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The Navier-Stokes-Vlasov-Fokker-Planck system near equilibrium

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Abstract

This paper is concerned with a system that couples the incompressible Navier-Stokes equations to the Vlasov-Fokker-Planck equation. Such a system arises in the modeling of sprays, where a dense phase interacts with a disperse phase. The coupling arises from the Stokes drag force exerted by a phase on the other. We study the global-in-time existence of classical solutions for data close to an equilibrium. We investigate further regularity properties of the solutions as well as their long time behavior. The proofs use energy estimates and the hypoelliptic structure of the system.

1 Introduction

We are concerned with the following PDEs system

$$u_t + u\cdot\nabla_x u + \nabla_x p - \triangle_x u = \int_{\mathbf{R}^3} (v-u)F\,dv, \qquad (t,x)\in\mathbf{R}^+\times\mathbf{T}^3, \tag{1.1}$$

$$\nabla_x\cdot u = 0, \tag{1.2}$$

$$F_t + v\cdot\nabla_x F + \mathrm{div}_v\big((u-v)F - \nabla_v F\big) = 0, \qquad (t,x,v)\in\mathbf{R}^+\times\mathbf{T}^3\times\mathbf{R}^3\,. \tag{1.3}$$

The system is completed by the initial data:

$$u|_{t=0} = u_0, \qquad \nabla_x\cdot u_0 = 0, \qquad F|_{t=0} = F_0, \tag{1.4}$$

and we assume periodic boundary conditions with respect to the variable $x \in [-\pi, \pi]^3 = \mathbf{T}^3$. The system is intended to describe the interactions of particles —- droplets or bubbles —- with a viscous and incompressible fluid. The fluid is described by its velocity field $u(t,x) \in \mathbf{R}^3$, and its pressure $p(t,x)$, which are both function of the time variable $t \geq 0$ and the space variable $x = (x_1, x_2, x_3) \in \mathbf{T}^3$. The particles are described by their distribution function in phase space which depends additionally on the velocity variable $v = (v_1, v_2, v_3) \in \mathbf{R}^3$: at time t, $F(t,x,v)\,dv\,dx$ gives the number of particles having their position in the infinitesimal domain centered on x with volume dx with velocity in the domain centered on v with volume dv. It is assumed that the presence of particles does not affect the density of the fluid, supposed to be constant, and collisions between particles are neglected. The coupling between the two phases is only due to the drag force, which is proportional to the relative velocity $(u - v)$. Here we restrict to the simplest situation where the drag force is linear with respect to the relative velocity. This framework corresponds to the modeling of the so-called thin sprays at moderate Reynolds number. As a matter of fact, we observe that certain quantities are conserved or dissipated:

$$\text{Mass conservation:} \quad \frac{d}{dt}\int_{\mathbf{T}^3 \times \mathbf{R}^3} F\,dx\,dv = 0, \tag{1.5}$$

$$\text{Momentum conservation:} \quad \frac{d}{dt}\Big(\int_{\mathbf{T}^3} u\,dx + \int_{\mathbf{T}^3 \times \mathbf{R}^3} vF\,dx\,dv\Big) = 0, \tag{1.6}$$

$$\begin{aligned}\text{Energy/Entropy dissipation:} \quad & \frac{d}{dt}\int_{\mathbf{T}^3}\Big(\frac{|u|^2}{2} + \int_{\mathbf{R}^3}(F\ln F + \frac{|v|^2 + M_0}{2}F)\,dv\Big)\,dx \\ & + \int_{\mathbf{T}^3 \times \mathbf{R}^3} \frac{|(u-v)F - \nabla_v F|^2}{F}\,dx\,dv + \int_{\mathbf{T}^3} |\nabla_x u|^2\,dx = 0,\end{aligned} \tag{1.7}$$

with $M_0 \in \mathbf{R}$ any constant. Of course, the analysis of (1.1)-(1.4) utilizes strongly these remarkable properties.

We refer to [28] or [31] for an introduction to the physical background. In fact a large variety of models can be used for modeling sprays, depending on the physical properties of the flows: compressible or incompressible fluid, viscous or inviscid fluid equations (which might sound strange since the viscosity enters in the definition of the drag force, but it can be justified on scaling arguments), with or without thermal diffusion acting on the particles... Anyway, the mathematical analysis remains difficult since the systems always couple nonlinear evolution equations for unknowns that do not depend on the same set of variables. Concerning the system (1.1)-(1.4), the global existence of weak solutions without the Fokker-Planck term is due to [18], in the case without convection, revisited in [3]. The compressible case is investigated in [25]. Scaling and stability issues are discussed in [14, 15, 5, 26]. Another viewpoint consists in investigating the local in time well-posedness; we refer to [1, 24] for results in this spirit. We also mention the traveling wave analysis in [11, 12]. Here, we adopt a different strategy. We start by remarking that $u = 0$, $f = Me^{-\frac{|v|^2}{2}}$ with $M \geq 0$, is a (equilibrium) solution of (1.1)-(1.4). Then we are interested in solutions which are perturbations of the equilibrium state. To be more specific, without loss of generality, we consider the normalized Maxwellian

$$\mu(v) = \frac{1}{(2\pi)^{3/2} \mid \mathbf{T} \mid^3} e^{-v^2/2}$$

and we look at solutions of (1.3) which read

$$F = \mu + \sqrt{\mu} f. \tag{1.8}$$

Plugging (1.8) into (1.1), we obtain the following new system for (u, f):

$$u_t + u \cdot \nabla_x u + \nabla_x p - \triangle_x u + u + u \int_{\mathbf{R}^3} \sqrt{\mu} f \, dv - \int_{\mathbf{R}^3} v \sqrt{\mu} f \, dv = 0, \tag{1.9}$$

$$\nabla_x \cdot u = 0, \tag{1.10}$$

$$f_t + v \cdot \nabla_x f + u \cdot (\nabla_v f - \frac{v}{2} f) - u \cdot v \sqrt{\mu} = -\frac{|v|^2}{4} f + \frac{3}{2} f + \triangle_v f. \tag{1.11}$$

In what follows, we shall consider the global existence of classical small solutions to (1.9)–(1.11) together with the initial datum

$$u|_{t=0} = u_0, \qquad f|_{t=0} = f_0, \tag{1.12}$$

which is requested to satisfy

$$\int_{\mathbf{T}^3} u_0 \, dx + \int_{\mathbf{T}^3 \times \mathbf{R}^3} v \sqrt{\mu} f_0 \, dv \, dx = 0, \quad \text{and} \quad \nabla \cdot u_0 = 0. \tag{1.13}$$

This assumption will be crucial to the analysis. According to (1.6), it means that the perturbation has a vanishing momentum since we have

$$\frac{d}{dt} \left(\int_{\mathbf{T}^3} u \, dx + \int_{\mathbf{T}^3 \times \mathbf{R}^3} v \sqrt{\mu} f \, dv \, dx \right) = 0.$$

Similarly, if we assume further

$$\int_{\mathbf{T}^3 \times \mathbf{R}^3} \sqrt{\mu} \; f_0 \, dv \, dx = 0, \tag{1.14}$$

then the perturbation does not affect the global mass.

The use of fine energy estimates will lead to the global existence of smooth solutions, at the price of a smallness condition on the perturbation. This approach is in the spirit of the striking results [16, 17] for the Boltzmann and Landau equations. We also mention the analysis of viscoelastic flows and polymeric fluids [20, 21, 22, 23]. We address three questions: firstly, the global existence of a smooth solution, small perturbation of an equilibrium; secondly, we discuss the asymptotic trend to the equilibrium, with an exponential rate and thirdly, we investigate further regularity issues. The paper is organized as follows. In section 2 we set up the needed notation and we give the statements of our main results. Section 3 is devoted to the existence theory. Section 4 deals with the large time behavior. The analysis relies on the dissipative properties of the system, or more precisely on its hypocoercive structure, which allows to appeal to the strategy detailed in [30]. This strategy has already been applied successfully to many situations, see e. g. [10, 27]. Eventually, we discuss in Section 5 the smoothing effect of the system. This Section is based on hypoellipticity arguments, according to methods presented in [2] and further developed for many applications in plasmas physics [6, 7, 8, 19].

2 Notation and statements of the main results

We start by introducing the notation that will be used throughout the paper. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{N}^3$ be a multi-index. The length of the multi-index is defined by $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We denote by ∂^α the corresponding space derivative

$$\partial^\alpha = \partial_{x_1}^{\alpha_1}\, \partial_{x_2}^{\alpha_2}\, \partial_{x_3}^{\alpha_3}.$$

Similarly, for the velocity variable, we denote

$$\partial_v^\alpha = \partial_{v_1}^{\alpha_1}\, \partial_{v_2}^{\alpha_2}\, \partial_{v_3}^{\alpha_3}.$$

Given two muti-indices α and β, with $\beta_i \leq \alpha_i$, we denote

$$\left(\begin{array}{c} \alpha \\ \beta \end{array} \right) = \prod_{i=1}^{3} \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!}.$$

The same notation $\langle \cdot, \cdot \rangle$ stands for the standard L^2 inner product on $\mathbf{R}^3$ or on $\mathbf{T}^3 \times \mathbf{R}^3$:

$$\langle f, g \rangle = \int_{\mathbf{T}^3} f g \, dx \qquad \text{or} \qquad \langle f, g \rangle = \int_{\mathbf{T}^3 \times \mathbf{R}^3} f g \, dv \, dx.$$

We denote by $\| \cdot \|_{L^2}$ the corresponding norms. Equally, given $s \in \mathbf{N}$, $\| \cdot \|_{H^s}$ represents the usual Sobolev norm either on $\mathbf{R}^3$ or on $\mathbf{T}^3 \times \mathbf{R}^3$, based on the L^2 norm of all derivatives, with respect to all variables, up to order s. For a function $\phi : \mathbf{T}^3 \times \mathbf{R}^3 \to \mathbf{R}$, we shall need the partial Sobolev norm

$$|\phi|_s = \left\{ \int_{\mathbf{T}^3 \times \mathbf{R}^3} \sum_{|\alpha| \leq s} |\partial^\alpha \phi|^2 \, dx \, dv \right\}^{\frac{1}{2}}.$$

Eventually, we use the convention that the same letter C represents constants the value of which might vary from a line to another, but bearing in mind it is uniform with respect to the data.

Coming back to (1.9)-(1.11), it is convenient to define the mean fluid velocity

$$\bar{u}(t) \stackrel{\text{def}}{=} \frac{1}{|\,\mathbf{T}\,|^3} \int_{\mathbf{T}^3} u(t, x) \, dx.$$

Averaging (1.9), we check that

$$\bar{u}_t + \bar{u} + \frac{1}{|\,\mathbf{T}^3\,|} \int_{\mathbf{T}^3} u \int_{\mathbf{R}^3} \sqrt{\mu} f \; dv \; dx - \frac{1}{|\,\mathbf{T}^3\,|} \int_{\mathbf{T}^3 \times \mathbf{R}^3} v \sqrt{\mu} f \; dv \; dx = 0.$$

However, the momentum conservation (1.6) together with the condition (1.13) imply that

$$- \int_{\mathbf{T}^3 \times \mathbf{R}^3} v \sqrt{\mu} f \; dv \; dx = \int_{\mathbf{T}^3} u \; dx.$$

Hence the evolution equation for $\bar{u}$ recasts as

$$\bar{u}_t + 2\bar{u} + \frac{1}{|\,\mathbf{T}^3\,|}\int_{\mathbf{T}^3} u \int_{\mathbf{R}^3} \sqrt{\mu} f \; dv \; dx = 0. \tag{2.15}$$

We are now ready to state our main results. We begin with the global existence of classical solutions for small initial data.

Theorem 2.1 *Let $s \geq 2$ be an integer. Let (u_0, f_0) satisfy* (1.13). *Then, there exists a sufficiently small constant ε such that if*

$$\|u_0\|_{H^s}^2 + |f_0|_s^2 \leq \varepsilon, \tag{2.16}$$

holds, then (1.1)-(1.4) *has a unique global classical solution (u, F) with $F = \mu + \sqrt{\mu} f \geq 0$ satisfying*

$$\sup_{t\geq 0}(\|u(t)\|_{H^s}^2 + |f(t)|_s^2) + \int_0^t \left[|\bar{u}|^2 + \|\nabla_x u\|_{H^s}^2 + \left|\sqrt{\mu} u - (\nabla_v f + \frac{v}{2} f)\right|_s^2 \right] d\tau \leq C\varepsilon. \tag{2.17}$$

Furthermore, if (2.16) *holds with $s \geq 3$, then for any positive time $t \geq t_0 > 0$, we have*

$$\begin{aligned} &\sup_{t\geq t_0}\left(\|u(t)\|_{H^s}^2 + \|f(t)\|_{H^s}^2 + \|vf(t)\|_{H^{s-1}}^2\right) \\ &+ \int_t^{t+1} \left[\left\|\nabla_v f + \frac{v}{2} f\right\|_{H^s}^2 + \left\|v\otimes\nabla_v f + \frac{v\otimes v}{2} f\right\|_{H^{s-1}}^2 \right] d\tau \leq C(t_0, \varepsilon), \end{aligned} \tag{2.18}$$

where $C(t_0, \varepsilon)$ blows up as t_0 goes to 0.

The second result is concerned with the large time convergence to equilibrium.

Theorem 2.2 *There exists $\varepsilon_0 > 0$ and $\lambda > 0$ such that for any initial data verifying* (1.13), (1.14) *and* (2.16) *with $s \geq 3$ and $0 < \varepsilon < \varepsilon_0$, the following estimate of exponential convergence holds:*

$$\|u(t)\|_{L^2}^2 + \|f(t)\|_{L^2}^2 \leq C(t_0, \varepsilon) e^{-\lambda t},$$

for any $t \geq t_0 > 0$, with a positive constant $C(t_0, \varepsilon)$.

Estimate (2.18) already indicates that the system has an instantaneous smoothing effect. A statement with a slightly different flavor can be obtained, by reasoning directly on the original system (1.1)-(1.3) and by exploiting the hypoelliptic structure of the Fokker-planck equation, in the spirit of [2].

Theorem 2.3 *Let (u, F) be a (local) solution to* (1.1)-(1.4). *If the initial data u_0 and F_0 satisfy for some integers $n \geq 3$ and $q \geq 2$ the following estimate*

$$\|u_0\|_{H^q}^2 + |F_0\langle v\rangle^n|_q^2 < \infty,$$

where $\langle v\rangle = (1 + |v|^2)^{\frac{1}{2}}$, then, there exists a time $T > 0$ such that for all $0 < t_\star < T_\star < T$, the solution (u, F) of (1.1)-(1.4) *satisfies:*

$$\begin{aligned} \sup_{t_\star\leq t\leq T_\star} &(\|u(t)\|_{H^q}^2 + \|F(t)\langle v\rangle^{n-3}\|_{H^q}^2) + \int_{t_\star}^{T_\star} \left[\|u\|_{H^{q+1}}^2 + \|F\langle v\rangle^{n-3}\|_{H^{q+1}}^2\right] d\tau \\ &\leq C(T, \|u_0\|_{H^q}, |F_0\langle v\rangle^n|_q) < \infty. \end{aligned} \tag{2.19}$$

Remark 2.1 *This statement can be applied to the solutions obtained in Theorem 2.1 and it shows that these solutions become immediately smooth with respect to all variables. Indeed,* (2.17) *implies that, for $s \geq 3$ and for any integer n and some $T > 0$,*

$$\sup_{0\leq t\leq T} (\|u(t)\|_{H^s}^2 + |F(t)\langle v\rangle^n|_s^2) \leq C(1+\varepsilon).$$

Thus, (2.19) *implies the smoothness of the solution for any positive time.*

3 Global existence and regularity theory

In this section we present the proof of Theorem 2.1. It uses the conservation and dissipation properties (1.5)-(1.7). The proof splits into two parts. Firstly, we detail the derivation of (2.17) which is the key estimate for justifying the global existence of solutions. Secondly, we prove the strengthened regularity estimate (2.18).

3.1 Existence of global solutions to (1.9)-(1.12)

It is well-known that the existence of solutions to a nonlinear PDE can be obtained by constructing solutions to approximated problems and proving estimates which are uniform with respect to the approximation parameter. For (1.9)-(1.13), one can construct such approximate solutions via Galerkin's approximation, like in [22]. For simplicity, we do not detail this part. Instead, we assume that there is a positive time T such that (1.9)-(1.12) has a unique smooth enough solution on $[0,T]$, and we shall present the *a priori* estimates for such solutions. Obtaining estimates uniform with respect to T allows to construct global solutions. The main issue is summarized in the following claim.

Proposition 3.1 *Let $s \geq 2$. Let (u,f) be a solution of* (1.9)-(1.13)*. We have*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|u\|_{H^s}^2 + |f|_s^2 + |\bar{u}|^2\big) + \|\nabla u\|_{H^s}^2 + 2|\bar{u}|^2 + \Big|u\sqrt{\mu} - \nabla_v f - \frac{v}{2}f\Big|_s^2 \\ \leq C(|f|_s + \|u\|_{H^s})\,\Big(\|\nabla u\|_{H^s}^2 + |\bar{u}|^2 + \Big|u\sqrt{\mu} - \nabla_v f - \frac{v}{2}f\Big|_s^2\Big).\end{aligned} \tag{3.20}$$

Proof. The proof is based on energy estimates. Let $s \geq 2$ be a positive integer and let $\alpha \in \mathbf{N}^3$ with $|\alpha| \leq s$. We first apply ∂^α to (1.9), then multiply the resulting equation by $\partial^\alpha u$, and integrate over $\mathbf{T}^3$. We get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u\|_{L^2}^2 + \langle \partial^\alpha(u\cdot\nabla u), \partial^\alpha u\rangle + \|\nabla\partial^\alpha u\|_{L^2}^2 + \|\partial^\alpha u\|_{L^2}^2 \\ +\Big\langle \partial^\alpha\Big(u\int_{\mathbf{R}^3}\sqrt{\mu}f\,dv\Big), \partial^\alpha u\Big\rangle - \Big\langle \int_{\mathbf{R}^3} v\sqrt{\mu}\partial^\alpha f\,dv, \partial^\alpha u\Big\rangle = 0.\end{aligned} \tag{3.21}$$

Similar arguments applied to (1.11) yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|_{L^2}^2 + \Big\langle \partial^\alpha\Big(u\cdot\big(\nabla_v f - \frac{v}{2}f\big)\Big), \partial^\alpha f\Big\rangle - \langle \partial^\alpha u\cdot v\sqrt{\mu}, \partial^\alpha f\rangle \\ = -\Big\|\nabla_v\partial^\alpha f + \frac{v}{2}\partial^\alpha f\Big\|_{L^2}^2.\end{aligned} \tag{3.22}$$

Note that $\nabla\cdot u = 0$, and thus integrating by parts we have $\langle u\cdot\nabla_x u, u\rangle = 0$. Since $s \geq 2 > 3/2$, we can estimate as follows

$$\left|\left\langle \partial^\alpha(u\cdot\nabla_x u), \partial^\alpha u\right\rangle\right| = \left|\left\langle \partial^\alpha(u\cdot\nabla u) - u\cdot\partial^\alpha\nabla_x u, \partial^\alpha u\right\rangle\right| \leq C\|u\|_{H^s}\|\nabla u\|_{H^s}^2.$$

Furthermore, since μ is normalized and satisfies $\nabla_v\sqrt{\mu} = -\frac{v}{2}\sqrt{\mu}$, by using an integration by parts we obtain

$$\begin{aligned}&\|\partial^\alpha u\|_{L^2}^2 - 2\langle\partial^\alpha u\cdot v\sqrt{\mu}, \partial^\alpha f\rangle + \left\|\nabla_v\partial^\alpha f + \frac{v}{2}\partial^\alpha f\right\|_{L^2}^2\\ &= \left\|\partial^\alpha u\sqrt{\mu} - \nabla_v\partial^\alpha f - \frac{v}{2}\partial^\alpha f\right\|_{L^2}^2.\end{aligned}$$

We can also write

$$\begin{aligned}&\left\langle\partial^\alpha\Big(u\int_{\mathbf{R}^3}\sqrt{\mu}f\,dv\Big), \partial^\alpha u\right\rangle + \left\langle\partial^\alpha\Big(u\cdot\big(\nabla_v f - \frac{v}{2}f\big)\Big), \partial^\alpha f\right\rangle\\ &= \left\langle\partial^\alpha(uf), \partial^\alpha u\sqrt{\mu} - \nabla_v\partial^\alpha f - \frac{v}{2}\partial^\alpha f\right\rangle.\end{aligned}$$

Then owing to (3.21) and (3.22), we are led to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\partial^\alpha u\|_{L^2}^2 + \|\partial^\alpha f\|_{L^2}^2\big) + \|\nabla\partial^\alpha u\|_{L^2}^2 + \left\|\partial^\alpha u\sqrt{\mu} - \nabla_v\partial^\alpha f - \frac{v}{2}\partial^\alpha f\right\|_{L^2}^2\\ &\quad\leq C\|u\|_{H^s}\|\nabla_x u\|_{H^s}^2 + \left|\left\langle\partial^\alpha(uf), \partial^\alpha u\sqrt{\mu} - \nabla_v\partial^\alpha f - \frac{v}{2}\partial^\alpha f\right\rangle\right|.\end{aligned}\tag{3.23}$$

Still using $s \geq 2$, we get (see Lemma 3.1 of [23] for a similar estimate)

$$\|\partial^\alpha(uf)\|_{L^2} \leq |uf|_s \leq C\|u\|_{H^s}|f|_s \leq C(\|u\|_{L^2} + \|\nabla_x u\|_{H^{s-1}})|f|_s.$$

Now, we make use of the mean velocity $\bar{u}$. By the Poincaré-Wirtinger inequality, there exists a constant C_P such that

$$\|u\|_{L^2} \leq \|u-\bar{u}\|_{L^2} + \sqrt{|\,\mathbf{T}^3\,|}\;|\bar{u}| \leq C_P(\|\nabla_x u\|_{L^2} + |\bar{u}|) \leq C_P(\|\nabla_x u\|_{H^{s-1}} + |\bar{u}|),$$

since $s-1 \geq 0$. Hence, we obtain

$$\|\partial^\alpha(uf)\|_{L^2} \leq C(\|\nabla_x u\|_{H^{s-1}} + |\bar{u}|)|f|_s.$$

Summing over α, (3.23) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|_{H^s}^2 + |f|_s^2) + \|\nabla u\|_{H^s}^2 + \left|u\sqrt{\mu} - \nabla_v f - \frac{v}{2}f\right|_s^2\\ &\quad\leq C(|f|_s + \|u\|_{H^s})\Big(\|\nabla u\|_{H^s}^2 + |\bar{u}|^2 + \left|u\sqrt{\mu} - \nabla_v f - \frac{v}{2}f\right|_s^2\Big).\end{aligned}\tag{3.24}$$

It remains to derive an estimate for the mean fluid velocity. To this end, we go back to (2.15). We deduce the following estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\bar{u}|^2 + 2|\bar{u}|^2 &= -\frac{\bar{u}}{|\,\mathbf{T}^3\,|}\cdot\int_{\mathbf{T}^3}u\int_{\mathbf{R}^3}\sqrt{\mu}f\;dv\;dx\\ &\leq \frac{1}{|\,\mathbf{T}^3\,|}\|u\|_{L^2}\|f\|_{L^2}|\bar{u}|\\ &\leq C\|f\|_{L^2}(\|\nabla_x u\|_{L^2}^2 + |\bar{u}|^2),\end{aligned}\tag{3.25}$$

where the last line follows from the Poincaré-Wirtinger inequality. Combining (3.24) with (3.25), we obtain (3.20). This completes the proof of Proposition 3.1. ■

Having disposed of this preliminary step, we are in position to present the proof to the existence part of Theorem 2.1.

Proof of Theorem 2.1 (Existence and estimate (2.17)). According to what is stated at the beginning of this subsection, the crucial point consists in proving (2.17). Indeed, this estimate provides all the necessary compactness on the sequence of approximations which allows to pass to the limit in the equations, and thus we obtain a solution which still satisfies (2.17). We skip the discussion of this point and switch to the proof of (2.17), assuming a smallness condition on the initial data. Clearly (2.16) combined to the Sobolev imbedding $H^2(\mathbf{T}^3) \subset L^1(\mathbf{T}^3)$ yields

$$\|u_0\|_{H^s}^2 + |\bar{u}_0|^2 + |f_0|_s^2 \leq C_0\varepsilon, \tag{3.26}$$

for some $C_0 > 0$. Then using continuity with respect to time, we define

$$T^* \stackrel{\text{def}}{=} \sup\Big\{\tilde{T} \geq 0 : \sup_{0\leq t<\tilde{T}} \big(\|u(t)\|_{H^s}^2 + |\bar{u}|^2 + |f(t)|_s^2\big) \leq 2C_0\varepsilon\Big\}. \tag{3.27}$$

However, (3.20) can be recast as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|u\|_{H^s}^2 + |f|_s^2 + |\bar{u}|^2\big)\\ &\qquad +\big(1 - C\ (|f|_s + \|u\|_{H^s})\big)\Big(\|\nabla u\|_{H^s}^2 + 2|\bar{u}|^2 + \big|u\sqrt{\mu} - \nabla_v f - \tfrac{v}{2}f\big|_s^2\Big) \leq 0.\end{aligned}$$

Let us fix ε such that $0 < \varepsilon < 1/(2C_0C)$. Hence, on $0 \leq t \leq T^*$ we have $1-C\ (|f|_s+\|u\|_{H^s}) > 0$ and it follows that

$$\|u(t)\|_{H^s}^2 + |f(t)|_s^2 + |\bar{u}(t)|^2 \leq \|u_0\|_{H^s}^2 + |\bar{u}_0|^2 + |f_0|_s^2 \leq C_0\varepsilon,$$

holds on $0 \leq t \leq T^*$. It prevents T^* for being finite. Thus, the proof to the existence part of Theorem 2.1 is complete. ■

3.2 Estimates of full Sobolev norms

Up to now, we have only obtained a partial regularity for the particles distribution function, since only space derivatives are involved in the norm $|f|_s$. We wish to strengthen the regularity analysis, showing that for positive time, the regularity of f with respect to the space variables can be transferred to the velocity variables. We make use of the nice structure of the Fokker-Planck operator, which allows to avoid any estimates on the moment of the distribution, like it is done in [21] for the Dumbbell model. We start by introducing convenient functions spaces and justify a useful statement on real variable functions.

Definition 3.1 *We consider the cone of non negative continuous functions*

$$C_+(\mathbf{R}_+) := \Big\{f \in C(\mathbf{R}_+) :\ f \geq 0\Big\}.$$

Then, for any $K, r > 0$, we set

$$\mathscr{B}(K,r) := \left\{ f \in C_+(\mathbf{R}_+) : \int_t^{t+1} f(\tau)\, d\tau \le K, \, \forall\, t \ge r \right\}.$$

Lemma 3.1 *Let $\zeta(t), \xi(t) \in \mathscr{B}(K,r)$, and $\eta(t) \in C_+(\mathbf{R}_+)$ satisfy*

$$\zeta'(t) + \eta(t) \le K(1 + \xi(t)). \tag{3.28}$$

Then for all $t_0 > r$, there exists a constant $\tilde{K}$ depending on t_0, r and K, such that

$$\sup_{t \ge t_0} \zeta(t) \quad \le \quad \tilde{K} \qquad \text{and} \qquad \eta(t) \in \mathscr{B}(\tilde{K}, t_0).$$

Proof. We first prove the uniform estimate on ζ. Let n be the smallest integer such that $t_0 < r + n$. Since $\zeta \in \mathscr{B}(K,r)$, we have

$$\int_r^{t_0} \zeta(\tau)\, d\tau \le \int_r^{r+n} \zeta(\tau)\, d\tau \le nK.$$

Hence the mean value theorem allows us to find some $\tau_0 \in]r, t_0[$ such that:

$$\zeta(\tau_0) \le \frac{nK}{t_0 - r}.$$

Then for $t \in [t_0, \tau_0 + n + 1]$ we have

$$\begin{aligned} \zeta(t) &= \zeta(\tau_0) + \int_{\tau_0}^{t} \zeta'(\tau)\, d\tau \\ &\le \zeta(\tau_0) + \int_{\tau_0}^{\tau_0+n+1} K(1 + \xi(\tau))\, d\tau \\ &\le \frac{nK}{t_0 - r} + (n+1)(K + K^2), \end{aligned}$$

by using (3.28) and the fact that ξ is non negative.

For $t \ge \tau_0 + n + 1 > r + 1$, since $\zeta \in \mathscr{B}(K,r)$ we have

$$\int_{t-1}^{t} \zeta(\tau)\, d\tau \le K.$$

Thus there exists $\tilde{t} \in [t-1, t[$ such that $\zeta(\tilde{t}) \le K$. Integration of (3.28) over $[\tilde{t}, t]$ yields

$$\begin{aligned} \zeta(t) &\le \zeta(\tilde{t}) + K \int_{\tilde{t}}^{t} (1 + \xi(\tau))\, d\tau \\ &\le K + K \int_{t-1}^{t} (1 + \xi(\tau))\, d\tau \\ &\le K + K(1 + K), \end{aligned}$$

where we used the definition of $\tilde{t}$ and again the fact that $\xi \in \mathscr{B}(K,r)$ and $\eta \ge 0$.

Summarizing the obtained estimates, we get

$$\sup_{\tau\geq t_0}\zeta(\tau)\leq \max\left(2K+K^2,\frac{nK}{t_0-r}+(n+1)(K+K^2)\right):=K_1.$$

Finally, let us integrate (3.28) over $[t,t+1]$. It follows that, for any $t\geq t_0$,

$$\int_t^{t+1}\eta(\tau)\,d\tau\leq \zeta(t)+K\left[1+\int_t^{t+1}\xi(\tau)\,d\tau\right]\leq K_1+K(1+K):=\tilde{K}.$$

This concludes the proof of the Lemma 3.1. ∎

Lemma 3.1 will be useful for proving the transfer of regularity. The argument is based on an induction reasoning.

Lemma 3.2 *Let $s\geq 3$. Let (u,f) be a solution to* (1.9)-(1.11)*, satisfying for any $t\geq 0$*

$$\sup_{t\geq 0}(\|u(t)\|_{H^s}^2+|f(t)|_s^2)+\int_t^{t+1}\left|\nabla_v f+\frac{v}{2}f\right|_s^2 d\tau\leq A. \tag{3.29}$$

Then for any $t_0>0$ and $t\geq t_0$, there holds:

$$\sup_{t\geq t_0}(|\partial_{v_i}f|_{s-1}^2+|v_i f|_{s-1}^2)+\int_t^{t+1}\left[\left|\nabla_v\partial_{v_i}f+\frac{v}{2}\partial_{v_i}f\right|_{s-1}^2 +\left|v\otimes\nabla_v f+\frac{v\otimes v}{2}f\right|_{s-1}^2\right]d\tau\leq C(t_0,A). \tag{3.30}$$

Proof. Let $h\stackrel{\text{def}}{=}\nabla_v f+\frac{v}{2}f$. Then a simple calculation shows that

$$\partial_t h+v\cdot\nabla_x h+u\cdot(\nabla_v h-\frac{v}{2}h)-uf-u\sqrt{\mu}=-\frac{|v|^2}{4}h+\frac{3}{2}h+\triangle_v h-h-\nabla_x f.$$

Let $|\alpha|\leq s-1$. We remark that

$$\begin{aligned}&\int_{\mathbf{R}^3}\partial^\alpha\Big(-\frac{|v|^2}{4}h+\frac{3}{2}h+\triangle_v h\Big)\,\partial^\alpha h\,dv=\int_{\mathbf{R}^3}\Big(|\partial^\alpha\nabla_v h|^2-\frac{v^2}{4}|\partial^\alpha h|^2+\frac{3}{2}|\partial^\alpha h|^2\Big)\,dv\\ &=\int_{\mathbf{R}^3}\Big(|\partial^\alpha\nabla_v h|^2-\frac{v^2}{4}|\partial^\alpha h|^2+\frac{1}{2}\nabla_v\cdot v|\partial^\alpha h|^2\Big)\,dv\\ &=\int_{\mathbf{R}^3}\Big(\nabla_v\partial^\alpha\ h+\frac{v}{2}\partial^\alpha h\Big)^2\,dv.\end{aligned}$$

Thus, the following energy estimate holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha h\|_{L^2}^2+\big\|\partial^\alpha[\nabla_v h+\frac{v}{2}h]\big\|_{L^2}^2+\|\partial^\alpha h\|_{L^2}^2\\ &=-\Big\langle\partial^\alpha\Big[u\cdot\big(\nabla_v h-\frac{v}{2}h\big)\Big],\partial^\alpha h\Big\rangle+\big\langle\partial^\alpha[uf],\partial^\alpha h\big\rangle+\big\langle\partial^\alpha u\sqrt{\mu},\partial^\alpha h\big\rangle-\big\langle\partial^\alpha\nabla_x f,\partial^\alpha h\big\rangle.\end{aligned}$$

The third and the fourth term in the right hand side can be estimated by $C|h|_{s-1}\|u\|_{H^{s-1}}$ and $C|f|_s\|u\|_{H^{s-1}}$ respectively. Moreover, we observe that

$$\begin{aligned}\Big|\Big\langle\partial^\alpha\big[u\cdot\big(\nabla_v h-\frac{v}{2}h\big)\big],\partial^\alpha h\Big\rangle\Big|&=\Big|\Big\langle\partial^\alpha(uh),\partial^\alpha\big(\nabla_v h+\frac{v}{2}h\big)\Big\rangle\Big|\\ &\leq C\|u\|_{H^{s-1}}\big|\nabla_v h+\frac{v}{2}h\big|_{s-1}|h|_{s-1}\\ &\leq C\|u\|_{H^{s-1}}^2|h|_{s-1}^2+\frac{1}{2}\big|\nabla_v h+\frac{v}{2}h\big|_{s-1}^2,\end{aligned}$$

and, since $s-1 \geq 3/2$,

$$\left|\left\langle \partial^\alpha(uf), \partial^\alpha h\right\rangle\right| \leq C\|u\|_{H^{s-1}}|f|_{s-1}|h|_{s-1} \leq C\|u\|_{H^{s-1}}\left(|f|_{s-1}^2 + |h|_{s-1}^2\right).$$

Therefore, by using (3.29), we obtain

$$\frac{d}{dt}|h|_{s-1}^2 + \left|\nabla_v h + \frac{v}{2}h\right|_{s-1}^2 + |h|_{s-1}^2 \leq C_A(1 + |h|_{s-1}^2).$$

We apply lemma 3.1 with $\zeta = \xi = |h|_{s-1}^2$ which belongs to $\mathscr{B}(A,0)$ by virtue of (3.29), and $\eta = \left|\nabla_v h + \frac{v}{2}h\right|_{s-1}^2$. We get for $t \geq t_0 > 0$

$$\sup_{t\geq t_0}|h(t)|_{s-1}^2 + \int_t^{t+1} |\nabla_v h + \frac{v}{2}h|_{s-1}^2\, d\tau \leq C(t_0, A). \tag{3.31}$$

Now we make use of the following formulae

$$\begin{aligned}
&|h|_{s-1}^2 = |\nabla_v f|_{s-1}^2 + |\frac{v}{2}f|_{s-1}^2 - \frac{3}{2}|f|_{s-1}^2,\\
&\left|\nabla_v h + \frac{v}{2}h\right|_{s-1}^2 = |\nabla_v h|_{s-1}^2 + |\frac{v}{2}h|_{s-1}^2 - \frac{3}{2}|h|_{s-1}^2,\\
&\partial_{v_j}h_i = \partial_{v_i}(\partial_{v_j}f) + \frac{v_i}{2}(\partial_{v_j}f) + \delta_{ij}\frac{f}{2},\\
&\frac{v}{2}h_i = \frac{v}{2}\partial_{v_i}f + \frac{v}{2}\frac{v_i}{2}f,
\end{aligned}$$

consequently, (3.31) leads to

$$\begin{aligned}
\sup_{t\geq t_0}\left(|\nabla_v f|_{s-1}^2 + \left|\frac{v}{2}f\right|_{s-1}^2\right) &\leq \sup_{t\geq t_0}|h|_{s-1}^2 + \frac{3}{2}\sup_{t\geq t_0}|f|_{s-1}^2\\
&\leq C(t_0,A) + 3A,
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^{t+1}\sum_{i,j=1}^{3}\left[\left|\partial_{v_i}(\partial_{v_j}f) + \frac{v_i}{2}(\partial_{v_j}f)\right|_{s-1}^2 + \left|v_j\partial_{v_i}f + v_j\frac{v_i}{2}f\right|_{s-1}^2\right] d\tau\\
&\leq 4\int_t^{t+1}\left[|\nabla_v h|_{s-1}^2 + \left|\frac{v}{2}h\right|_{s-1}^2 + \frac{3}{2}|f|_{s-1}^2\right] d\tau\\
&\leq 4\int_t^{t+1}\left[\left|\nabla_v h + \frac{v}{2}h\right|_{s-1}^2 + 3|f|_{s-1}^2\right] d\tau \leq 4C(t_0,A) + 12A.
\end{aligned}$$

It completes the proof Lemma 3.2. ∎

Thanks to Lemma 3.2, we now can present the estimate for mixed derivatives of f via an inductive argument.

Lemma 3.3 *Under the assumptions of Lemma 3.2, for any $t \geq t_0 > 0$ we have*

$$\begin{aligned}
&\sup_{t\geq t_0}\left(\|u(t)\|_{H^s}^2 + \|f(t)\|_{H^s}^2 + \|vf(t)\|_{s-1}^2\right)\\
&+\int_t^{t+1}\left[\left\|\nabla_v f + \frac{v}{2}f\right\|_{H^s}^2 + \left\|v\otimes\nabla_v f + \frac{v\otimes v}{2}f\right\|_{s-1}^2\right] d\tau \leq C(t_0,A).
\end{aligned} \tag{3.32}$$

Proof. We wish to estimate the mixed derivatives $\partial_x^\alpha \partial_v^\beta f$ with $|\alpha| + |\beta| \le s$. Lemma 3.2 already tells us that for any $|\alpha| + |\beta| \le s$ with $|\beta| \le 1$ we have

$$\sup_{t \ge t_0 > 0} \|\partial_x^\alpha \partial_v^\beta f\|_{L^2}^2 + \int_t^{t+1} \left\|\nabla_v(\partial_x^\alpha \partial_v^\beta f) + \frac{v}{2}\partial_x^\alpha \partial_v^\beta f\right\|_{L^2}^2 d\tau \le C(t_0, A). \tag{3.33}$$

For $N \in \{1, ..., s\}$, we define $P(N)$ as the following property:

For all $t_1 > 0$ there exists a constant $C(t_1, A)$ such that:

1. For all multi-indices α and β such that $|\alpha| + |\beta| \le s,\ 0 \le |\beta| \le N < s$,

$$\sup_{t \ge t_1 > 0} \|\partial_x^\alpha \partial_v^\beta f\|_{L^2}^2 + \int_t^{t+1} \left\|\nabla_v(\partial_x^\alpha \partial_v^\beta f) + \frac{v}{2}\partial_x^\alpha \partial_v^\beta f\right\|_{L^2}^2 d\tau \le C(t_1, A). \tag{3.34}$$

2. For all multi-indices α and β such that $|\alpha| + |\beta| \le s - 1,\ \ 0 \le |\beta| \le N - 1 < s$ and $t \ge t_1 > 0$,

$$\sup_{t \ge t_1 > 0} \|v\partial_x^\alpha \partial_v^\beta f\|_{L^2}^2 + \int_t^{t+1} \left\|v \otimes \nabla_v(\partial_x^\alpha \partial_v^\beta f) + \frac{v \otimes v}{2}\partial_x^\alpha \partial_v^\beta f\right\|_{L^2}^2 d\tau \le C(t_1, A). \tag{3.35}$$

Property $P(1)$ holds true, due to Lemma 3.2: see (3.30) and (3.33). Let us now assume that $P(N)$ is satisfied.

Let us set

$$g_{\alpha,\beta} \overset{\text{def}}{=} \partial_x^\alpha \partial_v^\beta f$$

with $|\alpha| + |\beta| \le s - 1,\ \ |\beta| = N < s$. We use the generalized Leibniz formula

$$\partial^\alpha(\varphi\psi) = \sum_{0 \le \gamma \le \alpha} \binom{\alpha}{\gamma} \partial^\gamma \varphi\ \partial^{\alpha-\gamma}\psi.$$

Let us define the operators

$$K = \nabla_v + \frac{v}{2}, \qquad L = \nabla_v - \frac{v}{2}.$$

We apply $\partial_x^\alpha \partial_v^\beta$ to (1.11). Then $g_{\alpha,\beta}$ satisfies:

$$\begin{aligned}
&\partial_t g_{\alpha,\beta} + v \cdot \nabla_x g_{\alpha,\beta} \\
&+ \sum_i \binom{\beta}{\delta_i} \partial_x^{\alpha+\delta_i} \partial_v^{\beta-\delta_i} f - \partial_x^\alpha u \cdot \partial_v^\beta (v\sqrt{\mu}) + \partial_x^\alpha \left[u \cdot L(\partial_v^\beta f)\right] - \frac{1}{2}\sum_i \binom{\beta}{\delta_i} \partial_x^\alpha \left[u_i \partial_v^{\beta-\delta_i} f\right] \\
&= -\frac{|v|^2}{4} g_{\alpha,\beta} + \frac{3}{2} g_{\alpha,\beta} + \triangle_v g_{\alpha,\beta} - \frac{1}{2}\sum_i \left[\binom{\beta}{2\delta_i} \partial_x^\alpha \partial_v^{\beta-2\delta_i} f + \binom{\beta}{\delta_i} v_i \partial_x^\alpha \partial_v^{\beta-\delta_i} f\right],
\end{aligned}$$

where δ_i is the multi-index whose ith component is 1, and the others are 0.

Before going further, we set $H_{\alpha,\beta} \overset{\text{def}}{=} \nabla_v g_{\alpha,\beta} + \frac{v}{2} g_{\alpha,\beta}$. Noticing that:

$$H_{\alpha,\beta}^j = K_j(g_{\alpha,\beta}) \quad \text{and} \quad K_j \partial_x^\alpha = \partial_x^\alpha K_j,$$

we apply K_j to the previous equation so that $H^j_{\alpha,\beta}$ satisfies

$$\partial_t H^j_{\alpha,\beta} + v\cdot\nabla H^j_{\alpha,\beta} + \partial_{x_j} g_{\alpha,\beta} + I_j = -\frac{|v|^2}{4}H^j_{\alpha,\beta} + \frac{3}{2}H^j_{\alpha,\beta} + \triangle_v H^j_{\alpha,\beta} - H^j_{\alpha,\beta} + II_j,$$

where

$$\begin{aligned} I_j = & \sum_i \left[\binom{\beta}{\delta_i} K_j\left(\partial_x^{\alpha+\delta_i}\partial_v^{\beta-\delta_i} f\right)\right] - \partial_x^\alpha u\cdot\left[K_j\left(\partial_v^\beta(v\sqrt{\mu})\right)\right] + \partial_x^\alpha\left[u\cdot K_j L(\partial_v^\beta f)\right] \\ & -\frac{1}{2}\sum_i \binom{\beta}{\delta_i}\partial_x^\alpha\left[u_i K_j\left(\partial_v^{\beta-\delta_i} f\right)\right], \end{aligned}$$

and

$$\begin{aligned} II_j = & -\frac{1}{2}\sum_i\left[\binom{\beta}{2\delta_i}K_j\left(\partial_x^\alpha\partial_v^{\beta-2\delta_i} f\right) + \binom{\beta}{\delta_i} v_i K_j\left(\partial_x^\alpha\partial_v^{\beta-\delta_i} f\right)\right] \\ & -\frac{1}{2}\binom{\beta}{\delta_j}\partial_x^\alpha\partial_v^{\beta-\delta_j} f. \end{aligned}$$

We now multiply the previous equation by $H^j_{\alpha,\beta}$ and standard energy estimates will tell us that

$$\frac{1}{2}\frac{d}{dt}\|H_{\alpha,\beta}\|^2_{L^2} + \|\nabla_v H_{\alpha,\beta} + \frac{v}{2}H_{\alpha,\beta}\|^2_{L^2} + \frac{1}{2}\|H_{\alpha,\beta}\|^2_{L^2} \le \frac{1}{2}\|\nabla g_{\alpha,\beta}\|^2_{L^2} + \sum_j\left|\langle I_j + II_j, H^j_{\alpha,\beta}\rangle\right|$$

holds.

Let us estimate the components of the last sum, term by term. By Young's inequality, we only need to bound the L^2 norm of each term contained in the I_j and II_j's. To this end, we use the estimates contained in $P(N)$. More precisely, we have for $t\ge t_1$:

$$\left\|K_j\left(\partial_x^{\alpha+\delta_i}\partial_v^{\beta-\delta_i} f\right)\right\|^2_{L^2} + \left\|K_j\left(\partial_x^\alpha\partial_v^{\beta-2\delta_i} f\right)\right\|^2_{L^2} \le \sum_{\substack{|a|+|b|\le s-1,\\ 1\le|b|\le N}} \left\|\nabla_v(\partial_x^a\partial_v^b f) + \frac{v}{2}\partial_x^a\partial_v^b f\right\|^2_{L^2},$$

and

$$\begin{aligned} \left\|v_i K_j\left(\partial_x^\alpha\partial_v^{\beta-\delta_i} f\right)\right\|^2_{L^2} &\le \sum_{\substack{|a|+|b|\le s-2,\\ 1\le|b|\le N-1}} \left\|v\otimes\nabla_v(\partial_x^a\partial_v^b f) + \frac{v\otimes v}{2}\partial_x^a\partial_v^b f\right\|^2_{L^2}, \\ \left\|\partial_x^\alpha u\cdot K_j\left(\partial_v^\beta(v\sqrt{\mu})\right)\right\|^2_{L^2} &\le C\|u\|_s^2, \\ \left\|\partial_x^\alpha\partial_v^{\beta-\delta_j} f\right\|^2_{L^2} &\le C(t_1, A). \end{aligned}$$

The last non linear term in I_j reads

$$\left\|\partial_x^\alpha\left[u_i K_j\left(\partial_v^{\beta-\delta_i} f\right)\right]\right\|^2_{L^2} = \sum_{\alpha_1+\alpha_2=\alpha}\left\|\partial_x^{\alpha_1} u_i K_j\left(\partial_x^{\alpha_2}\partial_v^{\beta-\delta_i} f\right)\right\|^2_{L^2}.$$

When $|\alpha| \le 1$, it can be estimated as follows

$$\begin{aligned}\left\|\partial_x^\alpha\left[u_i K_j\left(\partial_v^{\beta-\delta_i} f\right)\right]\right\|_{L^2}^2 &\le C\|u\|_{H^{|\alpha_1|+2}}^2\left|K_j\left(\partial_v^{\beta-\delta_i} f\right)\right|_{|\alpha_2|}^2\\ &\le C\|u\|_{H^s}^2 \sum_{\substack{|a|+|b|\le s-1,\\ 1\le|b|\le N}} \left\|\nabla_v(\partial_x^a\partial_v^b f)+\frac{v}{2}\partial_x^a\partial_v^b f\right\|_{L^2}^2.\end{aligned}$$

Now we turn to the case $|\alpha| \ge 2 > 3/2$. We have

$$\begin{aligned}\left\|\partial_x^\alpha\left[u_i K_j\left(\partial_v^{\beta-\delta_i} f\right)\right]\right\|_{L^2}^2 &\le C\|u\|_{H^{|\alpha|}}^2\left|K_j\left(\partial_v^{\beta-\delta_i} f\right)\right|_{|\alpha|}^2\\ &\le C\|u\|_{H^s}^2 \sum_{\substack{|a|+|b|\le s-1,\\ 1\le|b|\le N}} \left\|\nabla_v(\partial_x^a\partial_v^b f)+\frac{v}{2}\partial_x^a\partial_v^b f\right\|_{L^2}^2.\end{aligned}$$

Therefore, we conclude that

$$\left\|\partial_x^\alpha\left[u_i K_j\left(\partial_v^{\beta-\delta_i} f\right)\right]\right\|_{L^2}^2 \le C\|u\|_s^2 \sum_{\substack{|a|+|b|\le s-1,\\ 1\le|b|\le N}} \left\|\nabla_v(\partial_x^a\partial_v^b f)+\frac{v}{2}\partial_x^a\partial_v^b f\right\|_{L^2}^2.$$

Eventually, for the non linear terms of I_j which involve L, we proceed as follows

$$\begin{aligned}&\left|\left\langle\partial_x^\alpha\left[u\cdot K_j L\left(\partial_v^\beta f\right)\right], H_{\alpha,\beta}^j\right\rangle\right| = \sum_m \left|\left\langle\partial_x^\alpha\left[u_m K_j L_m\left(\partial_v^\beta f\right)\right], H_{\alpha,\beta}^j\right\rangle\right|\\ &\le \sum_m \left|\left\langle\partial_x^\alpha\left[u_m L_m K_j\left(\partial_v^\beta f\right)\right], H_{\alpha,\beta}^j\right\rangle\right| + \left|\left\langle\partial_x^\alpha\left[u_j K_j\left(\partial_v^\beta f\right)\right], H_{\alpha,\beta}^j\right\rangle\right|\\ &\le \sum_m \left|\left\langle\partial_x^\alpha\left[u_m K_j\left(\partial_v^\beta f\right)\right], K_m H_{\alpha,\beta}^j\right\rangle\right| + \left|\left\langle\partial_x^\alpha\left[u_j K_j\left(\partial_v^\beta f\right)\right], H_{\alpha,\beta}^j\right\rangle\right|\\ &\le C_\epsilon\|u\|_s^2\Bigg(\sum_{\substack{|a|+|b|\le s-1,\\ 1\le|b|\le N}} \left\|\nabla_v(\partial_x^a\partial_v^b f)+\frac{v}{2}\partial_x^a\partial_v^b f\right\|_{L^2}^2\Bigg) + \epsilon(\|K_m H_{\alpha,\beta}^j\|_{L^2}^2 + \|H_{\alpha,\beta}^j\|_{L^2}^2),\end{aligned}$$

with $\epsilon > 0$, where we use the fact that $K_i^* = -L_i$, and $[K_j, L_i] = -\delta_{i,j}$ in the second inequality.

Combining together all the above estimates, we end up with

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|H_{\alpha,\beta}\|_{L^2}^2 + \frac{1}{2}\|\nabla_v H_{\alpha,\beta} + \frac{v}{2}H_{\alpha,\beta}\|_{L^2}^2 + \frac{1}{4}\|H_{\alpha,\beta}\|_{L^2}^2\\ &\quad\le C\Bigg(\|\nabla_x g_{\alpha,\beta}\|_{L^2}^2 + \sum_{\substack{|a|+|b|\le s-2,\\ 1\le|b|\le N-1}} \left\|v\otimes\nabla_v(\partial_x^a\partial_v^b f)+\frac{v\otimes v}{2}\partial_x^a\partial_v^b f\right\|_{L^2}^2\\ &\qquad + \sum_{\substack{|a|+|b|\le s-1,\\ 1\le|b|\le N}} \left\|\nabla_v(\partial_x^a\partial_v^b f)+\frac{v}{2}\partial_x^a\partial_v^b f\right\|_{L^2}^2 + 1\Bigg). \qquad (3.36)\end{aligned}$$

Let us now prove that $P(N+1)$ holds true. Estimate (3.34) in $P(N)$ implies for all $t \geq t_1$

$$\int_t^{t+1} \|H_{\alpha,\beta}\|_{L^2}^2 \, d\tau \;\leq\; C(t_1, A) \qquad \text{and} \qquad \int_t^{t+1} \|H_{a,b}\|_{L^2}^2 \, d\tau \leq C(t_1, A).$$

We also have

$$\|\nabla_x g_{\alpha,\beta}\|_{L^2}^2 = \|\nabla_x \partial_x^\alpha \partial_v^\beta f\|_{L^2}^2 \;=\; \sum_i \|\partial_x^{\alpha+\delta_i} \partial_v^\beta f\|_{L^2}^2 \leq C(t_1, A).$$

Since now $|\alpha| + |\beta| \leq s - 1$ and $|\beta| = N < s$ so that (3.34) in $P(N)$ can be used again. Therefore, we can exhibit a a constant (still denoted by $C(t_1, A)$) such that (3.36) becomes

$$\frac{d}{dt}\|H_{\alpha,\beta}\|_{L^2}^2 + \|\nabla_v H_{\alpha,\beta} + \frac{v}{2} H_{\alpha,\beta}\|_{L^2}^2 \leq C(t_1, A)(1 + \xi).$$

where by using now the second assumption (3.35) in $P(N)$, the non negative function ξ belongs to $\mathscr{B}(C(t_1, A), t_1)$.

We use Lemma 3.1 with $\zeta(t) = \|H_{\alpha,\beta}\|_{L^2}^2$, which lies in $\mathscr{B}(C(t_1, A), t_1)$, and $\eta(t) = \|\nabla_v H_{\alpha,\beta} + \frac{v}{2} H_{\alpha,\beta}\|_{L^2}^2$. Then for all $t \geq t_2 > t_1$, there exists a constant denoted $C(t_2, A)$ such that:

$$\|H_{\alpha,\beta}(t)\|_{L^2}^2 + \int_t^{t+1} \|\nabla_v H_{\alpha,\beta} + \frac{v}{2} H_{\alpha,\beta}\|_{L^2}^2 \, d\tau \leq C(t_2, A). \tag{3.37}$$

Observe that

$$\begin{aligned}
&\|H_{\alpha,\beta}^i\|_{L^2}^2 = \|\partial_{v_i} g_{\alpha,\beta}\|_{L^2}^2 + \|\frac{v_i}{2} g_{\alpha,\beta}\|_{L^2}^2 - \frac{1}{2}\|g_{\alpha,\beta}\|_{L^2}^2, \\
&\|\nabla_v H_{\alpha,\beta} + \frac{v}{2} H_{\alpha,\beta}\|_{L^2}^2 = \|\nabla_v H_{\alpha,\beta}\|_{L^2}^2 + \|\frac{v}{2} H_{\alpha,\beta}\|_{L^2}^2 - \frac{3}{2}\|H_{\alpha,\beta}\|_{L^2}^2, \\
&\partial_{v_j} H_{\alpha,\beta}^i = \partial_{v_i}(\partial_{v_j} g_{\alpha,\beta}) + \frac{v_i}{2}\partial_{v_j} g_{\alpha,\beta} + \delta_{ij}\frac{g_{\alpha,\beta}}{2}, \\
&\frac{v_j}{2} H_{\alpha,\beta}^i = \frac{v_j}{2}\partial_{v_i} g_{\alpha,\beta} + \frac{v_j}{2}\frac{v_i}{2} g_{\alpha,\beta}.
\end{aligned}$$

Hence, we rewrite (3.37) as follows

$$\begin{aligned}
&\sup_{t \geq t_2}\left(\sum_i \|\partial_{v_i} g_{\alpha,\beta}\|_{L^2}^2 + \|\frac{v_i}{2} g_{\alpha,\beta}\|_{L^2}^2\right) \\
&+ \int_t^{t+1} \sum_{i,j} \left\|\partial_{v_i}(\partial_{v_j} g_{\alpha,\beta}) + \frac{v_i}{2}\partial_{v_j} g_{\alpha,\beta}\right\|_{L^2}^2 + \left\|\frac{v_j}{2}\partial_{v_i} g_{\alpha,\beta} + \frac{v_j}{2}\frac{v_i}{2} g_{\alpha,\beta}\right\|_{L_2}^2 d\tau \\
&\leq C(t_2, A).
\end{aligned}$$

As a consequence, $P(N+1)$ holds. More precisely, we have

1. For all multi-indices α and β such that $|\alpha| + |\beta| \leq s$, $|\beta| \leq N + 1$, and $t \geq t_2 > 0$,

$$\sup_{t \geq t_2 > 0} \|\partial_x^\alpha \partial_v^\beta f\|_{L^2}^2 + \int_t^{t+1} \left\|\nabla_v(\partial_x^\alpha \partial_v^\beta f) + \frac{v}{2}\partial_x^\alpha \partial_v^\beta f\right\|_{L^2}^2 d\tau \leq C(t_2, A).$$

2. For all multi-indices α and β such that $|\alpha| + |\beta| \leq s - 1, |\beta| \leq N$ and $t \geq t_2 > 0$,

$$\sup_{t \geq t_2 > 0} \|v\partial_x^\alpha \partial_v^\beta f\|_{L^2}^2 + \int_t^{t+1} \left\|v \otimes \nabla_v(\partial_x^\alpha \partial_v^\beta f) + \frac{v \otimes v}{2}\partial_x^\alpha \partial_v^\beta f\right\|_{L^2}^2 d\tau \leq C(t_2, A).$$

The induction is proved and this procedure stops when $|\beta| = s$, which gives (3.32). This completes the proof of Lemma 3.3. ■

Remark 3.1 *The proof does not involve the commutator $[v \cdot \nabla_x, \triangle_v]$. The main reason for this lies in the fact that the inductive method depends only on the structure of the Fokker-Planck operator (and the basic Lemma 3.1), which also can be applied to treat other models (see[20]).*

End of proof of Theorem 2.1: proof of (2.18)**).** Owing to Lemma 3.3, we are left with the task of justifying that (3.29) holds. Actually, due to (2.17), it only remains to exhibit a positive constant A such that

$$\int_t^{t+1} \big|\nabla_v f + \frac{v}{2} f\big|_s^2 \, d\tau \leq A, \tag{3.38}$$

holds for any $t \geq 0$. Remarking that $\big|\nabla_v f \pm \frac{v}{2} f\big|_s^2 = \big|\nabla_v f \mp \frac{v}{2} f\big|_s^2 \mp 3|f|_s^2$, we deduce from (3.22) that

$$\frac{1}{2}\frac{d}{dt}|f|_s^2 + \big|\nabla_v f + \frac{v}{2} f\big|_s^2 \leq C\|u\|_{H^s}|f|_s\big(1 + \big|\nabla_v f + \frac{v}{2} f\big|_s\big)$$

holds. Using the basic inequality $|ab| \leq a^2/2 + b^2/2$, we conclude that

$$\frac{d}{dt}|f|_s^2 + \big|\nabla_v f + \frac{v}{2} f\big|_s^2 \leq C\|u\|_{H^s}|f|_s(1 + \|u\|_{H^s}|f|_s).$$

Integrating this inequality over $[t, t+1]$ yields that

$$\int_t^{t+1} \big|\nabla_v f + \frac{v}{2} f\big|_s^2 \, d\tau \leq C\varepsilon(1 + \varepsilon)$$

as a consequence of (2.17). It proves (3.38) and concludes the proof of Theorem 2.1. ■

4 Large time behavior

In this section, we consider the large time behavior of the solutions to the nonlinear system (1.9)-(1.11) and prove Theorem 2.2. The key obstacle is that there is no dissipation terms to control the microscopic solution f. However, the diffusion operator in (1.11) takes the abstract form $K^*K + P$, with P a skew-symetric operator $P^* = -P$. This specific form makes appealing the use of the hypocoercivity method, see [30]. ¿From now on, we denote

$$K = \nabla_v + \frac{v}{2}, \qquad P = v \cdot \nabla_x, \qquad S_i = [K_i, P] = \partial_{x_i}$$

We remark that $K^*Kf = (-\nabla_v + v/2) \cdot \big[(\nabla_v + v/2) f\big] = -\triangle_v f - \nabla_v \cdot (\frac{v}{2} f) + \frac{v}{2} \cdot \nabla_v f + \frac{v^2}{4} f = -\triangle_v f - \frac{3}{2} f - \frac{v}{2} \cdot \nabla_v f + \frac{v}{2} \cdot \nabla_v f + \frac{v^2}{4} f$ so that

$$K^*Kf = -\triangle_v - \frac{3}{2} + \frac{v^2}{4}, \qquad P^* = -P, \qquad [S, P] = 0 = [S, K].$$

Accordingly, the microscopic equation (1.11) can be rewritten as

$$\partial_t f + Pf + K^*Kf = g \tag{4.39}$$

where

$$g = u \cdot v\sqrt{\mu} + u \cdot K^* f.$$

Note also that

$$\mathrm{Ker}(K^*K + P) = \mathrm{Span}\big\{\mathbb{1}(x)\sqrt{\mu(v)}\big\},$$

since $\langle (K^*K+P)f, f\rangle = \|Kf\|_{L^2}^2 = \int |\nabla_v(f/\sqrt{\mu})|^2\ \mu\, dv\, dx$. Let us set

$$\mathscr{N} = \mathrm{Ker}^{\perp}(K^*K+P) = \Big\{g \in L^2(\mathbf{T}^3 \times \mathbf{R}^3);\ \int_{\mathbf{T}^3 \times \mathbf{R}^3} g\sqrt{\mu}\, dx\, dv = 0\Big\}.$$

Bearing in mind Propositon 4.2, we introduce a new inner product $((\cdot,\cdot))$ on $\mathscr{N}$ defined as follows

$$((f,g)) \stackrel{\text{def}}{=} 2\langle Kf, Kg\rangle + \langle Kf, Sg\rangle + \langle Sf, Kg\rangle + \langle Sf, Sg\rangle.$$

We can find two constants $C^* > C_* > 0$ such that

$$C_*\big(\|Kf\|_{L^2}^2 + \|Sf\|_{L^2}^2\big) \le ((f,f)) \le C^*\big(\|Kf\|_{L^2}^2 + \|Sf\|_{L^2}^2\big). \tag{4.40}$$

Then, the key ingredient for obtaining the exponential convergence to equilibrium relies on the following statement.

Proposition 4.1 *Let the assumptions of Theorem 2.1 be fulfilled with $s \ge 3$ and $0 < \varepsilon < \varepsilon_0$ small enough. Furthermore, we assume that* (1.14) *holds. Then, there exists a strictly positive constant λ_1 such that*

$$\begin{aligned}\frac{d}{dt}((f,f)) &+ \lambda_1(\|Kf\|_{L^2}^2 + \|Sf\|_{L^2}^2 + \|K^2 f\|_{L^2}^2 + \|KSf\|_{L^2}^2)\\ &\le C(\lambda_1)(\|u\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + \|Kf\|_{L^2}^2).\end{aligned} \tag{4.41}$$

We begin with the proof of a weighted Poincaré inequality.

Proposition 4.2 *There exists a constant $C_P^* > 0$ such that for any $f \in L^2(\mathbf{T}^3 \times \mathbf{R}^3)$ verifying $\int_{\mathbf{T}^3 \times \mathbf{R}^3} f\sqrt{\mu}\, dv\, dx = 0$, we have*

$$\|f\|_{L^2}^2 \le C_P^*(\|Kf\|_{L^2}^2 + \|Sf\|_{L^2}^2). \tag{4.42}$$

Proof. We argue by contradiction: suppose that for any integer n, there exists a function f_n such that $\|f_n\|_{L^2} = 1$ and

$$\|Kf_n\|_{L^2}^2 + \|Sf_n\|_{L^2}^2 \le \frac{1}{n}. \tag{4.43}$$

Since

$$\|\nabla_v f\|_{L^2}^2 + \Big\|\frac{v}{2} f\Big\|_{L^2}^2 = \|Kf\|_{L^2}^2 + \frac{3}{2}\|f\|_{L^2}^2,$$

we immediately deduce that

$$\|\nabla_v f_n\|_{L^2}^2 + \big\|\frac{v}{2} f_n\big\|_{L^2}^2 + \|\nabla_x f_n\|_{L^2}^2 \leq \frac{1}{n} + \frac{3}{2}.$$

Since this estimate controls both the derivatives of f_n and the tails for large velocities, we can assume, as a consequence of the Rellich-Kondrazhov theorem, that a subsequence satisfies

$$\begin{array}{rl} f_{n_k} \longrightarrow f & \text{strongly in } L^2(\mathbf{T}^3 \times \mathbf{R}^3), \\ \nabla_x f_{n_k} \rightharpoonup \nabla_x f \text{ and } \nabla_v f_{n_k} \rightharpoonup \nabla_v f & \text{weakly in } L^2(\mathbf{T}^3 \times \mathbf{R}^3) \end{array}$$

with furthermore $\|f\|_{L^2} = 1$. Coming back to (4.43), we obtain

$$\big\|\nabla_v f + \frac{v}{2} f\big\|_{L^2}^2 + \|\nabla_x f\|_{L^2}^2 \leq \liminf_{k\to\infty}(\|K f_{n_k}\|_{L^2}^2 + \|S f_{n_k}\|_{L^2}^2) = 0.$$

We deduce that $f(x,v) = M\sqrt{\mu(v)}$ for some $M \in \mathbf{R}$. Eventually, assuming that the f_n's are orthogonal to $\sqrt{\mu}$ we get

$$\int_{\mathbf{T}^3\times\mathbf{R}^3} f\sqrt{\mu}\, dv\, dx = \lim_{k\to\infty} \int_{\mathbf{T}^3\times\mathbf{R}^3} f_{n_k}\sqrt{\mu}\, dv\, dx = 0.$$

Hence $f = 0$, which contradicts the fact that f is normalized. This completes the proof of Proposition 4.2. ■

Proof of Proposition 4.1. Multiply (4.39) by f and use the new scalar product. It yields

$$\frac{1}{2}\frac{d}{dt}((f,f)) + ((Pf + K^*Kf, f)) = ((g,f)). \tag{4.44}$$

We shall estimate the quantities of this equality, term by term.

Firstly, by definition of the scalar product, we have

$$((Pf,f)) \;=\; 2\langle KPf, Kf\rangle + \langle KPf, Sf\rangle + \langle SPf, Kf\rangle + \langle SPf, Sf\rangle.$$

Since P is skew-symmetric, for any u we have $\langle Pu, u\rangle = -\langle u, Pu\rangle = 0$ so that $\langle SPf, Sf\rangle = \langle PSf, Sf\rangle = 0$ and $\langle KPf, Kf\rangle = \langle [K,P]f, Kf\rangle$. We can also write $\langle SPf, Kf\rangle = \langle PSf, Kf\rangle = -\langle PKf, Sf\rangle$. We thus arrive at

$$\begin{aligned} ((Pf,f)) &= 2\langle [K,P]f, Kf\rangle + \langle [K,P]f, Sf\rangle \\ &= 2(Sf, Kf) + \|Sf\|_{L^2}^2 \\ &\geq \frac{3}{4}\|Sf\|_{L^2}^2 - 4\|Kf\|_{L^2}^2. \end{aligned}$$

Secondly, we get

$$\begin{aligned} ((K^*Kf, f)) &= 2\langle KK^*Kf, Kf\rangle + \langle KK^*Kf, Sf\rangle + \langle Kf, SK^*Kf\rangle + \langle SK^*Kf, Sf\rangle \\ &= 2\langle [K_i, K_j^*]K_j f, K_i f\rangle + 2\langle K_iK_j f, K_jK_i f\rangle + \langle [K_i, K_j^*]K_j f, S_i f\rangle \\ &\quad + 2\langle K_iK_j f, K_jS_i f\rangle + \langle S_iK_j f, K_jS_i f\rangle \\ &= 2\|Kf\|_{L^2}^2 + 2\|K^2 f\|_{L^2}^2 + \|SKf\|_{L^2}^2 + \langle Kf, Sf\rangle + 2\langle K^2 f, SKf\rangle \\ &\geq \frac{3}{2}\|Kf\|_{L^2}^2 + \frac{1}{2}\|K^2 f\|_{L^2}^2 + \frac{1}{3}\|SKf\|_{L^2}^2 - \frac{1}{2}\|Sf\|_{L^2}^2, \end{aligned}$$

where we used the identity

$$[K_i, K_j^*] = \delta_{ij}. \tag{4.45}$$

We treat now the right hand side of (4.44). We have, for any $\epsilon > 0$,

$$\begin{aligned} |((u \cdot v\sqrt{\mu}, f))| &= \Big| 2\big\langle K(u \cdot v\sqrt{\mu}), Kf\big\rangle + \big\langle K(u \cdot v\sqrt{\mu}), Sf\big\rangle \\ &\quad + \big\langle S(u \cdot v\sqrt{\mu}), Kf\big\rangle + \big\langle S(u \cdot v\sqrt{\mu}), Sf\big\rangle\Big| \\ &\leq \epsilon(\|Kf\|_{L^2}^2 + \|Sf\|_{L^2}^2) + C_\epsilon(\|u\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2), \end{aligned}$$

since $\|K(u \cdot v\sqrt{\mu})\|_{L^2} \leq C\|u\|_{L^2}$ and $\|S(u \cdot v\sqrt{\mu})\|_{L^2} \leq C\|\nabla_x u\|_{L^2}$. Next, by virtue of of (4.45), we can write

$$\big\langle K(u \cdot K^* f), Kf\big\rangle = \big\langle K^*(u \cdot Kf), Kf\big\rangle + \big\langle [K, K^*]uf, Kf\big\rangle = \big\langle u \cdot Kf, K^2 f\big\rangle + \big\langle uf, Kf\big\rangle.$$

Therefore, we are led to the following estimate

$$\begin{aligned} \big|((u \cdot K^* f, f))\big| &= \Big| 2\big\langle u \cdot K^* f, K^2 f\big\rangle + 2\big\langle uf, Kf\big\rangle + \big\langle K(u \cdot K^* f), Sf\big\rangle \\ &\quad + \big\langle S(u \cdot K^* f), Kf\big\rangle + \big\langle S(u \cdot K^* f), Sf\big\rangle\Big| \\ &\leq \|u\|_{L^\infty}\big(2\|Kf\|_{L^2}\|K^2 f\|_{L^2} + 2\|f\|_{L^2}\|Kf\|_{L^2} \\ &\qquad + 2\|Kf\|_{L^2}\|KSf\|_{L^2} + \|KSf\|_{L^2}\|Sf\|_{L^2}\big) \\ &\quad + \|\nabla_x u\|_{L^\infty}\big(\|Kf\|_{L^2}^2 + \|Kf\|_{L^2}\|Sf\|_{L^2}\big). \end{aligned}$$

We combine now (4.42), the Sobolev embedding $H^3(\mathbf{T}^3) \subset W^{1,\infty}(\mathbf{T}^3)$ and the Young inequality, so that this relation becomes

$$\begin{aligned} \big|((u \cdot K^* f, f))\big| &\leq C\|u\|_{H^3}(\|Kf\|_{L^2}^2 + \|Sf\|_{L^2}^2 + \|KSf\|_{L^2}^2 + \|K^2 f\|_{L^2}^2) \\ &\leq C\epsilon(\|Kf\|_{L^2}^2 + \|Sf\|_{L^2}^2 + \|KSf\|_{L^2}^2 + \|K^2 f\|_{L^2}^2), \end{aligned}$$

where the last line uses (2.17). Combining all together the estimates concludes the proof of Proposition 4.1. ■

We now are in position to prove Theorem 2.2.

Proof of Theorem 2.2. Thanks to Theorem 2.1, we can revisit the basic energy estimates. Coming back to (3.23) with $\alpha = 0$ (so that there is no contribution from the convection term in (1.9)) we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|f\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \big\|u\sqrt{\mu} - \nabla_v f - \frac{v}{2}f\big\|_{L^2}^2 \\ &\quad \leq \big|\big\langle uf, u\sqrt{\mu} - \nabla_v f - \frac{v}{2}f\big\rangle\big| \leq C\|u\|_{L^2}|f|_s\big\|u\sqrt{\mu} - \nabla_v f - \frac{v}{2}f\big\|_{L^2} \\ &\leq C\epsilon\big(\|u\|_{L^2}^2 + \big\|u\sqrt{\mu} - \nabla_v f - \frac{v}{2}f\big\|_{L^2}^2\big). \end{aligned}$$

We have used successively the Sobolev embedding $H^s(\mathbf{T}^3) \subset L^\infty(\mathbf{T}^3)$ for $s > 3/2$ and (2.17). Similarly, (2.17) allows to deduce from (3.25)

$$\frac{1}{2}\frac{d}{dt}|\bar{u}|^2 + 2|\bar{u}|^2 \leq C\epsilon(\|\nabla u\|_{L^2}^2 + |\bar{u}|^2).$$

Since $K = \nabla_v + \frac{v}{2}$ we expand $\|u\sqrt{\mu} - \nabla_v f - \frac{v}{2} f\|_{L^2}^2 = \|u\|_{L^2}^2 + \|Kf\|_{L^2}^2 - 2\langle u\sqrt{\mu}, Kf\rangle$. Hence, we deduce that

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^2}^2 + \|f\|_{L^2}^2 + |\bar{u}|^2) + 2\|\nabla_x u\|_{L^2}^2 + 2|\bar{u}|^2 + 2\|u\|_{L^2}^2 + 2\|Kf\|_{L^2}^2 \\
&\leq 4\langle u\sqrt{\mu}, Kf\rangle + C\epsilon\big(2\|u\|_{L^2}^2 - 2(u\sqrt{\mu}, Kf) + \|Kf\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + |\bar{u}|^2\big)
\end{aligned}$$

holds. Let $\alpha > 1$ to be determined later. By using the Young inequality, we arrive at

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^2}^2 + \|f\|_{L^2}^2 + |\bar{u}|^2) + (2 - C\epsilon)\big(\|\nabla_x u\|_{L^2}^2 + |\bar{u}|^2\big) \\
&\quad + \big((2 - C\epsilon)(1 - \alpha) - 2C\epsilon\big)\|u\|_{L^2}^2 + (2 - C\epsilon)(1 - 1/\alpha)\|Kf\|_{L^2}^2 \leq 0.
\end{aligned} \tag{4.46}$$

The last step uses the Poincaré-Wirtinger inequality which tells us that for any $\kappa > 0$,

$$\kappa\|u\|_{L^2}^2 - \kappa C_P\big(\|\nabla_x u\|_{L^2}^2 + |\bar{u}|^2\big) \leq 0.$$

Therefore, we can modify (4.46) as follows

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^2}^2 + \|f\|_{L^2}^2 + |\bar{u}|^2) + (2 - C\epsilon - \kappa C_P)\big(\|\nabla_x u\|_{L^2}^2 + |\bar{u}|^2\big) \\
&\quad + \big(\kappa + (2 - C\epsilon)(1 - \alpha) - 2C\epsilon\big)\|u\|_{L^2}^2 + (2 - C\epsilon)(1 - 1/\alpha)\|Kf\|_{L^2}^2 \leq 0.
\end{aligned}$$

Let us now choose the parameters in a suitable way. We proceed as follows:

- Firstly we pick κ small enough to ensure $2 - \kappa C_P > 0$,
- Secondly, we pick ϵ small enough to ensure both $2 - \kappa C_P - C\epsilon > 0$ and $\kappa - 2C\epsilon > 0$, that is $0 < \epsilon \leq \frac{1}{C}\min(2 - \kappa C_P, \frac{\kappa}{2})$,
- Thirdly, we pick $\alpha > 1$ such that the coefficient in front of $\|u\|_{L^2}^2$ is positive that is $1 < \alpha < \frac{\kappa - 2C\epsilon + 2 - C\epsilon}{2 - C\epsilon} = 1 + \frac{\kappa - 2C\epsilon}{2 - C\epsilon}$.

Summarizing, we exhibit a constant $\lambda_2 > 0$ such that

$$\frac{d}{dt}(\|u\|_{L^2}^2 + \|f\|_{L^2}^2 + |\bar{u}|^2) + \lambda_2(\|\nabla_x u\|_{L^2}^2 + |\bar{u}|^2 + \|Kf\|_{L^2}^2) \leq 0. \tag{4.47}$$

To finish the proof of Theorem 2.2, we pick $\lambda > 0$ such that

$$2C(\lambda_1) + 2C(\lambda_1)C_P < \lambda\lambda_2,$$

with λ_1 and $C(\lambda_1)$ defined in Proposition 4.1, and we introduce

$$\mathscr{E}(t) := \lambda(\|u(t)\|_{L^2}^2 + \|f(t)\|_{L^2}^2 + |\bar{u}(t)|^2) + ((f(t), f(t))).$$

Thanks to Proposition 4.2, (4.40) and Poincaré-Wirtinger's inequality, we see that

$$\mathscr{E} \leq C(\|\nabla u\|_{L^2}^2 + |\bar{u}|^2 + \|Kf\|_{L^2}^2 + \|Sf\|_{L^2}^2).$$

Combining estimates (4.41) and (4.47), the Poincaré-Wirtinger inequality again, and the definition of λ, we have

$$\begin{aligned}
\frac{d}{dt}\mathscr{E}(t) &\leq -\lambda\lambda_2(\|\nabla u\|_{L^2}^2 + |\bar{u}|^2 + \|Kf\|_{L^2}^2) + C(\lambda_1)(\|u\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + \|Kf\|_{L^2}^2) - \lambda_1\|Sf\|_{L^2}^2 \\
&\leq -C(\lambda_1)(\|\nabla u\|_{L^2}^2 + |\bar{u}|^2 + \|Kf\|_{L^2}^2) - \lambda_1\|Sf\|_{L^2}^2 \\
&\leq -\min(C(\lambda_1), \lambda_1)(\|\nabla_x u\|_{L^2}^2 + |\bar{u}|^2 + \|Kf\|_{L^2}^2 + \|Sf\|_{L^2}^2).
\end{aligned}$$

At last, one obtains the existence of a constant λ_3 $(:=\frac{\min(C(\lambda_1),\lambda_1)}{C+1})$ such that

$$\frac{d}{dt}\mathscr{E}(t) + \lambda_3\mathscr{E}(t) \leq 0, \tag{4.48}$$

so that, by using (2.18),

$$\mathscr{E}(t) \leq C(t_0, \epsilon)e^{-\lambda_3 t}.$$

This completes the proof of Theorem 2.2.

5 Smoothing effect

In this section, we wish to investigate the smoothing effect of system (1.1)-(1.3). The analysis is based on the hypoellipticity property of (1.3) We start the proof of Theorem 2.3 with the following claim.

Proposition 5.1 *Let $s \geq 2$. Let $f, g \in H^{s+1}(\mathbf{T}^3)$ and set $D_x^\gamma = (-\triangle_x)^{\gamma/2}$ with $0 < \gamma < 1$. Then*

$$\|D_x^\gamma(fg)\|_{H^s} \leq \|D_x^\gamma f\|_{H^s}\|g\|_{H^s} + \|f\|_{H^s}\|D_x^\gamma g\|_{H^s}.$$

Proof. We introduce the operator $\triangle_{\gamma,k}$ defined by

$$\triangle_{\gamma,k} f = \frac{f(x+k) - f(x)}{|k|^{\gamma+\frac{3}{2}}},$$

so that (see e. g. [29], Lemma 16.3)

$$\|D_x^\gamma f\|_{L^2(x)} = \|\triangle_{\gamma,k} f\|_{L^2(x,k)}.$$

It implies

$$\begin{aligned}
\|D_x^\gamma(fg)\|_{H^s} &= \|\triangle_{\gamma,k}(fg)\|_{L^2(k;H^s(x))} \\
&\leq \|(\triangle_{\gamma,k} f)g(x+k)\|_{L^2(k;H^s(x))} + \|f(x)\triangle_{\gamma,k} g\|_{L^2(k;H^s(x))} \\
&\leq \|\triangle_{\gamma,k} f\|_{L^2(k;H^s(x))}\|g\|_{H^s} + \|f\|_{H^s}\|\triangle_{\gamma,k} g\|_{L^2(k;H^s(x))} \\
&= \|D_x^\gamma f\|_{H^s}\|g\|_{H^s} + \|f\|_{H^s}\|D_x^\gamma g\|_{H^s},
\end{aligned}$$

so that Proposition 5.1 is proven. ■

Proof of Theorem 2.3. Let $s \geq 2$ be a positive integer and $|\alpha| \leq s$. For every integer $n \geq 3$, we can derive from (1.1) the following energy estimate

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u\|_{L^2}^2 + \|\nabla_x \partial^\alpha u\|_{L^2}^2 \leq C\|u\|_{H^s}^2 \|\nabla u\|_{H^s} + C|F\langle v\rangle^n|_s \|u\|_{H^s}^2 + C|F\langle v\rangle^n|_s \|u\|_{H^s}.$$

By using the Young inequality, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 + \frac{1}{2}\|\nabla_x u\|_{H^s}^2 \leq C\big(\|u\|_{H^s}^4 + \|u\|_{H^s}^2 + |F\langle v\rangle^n|_s^2\big) \tag{5.49}$$

holds. Treating similarly (1.3), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|F\langle v\rangle^n|_s^2 + \frac{1}{2}|(\nabla_v F)\langle v\rangle^n|_s^2 &\leq C\big(\|u\|_{H^s}^2 + 1\big)|F\langle v\rangle^n|_s^2 \\ &\leq C\big(\|u\|_{H^s}^4 + |F\langle v\rangle^n|_s^2 + |F\langle v\rangle^n|_s^4\big). \end{aligned} \tag{5.50}$$

Let us set

$$X(t) = \|u(t)\|_{H^s}^2, \qquad \text{and} \qquad Y(t) = |F(t)\langle v\rangle^n|_s^2.$$

Then X and Y satisfy the following system of differential inequalities

$$\begin{cases} \dfrac{d}{dt}X(t) \leq C\big(X(t)^2 + X(t) + Y(t)\big), \\ \dfrac{d}{dt}Y(t) \leq C\big(X(t)^2 + Y(t) + Y(t)^2\big), \end{cases}$$

so that

$$\frac{d}{dt}\big(X(t)+Y(t)\big) \leq C\Big(\big(X(t)+Y(t)\big)^2 + \big(X(t)+Y(t)\big)\Big).$$

Therefore we can estimate

$$X(t) + Y(t) \leq C(T, X(0)+Y(0)) < \infty,$$

at least on a (small enough) time interval $0 \leq t \leq T$ (related to the life time of the solution of $\frac{d}{dt}z = C(z^2+z)$ with initial data $z(0) = X(0)+Y(0)$). Accordingly, coming back to (5.49), (5.50), we have, for every integer $n \geq 3$,

$$\begin{aligned} \sup_{0\leq t\leq T}(\|u(t)\|_{H^s}^2 + |F(t)\langle v\rangle^n|_s^2) + \int_0^T \Big[\|u\|_{H^{s+1}}^2 + |(\nabla_v F)\langle v\rangle^n|_s^2\Big]\,d\tau \\ \leq C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^n|_s). \end{aligned} \tag{5.51}$$

Let us introduce

$$p \in \mathbf{N} \qquad \text{and} \qquad \phi \in C_c^\infty((0,T]).$$

We set

$$g^p = \phi(t)\partial_x^\alpha F\langle v\rangle^p,$$

which satisfies

$$\partial_t g^p + v\cdot\nabla_x g^p = \mathrm{div}_v(\nabla_v g^p + h_1^p) + h_2^p, \tag{5.52}$$

where

$$
\begin{aligned}
h_1^p &= \phi(t)\partial^\alpha[(v-u)F]\langle v\rangle^p - 2\phi(t)\partial^\alpha F p\langle v\rangle^{p-2}v,\\
h_2^p &= -p\langle v\rangle^{p-2}\phi(t)\partial^\alpha[(v-u)F]\cdot v + 2\phi(t)\partial^\alpha F \mathrm{div}_v[p\langle v\rangle^{p-2}v]\\
&\quad -\phi(t)\partial^\alpha F\triangle_v(\langle v\rangle^p) + \phi'(t)\partial^\alpha F\langle v\rangle^p.
\end{aligned}
$$

Owing to (5.51), we check that

$$
\|g^p\|_{L^2(\mathbf{R}\times\mathbf{T}^3\times\mathbf{R}^3)} + \|h_1^p\|_{L^2(\mathbf{R}\times\mathbf{T}^3\times\mathbf{R}^3)} + \|h_2^p\|_{L^2(\mathbf{R}\times\mathbf{T}^3\times\mathbf{R}^3)} < C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^{p+1}|_s).
$$

Then the basic energy estimate for (5.52) yields

$$
\|\nabla_v g^p\|_{L^2(\mathbf{R}\times\mathbf{T}^3\times\mathbf{R}^3)} \le C(\|g^p\|_{L^2(\mathbf{R}\times\mathbf{T}^3\times\mathbf{R}^3)} + \|h_1^p\|_{L^2(\mathbf{R}\times\mathbf{T}^3\times\mathbf{R}^3)} + \|h_2^p\|_{L^2(\mathbf{R}\times\mathbf{T}^3\times\mathbf{R}^3)}).
$$

Thus, we can make use of Theorem 2.1 from [2]. It allows to control the L^2 norm, with respect to time, space and velocity variables, of $D_x^{\frac{1}{3}}g^p$ by quantities depending on the L^2 norms of $g^p, \nabla_v g^p, h_1^p, h_2^p$. We obtain

$$
\|D_x^{\frac{1}{3}}g^p\|_{L^2(\mathbf{R}\times\mathbf{T}^3\times\mathbf{R}^3)} \le C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^{p+1}|_s). \tag{5.53}
$$

We shall repeat the argument in order to estimate a full space derivative of g^p. We introduce another cut-off function

$$
\varphi_1 \in C_c^\infty((0,T]), \qquad \mathrm{supp}(\varphi_1) \subset \mathrm{supp}(\phi),
$$

and we set

$$
\widetilde{g^p} = \varphi_1(t) D_x^{\frac{1}{3}} g^p.
$$

The function $\widetilde{g^p}$ verifies

$$
\partial_t\widetilde{g^p} + v\cdot\nabla_x\widetilde{g^p} = \mathrm{div}_v\big(\nabla_v\widetilde{g^p} + \varphi_1(t)D_x^{\frac{1}{3}}h_1^p\big) + \varphi_1(t)D_x^{\frac{1}{3}}h_2^p + \varphi_1'(t)D_x^{\frac{1}{3}}g^p. \tag{5.54}
$$

Due to the Proposition 5.1, we have

$$
\Big|D_x^\gamma\big[\phi(t)uF\langle v\rangle^p\big]\Big|_s^2 \le C\Big(\|D_x^\gamma u\|_{H^s}^2\ \Big|\phi(t)F\langle v\rangle^p\Big|_s^2 + \|u\|_{H^s}^2\ \Big|D_x^\gamma\big[\phi(t)F\langle v\rangle^p\big]\Big|_s^2\Big).
$$

Integrating with respect to time, it follows that

$$
\begin{aligned}
\int_0^T \Big|D_x^\gamma\big[\phi(t)uF\langle v\rangle^p\big]\Big|_s^2 dt \quad &\le \quad C\Big(\|D_x^\gamma u\|_{L^2(0,T;H^s(\mathbf{T}^3))}^2 \sup_{0\le t\le T}\Big|\phi(t)F\langle v\rangle^p\Big|_s^2\\
&\qquad + \|u\|_{L^\infty(0,T;H^s(\mathbf{T}^3))}^2 \int_0^T \Big|D_x^\gamma\big[\phi(t)F\langle v\rangle^p\big]\Big|_s^2 dt\Big).
\end{aligned}
$$

By using this formula, we obtain the following estimate

$$
\begin{aligned}
\int_0^T \Big\|\varphi_1(t)D_x^\gamma h_1^p\Big\|_{L^2(\mathbf{T}^3\times\mathbf{R}^3)}^2 dt \quad &\le \quad C\Big(\|D_x^\gamma u\|_{L^2(0,T;H^s(\mathbf{T}^3))}^2 \sup_{0\le t\le T}\Big|\varphi_1(t)F\langle v\rangle^p\Big|_s^2\\
&\qquad + \big(1+\|u\|_{L^\infty(0,T;H^s(\mathbf{T}^3))}^2\big) \int_0^T \Big|D_x^\gamma\big[\varphi_1(t)F\langle v\rangle^{p+1}\big]\Big|_s^2 dt\Big),
\end{aligned}
$$

since h_1^p involves a higher moment with respect to v. Therefore, coming back to (5.53) we are led to

$$\left\|\varphi_1(t) D_x^{\frac{1}{3}} h_1^p\right\|^2_{L^2((0,T)\times \mathbf{T}^3 \times \mathbf{R}^3)} \le C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^{p+2}|_s). \tag{5.55}$$

By the same calculation, we also get

$$\begin{aligned}\left\|\varphi_1(t) D_x^{\frac{1}{3}} h_2^p\right\|^2_{L^2((0,T)\times \mathbf{T}^3 \times \mathbf{R}^3)} + \left\|\varphi_1'(t) D_x^{\frac{1}{3}} g^p\right\|^2_{L^2((0,T)\times \mathbf{T}^3 \times \mathbf{R}^3)} \\ \le C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^{p+2}|_s).\end{aligned} \tag{5.56}$$

Consequently, the basic energy estimate for (5.54) yields

$$\|\nabla_v \widetilde{g^p}\|_{L^2(t,x,v)} \le C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^{p+2}|_s).$$

With this estimate, together with (5.55) and (5.56), we can apply Theorem 2.1 of [2] again to equation (5.54). It allows to estimate $D_x^{\frac{1}{3}}\widetilde{g^p}$ by means of h_1^p, h_2^p, g^p, $\widetilde{g^p}$ and $\nabla_v \widetilde{g^p}$ and thus it finally leads to

$$\left\|D_x^{\frac{1}{3}} \widetilde{g^p}\right\|_{L^2(\mathbf{R}\times \mathbf{T}^3 \times \mathbf{R}^3)} \le C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^{p+2}|_s). \tag{5.57}$$

Let us consider

$$\varphi_2 \in C_c^\infty((0,T]), \qquad \mathrm{supp}(\varphi_2) \subset \mathrm{supp}(\varphi_1) \subset \mathrm{supp}(\phi).$$

Set

$$\widehat{g^p} = \varphi_2(t) D_x^{\frac{2}{3}} g^p.$$

Then $\widehat{g^p}$ verifies

$$\partial_t \widehat{g^p} + v\cdot\nabla_x \widehat{g^p} = \mathrm{div}_v\big(\nabla_v \widehat{g^p} + \varphi_2(t) D_x^{\frac{2}{3}} h_1^p\big) + \varphi_2(t) D_x^{\frac{2}{3}} h_2^p + \varphi_2'(t) D_x^{\frac{2}{3}} g^p.$$

Repeating the previous argument and using (5.57) we show that

$$\begin{aligned}\left\|\varphi_2(t) D_x^{\frac{2}{3}} h_1^p\right\|^2_{L^2((0,T)\times \mathbf{T}^3 \times \mathbf{R}^3)} &\le \left\|D_x^{\frac{2}{3}} u\right\|^2_{L^2(0,T;H^s(\mathbf{R}^3))} \sup_{0\le t\le T}\left|\varphi_2(t) F\langle v\rangle^p\right|_s^2 \\ &\quad + \Big(\|u\|^2_{L^\infty(0,T;H^s(\mathbf{R}^3))} + 1\Big)\int_0^T \left|D_x^{\frac{2}{3}}\left[\varphi_2(t) F\langle v\rangle^{p+1}\right]\right|_s^2 dt \\ &\le C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^{p+3}|_s)\end{aligned}$$

holds. Therefore, we can prove

$$\left\|D_x^{\frac{1}{3}} \widehat{g^p}\right\|_{L^2((0,T)\times\times \mathbf{T}^3 \times \mathbf{R}^3)} \le C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^{p+3}|_s).$$

In other words, for all $0 < t_\star < T_\star < T$, we have

$$\int_{t_\star}^{T_\star} |F(\tau)\langle v\rangle^p|^2_{s+1}\, d\tau < C(T, \|u_0\|_{H^s}, |F_0\langle v\rangle^{p+3}|_s). \tag{5.58}$$

Once this estimate of a higher full space derivative has been obtained, it becomes quite standard to justify (2.19), by using (5.51) and (5.58), and the parabolic structure with respect to the variable v of the Fokker-Planck equation (1.3). ■

Remark 5.1 *As a final comment, let us mention that an alternative proof of the smoothness of the solution in Theorem 2.1 can be proposed. This proof is based on the averaging lemma, see [13] and [4], and does not use explicitly the hypoellipticity result of [2]. Let us sketch the proof, referring for details to [8] and [9] where this approach has been successfully used. Let us assume we already know that, for all* $t > t_1 > 0$, $u(t)$ *and* $F(t)$ *belong to* $H^s(\mathbf{T}^3)$ *and* $H^s(\mathbf{T}^3 \times \mathbf{R}^3)$, *respectively, with* $s \geq 3$.

- *Firstly, thanks to the diffusion term in* (1.1), *we see that, for all* $t > t_1$, $u(t) \in H^{s+1}(\mathbf{T}^3)$.
- *Secondly, thanks to the diffusion term with respect to* v *in* (1.3), *we get* $\nabla_v F(t) \in H^s(\mathbf{T}^3 \times \mathbf{R}^3)$, *still for* $t > t_1$.
- *Writing the equation satisfied by any derivative of order* $\leq s$ *of* F, *we can apply the averaging lemma (see [13] and [4]) and get that the averages with respect to the* v *variable of this derivative lie in* $H^{1/6}(\mathbf{T}^3)$.
- *Interpolating this result with the estimate of regularity that we obtained for the* v *variable, we get that* $F(t) \in H^{s+1/20}(\mathbf{T}^3 \times \mathbf{R}^3)$ *for* $t > t_1$.
- *Iterating this result* 20 *times (it requires to write down the equation satisfied by translations of the derivatives of* F *of order less than* s*), we end up with* $F(t) \in H^{s+1}(\mathbf{T}^3 \times \mathbf{R}^3)$, *for all* $t > t_1$ *and, finally, that* f *is of class* $\mathscr{C}^\infty$.

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