 3.1. *Let $r_n(c,\beta)$ be as defined in (3.2) with $\beta > 0$ and $0 < c < c_0$. Then as $n \to \infty$,*

$$W_n(r_n(c,\beta)) \stackrel{d}{\to} Po(\beta), \tag{3.6}$$

$$\mathsf{P}\left(M_n \leq r_n(c,\beta)\right) \to e^{-\beta}. \tag{3.7}$$

Remark 3.1. *Let $B_x(r)$ denote the closed ball of radius r centred at $x \in \mathbb{R}^d$. For any locally finite point process $\mathcal{X}$ (for example $\mathcal{P}_n^{(1)}$ or $\mathcal{P}_n^{(2)}$), we denote the number of points of $\mathcal{X}$ in A, $A \subset \mathbb{R}^d$ by $\mathcal{X}(A)$. Define*

$$W_n^0(c,r) = \sum_{Y_i \in \mathcal{P}_{cn}^{(2)}} \mathbf{1}[\mathcal{P}_n^{(1)}(B_{Y_i}(r)) = 0],$$

that is, $W_n^0(c,r)$ is the number of $\mathcal{P}_{cn}^{(2)}$ nodes isolated from $\mathcal{P}_n^{(1)}$ nodes. From Palm calculus for Poisson point processes (Theorem 1.6, [11]) and the fact that the metric is toroidal, we have

$$\mathsf{E}\big(W_n^0(c, r_n(c,\beta))\big) = cn \int_U \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_x(r)) = 0\right) dx = cn\exp(-n\theta_d r_n(c,\beta)^d).$$

Substituting from (3.2) we get

$$\lim_{n\to\infty} \mathsf{E}\big(W_n^0(c, r_n(c,\beta))\big) = \begin{cases} 0 & \text{if } c < 1 \\ \beta & \text{if } c = 1 \\ \infty & \text{if } c > 1. \end{cases} \tag{3.8}$$

Thus there is a trade off between the relative density of the nodes and the radius required to stabilise the expected number of isolated nodes.

The next theorem gives asymptotic bounds for strong connectivity threshold in the AB random geometric graphs. Asymptotics of the strong connectivity threshold was one of the more difficult problems in the theory of random geometric graphs. While the lower bound can be derived using

Theorem 3.1, for the upper bound, we couple the AB random geometric graph with the usual random geometric graph and use the connectivity threshold for the usual random geometric graph (see Theorem 5.1). As will become obvious, the bounds are very tight for small c. We will take $\beta = 1$ in (3.2) and work with the cut-off functions $r_n(c)$ as defined in (3.3). Define the function $\eta : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$\eta(a,c) = \begin{cases} \frac{1}{\pi}\left[2\phi\left(\frac{1}{2}\left(\frac{c}{a}\right)^{\frac{1}{2}}\right) - \sin\left(2\phi\left(\frac{1}{2}\left(\frac{c}{a}\right)^{\frac{1}{2}}\right)\right)\right] & \text{if } d = 2 \\ \left(1 - \frac{1}{2}\left(\frac{c}{a}\right)^{\frac{1}{d}}\right)^{d} & \text{if } d \geq 3, \end{cases} \tag{3.9}$$

where $\phi(a) = \arccos(a)$. Define the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ by

$$\alpha(c) := \inf\{a : a\eta(a,c) > 1\}. \tag{3.10}$$

It is easily seen that $\alpha(c) \leq \left(1 + \frac{c^{\frac{1}{d}}}{2}\right)^{d}$ for $d \geq 2$ with equality for $d \geq 3$.

Theorem 3.2. *Let $\alpha(c)$ be as defined in (3.10), $r_n(c)$ be as defined in (3.3) and c_0 be as in (3.5). Define $\alpha_n^*(c) := \inf\{a : G_n(cn, a^{\frac{1}{d}} r_n(c))$ is connected$\}$. Then almost surely,*

$$\liminf_{n\to\infty} \alpha_n^*(c) \geq 1, \tag{3.11}$$

for any $c < c_0$, and for any $c > 0$,

$$\limsup_{n\to\infty} \alpha_n^*(c) \leq \alpha(c). \tag{3.12}$$

4 Proofs for Section 2

Proof of Proposition 2.1

(1). Recall from Definition 2.2 the graph $G(\lambda, \mu, r)$ with vertex set $\Phi^{(1)}$ and edge set $E(\lambda, \mu, r)$. Consider the graph $\tilde{G}(\lambda + \mu, r)$ (see Definition 2.1), where the vertex set is taken to be $\Phi^{(1)} \cup \Phi^{(2)}$ and let the edge set of this graph be denoted $\tilde{E}(\lambda + \mu, r)$.

If $< X_i, X_j > \in E(\lambda, \mu, r)$, then there exists a $Y \in \Phi^{(2)}$ such that $< X_i, Y >, < X_j, Y > \in \tilde{E}(\lambda + \mu, r)$. It follows that $G(\lambda, \mu, r)$ has an infinite component only if $\tilde{G}(\lambda + \mu, r)$ has an infinite component. Consequently, for any $\mu > \mu_c(\lambda, r)$ we have $\mu + \lambda > \lambda_c(r)$, and hence $\mu_c(\lambda, r) + \lambda \geq \lambda_c(r)$. Thus for any $\lambda < \lambda_c(r)$, we obtain the (non-trivial) lower bound $\mu_c(\lambda, r) \geq \lambda_c(r) - \lambda$.

(2). Again $< X_i, X_j > \in E(\lambda, \mu, r)$ implies that $|X_i - X_j| \leq 4r$. Hence, $G(\lambda, \mu, r)$ has an infinite component only if $\tilde{G}(\lambda, 2r)$ has an infinite component. Thus $\mu_c(\lambda, r) = \infty$ if $\lambda \leq \lambda_c(2r)$. $\square$

Proof of Theorem 2.1

Fix $\lambda > \lambda_c(2r)$. The proof adapts the idea used in [3] of coupling the continuum percolation model to a discrete percolation model. For $l > 0$, let $l\mathbb{L}^2$ be the graph with vertex set $l\mathbb{Z}^2$, the expanded two-dimensional integer lattice, and endowed with the usual graph structure, that is, $x, y \in l\mathbb{Z}^2$ share an edge if $|x - y| = l$. Denote the edge-set by $l\mathbb{E}^2$. For any edge $e \in l\mathbb{E}^2$ denote the mid-point of e by (x_e, y_e). For every horizontal edge e, define three rectangles $R_{ei}, i = 1, 2, 3$ as follows : R_{e1} is the rectangle $[x_e - 3l/4, x_e - l/4] \times [y_e - l/4, y_e + l/4]$; R_{e2} is the rectangle $[x_e - l/4, x_e + l/4] \times [y_e - l/4, y_e + l/4]$ and R_{e3} is the rectangle $[x_e + l/4, x_e + 3l/4] \times [y_e - l/4, y_e - l/4]$. Let $R_e = \cup_i R_{ei}$. The corresponding rectangles for vertical edges are defined similarly. The reader can refer to Figure 2.

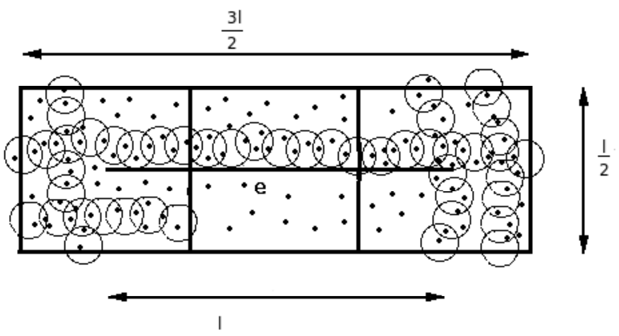

Figure 2: An horizontal edge e that satisfies the condition for $B_e = 1$. The balls are of radius $2r$, centered at points of $\Phi^{(1)}$ and the adjacent centers are of at most distance r_1. The dots are the points of $\Phi^{(2)}$.

Due to continuity of $\lambda_c(2r)$ (see [10, Theorem 3.7]), there exists $r_1 < r$ such that $\lambda > \lambda_c(2r_1)$. We shall now define some random variables associated with horizontal edges and the corresponding definitions for vertical edges are similar. Let A_e be the indicator random variable for the event that there exists a left-right crossing of R_e by a component of $\tilde{G}(\lambda, 2r_1)$ and top-down crossings of R_{e1} and R_{e3} by a component of $\tilde{G}(\lambda, 2r_1)$. Suppose that $A_e = 1$. Draw balls of radius $2r_1$ around each vertex of any left-right crossing of R_e and every top-down and left-right crossing of R_{e1} and R_{e3}. Let C_e be the indicator random variable of the event that, for each pair of balls drawn above that have non-empty intersection, when expanded to balls of radius $2r$ contain atleast one point of $\Phi^{(2)}$. Let B_e be the indicator random variable for the event that $\{A_e = 1\} \cap \{C_e = 1\}$.

Declare an edge $e \in l\mathbb{E}^2$ to be open if $B_e = 1$. We first show that for $\lambda > \lambda_c(2r)$ there exists a μ, l such that $l\mathbb{L}^2$ percolates (Step 1). The next step is to show that this implies percolation in the continuum model $G(\lambda, \mu, r)$. (Step 2).

STEP 1: The random variables $\{B_e\}_{e \in l\mathbb{E}^2}$ are 1-dependent, that is, B_e's indexed by two non-adjacent edges are independent. Hence, given edges $e_1, \ldots, e_n \in l\mathbb{E}^2$, there exists $\{k_j\}_{j=1}^m \subset \{1, \ldots, n\}$ with $m \geq n/4$ such that $\{B_{e_{k_j}}\}_{1 \leq j \leq m}$ are i.i.d. Bernoulli random variables. Hence,

$$\mathsf{P}\left(B_{e_i} = 0,\ 1 \leq i \leq n\right) \leq \mathsf{P}\left(B_{e_{k_j}} = 0, 1 \leq j \leq m\right) \leq \mathsf{P}\left(B_e = 0\right)^{n/4}. \tag{4.1}$$

We need to show that for a given $\epsilon > 0$ there exists l, μ, for which $\mathsf{P}\left(B_e = 0\right) < \epsilon$ for any $e \in l\mathbb{E}^2$. Fix an edge e. Observe that

$$\begin{aligned} \mathsf{P}\left(B_e = 0\right) &= \mathsf{P}\left(A_e = 0\right) + \mathsf{P}\left(B_e = 0 | A_e = 1\right) \mathsf{P}\left(A_e = 1\right) \\ &\leq \mathsf{P}\left(A_e = 0\right) + \mathsf{P}\left(B_e = 0 | A_e = 1\right). \end{aligned} \tag{4.2}$$

Since $\lambda > \lambda_c(2r_1)$, $\tilde{G}(\lambda, 2r_1)$ percolates. Hence by [10, Corollary 4.1], we can and do choose a l large enough so that

$$\mathsf{P}\left(A_e = 0\right) < \frac{\epsilon}{2}. \tag{4.3}$$

Now consider the second term on the right in (4.2). Given $A_e = 1$, there exist crossings as specified in the definition of A_e in $\tilde{G}(\lambda, 2r_1)$. Draw balls of radius $2r (> 2r_1)$ around each vertex. Any two vertices that share an edge in $\tilde{G}(\lambda, 2r_1)$ are centered at a distance of at most $4r_1$. The width of the lens of intersection of two balls of radius $2r$ whose centers are at most $4r_1 (< 4r)$ apart is bounded below by a constant, say $b(r, r_1) > 0$. Hence if we cover R_e with disjoint squares of diagonal-length $b(r, r_1)/3$, then every lens of intersection will contain at least one such square. Let $S_j, j = 1, \ldots, N(b)$, be the disjoint squares of diagonal-length $b(r, r_1)/3$ that cover R_e. Note that

$$\begin{aligned} \mathsf{P}\left(B_e = 1 | A_e = 1\right) &\geq \mathsf{P}\left(\Phi^{(2)} \cap S_j \neq \emptyset, 1 \leq j \leq N(b)\right) \\ &= (1 - \exp(-\frac{\mu b(r, r_1)^2}{18}))^{N(b)} \to 1, \text{ as } \mu \to \infty. \end{aligned}$$

Thus for the choice of l satisfying (4.3), we can choose a μ large enough so that

$$\mathsf{P}\left(B_e = 0 | A_e = 1\right) < \frac{\epsilon}{2}. \tag{4.4}$$

From (4.2) - (4.4), we get $\mathsf{P}\left(B_e = 0\right) < \epsilon$. Hence given any $\epsilon > 0$, it follows from (4.1) that there exists l, μ large enough so that $\mathsf{P}\left(B_{e_i}, 1 \leq i \leq n\right) \leq \epsilon^{n/4}$. That $l\mathbb{L}^2$ percolates now follows from a standard Peierl's argument as in [5, pp. 17, 18].

Step 2: By Step 1, choose l, μ so that $l\mathbb{L}^2$ percolates. Consider any infinite component in $l\mathbb{L}^2$. Let e, f be any two adjacent edges in the infinite component. In particular $B_e = B_f = 1$. This has two implications, the first one being that there exists crossings I_e and I_f of R_e and R_f respectively in $\tilde{G}(\lambda, 2r_1)$. Since e, f are adjacent, $R_{ei} = R_{fj}$ for some $i, j \in \{1, 3\}$. Hence there exists a crossing J of R_{ei} in $\tilde{G}(\lambda, 2r_1)$ that intersects both I_e and I_f. Draw balls of radius $2r$ around each vertex of the crossings J, I_e, I_f. The second implication is that every pairwise intersection of these balls will contain atleast one point of $\Phi^{(2)}$. This implies that I_e and I_f belong to the same AB component in $G(\lambda, \mu, r)$. Therefore $G(\lambda, \mu, r)$ percolates when $l\mathbb{L}^2$ does. $\square$.

Proof of Proposition 2.2. Recall Definition 2.4. For $d = 2$, let $\mathbb{T}$ be the triangular site percolation model with edge length $r_0/2$, and let Q_z be the flower centred at $z \in \mathbb{T}$ as shown in Figure 1. For

$d \geq 3$, let $\mathbb{Z}^{*d} = \frac{r_0}{\sqrt{3+d}}\mathbb{Z}^d$, and Q_z be the cube of side-length $\frac{r_0}{\sqrt{3+d}}$ centred at $z \in \mathbb{Z}^{*d}$. Note that the flowers or cubes are disjoint. We declare z open, if $Q_z \cap \Phi^{(i)} \neq \emptyset$, $1 \leq i \leq k$. This is clearly an independent site percolation model on $\mathbb{T}$ $(d = 2)$ or $\mathbb{Z}^{*d}$ $(d \geq 3)$ with probability $\prod_{i=1}^{k}(1 - e^{-\lambda_i a(d,r_0)})$ of z being open. By hypothesis, $\prod_{i=1}^{k}(1 - e^{-\lambda_i a(d,r_0)}) > p_c(d)$, the critical probability for site percolation on $\mathbb{T}$ $(d = 2)$ or $\mathbb{Z}^{*d}$ $(d \geq 3)$ and hence the corresponding graphs percolate. Let $< z_1, z_2, ... >$ denote the infinite percolating path in $\mathbb{T}$ $(d = 2)$ or $\mathbb{Z}^{*d}$ $(d \geq 3)$. Since it is a percolating path, *almost surely,* for all $i \geq 1$, and every $j = 1, 2, \ldots, k$, $\Phi^{(i)}(Q_{z_i}) > 0$, that is, each (flower or cube) Q_{z_i} contains a point of $\Phi^{(i)}$. Hence almost surely, for every word $\{w(i)\}_{i\geq 1}$ we can find a sequence $\{X_i\}_{i\geq 1}$ such that for all $i \geq 1$, $X_i \in \Phi^{(w(i))} \cap Q_{z_i}$. Further, $|X_i - X_{i+1}| \leq r_0 \leq r_{w(i)} + r_{w(i+1)}$. Thus, almost surely, every word occurs. □

Proof of Corollary 2.1. Apply Proposition 2.2 with $k = 2$, $\lambda_1 = \lambda$, $\lambda_2 = \mu$, $r_1 = r_2 = r$, and so $r_0 = 2r$. It follows that almost surely, every word occurs provided $(1 - e^{-\lambda a(d,2r)})(1 - e^{-\mu a(d,2r)}) > p_c(d)$. In particular, under the above condition, almost surely, the word $(1, 2, 1, 2, \ldots)$ occurs. This implies that there is a sequence $\{X_i\}_{i\geq 1}$ such that $X_{2j-1} \in \Phi^{(1)}$, $X_{2j} \in \Phi^{(2)}$, and $|X_{2j} - X_{2j-1}| \leq 2r$, for all $j \geq 1$. But this is equivalent to percolation in $G(\lambda, \mu, r)$. This proves the corollary once we note that there exists a $\mu < \infty$ satisfying the condition above only if $(1 - e^{-\lambda a(d,2r)}) > p_c(d)$, or equivalently $a(d, 2r)\lambda > \log(\frac{1}{1-p_c(d)})$ and the least such μ is given in the RHS of (2.2). □

Proof of Corollary 2.2. By the given condition $(1 - e^{-\lambda a(d,r)/2}) > \sqrt{p_c(d)}$, and continuity, there exists an $\epsilon > 0$ such that for all $p \in (1/2 - \epsilon, 1/2 + \epsilon)$, we have $(1 - e^{-\lambda p a(d,r)}) > \sqrt{p_c(d)}$. Thus for all $p \in (1/2 - \epsilon, 1/2 + \epsilon)$, we get that $(1 - e^{-\lambda p a(d,r)})(1 - e^{-\lambda(1-p)a(d,r)}) > p_c(d)$. Hence by invoking Proposition 2.2 as in the proof of Corollary 2.1 with $\lambda_1 = \lambda p, \lambda_2 = \lambda(1 - p), r_1 = r_2 = r$, we get that $\widehat{\theta}(\lambda, p, r) = 1$. □

5 Proofs for Section 3

For any locally finite point process $\mathcal{X} \subset U$, the coverage process is defined as

$$\mathcal{C}(\mathcal{X}, r) := \cup_{X_i \in \mathcal{X}} B_{X_i}(r), \tag{5.5}$$

and we abbreviate $\mathcal{C}(\mathcal{P}_n^{(1)}, r)$ by $\mathcal{C}(n, r)$. Recall that for any $A \subset \mathbb{R}^d$, we write $\mathcal{X}(A)$ to be the number of points of $\mathcal{X}$ that lie in the set A. We will need the following vacancy estimate similar to [7, Theorem 3.11] for the proof of Lemma 3.1. $\| \cdot \|$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Lemma 5.1. *For $d = 2$ and $0 < r < \frac{1}{2}$, define $V(r) := 1 - \frac{\|B_O(r) \cap \mathcal{C}(n,r)\|}{\pi r^2}$, the normalised vacancy in the r-ball. Then*

$$\mathsf{P}\left(V(r) > 0\right) \leq \left(1 + n\pi r^2 + 3(n\pi r^2)^2\right) \, \exp(-n\pi r^2).$$

Proof of Lemma 5.1. Write $\mathsf{P}\left(V(r) > 0\right) = p_1 + p_2 + p_3$, where

$$\begin{aligned}
p_1 &= \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_O(r)) = 0\right) = \exp(-n\pi r^2), \\
p_2 &= \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_O(r)) = 1\right) = n\pi r^2 \exp(-n\pi r^2), \\
p_3 &= \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_O(r)) > 1, V(r) > 0\right).
\end{aligned}$$

We shall now upper bound p_3 to complete the proof. A *crossing* is defined as a point of intersection of two r-balls centred at points of $\mathcal{P}_n^{(1)}$. A crossing is said to be *covered* if it lies in the interior of another r-ball centred at a point of $\mathcal{P}_n^{(1)}$, else it is said to be *uncovered*. If there is more than one point of $\mathcal{P}_n^{(1)}$ in $B_O(r)$, then there exists atleast one crossing in $B_O(r)$. If $V(r) > 0$ and there exists more than one r-ball centred at a point of $\mathcal{P}_n^{(1)}$ in $B_O(r)$, then there exists atleast one such r-ball with two uncovered crossings on its boundary. Denoting the number of uncovered crossings by M, we have that

$$p_3 \leq \mathsf{P}\left(M \geq 2\right) \leq \frac{\mathsf{E}(M)}{2}.$$

Given a disk, the number of crossings is twice the number of r-balls centred at a distance within $2r$. This number is $2\int_0^{2r} 2n\pi(r+x)dx = 6n\pi r^2$, where $2n\pi(r+x)dx$ is the expected number of r-balls whose centers lie between $r+x$ and $r+x+dx$ of the center of the given r-ball. Thus,

$$\mathsf{E}(M) = \mathsf{E}\Big(\mathcal{P}_n^{(1)}(B_O(r))\Big)\, 6n\pi r^2 \mathsf{P}\,(\text{a crossing is uncovered}) = 6(n\pi r^2)^2 \exp(-n\pi r^2). \quad \square$$

Lemma 5.2. *For any $r > 0$ and $x \in \mathbb{R}^d$ with $0 \le R = \|x\| \le 2r$, define $L(r,R) := \|B_O(r) \cap B_x(r)\|$. Then*

$$\begin{aligned} L(r,R) &= \left(2\phi\left(\frac{R}{2r}\right) - \sin\left(2\phi\left(\frac{R}{2r}\right)\right)\right) r^2, \qquad \textit{if } d = 2, \\ L(r,R) &\ge \theta_d \left(r - \frac{R}{2}\right)^d, \qquad \textit{if } d \ge 3, \end{aligned} \tag{5.6}$$

where $\phi(a) = \arccos(a)$.

Proof of Lemma 5.2.

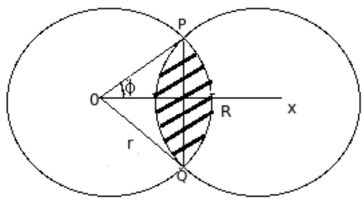

Figure 3: $|x| = R, \phi = \phi(r,R)$ and $L(r,R)$ is the area of the lens of intersection, the shaded region.

Let $d = 2$. From Figure 3, it is clear that $L(r,R)$ is cut into two equal halves by the line PQ and the area of each of those halves is the area enlosed between the chord PQ in the circle $B_O(r)$ and its circumference. The area of the segment OPQ (with PQ considered as the arc along the circumference of the circle) is $\phi\left(\frac{R}{2r}\right) r^2$. The area of the triangle OPQ is

$$r \sin\left(\phi\left(\frac{R}{2r}\right)\right) \times r\cos\left(\phi\left(\frac{R}{2r}\right)\right) = \frac{r^2}{2}\sin\left(2\phi\left(\frac{R}{2r}\right)\right).$$

Hence $L(r,R) = \left(2\phi\left(\frac{R}{2r}\right) - \sin\left(2\phi\left(\frac{R}{2r}\right)\right)\right) r^2$. Consider the case $d \geq 3$. The width of the lens of intersection of the balls $B_O(r)$ and $B_x(r)$ is $2r - R$. Thus the lens of intersection contains a ball of diameter $2r - R$. Hence the volume of such a ball, $\theta_d(r - \frac{R}{2})^d$, is a lower bound for $L(r,R)$. $\square$

Proof of Lemma 3.1. We first prove the second part of the Lemma which is easier.

(2). Let $\widehat{W}_n(r)$ be the number of $\mathcal{P}_n^{(1)}$ nodes for which there are no other $\mathcal{P}_n^{(1)}$ node within a distance r. Note that $\widehat{W}_n(2r) \leq W_n(r)$. By this inequality and the Palm calculus, we get

$$
\begin{aligned}
\mathsf{E}(W_n(r_n(c,\beta))) &\geq \mathsf{E}\Big(\widehat{W}_n(2r_n(c,\beta))\Big) \\
&= n\int_U \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_x(2r_n(c,\beta))) = 0\right) dx \\
&= n\,\exp(-2^d n\theta_d r_n^d(c,\beta)) \;=\; n\,\exp\left(-\frac{2^d}{c}\log n\right) \;\to\; \infty,
\end{aligned}
$$

as $n \to \infty$ since $c > 2^d$.

(1). We prove the cases $d = 2$ and $d \geq 3$ separately. Let $d \geq 3$ and fix $c < 1$. Define $\widetilde{W}_n(c,r)$ to be the number of $\mathcal{P}_n^{(1)}$ nodes for which there are no $\mathcal{P}_{cn}^{(2)}$ nodes within a distance r and $\overline{W}_n(c,r)$ be the number of $\mathcal{P}_{cn}^{(2)}$ nodes with only one $\mathcal{P}_n^{(1)}$ node within a distance r. Note that

$$
\widetilde{W}_n(c,r) \leq W_n(r) \leq \widetilde{W}_n(c,r) + \overline{W}_n(c,r). \tag{5.7}
$$

By Palm calculus for Poisson point processes, we have

$$
\begin{aligned}
\mathsf{E}\Big(\widetilde{W}_n(c,r_n(c,\beta))\Big) &= n\int_U \mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(B_x(r_n(c,\beta))) = 0\right) dx \\
&= n\exp(-cn\theta_d r_n^d(c,\beta)) \;=\; \beta, \qquad (5.8) \\
\mathsf{E}(\overline{W}_n(c,r_n(c,\beta))) &= cn\int_U \mathsf{P}\left(\mathcal{P}_n^{(1)}(B_x(r_n(c,\beta))) = 1\right) dx \\
&= c\,n\exp(-n\theta_d r_n^d(c,\beta))\,n\,\theta_d\,r_n^d(c,\beta) \to 0, \qquad (5.9)
\end{aligned}
$$

since $c < 1$. It follows from (5.7), (5.8) and (5.9) that $\mathsf{E}(W_n(r_n(c,\beta))) \to \beta$, as $n \to \infty$, if $d \geq 3$ and $c < 1$.

Now let $d = 2$, fix $c < c_0$, where c_0 is as defined in (3.5) and choose n large enough such that $r_n(c, \beta) < \frac{1}{2}$. For any $X \in \mathcal{P}_n^{(1)}$, using (5.5), the degree of X in the graph $G_n(cn, r)$ can be written as

$$deg_n(cn, X) := \sum_{X_j \in \mathcal{P}_n^{(1)}} \mathbf{1}\{< X_j, X >\in E_n(cn, r)\} = \mathcal{P}_n^{(1)}(\mathcal{C}((\mathcal{P}_{cn}^{(2)} \cap B_X(r)), r) \setminus \{X\}),$$

Since

$$\{\mathcal{P}_n^{(1)}(\mathcal{C}((\mathcal{P}_{cn}^{(2)} \cap B_X(r)), r) \setminus \{X\}) = 0\} = \{\mathcal{P}_{cn}^{(2)}(B_X(r) \cap \mathcal{C}(\mathcal{P}_n^{(1)} \setminus \{X\}, r)) = 0\}, \tag{5.10}$$

we have

$$W_n(r) = \sum_{X_i \in \mathcal{P}_n^{(1)}} \mathbf{1}\{deg_n(cn, X_i) = 0\} = \sum_{X_i \in \mathcal{P}_n^{(1)}} \mathbf{1}\{\mathcal{P}_{cn}^{(2)}(B_{X_i}(r) \cap \mathcal{C}(\mathcal{P}_n^{(1)} \setminus \{X\}, r)) = 0\}, \tag{5.11}$$

By Palm calculus for Poisson point processes (and the metric being toroidal) we have,

$$\mathsf{E}(W_n(r)) = n \int_U \mathsf{E}(\mathbf{1}\{deg_n(cn, x) = 0\})\, dx = n\mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(B_O(r) \cap \mathcal{C}(n, r)) = 0\right), \tag{5.12}$$

where $\mathcal{C}(n, r) = \mathcal{C}(\mathcal{P}_n^{(1)}, r)$. For any bounded random closed set F, conditioning on F and then taking expectation, we have

$$\mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(F) = 0\right) = \mathsf{E}(\exp(-cn\|F\|)). \tag{5.13}$$

Thus from (5.12), (5.13) we get

$$\mathsf{E}(W_n(r)) = n\ \mathsf{E}(\exp(-cn\|B_O(r) \cap \mathcal{C}(n, r)\|)) = n\ \mathsf{E}\big(\exp(-cn\pi r^2(1 - V(r)))\big), \tag{5.14}$$

where $V(r)$ is as defined in Lemma 5.1. Let $A(c)$ be as defined in (3.4) and $e_1 = (1, 0)$. Since

$\frac{r_n(1,\beta)}{2r_n(c,\beta)} = \frac{c^{\frac{1}{2}}}{2}$, by Lemma 5.2, we have

$$\begin{aligned}
\frac{\|B_O(r_n(c,\beta)) \cap B_{r_n(1,\beta)e_1}(r_n(c,\beta))\|}{\pi r_n(c,\beta)^2} &= \pi^{-1}\left(2\phi\left(\frac{r_n(1,\beta)}{2r_n(c,\beta)}\right) - \sin\left(2\phi\left(\frac{r_n(1,\beta)}{2r_n(c,\beta)}\right)\right)\right) \\
&= A(c).
\end{aligned}$$

Given $c < c_0$, by continuity, we can choose an $\epsilon \in (0,1)$, such that

$$A(c,\epsilon) = \frac{\|B_O(r_n(c,\beta)) \cap B_{r_n(1-\epsilon,\beta)e_1}(r_n(c,\beta))\|}{\pi r_n(c,\beta)^2} \quad \text{satisfies} \quad A(c,\epsilon) + \frac{1}{c} > 1. \tag{5.15}$$

From Lemma 5.1, we obtain the bound,

$$\mathsf{P}\left(V(r_n(c,\beta)) > 0\right) \leq D(1 + \log n + 3(\log n)^2)n^{-\frac{1}{c}}, \tag{5.16}$$

for some constant D. Let $N_n = \mathcal{P}_n^{(1)}(B_O(r_n(1-\epsilon,\beta)))$. On the event $\{N_n > 0\}$, we have

$$1 - V(r_n(c,\beta)) > A(c,\epsilon). \tag{5.17}$$

From (5.14), we get

$$\begin{aligned}
\mathsf{E}(W_n(r_n(c,\beta))) &= n\,\mathsf{E}\Big(e^{-cn\pi r_n^2(c,\beta)(1-V(r_n(c,\beta)))}\mathbf{1}\{V(r_n(c,\beta)) = 0\}\Big) \\
&+ n\,\mathsf{E}\Big(e^{-cn\pi r_n^2(c,\beta)(1-V(r_n(c,\beta)))}\mathbf{1}\{V(r_n(c,\beta)) > 0, N_n = 0\}\Big) \\
&+ n\,\mathsf{E}\Big(e^{-cn\pi r_n^2(c,\beta)(1-V(r_n(c,\beta)))}\mathbf{1}\{V(r_n(c,\beta)) > 0, N_n > 0\}\Big). \qquad (5.18)
\end{aligned}$$

Consider the first term in (5.18).

$$\begin{aligned}
n\,\mathsf{E}\Big(e^{-cn\pi r_n^2(c,\beta)(1-V(r_n(c,\beta)))}\mathbf{1}\{V(r_n(c,\beta)) = 0\}\Big) &= n\exp(-cn\pi r_n(c,\beta)^2)\mathsf{P}\left(V(r_n(c,\beta)) = 0\right)) \\
&= \beta\,\mathsf{P}\left(V(r_n(c,\beta)) = 0\right) \to \beta, \qquad (5.19)
\end{aligned}$$

as $n \to \infty$, since $\mathsf{P}\left(V(r_n(c,\beta)) = 0\right) \to 1$ by (5.16). The second term in (5.18) is bounded by

$$n\,\mathsf{P}\left(N_n = 0\right) = n\exp(-n\pi r_n(1-\epsilon,\beta)^2) = n^{1-\frac{1}{1-\epsilon}}\beta^{\frac{1}{1-\epsilon}} \to 0, \tag{5.20}$$

as $n \to \infty$. Using (5.17) first and then (5.16), the third term in (5.18) can be bounded by

$$\begin{aligned} ne^{-cn\pi r_n(c,\beta)^2 A(c,\epsilon)}\mathsf{P}\left(V(r_n(c,\beta)) > 0, N_n > 0\right) &\leq n^{1-A(c,\epsilon)}\beta^{A(c,\epsilon)}\mathsf{P}\left(V(r_n(c,\beta)) > 0\right) \\ &\leq D\,n^{1-A(c,\epsilon)-\frac{1}{c}}(1+\log n + 3(\log n)^2)\beta^{A(c,\epsilon)} \\ &\to 0, \end{aligned} \tag{5.21}$$

as $n \to \infty$ by (5.15).

It follows from (5.18) - (5.21) that

$$\mathsf{E}(W_n(r_n(c,\beta))) \to \beta, \qquad \text{as } n \to \infty. \quad \square$$

The *total variation distance* between two integer valued random variables ψ, ζ is given as follows:

$$d_{TV}(\psi,\zeta) = \sup_{A\subset\mathbb{Z}} |\mathsf{P}\left(\psi \in A\right) - \mathsf{P}\left(\zeta \in A\right)|. \tag{5.22}$$

The following estimate in the spirit of Theorem 6.7([11]) will be our main tool in proving Poisson convergence of $W_n(r_n(c,\beta))$. We denote the Palm version $\mathcal{P}_n^{(1)} \cup \{x\}$ of $\mathcal{P}_n^{(1)}$ by $\mathcal{P}_n^{(1,x)}$.

Lemma 5.3. *Let $0 < r < 1$ and let $\mathcal{C}(.\,,.)$ be the coverage process defined by (5.5). Define the integrals $I_{in}(r)$, $i = 1,2$, and $n \geq 1$ by*

$$\begin{aligned} I_{1n}(r) &:= n^2 \int_U dx \int_{B_x(5r)\cap U} dy\ \mathsf{P}\left(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_x(r), r)) = 0\right)\mathsf{P}\left(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_y(r), r)) = 0\right), \\ I_{2n}(r) &:= n^2 \int_U dx \int_{B_x(5r)\cap U} dy\ \mathsf{P}\left(\mathcal{P}_n^{(1,x)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_y(r), r)) = 0 = \mathcal{P}_n^{(1,y)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_x(r), r))\right). \end{aligned} \tag{5.23}$$

Then,

$$d_{TV}\left(W_n(r), Po(\mathsf{E}(W_n(r)))\right) \leq \min\left(3, \frac{1}{\mathsf{E}(W_n(r))}\right)(I_{1n}(r) + I_{2n}(r)). \tag{5.24}$$

Proof of Lemma 5.3. The proof follows along the same lines as the proof of Theorem 6.7 ([11]). For every $m \in \mathbb{N}$, partition U into disjoint cubes of side-length m^{-1} and corners at $m^{-1}\mathbb{Z}^d$. Let the cubes and their centres be denoted by $H_{m,1}, H_{m,2}, ...$ and $a_{m,1}, a_{m,2}...$ respectively. Let

$$\xi_{m,i} := 1_{\{\mathcal{P}_n^{(1)}(H_{m,i})=1\}\cap\{\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)}\cap B_{a_{m,i}}(r),r)\cap H_{m,i}^c)=0\}}.$$

$\xi_{m,i} = 1$ provided there is exactly one point of $\mathcal{P}_n^{(1)}$ in the cube $H_{m,i}$ which is not connected to any other point of $\mathcal{P}_n^{(1)}$ that falls outside $H_{m,i}$ in the graph $G_n(cn, r)$. Let $W^m = \sum_{i\in I_m} \xi_{m,i}$. Then almost surely,

$$W_n(r) = \lim_{m\to\infty} W^m. \tag{5.25}$$

Let $p_{m,i} = \mathsf{E}(\xi_{m,i})$ and $p_{m,i,j} = \mathsf{E}(\xi_{m,i}\xi_{m,j})$. The remaining part of the proof is based on the notion of dependency graphs and the Stein-Chen method.

Define $I_m := \{i \in \mathbb{N} : H_{m,i} \subset [0,1]^d\}$ and $E_m := \{< i.j >: i, j \in I_m,\ 0 < \|a_{m,i} - a_{m,j}\| < 5r\}$. The graph $G_m = (I_m, E_m)$ forms a dependency graph (see [11, Chapter 2]) for the random variables $\{\xi_{m,i}\}_{i\in I_m}$. The dependency neighbourhood of a vertex i is $N_{m,i} = i \cup \{j :< i, j >\in E_m\}$. By Theorem 2.1 [11], we have

$$d_{TV}(W^m, Po(\mathsf{E}(W^m))) \leq min(3, \frac{1}{\mathsf{E}(W^m)})(b_1(m) + b_2(m)), \tag{5.26}$$

where $b_1(m) = \sum_{i\in I_m}\sum_{j\in N_{m,i}} p_{m,i}p_{m,j}$ and $b_2(m) = \sum_{i\in I_m}\sum_{j\in N_{m,i}/\{i\}} p_{m,i,j}$. The result follows if we show that the expressions on the left and right in (5.26) converge to the left and right hand expressions respectively in (5.24).

Let $w_m(x) = m^d p_{m,i}$ for $x \in H_{m,i}$. Then $\sum_{i\in I_m} p_{m,i} = \int_U w_m(x)\ dx$. Clearly,

$$\lim_{m\to\infty} w_m(x) = n\mathsf{P}\left(\mathcal{P}_n^{(1,x)}(\mathcal{C}((\mathcal{P}_{cn}^{(2)} \cap B_x(r))/\{x\}, r)) = 0\right) = n\mathsf{P}\left(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_x(r), r)) = 0\right).$$

Since $w_m(x) \le m^d \mathsf{P}\left(\mathcal{P}_n^{(1)}(H_{m,i}) = 1\right) \le n$,

$$\lim_{m\to\infty} \mathsf{E}(W^m) = n \int_U \mathsf{P}\left(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_x(r), r)) = 0\right)\ dx = \mathsf{E}(W_n(r))\,,$$

where the first equality is due to the dominated convergence theorem and the second follows from (5.10) - (5.12). Similarly by letting $u_m(x,y) = m^{2d} p_{m,i} p_{m,j} 1_{[j \in N_{m,i}]}$ and $v_m(x,y) = m^{2d} p_{m,i,j} 1_{[j \in N_{m,i}/\{i\}]}$ for $x \in H_{m,i},\ y \in H_{m,j}$, one can show that

$$\begin{aligned} b_1(m) &= \int_U u_m(x,y)\ dx\ dy \ \to\ I_{1n}(r), \\ b_2(m) &= \int_U v_m(x,y)\ dx\ dy \ \to\ I_{2n}(r). \quad \square \end{aligned}$$

Proof of Theorem 3.1. (3.7) follows easily from (3.6) by noting that

$$\mathsf{P}\left(M_n \le r\right) = \mathsf{P}\left(W_n(r) = 0\right).$$

Hence, the proof is complete if we show (3.6) for which we will use Lemma 5.3. Let $I_{in}(r_n(c,\beta))$, $i = 1, 2$, be the integrals defined in (5.23) with r taken to be $r_n(c,\beta)$ satisfying (3.2). From Lemma 3.1, $\mathsf{E}(W_n(r_n(c,\beta))) \to \beta$ as $n \to \infty$. Using (5.12) and Lemma 3.1, we get for some finite positive constant C that

$$I_{1n}(r_n(c,\beta)) = \int_U dx \int_{B_x(5r_n(c,\beta))\cap U} dy\ \left(\mathsf{E}(W_n(r_n(c,\beta)))\right)^2 \le C(5r_n(c,\beta))^d \to 0, \qquad \text{as } n \to \infty.$$

We now compute the integrand in the inner integral in $I_{2n}(r)$. Let $\Gamma(x,r) = \|B_O(r) \cap B_x(r)\|$. For

$x, y \in U$, using (5.13) we get

$$
\begin{aligned}
& \mathsf{P}\left(\{\mathcal{P}_n^{(1,x)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_y(r), r)) = 0\} \cap \{\mathcal{P}_n^{(1,y)}(\mathcal{C}(\mathcal{P}_{cn}^{(2)} \cap B_x(r), r)) = 0\}\right) \\
&= \mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(B_y(r) \cap (\mathcal{C}(n, r) \cup B_x(r))) = 0, \mathcal{P}_{cn}^{(2)}(B_x(r) \cap (\mathcal{C}(n, r) \cup B_y(r))) = 0\right) \\
&\leq \mathsf{P}\left(\mathcal{P}_{cn}^{(2)}(B_y(r) \cap \mathcal{C}(n, r)) = 0, \mathcal{P}_{cn}^{(2)}(B_x(r) \cap \mathcal{C}(n, r)) = 0\right) \\
&= \mathsf{P}\left(\mathcal{P}_{cn}^{(2)}((B_y(r) \setminus B_x(r)) \cap \mathcal{C}(n, r)) = 0, \mathcal{P}_{cn}^{(2)}(B_x(r) \cap \mathcal{C}(n, r)) = 0\right) \\
&= \mathsf{E}(\exp(-cn\|(B_y(r) \setminus B_x(r)) \cap \mathcal{C}(n, r)\|) \exp(-cn\|B_x(r) \cap \mathcal{C}(n, r)\|)) . \qquad (5.27)
\end{aligned}
$$

We can and do choose an $\eta > 0$ so that for any $r > 0$ and $\|y - x\| \leq 5r$ (see [11, Eqn 8.21]), we have

$$\|B_x(r) \setminus B_y(r)\| \;\geq\; \eta \, r^{d-1} \, \|y - x\|.$$

Hence if $\|y - x\| \leq 5r$, the left hand expression in (5.27) will be bounded above by

$$\mathsf{E}\left(\exp\left(-cn\eta r^{d-1}\|y - x\| \frac{\|(B_y(r) \setminus B_x(r)) \cap \mathcal{C}(n, r)\|}{\|B_y(r) \setminus B_x(r)\|}\right) \exp\left(-cn\|B_x(r) \cap \mathcal{C}(n, r)\|\right)\right).$$

Using the above bound, we get

$$
\begin{aligned}
I_{2n}(r_n(c, \beta)) \;\leq\; & \int_U \int_{B_O(5r_n^d(c,\beta)) \cap U} n^2 \mathsf{E}\Big(\exp\left(-cn\|B_O(r_n(c, \beta)) \cap \mathcal{C}(n, r_n(c, \beta))\|\right) \\
& \exp\left(-cn\eta r_n(c, \beta)^{d-1}\|y\| \frac{\|(B_y(r_n(c, \beta)) \setminus B_O(r_n(c, \beta))) \cap \mathcal{C}(n, r_n(c, \beta))\|}{\|B_y(r_n(c, \beta)) \setminus B_O(r_n(c, \beta))\|}\right)\Big) \, dx \, dy.
\end{aligned}
$$

Making the change of variable $w = nr_n(c, \beta)^{d-1}(y - x)$ and using (5.14), we get

$$
\begin{aligned}
I_{2n}(r_n(c, \beta)) \;\leq\; & \int_U dx \int_{B_x(5nr_n(c,\beta)^d) \cap U} (nr_n(c, \beta)^d)^{1-d} \mathsf{E}\Big(n \exp(-cn\|B_O(r_n(c, \beta)) \cap \mathcal{C}(n, r_n(c, \beta))\|) \\
& \exp\left(-c\eta\|w\| \frac{\|(B_{w(nr_n(c,\beta)^{d-1})^{-1}}(r_n(c, \beta)) \setminus B_O(r_n(c, \beta))) \cap \mathcal{C}(n, r_n(c, \beta))\|}{\|B_{w(nr_n(c,\beta)^{d-1})^{-1}}(r_n(c, \beta)) \setminus B_O(r_n(c, \beta))\|}\right)\Big) \, dw \\
\leq\; & (nr_n(c, \beta)^d)^{1-d} \mathsf{E}(W_n(r_n(c, \beta))) \to 0,
\end{aligned}
$$

as $n \to \infty$, since by Lemma 3.1, $\mathsf{E}(W_n(r_n(c, \beta))) \to \beta$ and $r_n(c, \beta) \to 0$ as $n \to \infty$. We have shown that for $i = 1, 2$, $I_{in}(r_n(c, \beta)) \to 0$, and hence by Lemma 3.1,

$$d_{TV}(W_n(r_n(c, \beta)), Po(\mathsf{E}(W_n(r_n(c, \beta))))) \to 0,$$

as $n \to \infty$. Again, since $\mathsf{E}(W_n(r_n(c, \beta))) \to \beta$, we have $Po(\mathsf{E}(W_n(r_n(c, \beta)))) \xrightarrow{d} Po(\beta)$. Consequently, $d_{TV}(W_n(r_n(c, \beta)), Po(\beta) \to 0$ as $n \to \infty$. As convergence in *total variation distance* implies convergence in distribution, we get (3.6). $\square$

We now prove Theorem 3.2. In the second part of this proof, we will couple our sequence of AB RGGs with a sequence of usual RGGs. By usual RGG we mean the sequence of graphs $\underline{G}_n(r)$ with vertex set $\mathcal{P}_n^{(1)}$ and edge set $\{\langle X_i, X_j \rangle : X_i, X_j \in \mathcal{P}_n^{(1)}, d(X_i, X_j) \leq r\}$, where d is the toroidal metric defined in (3.1). We will use the following well known result regarding strong connectivity in the graphs $\underline{G}_n(r)$.

Theorem 5.1 (Theorem 13.2, [11]). *For $R_n(A_0) = \left(\frac{A_0 \log n}{n\theta_d}\right)^{1/d}$, almost surely, the sequence of graphs $\underline{G}_n(R_n(A_0))$ is connected eventually if and only if $A_0 > 1$.*

Proof of Thm 3.2. Let $r_n = a^{\frac{1}{d}} r_n(c)$, where $r_n(c) = r_n(c, 1)$ is as defined in (3.3). It is enough to show the following :

$$\text{For all } c < c_0 \text{ and } a < 1, \qquad \lim_{n\to\infty} \mathsf{P}\left(G_n(cn, r_n) \text{ is not connected }\right) = 1. \tag{5.28}$$

$$\text{For all } c > 0 \text{ and } a > \alpha(c), \qquad \mathsf{P}\left(G_n(cn, r_n) \text{ is not connected i.o.}\right) = 0, \tag{5.29}$$

where i.o. stands for infinitely often. To show (5.28) note that

$$r_n^d = \frac{\log(\frac{n}{n^{1-a}})}{cn\pi} < \frac{\log(\frac{n}{\beta})}{cn\pi},$$

for any $\beta > 0$ and sufficiently large n. From Theorem 3.1, if $c < c_0$ and $a < 1$, then the largest nearest neighbour radius is asymptotically greater than r_n with probability tending to one. This gives (5.28) and thus we have proved the lower limit.

Let $R_n(A_0)$ be as in Theorem 5.1. We will show (using a subsequence argument) that if $a > \alpha(c)$, then we can find $A_0 > 1$, such that the probability of the event that every point of $\mathcal{P}_n^{(1)}$ is connected to all points of $\mathcal{P}_n^{(1)}$ that fall within a distance $R_n(A_0)$ in $G_n(cn, r_n)$, is summable. (5.29) then follows from Theorem 5.1 and the Borel-Cantelli Lemma.

Since $a > \alpha(c)$, by definition $a\eta(a,c) > 1$. By continuity, we can choose $A_0 > 1$ such that $a\eta(a, A_0c) > 1$. Choose $\epsilon \in (0,1)$ so that

$$(1-\epsilon)^2 a\eta(a, A_0c) > 1. \tag{5.30}$$

For each $X_i \in \mathcal{P}_n^{(1)}$, define the event

$$A_i(n,m,r,R) := \{X_i \text{ connects to all points of } \mathcal{P}_n^{(1)} \cap B_{X_i}(R) \text{ in } G_n(m,r)\},$$

and let

$$B(n,m,r,R) = \cup_{X_i \in \mathcal{P}_n^{(1)}} A_i(n,m,r,R)^c.$$

Observe that $B(n,m,r,R) \subset B(n_1,m_1,r_1,R_1)$, provided $n \leq n_1, m \geq m_1, r \geq r_1, R \leq R_1$. Let $n_j = j^b$ for some integer $b > 0$ that will be chosen later. Since $B(n_k, cn_k, r_{n_k}, R_{n_k}) \subset B(n_{j+1}, cn_j, r_{n_{j+1}}, R_{n_j})$, for $j \leq k \leq j+1$,

$$\cup_{k=j}^{j+1} B(n_k, cn_k, r_{n_k}, R_{n_k}) \subset B(n_{j+1}, cn_j, r_{n_{j+1}}, R_{n_j}). \tag{5.31}$$

Let $p_j = \mathsf{P}\left(A_i(n_{j+1}, cn_j, r_{n_{j+1}}, R_{n_j})^c\right)$. Let $N_n = \mathcal{P}_n^{(1)}([0,1]^2)$. From (5.31) and the union bound we get

$$\begin{aligned}
\mathsf{P}\left(\cup_{k=j}^{j+1} B(n_k, cn_k, r_{n_k}, R_{n_k})\right) &\leq \mathsf{P}\left(B(n_{j+1}, cn_j, r_{n_{j+1}}, R_{n_j})\right) \\
&\leq \mathsf{P}\left(\cup_{i=1}^{N_{n_{j+1}}} A_i(n_{j+1}, cn_j, r_{n_{j+1}}, R_{n_j})^c\right) \\
&\leq \sum_{i=1}^{n_{j+1}+n_{j+1}^{\frac{3}{4}}} \mathsf{P}\left(A_i(n_{j+1}, cn_j, r_{n_{j+1}}, R_{n_j})^c\right) + \mathsf{P}\left(|N_{n_{j+1}} - n_{j+1}| > n_{j+1}^{\frac{3}{4}}\right) \\
&\leq 2n_{j+1}p_j + \mathsf{P}\left(|N_{n_{j+1}} - n_{j+1}| > n_{j+1}^{\frac{3}{4}}\right).
\end{aligned} \tag{5.32}$$

We now estimate p_j. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$. Conditioning on the number of points of $\mathcal{P}_{n_{j+1}}$ in $B_O(R_{n_j})$ and then using the Boole's inequality, we get

$$\begin{aligned}
p_j &\leq \sum_{k=0}^{\infty} \frac{(n_{j+1}\theta_d R_{n_j}^d)^k e^{-n_{j+1}\theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} e^{-cn_j \|B_0(r_{n_{j+1}}) \cap B_x(r_{n_{j+1}})\|} dx \\
&\leq \sum_{k=0}^{\infty} \frac{(n_{j+1}\theta_d R_{n_j}^d)^k e^{-n_{j+1}\theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} e^{-cn_j \|B_0(r_{n_{j+1}}) \cap B_x(r_{n_{j+1}})\|} dx \\
&\leq \sum_{k=0}^{\infty} \frac{(n_{j+1}\theta_d R_{n_j}^d)^k e^{-n_{j+1}\theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} e^{-cn_j \|B_0(r_{n_{j+1}}) \cap B_{R_{n_j} e_1}(r_{n_{j+1}})\|} dx, \\
&= n_{j+1}\theta_d R_{n_j}^d e^{-cn_j L(r_{n_{j+1}}, R_{n_j})},
\end{aligned}$$

where $L(r, R)$ is as defined in Lemma 5.2. Since

$$\frac{R_{n_j}}{r_{n_{j+1}}} = \left(\frac{A_0 \log n_j}{\theta_d n_j} \frac{cn_{j+1}\theta_d}{a \log n_{j+1}} \right)^{\frac{1}{d}} \to \left(\frac{A_0 c}{a} \right)^{\frac{1}{d}},$$

by Lemma 5.2, we have

$$L(r_{n_{j+1}}, R_{n_j}) \geq (1-\epsilon)\, \eta(a, A_0 c)\, \theta_d r_{n_{j+1}}^d, \tag{5.33}$$

for all sufficiently large j, where η is as defined in (3.9). For all j sufficiently large, we have $(\frac{j}{j+1})^b \geq (1-\epsilon)$. Using (5.33) and simplifying by substituting for R_{n_j} and $r_{n_{j+1}}$, for all sufficiently large j, we have

$$\begin{aligned}
p_j &\leq \frac{(j+1)^b\, A_0\, b\, \log j}{j^b} e^{-\frac{j^b}{(j+1)^b}(1-\epsilon)\, \eta(a, A_0 c)\, a\, b\, \log(j+1)} \\
&\leq \frac{A_0\, b\, \log j}{(1-\epsilon)} e^{-(1-\epsilon)^2\, \eta(a, A_0 c)\, a\, b\, \log(j+1)} \\
&= \frac{A_0\, b\, \log j}{(1-\epsilon)(j+1)^{(1-\epsilon)^2\, \eta(a, A_0 c)\, a\, b}}.
\end{aligned}$$

Hence

$$n_{j+1}\, p_j \leq \frac{A_0\, b\, \log j}{(1-\epsilon)(j+1)^{((1-\epsilon)^2\, \eta(a, A_0 c)\, a\, -1) b}}. \tag{5.34}$$

Using (5.30), we can choose b large enough so that $((1-\epsilon)^2\, \eta(a, A_0 c)\, a\, -1)b > 1$. It then follows from (5.34) that the first term on the right in (5.32) is summable in j. From [11, Lemma 1.4], the second term on the right in (5.32) is also summable.

Hence by the Borel-Cantelli Lemma, almost surely, only finitely many of the events

$$\cup_{k=j}^{j+1} B(n_k, cn_k, r_{n_k}, R_{n_k})$$

occur, and hence only finitely many of the events $B(n, cn, r_n, R_n)$ occur. This implies that almost surely, every vertex in $G_n(cn, r_n)$ is connected to every other vertex that is within a distance $R_n(A_0)$ from it, for all large n. Since $A_0 > 1$, it follows from Theorem 5.1 that almost surely, $G_n(cn, r_n)$ is connected eventually. This proves (5.29). $\square$

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