Shape-from-Shading with discontinuous image brightness

Fabio Camilli, Emmanuel Prados

Shape from Shading with discontinuous image brightness

Fabio Camilli

Dip. di Matematica Pura e Applicata, Università dell'Aquila (Italy), camilli@ing.univaq.it

Emmanuel Prados

UCLA Vision Lab. (USA), eprados@cs.ucla.edu

Abstract

The Shape–from–Shading models in image analysis lead to first order Hamilton–Jacobi equations which may have several weak solutions (in the viscosity sense). Moreover, for real images, these equations are highly discontinuous in the space variable. The lack of uniqueness and the irregularity of the coefficients involve some troubles when we try to compute a solution. In order to avoid these difficulties, here we use recent results in the theory of viscosity solutions to characterize the maximal solution of these equations. Moreover we describe an approximation procedure via smooth equations with a unique viscosity solution.

Key words: Shape-from-Shading, measurable Hamilton-Jacobi equation, viscosity solution, stability

1 Introduction

Shape-from-Shading (SfS) is the problem of recovering the three dimensional shape of a surface from the brightness of a black and white image of it. In the PDE approach, it leads to first order Hamilton–Jacobi equations coupled with appropriate boundary conditions ([6], [11], [12], [13]). The analytical characterization of the solution of these equations involves some difficulties since they may have in general several weak solutions (to be understood in the viscosity sense, see [1]), all in between a minimal and a maximal solution.

The datum of the problem is the brightness I which, after normalization, verifies $0 \leq I(x) \leq 1$. The lack of uniqueness for SfS equations is due to the presence of points at maximal light intensity, i.e. $I(x) = 1$. This difficulty is general and applies to all the models considered in literature (see [15]).

This ambiguity, which has a direct counterpart in the fail of a strong maximum principle for the Hamilton-Jacobi equation, is a big trouble when trying to compute a solution since it also affects the convergence of numerical algorithms ([8]). In order to avoid this difficulty, before approximating the problem, one can either add some extrinsic information such as the height of the solution at maximal intensity points ([13]) or regularize the problem by cutting the intensity light at some level strictly less than 1.

On the other side a brightness I corresponding to a real life image is in general highly discontinuous and corrupted by noise. To remove the noise, the images are often regularized [19]. Moreover most of the CCD sensors slightly smooth the images and defocus effects can strongly diffuse the brightness information. In other respects, in solving numerically a SfS equation we replace the irregular image brightness with some regular approximation, f.e. a piecewise linear or piecewise polynomial function on the mesh of the grid.

One therefore may wonder what the various numerical algorithms in SfS literature really compute after this double process of regularization. Aim of this paper is to give a solution, at least partial, to this problem. We use recent results in the theory of viscosity solutions ([2]-[4], [14]) to characterize the maximal solution of eikonal equations with measurable coefficients without extra information besides the equation. A feature of our method is that if some partial information about the solution, f.e. the height of the solution in some subset of the singular set, is known it can be included in the model.

As a consequence of stability properties of this new definition we describe a regularization procedure which give a sequence of equations with regular coefficients and a unique viscosity solution converging to the maximal (minimal) solution of the SfS equation. To compute the solution of the approximating equations we can use therefore anyone of approximation schemes that it is possible to find in SfS literature. Note that, once that the maximal and the minimal solutions of the problem are known all the other solutions of the problem can be recovered by taking the appropriate values on the singular set.

There are other papers dealing with discontinuous SfS equations ([5], [12], [18]). They are based on the notion of Ishii discontinuous viscosity solution (see [1] for the definition) and deal with piecewise continuous or lower semicontinuous image brightness. Observe that the solution they characterize in general does not coincide with maximal (minimal) solution of the problem.

Notations: Throughout the paper, measurable is intended in the sense of the Lebesgue measure. If $E \subset \mathbb{R}^d$ is a measurable set, then $|E|$ denotes its measure. If $|E| = 0$, then E is told a null set. For a measurable function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$, we define the *essential sup* (respectively, the *essential inf*) of f in a set E as

$$\begin{gathered}\inf\{C : |\{f \leq C\} \cap E| = |E|\} \\ (\text{resp., } \sup\{C : |\{f \geq C\} \cap E| = |E|\})\end{gathered}$$

and the *essential limsup* (respectively, the *essential liminf*) of f at x_0 as

$$\begin{gathered}\operatorname{ess}\limsup_{x \to x_0} f(x) = \lim_{r \to 0}[\operatorname{ess}\sup_{B(x_0,r)} f] \\ (\text{resp., } \operatorname{ess}\liminf_{x \to x_0} f(x) = \lim_{r \to 0}[\operatorname{ess}\inf_{B(x_0,r)} f])\end{gathered}$$

2 Assumptions and preliminaries

The SfS Hamiltonians satisfy some basic properties independently of the various SfS models considered. They are convex and (in practice, generally) coercive in the gradient variable and they have the same regularity of $I(x)$ in the state variable (see [15] for a detailed discussion of this point). Moreover they admit a subsolution which plays a key role in the uniqueness of the solution.

Let Ω be a bounded, open subset of $\mathbb{R}^N$ (in the SfS problem $N = 2$, so Ω is the rectangular domain given by the image) with Lipschitz continuous boundary. We model a discontinuous image brightness with a function $I(x)$ measurable, bounded and positive (i.e. there exists $m > 0$ for which $I(x) \geq m$ a.e. in Ω) and we consider the corresponding SfS equation

$$H(x, Du) = 0 \qquad x \in \Omega, \tag{1}$$

where H is one of the SfS Hamiltonians considered in literature (see [6], [13], [15], [20]). Under the previous assumptions on I, the Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ turns out to be measurable in x for any p, continuous, strictly convex and coercive (i.e. for any compact subset K of $\mathbb{R}^N$ there exists $R > 0$ s.t. $\operatorname{ess}\inf\{H(x,p) : |p| > R,\ x \in K\} > 0$) in p for a.e. x. Moreover there exists a locally Lipschitz continuous function ψ

$$H(x, D\psi) \leq 0 \qquad \text{for a.e. } x \tag{2}$$

Remark 1 *For a continuous image brightness, the existence of C^1 subsolutions for the various SfS Hamiltonians is studied in details in [15]. If I is only L^∞, the subsolutions constructed in [15] are Lipschitz continuous.*

Remark 2 *A basic example of (1) is the SfS equation considered in [16]*

$$|Du(x)| = n(x) \qquad x \in \Omega \tag{3}$$

where $n(x) = (I(x)^{-2} - 1)^{\frac{1}{2}}$. *Note that if* $I(x) \leq M < 1$ *a.e. in* Ω, *then* $n(x) \geq m > 0$ *a.e. in* Ω.

More generally, we can consider the Hamilton-Jacobi-Bellman equation

$$\sup_{a \in A} \{-b(x,a) \cdot Du - f(x,a)\} = 0 \tag{4}$$

where A *is a compact metric space,* f *and* b *are bounded functions, measurable in* x, *continuous in* a *and such that there exists* $r > 0$ *for which*

$$B(0,r) \subset \overline{\mathrm{co}}\{b(x,a) : a \in A\}$$

a.e. in x *(*$\overline{\mathrm{co}}$ *stands for the closure of the convex hull). In [15], it is shown that a large class of SfS equations can be written in the form (4) for an appropriate choice of* f *and* b.

We complete (1) with the Dirichlet boundary condition

$$u = \varphi \quad \text{in } \partial\Omega \cup K \tag{5}$$

where φ is a continuous function defined on $\partial\Omega \cup K$, with K a proper, closed, possible empty subset of Ω which represents the part of the data available inside Ω.

We recall that a Lipschitz continuous function ψ is said a *strict subsolution* of (1) if

$$H(x, D\psi) \leq -\theta \qquad \text{a.e. in } \Omega$$

for some $\theta > 0$. If a strict subsolution to (1) exists, then (1)–(5) admits a unique viscosity solution (see [10] for the continuous case, [3] for the measurable one), otherwise in general uniqueness fails (see [2] for an example).

To simplify the presentation, from now on we assume that the subsolution ψ in (2) is identically null. This is true for some SfS models (see e.g. [6], [12] [13], [16], [18]), but not in general.

Consider equation (3). If $I(x) \leq M < 1$ a.e. in Ω, $\psi \equiv 0$ is a strict subsolution to (3) and therefore there exists a unique viscosity solution to (3)-(5).

If $\operatorname{ess\,sup}_\Omega I(x) = 1$, then $\psi \equiv 0$ is not a strict subsolution in all Ω. In particular, it fails to be strict subsolution in a neighborhood of a point x_0 such that $\operatorname{ess\,lim\,sup}_{x\to x_0} I(x) = 1$. We therefore define the *singular set* for (3) in the measurable case as

$$\mathcal{S} = \{x \in \mathbb{R}^N \,:\, \operatorname{ess}\limsup_{y\to x} I(y) = 1\}. \tag{6}$$

Since $\psi \equiv 0$ is not a strict subsolution to (3), one may wonder if it is possible to find another function with this property. But it is possible to prove, see [9], [4], that the set $\mathcal{S}$ (called the Aubry set in these papers) can be characterized as the set of points $x \in \Omega$ where any subsolution to (1) necessarily fails to be a strict subsolution.

Let us mention that, due to their similar structure, this discussion applies to the various SfS equations. Therefore we define the *singular set* for a generic Hamilton-Jacobi equation (1) as in (6). If I is continuous, $\mathcal{S}$ coincides with set of points with brightness equal to 1. Since the definition of viscosity solution we will give in the next section reduces to the classical one in the continuous case, if $\mathcal{S}$ is not empty, then uniqueness does not hold.

Proposition 3 *If* $\operatorname{ess\,sup}_\Omega I(x) = 1$*, then*

i) $\mathcal{S}$ *is nonempty and closed.*
ii) For a set $E \subset \Omega$ *with* $|E| > 0$ *and* $\inf_{x\in E} d(x, \mathcal{S}) > 0$*, we have*

$$\operatorname{ess\,sup}_E I < 1 \tag{7}$$

PROOF. See the Appendix.

Throughout the paper we assume that

$$\mathcal{S} \cap \partial\Omega = \emptyset. \tag{8}$$

3 A distance function, a weaker topology and the definition of viscosity solution

The subsolution part of the definition of solution is the easier one. In the continuous case, because of the convexity and coercitivity of the equation, viscosity subsolutions and Lipschitz-continuous a.e. subsolutions coincide. The concept of a.e. subsolution makes sense also in the measurable setting, while the corresponding definition of viscosity subsolution can be introduced as in

[3] using suitable measure-theoretic limits. This definition turns out to be equivalent to the one of a.e. subsolution, its advantage being that is of pointwise type. Since, for the purposes of this paper, it is sufficient to consider a.e. subsolutions, we refer the interested reader to [3] for the definition of viscosity subsolution in the measurable setting.

We aim to introduce a definition of viscosity supersolution which select the maximal a.e. subsolution of (1)–(5). We have to cope with two difficulties: the measurable setting and the presence of the singular set $\mathcal{S}$.

We proceed introducing a distance function and the associated weak topology. We set

$$\mathcal{Z}(x) := \Big\{p \in \mathbb{R}^N : \; H(x,p) \leq 0\Big\} \tag{9}$$

and we observe that by the properties of the Hamiltonian it follows that $\mathcal{Z}(x)$ is convex, compact, $0 \in \mathcal{Z}(x)$ and

$$\partial\mathcal{Z}(x) = \{p : \; H(x,p) = 0\}$$

for a.e. $x \in \Omega$ (this last property follows from the fact that $\mathcal{Z}(x)$ is strictly convex). We define for $x \in \Omega$, $q \in \mathbb{R}^N$

$$\sigma(x,q) = \sup\{p \cdot q : \; p \in \mathcal{Z}(x)\}, \tag{10}$$

i.e. σ is the support function of the convex set $\mathcal{Z}(x)$ at q.

Remark 4 *For (3), we have that $\mathcal{Z}(x) = B(0, n(x))$, $\sigma(x,q) = n(x)|q|$ and $\mathcal{S} = \{x \in \Omega : \; \operatorname{ess\,lim\,inf}_{x \to x_0} n(x) = 0\}$.*

Proposition 5 *The function $\sigma(x,q)$ is measurable in x for any q and continuous, convex, positive homogeneous in q for a.e. x and for a.e. x, for any q*

$$0 \leq \sigma(x,q) \leq R|q|. \tag{11}$$

Moreover, if $x \notin \mathcal{S}$, then $\sigma(y,q) \geq \delta|q|$ a.e. in a neighborhood of x, for some $\delta > 0$.

PROOF. The first part of statement comes from the definition of support function, the measurability of $\mathcal{Z}$, the a.e. convexity of $\mathcal{Z}(x)$ and $0 \in \mathcal{Z}(x)$, see [3]. If $x \notin \mathcal{S}$, since $\mathcal{S}$ is closed, there exists a neighborhood A of x where $\psi \equiv 0$

is a strict subsolution to (1). Hence there exists $\delta > 0$ such that $B(0,\delta) \subset \mathcal{Z}(x)$ a.e. in A and therefore $\sigma(x,q) \geq \delta|q|$.

For $A \subset \Omega$, we denote for any $x, y \in \bar{A}$,

$$
\begin{aligned}
S_A(x,y) = \sup\nolimits_{N\in\mathcal{N}_A} \Big[&\inf \{ \textstyle\int_0^1 \sigma(\xi(t), -\dot{\xi}(t))dt : \xi(t) \in W^{1,\infty}([0,1], A) \\
&\text{s.t.}\, \xi(0) = x,\ \xi(1) = y \,\text{and}\, \xi \pitchfork N\}\Big].
\end{aligned}
\tag{12}
$$

where $\mathcal{N}_A$ is that class of subsets of A of null Lebesgue-measure and $\xi \pitchfork N$ means that

$$
|\{t \in [0,1] : \xi(t) \in N\}| = 0 \tag{13}
$$

(in this case $|\cdot|$ stands for the 1-dimensional Lebesgue measure). If (13) holds, we say that ξ is *transversal* to N and we denote by $\mathcal{A}^N_{x,y}$ the set of the Lipschitz-continuous trajectories ξ joining x to y with this property. We also set $S(x,y) := S_\Omega(x,y)$.

Proposition 6 *Let $A \subset \Omega$ be a set with Lipschitz continuous boundary. Then*

i) A change of representative of the Hamiltonian H in (9) does not affect S_A.
ii) For any representative of H there exists a null set $F \subset A$ such that

$$
S_A(x,y) = \inf\{\int_0^1 \sigma(\xi(t), -\dot{\xi}(t))dt \ : \ \xi \in \mathcal{A}^F_{x,y}\}
$$

for every $x,\ y \in A$.
iii) $S_A(x,x) = 0$ and $S_A(x,z) \leq S_A(x,y) + S_A(y,z)$ for every $x,\ y,\ z$ in $\overline{A}$.
iv) $S(x,y) \leq R d_E(x,y)$ for any $x, y \in \overline{\Omega}$, where d_E is the Euclidean geodesic distance in $\overline{\Omega}$ and R as in (11).

PROOF. We refer to [3], [4].

Note that by *item iv)* it follows that $y \mapsto S(x,y)$ is *Lipschitz continuous* in $\overline{\Omega}$ for any fixed $x \in \overline{\Omega}$ (with a Lipschitz constant which does not depend on x). So S is a semi-distance on $\overline{\Omega}$, but not in general a distance. If $x_0 \in \mathcal{S}$, it may be possible that there points at null $S-$distance from x_0 (recall that is only $\sigma(x,q) \geq 0$ if $x \in \mathcal{S}$). The family of balls

$$
B_S(x_0, R) = \{x \in \overline{\Omega} \mid S(x_0,x) \leq R\}
$$

induces a topology τ_S in $\overline{\Omega}$ which is locally equivalent, see (7), to the Euclidean topology out of $\mathcal{S}$, but weaker on $\mathcal{S}$. We denote by $B_S(x_0)$ the subset

$$B_S(x_0) = \{x \in \overline{\Omega} \mid S(x_0, x) = 0\}.$$

We introduce the gauge function $\rho(x,p)$ of $\mathcal{Z}(x)$. We set for any $x \in \Omega$, $p \in \mathbb{R}^N$,

$$\rho(x,p) = \inf\{\lambda > 0 : \lambda^{-1}p \in \mathcal{Z}(x)\}. \tag{14}$$

(for equation (3), we have $\rho(x,p) = |p|n(x)$). The function ρ is measurable in x, l.s.c. (continuous in $\Omega \setminus \mathcal{S}$), convex and positive homogeneous in p and verifies the homogeneity condition:

$$\rho(x, \mu p) = \mu \rho(x,p) \qquad \text{for any } \mu > 0 \tag{15}$$

for a.e. $x \in \Omega$, for any $p \in \mathbb{R}^N$. Moreover the functions ρ and σ are correlated by the identity

$$\rho(x,p) = \sup_{q \in \mathbb{R}^N} \frac{p \cdot q}{\sigma(x,q)} \tag{16}$$

for a.e. $x \in \Omega$, any $p \in \mathbb{R}^N$.

Note that $H(x,p) \leq 0$ if and only if $\rho(x,p) \leq 1$, hence the set $\mathcal{Z}(x)$ in (9) can be equivalently defined as

$$\mathcal{Z}(x) = \{p \in \mathbb{R}^N : \rho(x,p) \leq 1\}. \tag{17}$$

If $x \in \mathcal{S}$, we have $\rho(x, 0) = 0$, so $\psi \equiv 0$ is a strict subsolution in all Ω of the equation

$$\rho(x, Du) = 1. \tag{18}$$

Introducing the new equation (18), which is equivalent to (1) because (17), we have recovered the fact that $\psi \equiv 0$ is a strict subsolution.

Definition 7 *For a l.s.c. function v, a Lipschitz continuous function ϕ is called S-subtangent to v at $x_0 \in \Omega$ if x_0 is a minimizer of $v - \phi$ in a τ_S-neighborhood of x_0 (or equivalently, in a neighborhood $\mathcal{O}$ of $B_S(x_0)$). The S-subtangent is called strict if the inequality*

$$(v - \phi)(x) > (v - \phi)(x_0)$$

holds for $x \in \mathcal{O} \setminus B_S(x_0)$.

Observe that a S-subtangent is also a subtangent in the standard viscosity solution sense, but the converse is not true.

Definition 8 *A l.s.c. v is said a* singular supersolution *of (1) at $x_0 \in \Omega$ if, given a Lipschitz-continuous function ϕ S-subtangent to v at x_0, then*

$$\operatorname{ess}\limsup_{x \to x_0} \rho(x, D\phi(x)) \geq 1.$$

Remark 9 *By the very definition of essential limit superior, we reformulate the supersolution condition as follows:*
A l.s.c. v is said a singular supersolution *of (1) at $x_0 \in \Omega$ if, given a Lipschitz-continuous function ϕ, $\theta \in (0,1)$ and a neighborhood $\mathcal{O}$ of $B_S(x_0)$ such that*

$$\rho(x, D\phi) \leq \theta \qquad \text{for a.e. } x \in \mathcal{O}$$

then ϕ cannot be S-subtangent to v at x_0.

Definition 10

i) A function $u \in W^{1,\infty}(\Omega) \cap C^0(\overline{\Omega})$ is said a subsolution of (1)-(5) if u is an a.e. subsolution of (1) in Ω and $u \leq \varphi$ on $\partial\Omega \cup K$.

ii) A l.s.c. function v is said a supersolution of (1)-(5) if
- *v is singular supersolution of (1) in $\Omega \setminus K$.*
- *For any $x_0 \in K$, either v is a singular supersolution of (1) at x_0 or there exists $x \in B_S(x_0)$ such that $v(x_0) \geq \varphi(x)$.*
- *For any $x_0 \in \partial\Omega$, $v(x_0) \geq \varphi(x_0)$.*

Finally, u is said a solution of (1)-(5) if it is a subsolution and a supersolution of the problem.

Remark 11 *We observe that if $I(x) \leq \eta < 1$ a.e. in Ω and therefore $\mathcal{S}$ is empty, the previous definition of solution coincides with the one given in [3]. Hence if I is continuous, see Prop 6.5 in [3], it is equivalent to the Crandall-Lions definition of viscosity solution. If I is continuous and $\mathcal{S}$ is not empty, the definition coincides with the one in [14].*

Remark 12 *The boundary condition on $\partial\Omega$ is given in a pointwise sense. This is restrictive, since it require a compatibility condition on the boundary datum for the existence of a solution (see (19)). We have preferred to avoid to introduce a boundary condition in viscosity sense to simplify the presentation. Also, this extension is quite direct.*

The boundary condition on K is given in the sense of the topology τ_S.

Remark 13 *Consider the case of the eikonal equation (3). In this case $\mathcal{S} = \{x \in \Omega : \operatorname{ess}\liminf_{x \to x_0} n(x) = 0\}$. If $\mathcal{S}$ has nonempty interior and $x_0 \in$*

$int(\mathcal{S})$, since $n(x) = 0$ a.e. in a (Euclidean) neighborhood $I_\delta(x_0)$ of x_0, a solution of (3) is constant in $I_\delta(x_0)$. Hence $\psi \equiv 0$ is subtangent (in the standard sense) at x_0 to any solution of (3) and a strict subsolution of (18). If we want to preserve the viscosity supersolution property, i.e. a strict subsolution cannot be subtangent to a supersolution, we have to use the weaker topology τ_S, which has the the property that a neighborhood of $x_0 \in \mathcal{S}$ contains all the connected component of $\mathcal{S}$ which contains x_0.

4 A representation formula for the maximal a.e. subsolution

In this section, we give a representation formula for the viscosity solution of (1)-(5) or, equivalently, for the maximal a.e. subsolution of (1) such that $u \leq \varphi$ on $\partial\Omega \cup K$. We assume that the boundary datum φ satisfies the compatibility condition

$$\varphi(x) - \varphi(y) \leq S(x, y) \qquad \text{for any } x, y \in \partial\Omega \tag{19}$$

We define for $x \in \overline{\Omega}$ the function

$$V(x) = \min_{y \in \partial\Omega \cup K} \{S(x, y) + \varphi(y)\}. \tag{20}$$

To prove that V is a solution we need some preliminary results

Proposition 14 *u is a subsolution of (1) in Ω if and only if*

$$u(x) - u(y) \leq S(x, y) \qquad \textit{for any } x, y \in \Omega \tag{21}$$

PROOF. See Prop.4.7 in [3].

Proposition 15 *Set $\Gamma_V = \{x_0 \in K : V(x_0) \geq \varphi(x) \text{ for some } x \in B_S(x_0)\}$. If $x_0 \in \Omega \setminus \Gamma_V$, then*

$$B_S(x_0) \cap \Gamma_V = \emptyset. \tag{22}$$

Moreover for any τ_S-neighborhood A of x_0 s.t. $A \cap (\Gamma_V \cup \partial\Omega) = \emptyset$,

$$V(x_0) = \min_{y \in \partial A} \{S(x_0, y) + V(y)\}. \tag{23}$$

PROOF. We first prove (22). The statement is obvious if $x_0 \notin \mathcal{S}$, since in this case $B_S(x_0) = \{x_0\}$. If $x_0 \in \mathcal{S} \cap K$, assume by contradiction that there exists $x_1 \in B_S(x_0) \cap \Gamma_V$. Hence there exists $y \in B_S(x_1)$ such that

$$V(x_1) \geq \varphi(y).$$

Since $S(x_0, y) \leq S(x_0, x_1) + S(x_1, y) = 0$, $y \in B_S(x_0)$. By (21) and $x_0 \notin \Gamma_V$ we have $V(x_1) \leq V(x_0) < \varphi(y)$ and so

$$V(x_1) < \varphi(y)$$

Hence a contradiction.

For the proof of (23), taking into account (23), we refer to [3].

In viscosity solution theory, given a subtangent ϕ_0 at a point x_0, by adding a quadratic term it is always possible to obtain another subtangent ϕ with the same gradient at x_0 which is a strict subtangent in a neighborhood of x_0. We need a similar property for S-subtangents.

Proposition 16 *Let u be a l.s.c. function. Given a Lipschitz-continuous function ϕ_0, S-subtangent to u at a point x_0 and a strict subsolution of (1) in a τ_S-neighborhood of x_0, it is possible to find a function ϕ which is strict S-subtangent to u at x_0 and a strict subsolution of (1) in a τ_S-neighborhood of x_0. Moreover if ϕ_0 and ϕ are differentiable at x and*

$$\rho(x, D\phi_0(x))) \leq \rho(x, D\phi(x))) + S(x_0, x) \tag{24}$$

PROOF. See the proof of Proposition 5.1 of [2].

Theorem 17 *V is a solution of (1)-(5).*

PROOF. ¿From (21), it follows that V is a subsolution of the equation and $V \leq \varphi$ on K.

To prove that V is a supersolution, we have only to prove that V is a singular supersolution out of the set Γ_V. We argue by contradiction assuming that there exists a function ϕ_0, a neighborhood A of $B_S(x_0)$ and $\theta \in (0, 1)$ s.t. ϕ_0 is a S-subtangent to V at x_0 with $\phi_0(x_0) = V(x_0)$ and

$$\rho(x, D\phi_0) \leq \theta, \quad \text{for any } x \in A \setminus E \tag{25}$$

where E is a suitable null set.
Let ϕ be a strict S-subtangent to V at x_0 verifying the statement of Proposition

16. By continuity of the function $x \mapsto S(x_0, x)$, we can select a neighborhood A' of $B_S(x_0)$ with $\bar{A}' \subset A$ satisfying

$$\begin{aligned} &\sup_{x \in A'} S(x_0, x) < 1 - \theta, \\ &\phi \leq V - \eta \quad \text{on} \quad \partial A' \\ &\quad A' \cap (\Gamma_V \cup \partial\Omega) = \emptyset \end{aligned} \tag{26}$$

for some $\eta > 0$. By (23), there exists $y_0 \in \partial A'$ such that

$$V(x_0) = S_{A'}(x_0, y_0) + V(y_0).$$

Since $V(x_0) = \phi(x_0)$ and $V(y_0) > \phi(y_0)$ we get

$$S_{A'}(x_0, y_0) + \phi(y_0) - \phi(x_0) < 0. \tag{27}$$

Consider a null set F, containing E and the set of the points where ϕ_0 and ϕ are not differentiable, such that

$$S_{A'}(x, y) = \inf\{\int_0^1 \sigma(\xi, -\dot{\xi}) dt : \ \xi \in \mathcal{A}^F_{xy}\}$$

for any x, $y \in \overline{A}'$. Take $\xi \in \mathcal{A}^F_{x_0, y_0}$ such that

$$\int_0^1 \sigma(\xi, -\dot{\xi}) dt \leq S_{A'}(x_0, y_0) + \frac{\eta}{2}.$$

By (27) we get

$$\int_0^1 \left[\sigma(\xi(t), -\dot{\xi}(t)) + \frac{d\phi}{dt}(\xi(t))) \right] dt < \frac{\eta}{2}.$$

Take into account that ϕ is differentiable at $\xi(t)$ for a.e. $t \in [0, 1]$ to derive

$$\int_0^1 \left[D\phi(\xi) \cdot \dot{\xi} - \sigma(\xi, \dot{\xi}) \right] dt \geq \frac{\eta}{2}.$$

Then in a subset of $[0, 1]$ of positive measure

$$D\phi(\xi)\dot{\xi} > \sigma(\xi, \dot{\xi})$$

and so, by (16),

$$\rho(\xi(t), D\phi(\xi(t))) > 1 \tag{28}$$

Since $\xi(t) \in A$ for any $t \in [0,1]$, $\xi \pitchfork F$, ϕ, ϕ_0 are differentiable in $A \setminus F$, by (26) and (28) we get a contradiction to (25).

Finally the compatibility condition (19) implies that $V(x) = \varphi(x)$ on $\partial\Omega$.

We now prove a uniqueness result for the solution of (1)-(5).

Theorem 18 *Let $u, v : \Omega \to \mathbb{R}$ be respectively a subsolution and a supersolution of (1)-(5). Then*

$$u(x) \leq v(x) \text{ for any } x \in \overline{\Omega}. \tag{29}$$

PROOF. We argue by contradiction and we assume that $M = \max_{\overline{\Omega}}\{u(x) - v(x)\} > 0$. Set

$$\Gamma_v = \{x \in K : \ v(x) \geq \varphi(y) \ \text{ for some } y \in B_S(x)\}.$$

If $x \in \Gamma_v$, let $y \in B_S(x)$ be such that $v(x) \geq \varphi(y)$. By (21) and because $\Gamma_v \subset K$

$$v(x) \geq \varphi(y) \geq u(y) \geq u(x).$$

So, for any $x \in \Gamma_v$, $u(x) \leq v(x)$. Since also $u \leq v$ on $\partial\Omega$,

$$M = \max_{x \in \overline{\Omega}}\{u(x) - v(x)\} = \max_{x \in \Omega \setminus \Gamma_v}\{u(x) - v(x)\}.$$

Given $\theta \in (0,1)$, the function $u_\theta = \theta u$ satisfies

$$\rho(x, Du_\theta) \leq \theta$$

in Ω by the homogeneity of ρ (see (15)). For θ sufficiently close to 1, a maximizer x_θ of $(u_\theta - v)$ is assumed in $\Omega \setminus \Gamma_v$. Therefore u_θ is S-subtangent to v at x_θ. This contradicts the fact that v is a singular supersolution at x_θ. So the minimizers of $(v - u_\theta)$ are in $\partial\Omega \cup \Gamma_v$. The assertion is obtained by letting θ go to 1.

It follows that

Corollary 19 *V is the maximal a.e. subsolution of (1)-(5).*

5 A regularization procedure

In this section, we analyze the stability properties of the solution to (1)–(5). In particular, we describe a regularization procedure which gives a sequence of regular (i.e. with smooth coefficients and no singular set) equations which approximate the irregular equation.

Let ϵ_n be a sequence of positive numbers converging to 0 and set

$$\mathcal{Z}_{\epsilon_n}(x) = \mathcal{Z}(x) \cup B(0, \epsilon_n).$$

Let $\rho_{\epsilon_n}(x, p)$ and $\sigma_{\epsilon_n}(x, q)$ be the corresponding gauge function (see (14)) and support function (see (10)). We have that $\sigma_{\epsilon_n}(x, q)$ satisfies the same properties of $\sigma(x, q)$, see Prop. 5, moreover, since $B(0, \epsilon_n) \subset \mathcal{Z}_{\epsilon_n}(x)$, we have

$$\sigma_{\epsilon_n}(x, q) \geq \epsilon_n |q| \qquad \text{for any } q \in \mathbb{R}^N, \text{ for a.e. } x \in \Omega. \tag{30}$$

So the distance defined as in (12) with $\sigma_{\epsilon_n}(x, q)$ in place of $\sigma(x, q)$ is locally equivalent to the Euclidean distance, i.e.

$$\epsilon_n d_E(x, y) \leq S_{\epsilon_n}(x, y) \leq R d_E(x, y) \qquad x,\, y \in \overline{\Omega}.$$

We define

$$\sigma_{\epsilon_n n}(x, q) = \sigma_{\epsilon_n}(\cdot, q) * \eta_n(x) \tag{31}$$

where $\eta_n(x)$ is a standard mollifier in $\mathbb{R}^N$, i.e. $\eta_n(x) = n^N \eta(nx)$ with $\eta : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a smooth, nonnegative function such that $\operatorname{supp}\{\eta\} \subset B(0, 1)$ and $\int_{\mathbb{R}^N} \eta(x)dx = 1$. The function $\sigma_{\epsilon_n n}(x, q)$ satisfies the same properties of $\sigma_{\epsilon_n}(x, q)$ (in particular (11) and (30)) with respect to q, moreover it is continuous in x. For any $n \in \mathbb{N}$, we consider the approximating equation

$$H_n(x, Du) = 0 \qquad x \in \Omega. \tag{32}$$

where $H_n(x, p) = \sup_{|q|=1} \{q \cdot p - \sigma_{\epsilon_n n}(x, q)\}$. The Hamiltonian H_n is continuous in (x, p), convex and coercive in p, moreover since $H_n(x, 0) \leq -\epsilon_n < 0$ for $x \in \Omega$, $\psi \equiv 0$ is a strict subsolution of the equation. Standard results in viscosity solution theory gives that for any $n \in \mathbb{N}$, the problem (32)-(5) admits a unique (Crandall-Lions) viscosity solution.

Remark 20 *For (3), we have* $\mathcal{Z}_{\epsilon_n}(x) = B(0, n(x) \vee \epsilon_n)$ *and* $\sigma_{\epsilon_n}(x, q) = (n(x) \vee$

$\epsilon_n)|q|$. *Hence the approximating equation (32) is*

$$|Du| = n_{\epsilon_n n}(x)$$

where $n_{\epsilon_n n}(x) = (n(\cdot) \vee \epsilon_n) * \eta_n(x)$.

Theorem 21 *Let u_n be the sequence of solutions of (32)-(5). Then u_n converges uniformly in $\overline{\Omega}$ to u, where u is the unique solution of (1)-(5).*

PROOF. We use the semi-relaxed limit technique introduced by Barles and Perthame (see [1]) . We set

$$\begin{aligned} \liminf{}_* \; u_n(x) &= \inf\{\liminf_{n\to\infty} u_n(x_n) : \; x_n \to x, \; x_n \in \overline{\Omega}\}, \\ \limsup{}^* \; u_n(x) &= \sup\{\limsup_{n\to\infty} u_n(x_n) : \; x_n \to x, \; x_n \in \overline{\Omega}\}. \end{aligned}$$

Set $\mathcal{Z}_{\epsilon_n n}(x) = \{p \in \mathbb{R}^N : \; H_n(x,p) \leq 0\}$. By (11), it follows that $\mathcal{Z}_{\epsilon_n n}(x) \subset B(0,R)$, for any $x \in \Omega$, so $\|Du_n\|_\infty \leq R$ and the sequence u_n is uniformly Lipschitz continuous and also uniformly bounded in $\overline{\Omega}$. Hence any subsequences of $(u_n)_{n\in\mathbb{N}}$ converging toward $\limsup^* u_n$ and $\liminf_* u_n$ converge uniformly and $\limsup^* u_n$ and $\liminf_* u_n$ are bounded and Lipschitz continuous on $\overline{\Omega}$.

As in [3], we can show that $\limsup^* u_n$ is a subsolution of (1)-(5), i.e. it is a a.e. subsolution in Ω and satisfies the constraint given by φ on $K \cup \partial\Omega$.

We now prove that all the limits u of subsequences of u_n uniformly convergent are singular supersolutions of (1)-(5).

Assume either that $x_0 \in K$ and

$$u(x_0) < \varphi(x) \qquad \text{for any } x \in B_S(x_0) \tag{33}$$

or that $x_0 \in \Omega \setminus K$, otherwise the conclusion is obvious. Note that, by continuity of u and φ, inequality (33) holds on a neighborhood of $B_S(x_0)$. We assume by contradiction that there is a strict S-subtangent ϕ to u at x_0 which is also a strict viscosity subsolution of (1) in a neighborhood A of $B_S(x_0)$, i.e.

$$\begin{aligned} (u-\phi)(x) > (u-\phi)(x_0) \quad & \text{for any } x \in A \setminus B_S(x_0), \\ \rho(x, D\phi) \leq \theta \quad & \text{a.e. in } A \end{aligned} \tag{34}$$

for some $\theta \in (0,1)$.

Since $\mathcal{Z}_{\epsilon_n}(x) \supset \mathcal{Z}(x)$, for any $n \in \mathbb{N}$, then

$$\rho_n(x, D\phi) \le \theta \qquad \text{a.e. in } A$$

and therefore (see Prop. 5.1 and Lemma 6.7 in [3]) there exists $\theta_n > 0$ such that

$$\sup_{|q|=1} \{q \cdot D\phi - \sigma_{\epsilon_n}(x, q)\} \le -\theta_n \qquad \text{a.e. in } A\ . \tag{35}$$

Set $\phi_n = \phi * \eta_n$. Since $u_n - \phi_n$ converges uniformly to $u - \phi$, a standard argument in viscosity solution theory gives the existence of a sequence x_n of minimizer of $u_n - \phi_n$ verifying $S(x_0, x_n) \to 0$. By (33) if $x_n \in K$, we have that $u_n(x_n) < \varphi(x_n)$ for n sufficiently large.

It results by (35),

$$\begin{aligned} H_n(x, D\phi_n(x)) = \quad & \sup_{|q|=1}\{D\phi_n(x) \cdot q - \sigma_{\epsilon_n n}(x, q)\} = \\ & \sup_{|q|=1}\{(D\phi(\cdot) \cdot q - \sigma_{\epsilon_n}(\cdot, q)) * \eta_n\} \le -\theta_n \end{aligned}$$

for $x \in A' \subset A$ and n large. This contradicts u_n being a singular supersolution of (32) at x_n.

Since $\liminf_* u_n$ is a singular supersolution of (1)-(5), then by Theorem 18, we get

$$\limsup{}^* u_n \le \liminf_* u_n.$$

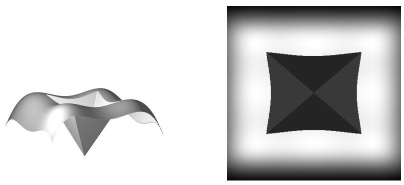
The reverse inequality is true by definition, thus we get that $\limsup^* u_n = \liminf_* u_n$ and the uniform convergence of the sequence u_n toward the solution (1)-(5).

Remark 22 *A more natural way to regularize a SfS equation would be to consider the same equation with an image brightness* $I_{\epsilon_n n}(x) = \min\{I(\cdot), 1 - \epsilon_n\} * \eta_n(x)$. *Unfortunately for the moment we are not able to prove the stability of singular solutions for this type of perturbation.*

6 Numerical experiments

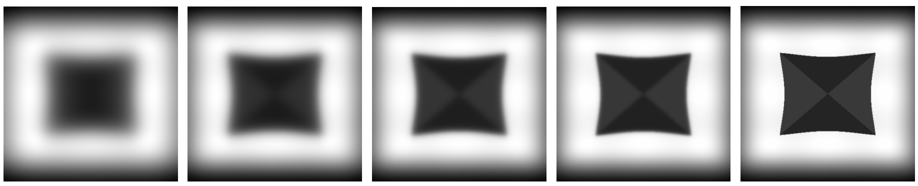
Here, we only consider the eikonal case and we deal with synthetic images. The numerical algorithm used in these experiments is the "Fast Marching Method" [17]. As the theory suggests [14], in our experiments we assume that we know

and we use the values of the solution (of the non-regularized equation) at all its local minima (which correspond to the maxima the following figures).
We are interested in testing the stability and the convergence of the regularization procedure, in particular when the image is discontinuous. In our experiments, the regularization is based on an isotropic Gaussian mollifier [19]. In Figure 1, we show in a) the considered (nonsmooth) original surface and in b) the image synthesized from a) by the eikonal process. The five images of Figure 2 are obtained from image Fig.1-b) after the regularization process associated with diffusion coefficients $\Sigma_1 = 0.4$, $\Sigma_2 = 2.0$, $\Sigma_3 = 6.4$, $\Sigma_4 = 12$, $\Sigma_5 = 24$. Figure 3 shows the reconstructions obtained from images of Figure 2. To better show the differences between these surfaces, we display vertical sections of them in Figures 4. In Figure 4, the black curves represent the sections of the computed approximations and the green curve is the section of the groundtruth. In this example, we see clearly the convergence of the computed solutions toward the original surface when the regularization (i.e. the Σ coefficient) vanishes.



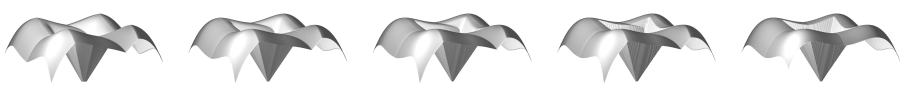
a) b)

Fig. 1. a) original surface (groundtruth); b) image synthesized from a).



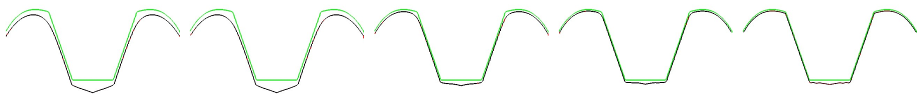
a) b) c) d) e)

Fig. 2. a)-e) image Fig.1-b) regularized with various Σ.



a) b) c) d) e)

Fig. 3. a)-e) surfaces reconstructed by the Fast Marching algorithm from the corresponding images of Fig.2.



a) b) c) d) e)

Fig. 4. a)-e) Vertical sections of the surfaces displayed in Figures 3.

7 Appendix

Proof of Prop 3 We first prove that the function

$$\overline{I}(x) := \operatorname{ess}\limsup_{y \longrightarrow x} I(y)$$

is a u.s.c. function in Ω. To see this, we take a sequence x_n converging to some x_0 and a positive ε, and we select r_ε such that

$$\operatorname{ess}\sup_{B(x_0,r_\varepsilon)} I \leq \operatorname{ess}\limsup_{y \longrightarrow x_0} I(y) + \varepsilon.$$

Since $B(x_n, r_\varepsilon/2) \subset B(x_0, r_\varepsilon)$, for n sufficiently large, then

$$\operatorname{ess}\limsup_{y \longrightarrow x_n} I(y) \leq \operatorname{ess}\sup_{B(x_n,r_\varepsilon/2)} I \leq \operatorname{ess}\sup_{B(x_0,r_\varepsilon)} I \leq \operatorname{ess}\limsup_{y \longrightarrow x_0} I(y) + \varepsilon$$

which proves the claim, since ε is arbitrary.

We proceed to show

$$\sup_\Omega \overline{I} = \operatorname{ess}\sup_\Omega I = 1 \tag{36}$$

We immediately see, since $\overline{I} \leq \operatorname{ess}\sup_\Omega I$ that $\leq$ holds in (36). For the converse, we prove the more general inequality

$$\sup_F \overline{I} \geq \operatorname{ess}\sup_F I \quad \text{for any measurable subset } F \text{ of } \Omega \tag{37}$$

If (37) were false, there would exist a constant m with

$$\sup_F \overline{I} < m \leq \operatorname{ess}\sup_F I \tag{38}$$

The inequality $I < m$ should consequently hold a.e. in some measurable subset G of F with positive measure. Let G^* be the measure theoretic interior of G

(see [7]) and take $x_0 \in G^* \cap F$ (one knows that $|G \setminus G^*| = 0$, see [7]). Then we should have $|G \cap B(x_0, r)| > 0$ for all $r > 0$, hence $m \leq \operatorname{ess\,sup}_{B(x_0,r)} I$, for all r, and passing to the limit for $r \longrightarrow 0$, $m \leq \overline{I}(x_0)$, in contrast with (38).

To prove ii), we put together (37), with $F = E$, the upper semicontinuity of $\overline{I}$ and the definition of $\mathcal{S}$, to get

$$\operatorname*{ess\,sup}_{E} I \leq \sup_{E} \overline{I} \leq \max_{\overline{E}} \overline{I} < \operatorname*{ess\,sup}_{\Omega} I = 1.$$

References

[1] M. Bardi and I. Capuzzo Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston, 1997.

[2] F.Camilli and A.Siconolfi, Maximal subsolutions for a class of degenerate Hamilton-Jacobi problems, Indiana Univ. Math. J. **48** (1999), 1111-1132.

[3] F.Camilli and A.Siconolfi, Hamilton–Jacobi equations with measurable dependence on the state variable, Adv. Differential Equations **8** (2003), 733-768.

[4] F.Camilli and A.Siconolfi, Effective Hamiltonian and homogenization of measurable Eikonal equations, preprint, 2005

[5] K.Deckelnick and C.M.Elliot, Uniqueness and error analysis for Hamilton-Jacobi equations with discontinuities, Interface Free Bound. **6** (2004), 329–349.

[6] P. Dupuis and J. Oliensis, An optimal control formulation and related numerical methods for a problem in shape reconstruction, Ann. Appl. Probab. , **4**, 287–346, 1994.

[7] L.C.Evans and R.F.Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992.

[8] M. Falcone and M. Sagona, An algorithm for the global solution of the Shape-from-Shading model, in A. Del Bimbo, ed., *"Image analysis and processing"*, 596–603, Lect. Notes in Computer Sciences 1310, Springer, 1997.

[9] A.Fathi and A.Siconolfi, PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians, Calc. Var **22** (2005), 185–228.

[10] H.Ishii, A simple, direct proof of uniqueness for solutions of Hamilton–Jacobi equations of Eikonal type, Proc. Am. Math. Soc. **100** (1987), 247–251.

[11] R. Kimmel, K. Siddiqi, B.B. Kimia and A. Bruckstein, Shape–from–Shading: level set propagation and viscosity solutions, Int. J. of Comp. Vision, **16** (1995), 107-133.

[12] Kain, J. and D.N. Ostrov , Numerical Shape-From-Shading for Discontinuous Photographic Images, Int. J. of Comp. Vision, **44** (2001), 163–173.

[13] P.L. Lions, E. Rouy, and A. Tourin. Shape-From-Shading, viscosity solutions and edges, Numer. Math. **64** (1993) 323–353.

[14] E. Prados, F.Camilli and O. Faugeras, A unifying and rigorous Shape From Shading method adapted to realistic data and applications, to appear on J. of Math. Imaging and Vision, Special Issue MIA'2004.

[15] E. Prados and O. Faugeras, A generic and provably convergent Shape-From-Shading Method for Orthographic and Pinhole Cameras, to appear on Int. J. of Computer Vision.

[16] E. Rouy and A. Tourin, A viscosity solutions approach to Shape-From-Shading. SIAM J. Numer. Anal. **29** (1992), 867–884.

[17] J.A. Sethian. *Level Set Methods and Fast Marching Methods.* Cambridge University Press, 1999.

[18] P.Soravia, Boundary value problems for Hamilton-Jacobi equations with discontinuous Lagrangian, Indiana Univ. Math. J. **51** (2002), 451–477.

[19] D. Tschumperlé. *PDE's Based Regularization of Multivalued Images and Applications.* PhD thesis, University of Nice-Sophia Antipolis, 2002.

[20] R. Zhang, P.-S. Tsai, J.-E. Cryer, and M. Shah, Shape From Shading: A survey, IEEE Trans. on Pattern Analysis and Machine Intelligence, **21** (1999), 690–706.