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On the behavior of upwind schemes in the low Mach number limit. IV : P0 Approximation on triangular and tetrahedral cells

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Abstract: Finite Volume upwind schemes for the Euler equations in the low Mach number regime face a problem of lack of convergence toward the solutions of the incompressible system. However, if applied to cell centered triangular grid, this problem disappears and convergence toward the incompressible solution is recovered. Extending the work of [3] that prove this result for regular triangular grid, we give here a general proof of this fact for arbitrary unstructured meshes. In addition, we also show that this result is equally valid for unstructured three dimensional tetrahedral meshes.

Key-words: Compressible flow solvers, Low Mach number, triangular cells, Finite Volume, Upwind schemes, cell centered scheme, P0 Approximation

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Comportement des schémas décentrés dans la limite des faibles nombres de Mach. IV : Approximation P0 sur des cellules triangulaires et tétrahèdrales.

Résumé : Les schémas de type volumes finis décentrés appliqués aux équations d'Euler sont confrontés à un problème de perte de convergence vers les solutions du système incompressible. Cependant si ses schémas sont appliqués à des maillages triangulaires centrés sur les cellules, ce problème disparait et on retrouve la convergence vers les solutions incompressibles. Etendant le travail de [3] qui ont montré ce résultat pour des maillages triangulaires réguliers, nous donnons ici une preuve générale de ce résultat pour des maillages non-structurés arbitraires. De plus, nous montrons que ce résultat est aussi vrai pour des maillages non-structurés tri-dimensionels de tétraèdres

Mots-clés : Solveurs pour les écoulements compressibles, Faible nombre de Mach, Volumes finis, schemas décentrés, Approximation P0, cellules triangulaires

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1 Introduction

Despite some real progresses made in the recent past, the computation of low speed flows with compressible solvers continues to rise a lot of questions. In the present work, among all these questions, we are interested by the question of the convergence toward the incompressible solutions : In general, on a fixed mesh, the solutions of the compressible flow discretized equations are not an accurate approximation of the solutions of the incompressible model. This problem was first pointed out in [5] and since, several works have tried to explain the reasons of this difficulty. In [2], this question was examined for Roe type solvers where the numerical flux between two cells takes the following form :

$$\Phi(q_L, q_R, \mathbf{n}_{LR}) = \frac{1}{2}(\mathbf{F}.\mathbf{n}_{LR}(q_L) + \mathbf{F}.\mathbf{n}_{LR}(q_R) + |\frac{\partial \mathbf{F}.\mathbf{n}_{LR}}{\partial q}|\Delta_{LR} q) \tag{1}$$

here $q = (\rho, \rho\vec{u}, \rho e)$ is the Euler state vector, $\mathbf{F}$ is the associated Euler flux, $\mathbf{n}_{LR}$ is the unit normal at the interface and $\Delta_{LR} q = q_L - q_R$ is the jump between the values q_L and q_R on each side of the interface. A rigorous asymptotic analysis, then shows that this behavior is linked to the appearance in the discrete solutions of pressure fluctuations of the wrong order of magnitude in the Mach number M_*. In the continuous model, for well-prepared initial data (Roughly speaking, well prepared data are initial data that are in the kernel of the acoustic operator, for a more accurate definition, see e.g [4]) the pressure p can be written as a thermodynamical space-independent pressure plus a dynamical correction of order M_*^2 :

$$p(\mathbf{x}, t) = P(t) + M_*^2 p_2(\mathbf{x}, t) \tag{2}$$

In contrast to this behavior, the discrete compressible approximation contains a pressure variation of order M_* :

$$p^{disc}(\mathbf{x}, t) = P^{disc}(t) + M_* \, p_1^{disc}(\mathbf{x}, t) \tag{3}$$

This point has been also revisited in [1] for Godunov-type schemes where the flux has the form :

$$\Phi(q_L, q_R, \mathbf{n}_{LR}) = \mathbf{F}.\mathbf{n}_{LR}(q_{LR}) \tag{4}$$

where q_{LR} is the solution of an exact or approximate Riemann problem defined by the two states q_L and q_R. This last work shows that the reason why upwind scheme fails to compute the incompressible limit is deeply linked to the behavior of the *continuous* solutions of the Euler equations in the low Mach number regime. Actually, the asymptotic behavior (2) is only valid for a special class of initial data (the well-prepared ones) but in general, in the low Mach number regime, the pressure writes :

$$p(\mathbf{x}, t) = P(t) + M_* p_1(\mathbf{x}, t/M_*) + M_*^2 p_2(\mathbf{x}, t) \tag{5}$$

where $p_1(\mathbf{x}, t/M_*)$ is a fast acoustic component that appears if the initial data are not well prepared. For finite volume upwind scheme, at each time step, even if the initial data are well prepared, their projections on piecewise constant fields are not. In consequence, the computed discrete pressure $p(q_{LR})$ at the interface between cells L and R has expression (5). The result being, that in general, the zero-Mach number limit of the discrete equations is not an approximation of the incompressible model but rather is characterized by an unphysical balance of the acoustic components $p_1^{disc}(q_{LR})$.

The asymptotic analysis in [2] and [1] were presented for Cartesian regular meshes. In a recent work, Rieper and Bader [3] show however that cell centered finite volume schemes on triangular grids (a P0 approximation in the Finite Element language) are not affected by this problem : In contrast to quadrangular or finite element dual meshes, the discrete solutions on triangular grid do not contain any pressure fluctuations of order M_* and the expression (2) is recovered at the discrete level. The asymptotic analysis performed in [3] gives the reason of this unexpected behavior : the only possible solution of the equation governing $p_1^{disc}(q_{LR})$ the discrete pressure at order M_* is

$$p_1^{disc} = \text{constant} \tag{6}$$

and therefore, no unphysical pressure variation at the wrong order in the Mach number are present on this type of mesh.

However, the proof of [3] is only given for structured triangular meshes obtained by dividing regular square Cartesian cells in two by introducing a diagonal as an additional cell edge. This proof uses heavily the geometrical properties of the obtained finite volume mesh. On the other hand, numerical experiments reported in [3] show the absence of order M_* pressure fluctuations on unstructured P0 meshes. This indicates that the source of this behavior is not related to geometrical regularity and that there must exist a more general proof than the one of [3].

In the present work, we give this general proof. Precisely we show that on unstructured triangular P0 meshes, upwind cell centered approximations do not give rise to pressure fluctuations of order M_*. Apart from the fact that the present proof is more general than the one given in [3] and do not use any possible geometrical regularity of the mesh, this proof also enlightens the “philosophical” reason why triangular P0 meshes do not support pressure fluctuations of order one in the Mach number. It shows that actually it is not the **geometry** of the cells that is important, but rather the number of neighbors of one cell that in cell centered approximations on triangular grid is constant and equal to three. Conversely, this proof also shows why order M_* pressure fluctuations are allowed on other type of meshes (structured or unstructured quadrangular meshes, dual finite volume meshes on unstructured triangulations). In addition, this “philosophical” reason is independent of the space dimension and then, it is also possible to extend the proof to three dimensional unstructured tetrahedral meshes. We also mention that in contrast with [3], it is not necessary to restrict the proof to steady state problems as will be shown in the sequel.

2 Asymptotic analysis of upwind schemes on P0 triangular meshes

2.1 Notations and preliminaries

Let Ω be a 2-D flow domain that for simplicity, we consider polygonal and let T_h be a triangulation of Ω. The present analysis does not assume any particular properties of the mesh T_h. In particular, no angle or regularity conditions are needed. The approximation we are dealing with, localizes the density, pressure and velocity degrees of freedom on the triangles in a co-localized manner (In the finite element language, we are considering a P0 approximation). We use the notations of [2]. The set of neighbours of cell $\boldsymbol{i}$, is denoted by $\mathcal{V}(\boldsymbol{i})$. The area of cell $\boldsymbol{i}$ is $|C_{\boldsymbol{i}}|$. The symbol $\boldsymbol{il}$ denotes an edge between cell $\boldsymbol{i}$ and cell $\boldsymbol{l}$ whose length is $\delta_{\boldsymbol{il}}$. The vector $\vec{n}_{\boldsymbol{il}}$ denotes the unit normal vector to edge $\boldsymbol{il}$ oriented from $\boldsymbol{i}$ to $\boldsymbol{l}$. For any variable ϕ, we define the jump operator $\Delta_{\boldsymbol{il}}\phi$ as

$$\Delta_{\boldsymbol{il}}\phi = \phi_{\boldsymbol{i}} - \phi_{\boldsymbol{l}} \tag{7}$$

Finally, the normal component of the velocity $\vec{u}$ jump at the interface between two cells, denoted by $\Delta_{\boldsymbol{il}}U$ is defined by

$$\Delta_{\boldsymbol{il}}U = (\vec{u}_{\boldsymbol{i}} - \vec{u}_{\boldsymbol{l}}).\vec{n}_{\boldsymbol{il}} \tag{8}$$

The set of discrete equations resulting from the application of Roe-type upwind schemes to the compressible Euler equations in the low Mach number limit is given in [2]. For ease of presentation, they are reproduced here

$$\frac{1}{2M_*}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}\frac{\Delta_{\boldsymbol{il}}p}{a_{\boldsymbol{il}}}+$$
$$|C_{\boldsymbol{i}}|\frac{d}{dt}\rho_{\boldsymbol{i}}+\frac{1}{2}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}(\rho_{\boldsymbol{l}}\vec{u}_{\boldsymbol{l}}.\vec{n}_{\boldsymbol{il}}+|U_{\boldsymbol{il}}|(\Delta_{\boldsymbol{il}}\rho-\frac{\Delta_{\boldsymbol{il}}p}{a^2_{\boldsymbol{il}}}))+$$
$$\frac{M_*}{2}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}\rho_{\boldsymbol{il}}\frac{U_{\boldsymbol{il}}}{a_{\boldsymbol{il}}}\Delta_{\boldsymbol{il}}U=0 \tag{9.a}$$

$$\frac{1}{2M_*^2}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}p_{\boldsymbol{l}}\vec{n}_{\boldsymbol{il}}+$$
$$\frac{1}{2M_*}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}(\frac{(U\vec{n}+\vec{u})_{\boldsymbol{il}}}{a_{\boldsymbol{il}}}\Delta_{\boldsymbol{il}}p+\rho_{\boldsymbol{il}}a_{\boldsymbol{il}}\Delta_{\boldsymbol{il}}U\vec{n}_{\boldsymbol{il}})+$$
$$|C_{\boldsymbol{i}}|\frac{d}{dt}\rho_{\boldsymbol{i}}\vec{u}_{\boldsymbol{i}}+\frac{1}{2}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}(\rho_{\boldsymbol{l}}\vec{u}_{\boldsymbol{l}}.\vec{n}_{\boldsymbol{il}}\vec{u}_{\boldsymbol{l}}+|U_{\boldsymbol{il}}|(\Delta_{\boldsymbol{il}}\rho-\frac{\Delta_{\boldsymbol{il}}p}{a^2_{\boldsymbol{il}}})\vec{u}_{\boldsymbol{il}}+\rho_{\boldsymbol{il}}|U_{\boldsymbol{il}}|\Delta_{\boldsymbol{il}}V\vec{n}^{\perp}_{\boldsymbol{il}})+$$
$$\frac{M_*}{2}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}\rho_{\boldsymbol{il}}\frac{U_{\boldsymbol{il}}}{a_{\boldsymbol{il}}}\Delta_{\boldsymbol{il}}U\vec{u}_{\boldsymbol{il}}=0 \tag{9.b}$$

$$\frac{1}{2M_*}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}\frac{h_{\boldsymbol{il}}}{a_{\boldsymbol{il}}}\Delta_{\boldsymbol{il}}p+$$
$$|C_{\boldsymbol{i}}|\frac{d}{dt}\rho_{\boldsymbol{i}}e_{\boldsymbol{i}}+\frac{1}{2}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}(\rho_{\boldsymbol{l}}e_{\boldsymbol{l}}+p_{\boldsymbol{l}})\vec{u}_{\boldsymbol{l}}.\vec{n}_{\boldsymbol{il}}+$$
$$\frac{M_*}{2}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}(\frac{U^2_{\boldsymbol{il}}}{a_{\boldsymbol{il}}}\Delta_{\boldsymbol{il}}p+\rho_{\boldsymbol{il}}a_{\boldsymbol{il}}U_{\boldsymbol{il}}\Delta_{\boldsymbol{il}}U+\rho_{\boldsymbol{il}}\frac{U_{\boldsymbol{il}}}{a_{\boldsymbol{il}}}h_{\boldsymbol{il}}\Delta_{\boldsymbol{il}}U)+$$
$$\frac{M_*^2}{2}\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})}\delta_{\boldsymbol{il}}(|U_{\boldsymbol{il}}|(\Delta_{\boldsymbol{il}}\rho-\frac{\Delta_{\boldsymbol{il}}p}{a^2_{\boldsymbol{il}}})(\frac{u^2_{\boldsymbol{il}}+v^2_{\boldsymbol{il}}}{2})+\rho_{\boldsymbol{il}}|U_{\boldsymbol{il}}|V_{\boldsymbol{il}}\Delta_{\boldsymbol{il}}V)=0 \tag{9.c}$$

where $\vec{n}^{\perp}_{\boldsymbol{il}}=(-(n_y)_{\boldsymbol{il}},(n_x)_{\boldsymbol{il}})^t$ is the unit tangential vector to the interface and $V=\vec{u}.\vec{n}^{\perp}$ denote the tangential component of the velocity. Also, note that in these equations, the energy is related to the pressure by the dimensionless equation of state given by

$$p=(\gamma-1)(\rho e-\frac{M_*^2}{2}\rho(u^2+v^2+w^2)) \tag{10}$$

and thus, in the low Mach number limit, the total energy is equivalent to the internal energy.
In order to identify the limit equations corresponding to this system in the low

Mach number limit, we consider the following asymptotic expansion :

$$\begin{aligned} \rho_{\boldsymbol{i}} &= \rho^0 + M_* \rho^1_{\boldsymbol{i}} + \mathcal{O}(M_*^2) \quad (11.a) \\ \vec{u}_{\boldsymbol{i}} &= \vec{u}^0_{\boldsymbol{i}} + M_* \vec{u}^1_{\boldsymbol{i}} + \mathcal{O}(M_*^2) \quad (11.b) \\ p_{\boldsymbol{i}} &= p^0_{\boldsymbol{i}} + M_* p^1_{\boldsymbol{i}} + \mathcal{O}(M_*^2) \quad (11.c) \end{aligned}$$

Note that as in [3], the analysis is restricted to flows with a **constant** background density. We then have the first following result that is shown in [2] :

Lemma 2.1 *The lowest order system obtained from (9) implies that* $p^0_{\boldsymbol{l}} = cte \ \forall \boldsymbol{l}$.

Our goal is now to study the next order approximation.

Using that $\Delta_{\boldsymbol{il}} p^0 = 0 \ \ \forall \boldsymbol{il}$ we obtain :

$$\sum_{\boldsymbol{l} \in \mathcal{V}(\boldsymbol{i})} \delta_{\boldsymbol{il}} \Big(\frac{\Delta_{\boldsymbol{il}} p^1}{a^0_{\boldsymbol{il}}} + \rho^0_{\boldsymbol{l}} \vec{u}^0_{\boldsymbol{l}} . \vec{n}_{\boldsymbol{il}} \Big) = 0 \qquad (12.a)$$

$$\sum_{\boldsymbol{l} \in \mathcal{V}(\boldsymbol{i})} \delta_{\boldsymbol{il}} (p^1_{\boldsymbol{l}} + \rho^0_{\boldsymbol{il}} a^0_{\boldsymbol{il}} \Delta_{\boldsymbol{il}} U^0) \vec{n}_{\boldsymbol{il}} = 0 \qquad (12.b)$$

$$\frac{1}{2} \sum_{\boldsymbol{l} \in \mathcal{V}(\boldsymbol{i})} \delta_{\boldsymbol{il}} \frac{h^0_{\boldsymbol{il}}}{a^0_{\boldsymbol{il}}} \Delta_{\boldsymbol{il}} p^1 + |C_{\boldsymbol{i}}| \frac{d}{dt} \rho^0_{\boldsymbol{i}} e^0_{\boldsymbol{i}} + \frac{1}{2} \sum_{\boldsymbol{l} \in \mathcal{V}(\boldsymbol{i})} \delta_{\boldsymbol{il}} (\rho^0_{\boldsymbol{l}} e^0_{\boldsymbol{l}} + p^0_{\boldsymbol{l}}) \vec{u}^0_{\boldsymbol{l}} . \vec{n}_{\boldsymbol{il}} = 0 \qquad (12.c)$$

As in [3], we take advantage of the fact that p^0 and ρ^0 are constant to scale the equations in such a way that $p^0 = 1/\gamma$ and $\rho^0 = 1$, thus $a^0 = \sqrt{\gamma p^0/\rho^0} = 1$. Now, using the non-dimensional state law (10) we deduce that

$$\frac{d}{dt} \rho_{\boldsymbol{i}} e_{\boldsymbol{i}} = \frac{1}{\gamma - 1} \frac{d}{dt} p^0_{\boldsymbol{i}} = 0$$

Then, again from (10) and the definition $h = e + p/\rho$, it is seen that the energy equation (12.c) is equivalent to the continuity equation (12.a) (that is not a surprise since in the low Mach number limit, the flows are essentially isentropic). Then subtracting $\vec{u}^0_{\boldsymbol{i}} . \sum_{\boldsymbol{l} \in \mathcal{V}} \delta_{\boldsymbol{il}} \vec{n}_{\boldsymbol{il}} = 0$ to the left hand side of (12.a) and $p^1_{\boldsymbol{i}} . \sum_{\boldsymbol{l} \in \mathcal{V}} \delta_{\boldsymbol{il}} \vec{n}_{\boldsymbol{il}} = 0$ to the left hand side of (12.b), it is seen that this system can be reduced to

$$\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})} (\Delta_{\boldsymbol{il}} p^1 - \Delta_{\boldsymbol{il}} U^0)\delta_{\boldsymbol{il}} = 0 \qquad (13.a)$$

$$\sum_{\boldsymbol{l}\in\mathcal{V}(\boldsymbol{i})} (\Delta_{\boldsymbol{il}} p^1 - \Delta_{\boldsymbol{il}} U^0)\delta_{\boldsymbol{il}} \vec{n}_{\boldsymbol{il}} = 0 \qquad (13.b)$$

The conclusion of [2] was that in general, this system admits non-constant solutions for the order one pressure p^1. This explains the appearance of spurious pressure fluctuations of order M_* in finite volume computations by upwind schemes of low Mach number flows. However, as noticed by [3] for the special case of regular triangular cells, miracles can occur and the solution of (13) can be $\Delta_{\boldsymbol{il}} p^1 = 0$, implying that on triangular meshes, in the low Mach number limit, pressure fluctuations have the right order of magnitude.
The aim of this work is to show that the conclusion of [3] is more general than proved in this paper and that it concerns in fact **all** P0 meshes regular or not. Moreover, we show that in fact this is not the **geometry** of the cells that is responsible of this result but rather the number of neighbors of one cell that results from this particular geometry.

2.2 The special case of P0 meshes

Figure 1: Triangular cell and neighbors

To prove this point, consider Figure 1 that shows a typical triangular cell. For any $\boldsymbol{i}$ the cell $\boldsymbol{i}$ has exactly three neighbors $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}$ and (13) takes the form

$$(\Delta_{\boldsymbol{ij}} p^1 - \Delta_{\boldsymbol{ij}} U^0)\delta_{\boldsymbol{ij}} + (\Delta_{\boldsymbol{ik}} p^1 - \Delta_{\boldsymbol{ik}} U^0)\delta_{\boldsymbol{ik}} + (\Delta_{\boldsymbol{il}} p^1 - \Delta_{\boldsymbol{il}} U^0)\delta_{\boldsymbol{il}} = 0 \tag{14.a}$$

$$(\Delta_{\boldsymbol{ij}} p^1 - \Delta_{\boldsymbol{ij}} U^0)\delta_{\boldsymbol{ij}}\vec{n}_{\boldsymbol{ij}} + (\Delta_{\boldsymbol{ik}} p^1 - \Delta_{\boldsymbol{ik}} U^0)\delta_{\boldsymbol{ik}}\vec{n}_{\boldsymbol{ik}} + (\Delta_{\boldsymbol{il}} p^1 - \Delta_{\boldsymbol{il}} U^0)\delta_{\boldsymbol{il}}\vec{n}_{\boldsymbol{il}} = 0 \tag{14.b}$$

Let us define $x_{\boldsymbol{im}} = (\Delta_{\boldsymbol{im}} p^1 - \Delta_{\boldsymbol{im}} U^0)$ for $\boldsymbol{m} = \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}$ and $x = (x_{\boldsymbol{ij}}, x_{\boldsymbol{ik}}, x_{\boldsymbol{il}})^t$, then (14) can be seen as a linear system for x. This system that we write $Ax = 0$ admits a unique zero solution iff the determinant of the matrix A is non zero.

Lemma 2.2 *Assume that the triangle* $\boldsymbol{i}$ *is non-degenerated then the determinant of the linear system (14) is non-zero*

Define $\Sigma_{\boldsymbol{im}} = (\delta_{\boldsymbol{im}}, \delta_{\boldsymbol{im}}\vec{n}^x_{\boldsymbol{im}}, \delta_{\boldsymbol{im}}\vec{n}^y_{\boldsymbol{im}})^t$, then A can be written

$$A = [\Sigma_{\boldsymbol{ij}}, \Sigma_{\boldsymbol{ik}}, \Sigma_{\boldsymbol{il}}]$$

using that $\delta_{\boldsymbol{ij}}\vec{n}_{\boldsymbol{ij}} + \delta_{\boldsymbol{ik}}\vec{n}_{\boldsymbol{ik}} + \delta_{\boldsymbol{il}}\vec{n}_{\boldsymbol{il}} = 0$, we can write

$$\begin{aligned}\det A = \det\,[\Sigma_{\boldsymbol{ij}}, \Sigma_{\boldsymbol{ik}}, \Sigma_{\boldsymbol{il}}] &= \det\,[\Sigma_{\boldsymbol{ij}}, \Sigma_{\boldsymbol{ik}}, \Sigma_{\boldsymbol{il}} + \Sigma_{\boldsymbol{ij}} + \Sigma_{\boldsymbol{ik}}] \\ &= \det \begin{bmatrix} \vdots & \vdots & \delta_{\boldsymbol{il}} + \delta_{\boldsymbol{ik}} + \delta_{\boldsymbol{ij}} \\ \Sigma_{\boldsymbol{ij}} & \Sigma_{\boldsymbol{ik}} & 0 \\ \vdots & \vdots & 0 \end{bmatrix}\end{aligned}$$

and then

$$\det\, A = 2(\delta_{\boldsymbol{il}} + \delta_{\boldsymbol{ik}} + \delta_{\boldsymbol{ij}})|C_{\boldsymbol{i}}|$$

where here $|C_{\boldsymbol{i}}|$ is the signed area of the triangle $\boldsymbol{i}$. This quantity can be zero only if the triangle $\boldsymbol{i}$ is degenerate. ∎

From the previous result, we therefore conclude that we have
$x_{\boldsymbol{im}} = (\Delta_{\boldsymbol{im}} p^1 - \Delta_{\boldsymbol{im}} U^0) = 0$ for $\boldsymbol{m} = \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}$.

Lemma 2.3 $\forall\ \boldsymbol{im} \quad x_{\boldsymbol{im}} = 0 \quad \Rightarrow \quad \forall\ \boldsymbol{im} \;\; p^1_{\boldsymbol{i}} = p^1_{\boldsymbol{m}}$

Consider the couple of triangles $\boldsymbol{i}$ and $\boldsymbol{j}$, writing (13) for the triangle $\boldsymbol{i}$ gives

$$\Delta_{\boldsymbol{ij}} p^1 - \Delta_{\boldsymbol{ij}} U^0 = 0$$

The same system for the triangle $\boldsymbol{j}$ gives

$$\Delta_{\boldsymbol{ji}} p^1 - \Delta_{\boldsymbol{ji}} U^0 = 0$$

Going back to the definition of the jump operator $\Delta_{\boldsymbol{ij}}$ we see that we have

$$\Delta_{\boldsymbol{ji}} p^1 = p^1_{\boldsymbol{j}} - p^1_{\boldsymbol{i}} = -\Delta_{\boldsymbol{ij}} p^1$$

but since $\vec{n}_{\boldsymbol{ji}} = -\vec{n}_{\boldsymbol{ij}}$ we have for the velocity jump

$$\Delta_{\boldsymbol{ji}} U = (\vec{u}_{\boldsymbol{j}} - \vec{u}_{\boldsymbol{i}}).\vec{n}_{\boldsymbol{ji}} = -(\vec{u}_{\boldsymbol{j}} - \vec{u}_{\boldsymbol{i}}).\vec{n}_{\boldsymbol{ij}} = \Delta_{\boldsymbol{ij}} U$$

and thus we conclude that $\forall \boldsymbol{ij}$

$$\Delta_{\boldsymbol{ij}} p^1 = \Delta_{\boldsymbol{ij}} U^0 = 0$$

and therefore the order M_* pressure p^1 is constant ∎

2.3 Extension to the 3-D case

The previous results shows that on simplicial meshes, the order M_* pressure is constant. Therefore, cell centered finite volume upwind schemes do not give rise to pressure fluctuations of the wrong order of magnitude. Looking at the proof, it is seen that the essential ingredient to establish this point is lemma 2.2; this result relies on the fact that the number of degrees of freedom per cell : 2 components of the velocity vector + the pressure is exactly equal to the number of neighbors of a cell. We remark that this argument is independent of the space dimension and can be transposed in 3D : a tetrahedral cell has exactly 4 neighbors. This is again exactely the number of degrees of freedom localized in one cell (3 components of the velocity vector + the pressure). The results of section 2 can therefore be extended to the 3D case. The details of the proof are as follows.

In 3D, any cell $\boldsymbol{i}$ has exactly four neighbors that we label $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$. The equivalent of (14) is now

$$\begin{aligned}
&(\Delta_{\boldsymbol{ij}} p^1 - \Delta_{\boldsymbol{ij}} U^0)\delta_{\boldsymbol{ij}} + (\Delta_{\boldsymbol{ik}} p^1 - \Delta_{\boldsymbol{ik}} U^0)\delta_{\boldsymbol{ik}} + \\
&(\Delta_{\boldsymbol{il}} p^1 - \Delta_{\boldsymbol{il}} U^0)\delta_{\boldsymbol{il}} + (\Delta_{\boldsymbol{im}} p^1 - \Delta_{\boldsymbol{im}} U^0)\delta_{\boldsymbol{im}} = 0 \qquad (15.a) \\
&(\Delta_{\boldsymbol{ij}} p^1 - \Delta_{\boldsymbol{ij}} U^0)\delta_{\boldsymbol{ij}}\vec{n}_{\boldsymbol{ij}} + (\Delta_{\boldsymbol{ik}} p^1 - \Delta_{\boldsymbol{ik}} U^0)\delta_{\boldsymbol{ik}}\vec{n}_{\boldsymbol{ik}} + \\
&(\Delta_{\boldsymbol{il}} p^1 - \Delta_{\boldsymbol{il}} U^0)\delta_{\boldsymbol{il}}\vec{n}_{\boldsymbol{il}} + (\Delta_{\boldsymbol{im}} p^1 - \Delta_{\boldsymbol{im}} U^0)\delta_{\boldsymbol{im}}\vec{n}_{\boldsymbol{im}} = 0 \qquad (15.b)
\end{aligned}$$

where now $\delta_{\boldsymbol{ij}}, \delta_{\boldsymbol{ik}}, \delta_{\boldsymbol{il}}, \delta_{\boldsymbol{im}}$ denote the surface of the triangular faces of the tetrahedron $\boldsymbol{i}$. Similarly, $\vec{n}_{\boldsymbol{ij}}, \vec{n}_{\boldsymbol{ik}}, \vec{n}_{\boldsymbol{il}}, \vec{n}_{\boldsymbol{im}}$ are now the unit normal vectors of these four faces. The proof can now proceed in the same way than in 2D.

Define $\Sigma_{\boldsymbol{im}} = (\delta_{\boldsymbol{im}}, \delta_{\boldsymbol{im}}\vec{n}^x_{\boldsymbol{im}}, \delta_{\boldsymbol{im}}\vec{n}^y_{\boldsymbol{im}}, \delta_{\boldsymbol{im}}\vec{n}^z_{\boldsymbol{im}})^t$, The matrix A expressing the linear system (15) can be written

$$A = [\Sigma_{\boldsymbol{ij}}, \Sigma_{\boldsymbol{ik}}, \Sigma_{\boldsymbol{il}}, \Sigma_{\boldsymbol{im}}]$$

and with the same argument than in 2D, we have

$$\det\, A = \det \begin{bmatrix} \vdots & \vdots & \vdots & \delta_{\boldsymbol{im}} + \delta_{\boldsymbol{il}} + \delta_{\boldsymbol{ik}} + \delta_{\boldsymbol{ij}} \\ \Delta_{\boldsymbol{ij}} & \Delta_{\boldsymbol{ik}} & \Delta_{\boldsymbol{il}} & 0 \\ \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & 0 \end{bmatrix}$$

and then

$$\det\, A = 6(\delta_{\boldsymbol{im}} + \delta_{\boldsymbol{il}} + \delta_{\boldsymbol{ik}} + \delta_{\boldsymbol{ij}})|C_{\boldsymbol{i}}|$$

where here $|C_{\boldsymbol{i}}|$ is the signed volume of the tetrahedron $\boldsymbol{i}$ and as in 2D we can conclude

Lemma 2.4 *Assume that tetrahedron $\boldsymbol{i}$ is non-degenerated then the determinant of the linear system (15) is non-zero*

The rest of the proof is identical to the 2D case by considering the face $\boldsymbol{ij}$ and writing (15) for the two tetrahedra $\boldsymbol{i}$ and $\boldsymbol{j}$.

3 Concluding remarks

When applied to triangular cells, Finite Volume upwind schemes do not exhibit the problem of lack of convergence to the incompressible solution in the zero Mach number limit that is known to exist for other type of meshes [2]. This was numerically proved in [3] for arbitrary unstructured meshes and theoretically for structured triangular regular grid in the same work. The aim of the present paper was to extend the theoretical work of Rieper and Bader [3] to **general unstructured** meshes. Apart from the fact that our proof is more general (and we believe simpler) that the one of their paper, it also enlightens the essential fact that is at the heart of the singular behavior of Finite Volume upwind schemes if applied to triangular cell meshes : the number of degree of freedom per cell is exactly equal to the number of neighbors of this cell. As this fact is also valid in 3D, we have been also able to extend the proof in the 3D case.

An interesting colateral result of Rieper and Bader's paper is that in the zero Mach limit, the discrete equations force the jumps of the normal component of the velocity to vanish. The velocity approximation space is thus characterized by this constraint. It does not seems that this type of approximation space is well known in the litterature and it would be interesting to look at this result from the point of view of approximation theory for incompressible flows.

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