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# A semantic method to prove strong normalization from weak normalization (draft)

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## 1 Minimal deduction modulo

### 1.1 Syntax

We consider a set ( $T$ ) of *sorts*, an infinite set of *variables* of each sort, a set ( $f$ ) of function symbols, and a set ( $P$ ) of predicate symbols, that come with their *rank*. The formation rules for objects and propositions are the usual ones.

- Variables of sort  $T$  are terms of sort  $T$ .
- If  $f$  is a function symbol of rank  $\langle T_1, \dots, T_n, U \rangle$  and  $t_1, \dots, t_n$  are respectively objects of sort  $T_1, \dots, T_n$ , then  $f(t_1, \dots, t_n)$  is a term of sort  $U$ .
- If  $P$  is a predicate symbol of rank  $\langle T_1, \dots, T_n \rangle$  and  $t_1, \dots, t_n$  are respectively objects of sort  $T_1, \dots, T_n$ , then  $P(t_1, \dots, t_n)$  is an *atomic proposition*.

Propositions are built-up from atomic propositions with the usual connective  $\Rightarrow$  and quantifier  $\forall$ . Remark that, implicitly, quantification in  $\forall x.A$  is restricted over the sort of the variable  $x$ .

#### Definition 1 (Terms and Propositions).

We call  $\mathcal{O}$  (as objects), the set of terms:  $t ::= x \mid f t \dots t$   
We call  $\mathcal{P}$ , the set of propositions:  $A ::= P t \dots t \mid A \Rightarrow A \mid \forall x.A$

In this language, proof-terms can contain both term variables (written  $x, y, \dots$ ) and proof variables (written  $\alpha, \beta, \dots$ ). Terms are written  $t, u, \dots$  while proof-terms are written  $\pi, \rho, \dots$ . We call  $\mathcal{X}$  the set of proof variables and  $\mathcal{Y}$  the set of term variables.

#### Definition 2 (Proof-terms).

We call  $\mathcal{T}$ , the set of proof-terms:  $\pi ::= \alpha \mid \lambda \alpha. \pi \mid \pi \pi' \mid \lambda x. \pi \mid \pi t$

Notice that variables  $\alpha$  and  $x$  are bound in the constructions  $\lambda \alpha. \pi$ , and  $\lambda x. \pi$ . Alphabetic equivalence, free and bound variables are defined as usual.

Each proof-term construction corresponds to a natural deduction rule: terms of the form  $\alpha$  express proofs built with the axiom rule, terms of the form  $\lambda \alpha. \pi$  and  $(\pi \pi')$  express proofs built respectively with the introduction and elimination

rules of the implication and terms of the form  $\lambda x.\pi$  and  $(\pi t)$  express proofs built with the introduction and elimination rules of the universal quantifier.

We call *neutral* those proof-terms of  $\mathcal{T}$  that are not abstractions *i.e.* of the form  $\alpha$ ,  $(\pi\pi')$  or  $(\pi t)$ . A proof-term is called *isolated* if it is neutral and only reduces on neutral terms.

## 1.2 Typing rules

We call *contexts*, lists of declarations  $[\alpha : A]$  where  $\alpha$  is a proof-variable and  $A$  is a proposition, such that each variable in a declaration is different from all the other variables of the context (in this way, we only consider *well formed* contexts, therefore we have to deal with alphabetic equivalence, when concatening them).

Given a congruence relation on propositions  $\equiv$ , we define typing rules as usual, in deduction modulo:

$$\begin{array}{c}
\frac{}{\Gamma, \alpha : A \vdash_{\equiv} \alpha : B} \quad A \equiv B \quad (\text{axiom}) \\
\frac{\Gamma, \alpha : A \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \lambda \alpha. \pi : C} \quad C \equiv A \Rightarrow B \quad (\Rightarrow\text{-intro}) \\
\frac{\Gamma \vdash_{\equiv} \pi : C \quad \Gamma \vdash_{\equiv} \pi' : A}{\Gamma \vdash_{\equiv} (\pi \pi') : B} \quad C \equiv A \Rightarrow B \quad (\Rightarrow\text{-elim}) \\
\frac{\Gamma \vdash_{\equiv} \pi : A}{\Gamma \vdash_{\equiv} \lambda x. \pi : B} \quad B \equiv \forall x. A, \quad x \notin FV(\Gamma) \quad (\forall\text{-intro}) \\
\frac{\Gamma \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \pi t : C} \quad B \equiv \forall x. A, \quad C \equiv (t/x)A, \quad t \text{ has the sort of } x \quad (\forall\text{-elim})
\end{array}$$

**Fig. 1.** Typing rules

## 1.3 Proof reduction rules and strong normalization

As usual in deduction modulo, the process of cut elimination is modeled by  $\beta$ -reduction. We consider the contextual closure of the reduction rules given figure 2. These rules correspond to proof reduction in natural deduction.

$$\begin{array}{c}
(\lambda \alpha. \pi \pi') \rightarrow_{\beta_{\pi}} (\pi' / \alpha) \pi \\
(\lambda x. \pi t) \rightarrow_{\beta_t} (t/x) \pi \quad (\text{if } x \text{ and } t \text{ have the same sort})
\end{array}$$

**Fig. 2.** Proof reduction rules

We write  $(\pi'/\alpha)\pi$  (resp.  $(t/x)\pi$ ) the substitution of  $\alpha$  (resp.  $x$ ) by  $\pi'$  (resp.  $t$ ) in  $\pi$ . We say that  $\pi$  reduces to  $\pi'$  if  $\pi \rightarrow_{\beta_i} \pi'$  or  $\pi \rightarrow_{\beta_\pi} \pi'$ . We write  $\pi \rightarrow \pi'$  if  $\pi$  reduces in one step to  $\pi'$ ,  $\pi \rightarrow^+ \pi'$  if  $\pi$  reduces in at least one step to  $\pi'$ , and  $\pi \rightarrow^* \pi'$  if  $\pi$  reduces in an arbitrary number of steps to  $\pi'$ .

A proof is said to be *normal* if it contains no redex. It is said to be weakly *normalizing* if it has a normal form and *strongly normalizing* if all reduction sequences issued from this proofs are finite. We write  $SN$  for the set of strongly normalizing proofs.

#### 1.4 Theories expressed in minimal deduction modulo

A theory expressed in minimal deduction modulo is defined by a many-sorted language in predicate logic  $\mathcal{L} = \langle (T), (F), (P) \rangle$  and a congruence relation  $\equiv$  on propositions of the associated many-sorted logic. We suppose  $\equiv$  not *ambiguous*, i.e. there does not exist  $x \in \mathcal{Y}$ ,  $A, B, C \in \mathcal{P}$  such that  $A \Rightarrow B \equiv \forall x.C$ . We will call  $\mathcal{L}_{\equiv}$  this theory.

*Remark 1.* Given a theory  $\mathcal{L}_{\equiv}$ , we will write  $\vdash$  for  $\vdash_{\equiv}$ .

**Proposition 1 (confluence and subject-reduction).**  $\rightarrow$  is confluent.

And for all contexts  $\Gamma$ , proof-terms  $\pi, \pi'$  and propositions  $A$ , if  $\Gamma \vdash \pi : A$  and  $\pi \rightarrow \pi'$ , then  $\Gamma \vdash \pi' : A$ .

**Example** As mentioned above deduction modulo permits to express (intentional) simple type theory [1] without any axiom. We show in the following, how minimal deduction modulo permits to express minimal (intentional) simple type theory, without any axiom (see [6] for details).

- The *sorts* are *simple types* inductively defined by:
- $\iota$  and  $o$  are sorts,
  - if  $T$  and  $U$  are sorts then  $T \rightarrow U$  is a sort.

- The language is composed of the individual symbols
- $S_{T,U,V}$  of sort  $(T \rightarrow U \rightarrow V) \rightarrow (T \rightarrow U) \rightarrow T \rightarrow V$ ,
  - $K_{T,U}$  of sort  $T \rightarrow U \rightarrow T$ ,
  - $\dot{\Rightarrow}$ , of sort  $o$ ,
  - $\dot{\forall}_T$  of sort  $(T \rightarrow o) \rightarrow o$ ,

the function symbols  $\alpha_{T,U}$  of rank  $\langle T \rightarrow U, T, U \rangle$ , and the predicate symbol  $\varepsilon$  of rank  $\langle o \rangle$ .

The combinators  $S_{T,U,V}$  and  $K_{T,U}$  are used to express functions. The objects  $\dot{\Rightarrow}$ , and  $\dot{\forall}_T$  allow to represent propositions as objects of sort  $o$ . Finally, the predicate  $\varepsilon$  allows to transform such an object  $t$  of type  $o$  into the actual corresponding proposition  $\varepsilon(t)$ .

$$\begin{aligned} \alpha(\alpha(\alpha(S_{T,U,V}, x), y), z) &\rightarrow \alpha(\alpha(x, z), \alpha(y, z)) \\ \alpha(\alpha(K_{T,U}, x), y) &\rightarrow x \\ \varepsilon(\alpha(\alpha(\dot{\Rightarrow}, x), y)) &\rightarrow \varepsilon(x) \Rightarrow \varepsilon(y) \\ \varepsilon(\alpha(\dot{\forall}_T, x)) &\rightarrow \forall y \varepsilon(\alpha(x, y)) \end{aligned}$$

## 2 Language-dependent truth values algebras

### 2.1 Definition

For all sets  $E$ , we call  $\mathbb{P}(E)$  the set of subsets of  $E$ .

For all sorts  $T$  of a language  $\mathcal{L}$ , we write  $\hat{T}$ , the set of closed terms of sort  $T$ .

#### Definition 3 (language-dependent tvas).

Let  $\mathcal{L} = \langle (T), (f), (P) \rangle$  be a many-sorted language in predicate logic.

$\langle \mathcal{B}, \Rightarrow, (\hat{\mathcal{A}}_T), (\hat{\forall}_T) \rangle$  is a LDTVA for  $\mathcal{L}$  if and only if:

- $\mathcal{B}$  is a set,
- $\Rightarrow$  is a function from  $\mathcal{B} \times \mathcal{B}$  to  $\mathcal{B}$ ,
- for all sorts  $T$ ,
  - $\hat{\mathcal{A}}_T$  is a set of functions from  $\hat{T}$  to  $\mathcal{B}$ :  $\hat{\mathcal{A}}_T \subseteq \hat{T} \mapsto \mathcal{B}$
  - $\hat{\forall}_T$  is a function from  $\hat{\mathcal{A}}_T$  to  $\mathcal{B}$ .

#### Definition 4 (Morphism).

Let  $\mathcal{B}^1 = \langle \mathcal{B}^1, \Rightarrow^1, (\hat{\mathcal{A}}_T^1), (\hat{\forall}_T^1) \rangle$  and  $\mathcal{B}^2 = \langle \mathcal{B}^2, \Rightarrow^2, (\hat{\mathcal{A}}_T^2), (\hat{\forall}_T^2) \rangle$  be two LDTVAs. A morphism from  $\mathcal{B}^1$  to  $\mathcal{B}^2$  is a function  $F$  from  $\mathcal{B}^1$  to  $\mathcal{B}^2$  such that:

- for all  $E, G \in \mathcal{B}^1$ ,  $F(E \Rightarrow^1 G) = F(E) \Rightarrow^2 F(G)$ ,
- for all sorts  $T$ ,  $x \in \hat{T}$  and  $f \in \hat{\mathcal{A}}_T$ ,  $F(\hat{\forall}_T^1 f) = \hat{\forall}_T^2 F \circ f$ .

#### Definition 5 (Valuation).

Given a LDTVA for  $\mathcal{L} = \langle (T), (f), (P) \rangle$ , a valuation  $\varphi$  is a substitution mapping term-variables of a sort to closed terms of the same sort. For all propositions  $A$  (resp. terms  $t$ ), we call  $\text{VAL}(A)$  (resp.  $\text{VAL}(t)$ ) the set of valuations whose domain contains the set of free variables of  $A$  (resp.  $t$ ).

We write  $x \notin \varphi$  for expressing the fact that  $\varphi(x)$  is not defined.

**Definition 6.** For all  $A \in \mathcal{P}$ , terms  $t$  and  $\varphi \in \text{VAL}(A)$ , we write  $|A|_\varphi$  the result of the substitution  $\varphi$  on  $A$ .

#### Definition 7 (Models).

Let  $\mathcal{L} = \langle (T), (f), (P) \rangle$  be a many-sorted language in predicate logic,

let  $\equiv$  be a congruence relation on propositions of minimal deduction based on  $\mathcal{L}$ ,

let  $\mathcal{B} = \langle \mathcal{B}, \Rightarrow, (\hat{\mathcal{A}}_T), (\hat{\forall}_T) \rangle$  be a LDTVA for  $\mathcal{L}$ .

1. We call  $\mathcal{B}$ -valued interpretations those functions which map every ordered pair of a proposition  $A$  and a valuation in  $\text{VAL}(A)$  to an element of  $\mathcal{B}$ .
2. A  $\mathcal{B}$ -valued interpretation  $\llbracket \cdot \rrbracket$  is a  $\mathcal{B}$ -valued model if and only if:
  - for all  $A, B \in \mathcal{P}$  and  $\varphi \in \text{VAL}(A \Rightarrow B)$ ,  $\llbracket A \Rightarrow B \rrbracket_\varphi = \llbracket A \rrbracket_\varphi \Rightarrow \llbracket B \rrbracket_\varphi$
  - for all  $A \in \mathcal{P}$ ,  $x$  of sort  $T$  and  $\varphi \in \text{VAL}(\forall x.A)$  such that  $x \notin \varphi$ ,  
 $\llbracket \forall x.A \rrbracket_\varphi = \hat{\forall}_T(t \mapsto \llbracket A \rrbracket_{\varphi + \langle x, t \rangle})$
  - for all  $A \in \mathcal{P}$ ,  $x$  of sort  $T$ ,  $t \in \hat{T}$  and  $\varphi \in \text{VAL}(\forall x.A)$  such that  $x \notin \varphi$ ,  
 $\llbracket (t/x)A \rrbracket_\varphi = \llbracket A \rrbracket_{\varphi + \langle x, t \rangle}$ .

3. A  $\mathcal{B}$ -valued model  $\llbracket \cdot \rrbracket$  is a model of the theory  $\mathcal{L}_{\equiv}$  if and only if:  
for all  $A, A' \in \mathcal{P}$ ,  $\varphi \in \text{VAL}(A)$  and  $\psi \in \text{VAL}(A')$ , if  $|A|_{\varphi} \equiv |A'|_{\psi}$ ,  
then  $\llbracket A \rrbracket_{\varphi} = \llbracket A' \rrbracket_{\psi}$

*Remark 2.* The previous conditions can be reformulated as: 2. Interpretations of propositions have to be adapted to the connectives to be a model. 3. Models have to be adapted to the congruence to be a model of the associated theory.

The following lemma explains that our definition of morphism is correct for the property of being a model of a theory  $\mathcal{L}_{\equiv}$ .

**Lemma 1.** For all LDTVAs  $\mathcal{B}_1$  and  $\mathcal{B}_2$  and morphisms  $F$  from  $\mathcal{B}_1$  to  $\mathcal{B}_2$ , if  $\llbracket \cdot \rrbracket$  is a  $\mathcal{B}_1$ -valued model of a theory  $\mathcal{L}_{\equiv}$ , then  $F \circ \llbracket \cdot \rrbracket$  is a  $\mathcal{B}_2$ -valued model of  $\mathcal{L}_{\equiv}$ .

### 3 About WN-reducibility candidates and typing

#### 3.1 $\mathcal{D}_{\equiv}$ , a ldtva of $(\equiv)$ well-typed WN-reducibility candidates

**Definition 8** ( $\mathcal{U}$ ).

$\mathcal{U} = \{(\Gamma, \pi) \text{ such that } \Gamma \text{ is a context and } \pi \text{ is a proof-term}\}$ .

**Definition 9.**

For all  $E \subseteq \mathcal{U}$ , we define the following properties :

- ( $P_{\equiv}$ ) There exists  $A_E$  such that  $\forall (\Gamma, \pi) \in E$ ,  $\Gamma \vdash \pi : A_E$
- ( $P_{1_{\equiv}}$ ) For all  $(\Gamma, \pi) \in E$ ,  $\pi \in \text{WN}$
- ( $P_{2_{\equiv}}$ ) For all  $(\Gamma, \pi) \in E$  and  $\pi' \in \text{WN}$  such that  $\pi \rightarrow \pi'$ ,  $(\Gamma, \pi') \in E$
- ( $P_{3_{\equiv}}$ ) For all  $(\Gamma, \pi) \in \mathcal{U}$ ,
  - If  $\pi \in \text{WN}$ ,  $\pi$  is isolated and  $\Gamma \vdash \pi : A_E$ , then  $(\Gamma, \pi) \in E$
  - If  $(\Gamma, \pi) \in E$  and  $\pi' \rightarrow_{\beta_t} \pi$ , then  $(\Gamma, \pi') \in E$ .

*Remark 3.* For all  $E \subseteq \mathcal{U}$ , if  $E$  satisfies ( $P_{\equiv}$ ) and ( $P_{3_{\equiv}}$ ), then for all proof-variables  $\alpha$ ,  $(\alpha : A_E, \alpha) \in E$ , as  $\alpha$  is isolated and in  $\text{WN}$ .

**Definition 10 (domain  $\mathcal{D}_{\equiv}$ ).** We call  $\mathcal{D}_{\equiv}$  the set of subsets of  $\mathcal{U}$  which satisfy ( $P_{\equiv}$ ), ( $P_{1_{\equiv}}$ ), ( $P_{2_{\equiv}}$ ) and ( $P_{3_{\equiv}}$ ).

**Definition 11 (leaves).**

The leaves of a proof-term  $\pi$  are its first reducts which are normal or not neutral. ( $\rho$  is a leaf of  $\pi$  if and only if it is normal or not neutral and there exists  $n \geq 0$  and  $\pi_1 \dots \pi_{n-1}$  neutral not normal terms such that  $\pi = \pi_1 \rightarrow \dots \rightarrow \pi_{n-1} \rightarrow \rho$ ). We call  $\mathcal{L}(\pi)$  the set of leaves of  $\pi$ ,  $\mathcal{L}_1(\pi)$  the set of neutral normal leaves of  $\pi$ , and  $\mathcal{L}_{\lambda}$  the set of not neutral leaves of  $\pi$ .

*Remark 4.* - The only leaf of a normal or not neutral proof-term is itself.  
- If  $\pi$  is a neutral non-normal proof-term, then  $\rho \in \mathcal{L}(\pi)$  if and only if there exists a one-step reduct  $\pi'$  of  $\pi$  such that  $\rho \in \mathcal{L}(\pi')$ .  
- If  $\pi \in \text{WN}$ , then  $\mathcal{L}(\pi) \neq \emptyset$ .

**Definition 12** ( $\Rightarrow$ ). For all  $E, F \in \mathcal{D}_{\equiv}$ ,  
 $E \Rightarrow F = \{(\Gamma, \pi) \in \mathcal{U} \text{ such that } \pi \in WN, \Gamma \vdash \pi : A_E \Rightarrow A_F \text{ and}$   
 $\quad - \forall \rho \in \mathcal{L}_\lambda(\pi), \rho = \lambda\alpha.\rho' \text{ with } (\Gamma, \alpha : A_E, \rho) \in F$   
 $\quad - \forall \rho \in \mathcal{L}_\downarrow(\pi) \text{ and } (\Gamma', \pi') \in E, (\Gamma\Gamma', \rho\pi') \in F\}$

*Remark 5.* We recall the fact that we only consider well-formed contexts, therefore the only variables  $\Gamma$  and  $\Gamma'$  can share have to be declared proofs of equivalent propositions, otherwise we have to deal with  $\alpha$ -conversion when concatenating  $\Gamma$  and  $\Gamma'$ .

**Lemma 2.**  $\Rightarrow$  is a function from  $\mathcal{D}_{\equiv} \times \mathcal{D}_{\equiv}$  to  $\mathcal{D}_{\equiv}$ .

*Proof.* Let  $E, F \in \mathcal{D}_{\equiv}$ , and  $(\Gamma, \pi) \in E \Rightarrow F$ ,

(P $_{\equiv}$ ) By definition,  $A_{E \Rightarrow F} \equiv A_E \Rightarrow A_F$ .

(P $_{1_{\equiv}}$ ) By definition.

(P $_{2_{\equiv}}$ ) By subject-reduction, the fact that  $F$  satisfies (P $_{2_{\equiv}}$ ) and the fact that all leaves of a reduct of a proof term  $\pi$  are also leaves or reducts of leaves of  $\pi$ .

(P $_{3_{\equiv}}$ )

- By the fact that an isolated term has only neutral leaves, and that if  $\pi$  is a neutral normal term, and  $\pi'$  is a term in  $WN$ , then  $\pi\pi'$  is isolated and in  $WN$ .
- By the fact that if  $\pi' \rightarrow_{\beta_t} \pi$ , then in a given context,  $\pi$  and  $\pi'$  have the same type, if  $\pi \in WN$  then so does  $\pi'$  and all leaves of  $\pi'$  are either leaves of  $\pi$ , either "  $\beta_t$ -expansions" of leaves of  $\pi$ .

**Definition 13** ( $\mathring{A}_T$ ). For all sorts  $T$ ,

$\mathring{A}_T = \{f : \hat{T} \mapsto \mathcal{D}_{\equiv}, \text{ such that there exists } A_f \in \mathcal{P} \text{ and } x_f \in \mathcal{X} \text{ such that}$   
 $\quad \text{for all } t \in \hat{T} \text{ and } (\Gamma, \pi) \in f(t), \Gamma \vdash \pi : (t/x_f)A_f\}$

**Definition 14** ( $\mathring{\forall}_T$ ). For all sorts  $T$  and functions  $f \in \mathring{A}_T$ ,

$\mathring{\forall}_T.f = \{(\Gamma, \pi) \in \mathcal{U} \text{ such that for all } t \in \hat{T}, (\Gamma, \pi t) \in f(t)\}$

**Lemma 3.** For all sorts  $T$ ,  $\mathring{\forall}_T$  is a function from  $\mathring{A}_T$  to  $\mathcal{D}_{\equiv}$ .

*Proof.* Let  $f \in \mathring{A}_T$ , and  $(\Gamma, \pi) \in \mathring{\forall}_T.f$

(P $_{\equiv}$ ) Let  $t \in \hat{T} (\neq \emptyset)$ . Then  $(\Gamma, \pi t) \in f(t)$ . As  $f \in \mathring{\forall}_T$ , we have  $\Gamma \vdash \pi t : (t/x_f)A_f$ . Therefore  $\Gamma \vdash \pi : \forall x_f.A_f$ , by case on the last rule used in the derivation of  $\Gamma \vdash \pi t : (t/x_f)A_f$ . Finally,  $A_{\mathring{\forall}_T.f} \equiv \forall x_f.A_f$ .

(P $_{1_{\equiv}}$ ) Let  $t \in \hat{T} (\neq \emptyset)$ . Then  $(\Gamma, \pi t) \in f(t) \in \mathcal{D}_{\equiv}$  therefore  $\pi t \in WN$  and so does  $\pi$ .

(P $_{2_{\equiv}}$ ) Let  $\pi'$  such that  $\pi \rightarrow \pi'$ . Then, for all  $t \in \hat{T}$ ,  $\pi t \rightarrow \pi' t$ , therefore  $\pi' t \in f(t) \in \mathcal{D}_{\equiv}$ .

(P $_{3_{\equiv}}$ )

- Let  $(\Gamma, \tau) \in \mathcal{U}$  such that  $\tau \in WN$ ,  $\tau$  is isolated and  $\Gamma \vdash \tau : \forall x_f.A_f$ . Let  $t \in \hat{T}$  then  $\Gamma \vdash \tau t : (t/x_f)A_f$ ,  $\tau t$  is isolated as  $\tau$  is, and  $\tau t \in WN$ , as  $\tau \in WN$ . Finally,  $\tau \in \mathring{\forall}_T.f$ , as  $f(t)$  satisfies (P $_{3_{\equiv}}$ ), for all  $t \in \hat{T}$ .

- Let  $\pi'$  such that  $\pi' \rightarrow_{\beta_t} \pi$ , then for all  $t \in \hat{T}$ ,  $\pi't \rightarrow_{\beta_t} \pi t$  therefore  $(\Gamma, \pi't) \in f(t)$  as it satisfies  $(P_{3_{\equiv}})$ . Hence  $(\Gamma, \pi') \in \check{\forall}_T f$ .

**Definition 15** ( $\mathcal{D}_{\equiv}$ ).  $\mathcal{D}_{\equiv}$  is the LDTVA  $\langle \mathcal{D}_{\equiv}, \overset{\circ}{\Rightarrow}, (\overset{\circ}{\mathcal{A}}_T), (\overset{\circ}{\forall}_T) \rangle$ .

### 3.2 Building a $\mathcal{D}_{\equiv}$ -valued interpretation of WN theories $\mathcal{L}_{\equiv}$

Let us now define a first  $\mathcal{D}_{\equiv}$ -valued model, by using directly definitions of  $\overset{\circ}{\Rightarrow}$  and  $\overset{\circ}{\forall}_T$ , and well-chosen interpretations of atomic propositions.

**Definition 16.** Let  $A$  be a proposition and  $\varphi \in \text{VAL}(A)$ .

We define the subset of  $\mathcal{U}$ ,  $[A]_{\varphi}$  by induction over the structure of  $A$ .

- $[P t_1 \dots t_n]_{\varphi} = \{(\Gamma, \pi) \in \mathcal{U} \text{ such that } \pi \in \text{WN and } \Gamma \vdash \pi : P \varphi(t_1) \dots \varphi(t_n)\}$
- $[B \Rightarrow C]_{\varphi} = [B]_{\varphi} \overset{\circ}{\Rightarrow} [C]_{\varphi}$
- $[\forall x.B]_{\varphi} = \overset{\circ}{\forall}_T (t \mapsto [B]_{\varphi+\langle x,t \rangle})$

**Lemma 4.** For all  $A \in \mathcal{P}$ ,  $x$  of sort  $T$ ,  $t \in \hat{T}$  and  $\varphi \in \text{VAL}(\forall x.A)$  such that  $x \notin \varphi$ , we have  $[(t/x)A]_{\varphi} = [A]_{\varphi+\langle x,t \rangle}$ .

*Proof.* By induction on  $A$ .

**Lemma 5.** For all  $A \in \mathcal{P}$ , and  $\varphi \in \text{VAL}(A)$ ,

$[A]_{\varphi} \in \mathcal{D}_{\equiv}$  with  $A_{[A]_{\varphi}} = A_{\varphi}$  (i.e.,  $\forall(\Gamma, \pi) \in [A]_{\varphi}, \Gamma \vdash \pi : A_{\varphi}$ ).

*Proof.* By induction on  $A$ .

- If  $A$  is an atomic proposition  $P t_1 \dots t_n$ ,
  - ( $P_{\equiv}$ ) By definition. (with  $A_{[P t_1 \dots t_n]_{\varphi}} \equiv P \varphi(t_1) \dots \varphi(t_n)$ ).
  - ( $P_{1_{\equiv}}$ ) By definition.
  - ( $P_{2_{\equiv}}$ ) By subject-reduction.
  - ( $P_{3_{\equiv}}$ ) By definition.
- If  $A = B \Rightarrow C$ , as  $\overset{\circ}{\Rightarrow} : \mathcal{D}_{\equiv} \times \mathcal{D}_{\equiv} \mapsto \mathcal{D}_{\equiv}$ , we conclude by induction hypothesis (with  $A_{[B \Rightarrow C]_{\varphi}} = A_{[B]_{\varphi} \overset{\circ}{\Rightarrow} [C]_{\varphi}} \equiv A_{[B]_{\varphi}} \Rightarrow A_{[C]_{\varphi}} \equiv B_{\varphi} \Rightarrow C_{\varphi} = (B \Rightarrow C)_{\varphi}$ ).
- If  $A = \forall x.B$ , let  $T$  be the sort of  $x$  and  $f = t \mapsto [B]_{\varphi+\langle x,t \rangle}$ . Then  $f$  is a function from  $\hat{T}$  to  $\mathcal{D}_{\equiv}$ , by induction hypothesis. Moreover, for all  $t \in \hat{T}$ ,  $A_{f(t)} = B_{\varphi+\langle x,t \rangle} = (t/x)B_{\varphi}$ , by induction hypothesis. Therefore  $f \in \overset{\circ}{\mathcal{A}}_T$  and  $\overset{\circ}{\forall}_T f \in \mathcal{D}_{\equiv}$  (with  $A_{[\forall x.B]_{\varphi}} = \forall x.A_f = \forall x.B_{\varphi}$ ).

At this point, we have  $\mathcal{D}_{\equiv}$ -valued model which is adapted to typing but not necessarily  $\equiv$ -adapted. Indeed, in a theory where we have two atomic proposition symbols  $P$  and  $Q$  such that  $P \equiv (Q \Rightarrow Q)$  (notice that such a theory can be weakly normalizing), then for all valuations  $\varphi \in \text{VAL}(P) \cap \text{VAL}(Q)$ ,  $[P]_{\varphi} \neq [Q]_{\varphi} \overset{\circ}{\Rightarrow} [Q]_{\varphi}$ . We have then to modify this interpretation to make it a  $\mathcal{D}_{\equiv}$ -valued model of  $\mathcal{L}_{\equiv}$ .



### 3.3 Adapting this interpretation to the congruence

**Definition 17.** We define a second interpretation  $[\cdot]_{\cdot}$ , as follows :  
for all  $A \in \mathcal{P}$  and  $\varphi \in \text{VAL}(A)$ ,

$$[A]_{\varphi} = \bigcap_{A \varphi \equiv A'_{\psi}} [A']_{\psi}$$

*Remark 6.* For all  $A, A' \in \mathcal{P}$ ,  $\varphi \in \text{VAL}(A)$  and  $\psi \in \text{VAL}(A')$  such that  $A \varphi \equiv A'_{\psi}$ , we have  $[A]_{\varphi} = [A']_{\psi}$ , by definition.

Then we prove that  $[\cdot]_{\cdot}$  is also a  $\mathcal{D}_{\equiv}$ -valued interpretation adapted to typing.

**Lemma 6.** For all  $A \in \mathcal{P}$ , and  $\varphi \in \text{VAL}(A)$ ,

$$[A]_{\varphi} \in \mathcal{D}_{\equiv} \quad \text{with } A_{[A]_{\varphi}} = A_{\varphi} \quad (\text{i.e.}, \forall (\Gamma, \pi) \in [A]_{\varphi}, \Gamma \vdash \pi : A_{\varphi}).$$

*Proof.* Let  $A \in \mathcal{P}$ , and  $\varphi \in \text{VAL}(A)$ ,

By lemma 5 and the fact that  $[A]_{\varphi} \subseteq [A']_{\psi}$ , for all  $A'_{\psi} \equiv A_{\varphi}$ .

**Lemma 7.** For all  $A \in \mathcal{P}$ ,  $x$  of sort  $T$ ,  $t \in \hat{T}$  and  $\varphi \in \text{VAL}(\forall x.A)$  such that  $x \notin \varphi$ , we have  $[(t/x)A]_{\varphi} = [A]_{\varphi + (x,t)}$ .

*Proof.* By lemma 7.

Finally, we proved, that  $[\cdot]_{\cdot}$  is a  $\mathcal{D}_{\equiv}$ -valued interpretation of propositions adapted to typing and to the congruence relation  $\equiv$ . Let us now show that  $[\cdot]_{\cdot}$  is also a  $\mathcal{D}_{\equiv}$ -valued model of weakly normalizing theories  $\mathcal{L}_{\equiv}$ , i.e. it is also adapted to connectives, when  $\mathcal{L}_{\equiv}$  is weakly normalizing.

### 3.4 $[\cdot]_{\cdot}$ is a $\mathcal{D}_{\equiv}$ -valued model of weakly normalizing theories $\mathcal{L}_{\equiv}$

In order to prove that  $[\cdot]_{\cdot}$  is a  $\mathcal{D}_{\equiv}$ -valued model of  $\mathcal{L}_{\equiv}$ , if it is weakly normalizing, we proceed by *reductio ad absurdum*, showing that if  $[\cdot]_{\cdot}$  is not connectives-adapted, then we can exhibit a typing judgement  $\Gamma \vdash \pi : A$  such that  $\pi \notin \text{WN}$ .

**Lemma 8.**

If there exists  $A, B \in \mathcal{P}$  and  $\varphi \in \text{VAL}(A \Rightarrow B)$ , such that  $[A \Rightarrow B]_{\varphi} \neq [A]_{\varphi} \overset{\Rightarrow}{\Rightarrow} [B]_{\varphi}$  then there exists  $\pi \in \mathcal{T}$ ,  $C \in \mathcal{P}$ ,  $\psi \in \text{VAL}(C)$  such that  $\Gamma \vdash \pi : C_{\psi}$  and  $(\Gamma, \pi) \notin [C]_{\psi}$ .

*Proof.* – If there exists  $(\Gamma, \pi) \in \mathcal{U}$  such that  $(\Gamma, \pi) \notin [A \Rightarrow B]_{\varphi}$  and

$$(\Gamma, \pi) \in [A]_{\varphi} \overset{\Rightarrow}{\Rightarrow} [B]_{\varphi}. \quad \text{Then } \Gamma \vdash \pi : A_{\varphi} \Rightarrow B_{\varphi} = (A \Rightarrow B)_{\varphi}.$$

We take  $C = A \Rightarrow B$  and  $\psi = \varphi$ .

– If there exists  $(\Gamma, \pi) \in \mathcal{U}$  such that  $(\Gamma, \pi) \in [A \Rightarrow B]_{\varphi}$  and  $(\Gamma, \pi) \notin [A]_{\varphi} \overset{\Rightarrow}{\Rightarrow} [B]_{\varphi}$ .

Notice that as  $\Gamma \vdash \pi : A_{\varphi} \Rightarrow B_{\varphi}$ ,  $\pi$  cannot reduce to a term-abstraction, by subject-reduction. Then, as  $\pi \in \text{WN}$  and  $\Gamma \vdash \pi : A_{\varphi} \Rightarrow B_{\varphi}$ , either there exists  $\lambda \alpha. \rho \in \mathcal{L}_{\lambda}(\pi)$  such that  $(\Gamma, \alpha : A_{\varphi}, \rho) \notin [B]_{\varphi}$ , with  $\Gamma, \alpha : A_{\varphi} \vdash \rho : B_{\varphi}$  by subject-reduction. Either there exists  $\rho \in \mathcal{L}_{\perp}(\pi)$  and  $(\Gamma', \pi') \in [A]_{\varphi}$  such that  $(\Gamma \Gamma', \rho \pi') \notin [B]_{\varphi}$ , with  $\Gamma \Gamma' \vdash \rho \pi' : B_{\varphi}$  by subject-reduction. We take  $C = B$  and  $\psi = \varphi$

**Lemma 9.**

If there exists  $A \in \mathcal{P}$ ,  $\varphi \in \text{VAL}(A)$ , and  $x$  of sort  $T$  such that  $x \notin \varphi$ , and

$$[\forall x.A]_{\varphi} \neq \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi+\langle x,t \rangle})$$

then there exists  $\pi \in \mathcal{T}$ ,  $C \in \mathcal{P}$ ,  $\psi \in \text{VAL}(C)$  such that  $\Gamma \vdash \pi : C_{\psi}$  and  $(\Gamma, \pi) \notin [C]_{\psi}$ .

*Proof.* – If there exists  $(\Gamma, \pi) \in \mathcal{U}$  such that  $(\Gamma, \pi) \notin [\forall x.A]_{\varphi}$  and

$$(\Gamma, \pi) \in \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi+\langle x,t \rangle}). \text{ Then } \Gamma \vdash \pi : (\forall x.A)_{\varphi}.$$

We take  $C = \forall x.A$  and  $\psi = \varphi$ .

– If there exists  $(\Gamma, \pi) \in \mathcal{U}$  such that  $(\Gamma, \pi) \in [\forall x.A]_{\varphi}$  and  $(\Gamma, \pi) \notin \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi+\langle x,t \rangle})$ .

Then there exists  $t \in \hat{T}$  such that  $(\Gamma, \pi t) \notin [A]_{\varphi+\langle x,t \rangle}$ . As  $\Gamma \vdash \pi : (\forall x.A)_{\varphi}$ , we have  $\Gamma \vdash \pi t : (t/x)A_{\varphi} = A_{\varphi+\langle x,t \rangle}$ . We take  $C = A$  and  $\psi = \varphi + \langle x, t \rangle$

**Lemma 10.**

If there exists  $A, B \in \mathcal{P}$ ,  $\varphi \in \text{VAL}(A \Rightarrow B)$  or  $\varphi' \in \text{VAL}(\forall x.A)$  with  $x$  of sort  $T$ ,  $x \notin \varphi'$  and

$$[A \Rightarrow B]_{\varphi} \neq [A]_{\varphi} \overset{\circ}{\Rightarrow} [B]_{\varphi} \quad \text{or} \quad [\forall x.A]_{\varphi'} \neq \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi'+\langle x,t \rangle})$$

then there exists  $D \in \mathcal{P}$ ,  $\pi \in \mathcal{T}$ ,  $\psi \in \text{VAL}(D)$  such that  $\Gamma \vdash \pi : D_{\psi}$  and  $(\Gamma, \pi) \notin [D]_{\psi}$ .

*Proof.* By lemmas 8 and 9, there exists  $C, \Gamma, \pi$  and  $\psi$  such that  $\Gamma \vdash \pi : C_{\psi}$  and  $(\Gamma, \pi) \notin [C]_{\psi}$ . Therefore, there exists a proposition  $D$  and  $\psi' \in \text{VAL}(D)$  such that  $D_{\psi'} \equiv C_{\psi}$  and  $(\Gamma, \pi) \notin [D]_{\psi'}$ . And  $\Gamma \vdash \pi : D_{\psi'}$ , by equivalence of  $C_{\psi}$  and  $D_{\psi'}$ .

**Lemma 11.**

If there exists  $A, B \in \mathcal{P}$ ,  $\varphi \in \text{VAL}(A \Rightarrow B)$  or  $\varphi' \in \text{VAL}(\forall x.A)$  with  $x$  of sort  $T$ ,  $x \notin \varphi'$

$$\text{and} \quad [A \Rightarrow B]_{\varphi} \neq [A]_{\varphi} \overset{\circ}{\Rightarrow} [B]_{\varphi} \quad \text{or} \quad [\forall x.A]_{\varphi'} \neq \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi'+\langle x,t \rangle})$$

then there exists a (term-closed) proposition  $E$ ,  $\pi \in \mathcal{T}$  and a context  $\Gamma$  such that

$$\Gamma \vdash \pi : E \text{ and } \pi \notin WN.$$

*Proof.* By lemma 10, there exists a proposition  $D$ , a context  $\Gamma$ , a proof  $\pi$  and  $\varphi \in \mathcal{V}(D)$  such that  $\Gamma \vdash \pi : D_{\varphi}$  and  $(\Gamma, \pi) \notin [D]_{\varphi}$ . By induction on  $D$ .

– if  $D$  is atomic, then as  $\Gamma \vdash \pi : D_{\varphi}$ , we have  $\pi \notin WN$ .

– if  $D = (F \Rightarrow G)_{\varphi}$ ,

then  $\Gamma \vdash \pi : (F \Rightarrow G)_{\varphi}$  and  $(\Gamma, \pi) \notin [F \Rightarrow G]_{\varphi} = [F]_{\varphi} \overset{\circ}{\Rightarrow} [G]_{\varphi}$ . If  $\pi \in WN$ , either there exists  $\lambda \alpha. \rho \in \mathcal{L}_{\lambda}(\pi)$  such that  $(\Gamma, \alpha : F_{\varphi}, \rho) \notin [G]_{\varphi}$ . Either there exists  $\rho \in \mathcal{L}_{\downarrow}(\pi)$  and  $(\Gamma', \pi') \in [F]_{\varphi}$  such that  $(\Gamma \Gamma', \rho \pi') \notin [G]_{\varphi}$ , with  $\Gamma \Gamma' \vdash \rho \pi' : G_{\varphi}$ . We conclude by induction hypothesis.

– if  $D = \forall x.F$ ,

then  $\Gamma \vdash \pi : (\forall x.F)_{\varphi}$  and  $(\Gamma, \pi) \notin [\forall x.F]_{\varphi}$ . Therefore there exists  $t \in \hat{T}$  such that  $(\Gamma, \pi t) \notin [F]_{\varphi+\langle x,t \rangle}$ , with  $\Gamma \vdash \pi t : A_{\varphi+\langle x,t \rangle}$ . We conclude by induction hypothesis.

**Proposition 2 (Completeness).** *If the theory  $\mathcal{L}_{\equiv}$  is weakly normalizing, then  $[\cdot]_{\cdot} = \langle A, \varphi \rangle \mapsto [A]_{\varphi}$  is a  $\mathcal{D}_{\equiv}$ -model of this theory.*

*Proof.* By remark 6 and lemmas 6 and 11.

## 4 From $\mathcal{D}_{\equiv}$ to $\mathcal{C}'$

### 4.1 $\mathcal{C}'$ , yet another algebra of candidates.

#### Definition 18.

For all sets  $E$  of proof-terms, we define the following properties :

- (CR<sub>1</sub>) For all  $\pi \in E$ ,  $\pi \in SN$ .
- (CR<sub>2</sub>) For all  $\pi \in E$ , for all  $\pi' \in \mathcal{T}$  such that  $\pi \rightarrow \pi'$ , then  $\pi' \in E$ .
- (CR'<sub>3</sub>) for all  $n \in \mathbb{N}$ , for all  $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$ , if
  - for all  $i \leq n$ ,  $\mu_i$  is neutral and not normal,
  - $\forall \rho_1, \dots, \rho_n \in \mathcal{T}$  such that for all  $i \leq n$ ,  $\mu_i \leq_n \rightarrow \rho_i$ ,  $(\rho_i/\alpha_i)_{i \leq n} \nu \in E$
 then  $(\mu_i/\alpha_i)\nu \in E$ .

*Remark 7.* If  $E$  satisfies (CR'<sub>3</sub>) then, in particular, all neutral *not normal* terms whose all one-steps reducts are in  $E$ , is in  $E$ . That is slightly different from the usual (CR<sub>3</sub>) of reducibility candidates, where the neutral term can be normal, therefore all neutral normal terms are in all reducibility candidates.

#### Definition 19 ( $\rightleftharpoons$ ).

For all  $E, F \subseteq \mathcal{T}$ ,  $E \rightleftharpoons F = \{\pi \in SN \text{ such that}$

- $\forall \rho \in \mathcal{L}_\lambda(\pi)$ ,  $\rho = \lambda\alpha.\rho'$  with  $\rho \in F$
- $\forall \rho \in \mathcal{L}_\downarrow(\pi)$  and  $\pi' \in E$ ,  $\rho\pi' \in F\}$

**Lemma 12.**  $\rightleftharpoons$  is a function from  $\mathcal{C}' \times \mathcal{C}'$  to  $\mathcal{C}'$ .

*Proof.* Let  $E, F \in \mathcal{C}'$  and  $\pi \in E \rightleftharpoons F$ ,

- (CR<sub>1</sub>)  $\pi \in SN$ , by definition.
- (CR<sub>2</sub>) If  $\pi'$  is a one-step reduct of  $\pi$ , then for all  $\pi' \in SN$  and all its leaves are leaves of  $\pi$ , or reducts of leaves of  $\pi$ .
- (CR'<sub>3</sub>) Let  $\pi = (\mu_i/\alpha_i)\nu$  with each  $\mu_i$  neutral not normal and such that for all  $(\rho_i)$  each respectively a one-step reduct of  $\mu_i$ ,  $(\rho_i/\alpha_i)\nu \in E \rightleftharpoons F$ . We can first notice that  $\pi$  cannot reduce to a term-abstraction, by confluence. Let us prove that  $\pi \in E \rightleftharpoons F$ , by induction on the length  $l$  of the maximal length of a reductions sequence from  $\pi$  to one of its leaves.
  - If  $l = 0$ , then  $\pi$  is either normal and neutral, either a proof-abstraction.
    - \* If  $\pi$  is neutral and normal then none of the  $\mu_i$  appears in  $\nu$ , hence  $\pi \in E \rightleftharpoons F$ .
    - \* If  $\pi = \lambda\alpha.\pi'$  then, as each  $\mu_i$  is neutral,  $\nu = \lambda\alpha.\nu'$ , with  $\pi' = (\mu_i/\alpha_i)\nu'$ . And for all  $(\rho_i)$  each respectively a one-step reduct of  $\mu_i$ ,  $(\rho_i/\alpha_i)\nu = \lambda\alpha.(\rho_i/\alpha_i)\nu' \in E \rightleftharpoons F$ , therefore  $(\rho_i/\alpha_i)\nu' \in F$ . Finally,  $\pi' \in F$  as it satisfies (CR'<sub>3</sub>), and  $\pi = \lambda\alpha.\pi' \in E \rightleftharpoons F$ .
  - If  $l > 0$ , then all its leaves are leaves of a one-step reduct of  $\pi$ , wich is in  $E \rightleftharpoons F$ , by induction hypothesis.

#### Definition 20 ( $\tilde{\mathcal{A}}_T$ ).

For all sorts  $T$ ,  $\tilde{\mathcal{A}}_T = \hat{T} \mapsto \mathcal{C}'$ .

**Definition 21** ( $\tilde{\forall}_T$ ). For all sorts  $T$  and function  $f \in \tilde{\mathcal{A}}_T$ ,  
 $\tilde{\forall}_T.f = \{\pi \in \mathcal{T} \text{ such that for all } t \in \hat{T}, \pi t \in f(t)\}$

**Lemma 13.** For all sorts  $T$ ,  $\tilde{\forall}_T$  is a function from  $\tilde{\mathcal{A}}_T$  to  $\mathcal{C}'$ .

*Proof.* Let  $T$  be a sort,  $f \in \tilde{\mathcal{A}}_T$  and  $\pi \in \tilde{\forall}_T.f$ .

(CR<sub>1</sub>) Let  $t \in \hat{T}$  ( $\neq \emptyset$ ), then  $\pi t \in f(t) \in \mathcal{C}'$ , therefore  $\pi t \in SN$  and so does  $\pi$ .

(CR<sub>2</sub>) Let  $\pi'$  such that  $\pi \rightarrow \pi'$ . Then for all  $t \in \hat{T}$ ,  $\pi' t$  is a one-step reduct of  $\pi t$ .

(CR<sub>3</sub>') If there exists  $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$ , such that each  $\mu_i$  is neutral not normal,  $\tau = (\mu_i/\alpha_i)_{i \leq n} \nu$  and for all  $(\rho_i)_{i \leq n} \subseteq \mathcal{T}$ , such that for all  $i \leq n$ ,  $\mu_i \leq n \rightarrow \rho_i$ , then  $(\rho_i/\alpha_i)_{i \leq n} \nu \in \tilde{\forall}_T.f$ . Then, for all  $t \in \hat{T}$ ,  $\tau t = (\mu_i/\alpha_i)_{i \leq n} \nu t = (\mu_i/\alpha_i)_{i \leq n} \nu'$  with  $\nu' = \nu t$ . And for all  $(\rho_i)_{i \leq n} \subseteq \mathcal{T}$ , such that for all  $i \leq n$ ,  $\mu_i \leq n \rightarrow \rho_i$ , we have  $(\rho_i/\alpha_i)_{i \leq n} \nu' = (\rho_i/\alpha_i)_{i \leq n} \nu t \in f(t)$  by hypothesis, therefore  $\tau t \in f(t)$  as it satisfies (CR<sub>3</sub>'). And finally,  $\tau \in \tilde{\forall}_T.f$ .

**Definition 22** ( $\mathcal{C}'$ ).  $\mathcal{C}'$  is the LDTVVA  $\langle \mathcal{C}', \Rightarrow, (\tilde{\mathcal{A}}_T), (\tilde{\forall}_T) \rangle$ .

## 4.2 Building a function from $\mathcal{D}_{\equiv}$ to $\mathcal{C}'$

**Definition 23** ( $\Delta$ ). We consider a context which contains an infinite number of variables for each proposition.  $\Delta = (\beta_i^A : A)_{A \in \mathcal{P}, i \in \mathbb{N}}$ .

**Definition 24** (Cl). For all  $E \subseteq \mathcal{U}$ , we define  $Cl(E)$  as follows :  
for all  $k \in \mathbb{N}$ ,

- $Cl^0(E) = \{\pi \in \mathcal{T} \text{ such that } (\Delta, \pi) \in E \text{ and } \pi \text{ is normal}\}$
- $Cl^{k+1}(E) = \{\pi \in \mathcal{T}, \text{ such that there exists } n \in \mathbb{N}:$   
 $\exists \nu_\pi \in \mathcal{T}, \exists (\mu_i)_{i \leq n} \subseteq SN, \text{ each neutral not normal s.t.}$   
 $\pi = (\mu_i/\alpha_i)_{i \leq n} \nu_\pi \text{ and } \forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}, \text{ s.t. } \forall i \leq n, \rho_i \in \mathcal{L}(\mu_i),$   
 $\text{we have } (\rho_i/\alpha_i)_{i \leq n} \nu_\pi \in Cl^k(E)\}$
- $Cl(E) = \cup_{n \in \mathbb{N}} Cl^n(E)$

*Remark 8.* For all  $E \in \mathcal{D}_{\equiv}$ ,

1. for all  $k \in \mathbb{N}$ ,  $Cl^k(E) \subseteq Cl^{k+1}(E)$ ,
2.  $Cl(E) \neq \emptyset$  as  $Cl^0(E)$  contains all variables  $\alpha$  such that  $\Delta \vdash \alpha : A_E$ .
3. if  $\pi \in Cl(E)$  and  $\pi$  is normal, then  $\pi \in Cl^0(E)$ .

**Proposition 3.**

For all  $E \in \mathcal{D}_{\equiv}$ ,  $Cl(E) \in \mathcal{C}'$ .

*Proof.* See [2]

### 4.3 Proving that $Cl(\cdot)$ is a morphism

#### $\Rightarrow$ -morphism

We prove now that for all  $E, F \in \mathcal{D}_{\equiv}$ , we have  $Cl(E \Rightarrow F) = Cl(E) \Rightarrow Cl(F)$ .

**Lemma 14.** *For all  $E \subseteq \mathcal{T}$  and  $\pi \in \mathcal{T}$ ,*

*If  $\pi \in SN$ ,  $\pi$  is neutral not normal and  $\forall \rho \in \mathcal{L}(\pi)$ ,  $\rho \in Cl(E)$ , then  $\pi \in Cl(E)$*

*Proof.* As  $\pi \in SN$ ,  $\mathcal{L}(\pi)$  is defined and finite.

And, if we call  $k_m = \max\{\min\{k, \rho \in Cl^k(E)\}, \rho \in \mathcal{L}(\pi)\}$ ,

then  $\pi \in Cl^{k_m+1}(E) \subseteq Cl(E)$ .

*Remark 9.* In the same way, if there exists  $\nu_\pi \in \mathcal{T}$ , and  $(\mu_i)_{i \leq n} \subseteq SN$ , each neutral not normal such that  $\pi = (\mu_i/\alpha_i)_{i \leq n} \nu_\pi$  and  $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$ , such that for all  $i \leq n$ ,  $\rho_i \in \mathcal{L}(\mu_i)$ , then  $(\rho_i/\alpha_i)_{i \leq n} \nu_\pi \in Cl(E)$ , we have  $\pi \in Cl(E)$ .

*Remark 10.* If  $\pi \in Cl(E \Rightarrow F)$ , then its normal form  $\rho$  is in  $Cl^0(E \Rightarrow F)$ , hence  $\Delta \vdash \rho : A_E \Rightarrow A_F$  and therefore,  $\pi$  cannot reduce to a term abstraction, by confluence.

**Proposition 4.** *For all  $E, F \in \mathcal{D}_{\equiv}$ , then  $Cl(E \Rightarrow F) = Cl(E) \Rightarrow Cl(F)$ .*

*Proof.*  $\subseteq$  Let  $\pi \in Cl(E \Rightarrow F)$ ,

then  $\pi \in SN$  as  $Cl(E \Rightarrow F)$  satisfies (CR<sub>1</sub>).

- Let  $\rho \in \mathcal{L}_\downarrow(\pi)$ , then  $\rho \in Cl(E \Rightarrow F)$  by (CR<sub>2</sub>), and as it is normal, it is, in particular, in  $Cl^0(E \Rightarrow F)$ , hence  $(\Delta, \rho) \in E \Rightarrow F$ . Let  $\pi' \in Cl(E)$ , then there exists (a minimal)  $j \in \mathbb{N}$ , such that  $\pi' \in Cl^j(E)$ . Let us show that  $\rho\pi' \in Cl(F)$  by induction on  $j$ .
  - \* If  $j = 0$ , then  $\pi'$  is normal and  $(\Delta, \pi') \in E$ , therefore  $(\Delta, \rho\pi') \in F$ , as  $(\Delta, \rho) \in E \Rightarrow F$ . Moreover,  $\rho\pi'$  is normal as  $\pi'$  is normal and  $\rho$  is neutral and normal. Finally  $\rho\pi' \in Cl^0(F)$ .
  - \* If  $j > 0$ , then there exists  $\nu_{\pi'} \in \mathcal{T}$ , and  $(\mu_i)_{i \leq n} \subseteq SN$ , each neutral not normal such that  $\pi' = (\mu_i/\alpha_i)_{i \leq n} \nu_{\pi'}$  and  $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$ , such that for all  $i \leq n$ ,  $\rho_i \in \mathcal{L}(\mu_i)$ , then  $(\rho_i/\alpha_i)_{i \leq n} \nu_{\pi'} \in Cl^{j-1}(E)$ , therefore  $\rho(\rho_i/\alpha_i)_{i \leq n} \nu_{\pi'} \in Cl(F)$ , by induction hypothesis. Finally,  $\rho\pi' = (\mu_i/\alpha_i)_{i \leq n} (\rho\nu_{\pi'}) \in Cl(F)$  by remark 9.
- Let  $\lambda\alpha.\rho \in \mathcal{L}_\lambda(\pi)$ , then  $\lambda\alpha.\rho \in Cl(E \Rightarrow F)$  by (CR<sub>2</sub>), and there exists (a minimal)  $k \in \mathbb{N}$ , such that  $\lambda\alpha.\rho \in Cl^k(E \Rightarrow F)$ . Let us prove that  $\rho \in Cl(F)$  by induction on  $k$ .
  - \* If  $k = 0$ , then  $\lambda\alpha.\rho \in SN$  and  $(\Delta, \lambda\alpha.\rho) \in E \Rightarrow F$ , therefore  $\rho \in SN$  and  $(\Delta, \rho) \in F$ , as we can choose  $\alpha$  such that  $\Delta \vdash \alpha : A_E$ , by  $\alpha$ -conversion. Finally,  $\rho \in Cl^0(F)$ .
  - \* If  $k > 0$ , then there exists  $\nu \in \mathcal{T}$ , and  $(\mu_i)_{i \leq n} \subseteq SN$ , each neutral not normal such that  $\lambda\alpha.\rho = (\mu_i/\alpha_i)_{i \leq n} \nu$  and  $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$ , such that for all  $i \leq n$ ,  $\rho_i \in \mathcal{L}(\mu_i)$ , then  $(\rho_i/\alpha_i)_{i \leq n} \nu \in Cl^{k-1}(E \Rightarrow F)$ . As each  $\mu_i$  is neutral, there exists  $\nu'$  such that  $\nu = \lambda\alpha.\nu'$ , therefore  $(\rho_i/\alpha_i)_{i \leq n} \nu' \in Cl(F)$ , by induction hypothesis. Finally,  $\rho = (\mu_i/\alpha_i)_{i \leq n} (\nu') \in Cl(F)$  by remark 9.

Finally,  $\pi \in Cl(E) \dot{\Rightarrow} Cl(F)$ .

- ⊇ Let  $\pi \in Cl(E) \dot{\Rightarrow} Cl(F)$ . then  $\pi \in SN$  and  $\pi$  cannot reduce to a term-abstraction, by definition of  $\dot{\Rightarrow}$ .
- If  $\pi$  is a proof-abstraction  $\lambda\alpha.\rho$ , then  $\rho \in Cl(F)$  and there exists (a minimal)  $k \in \mathbb{N}$ , such that  $\rho \in Cl^k(F)$ . Let us prove that  $\pi \in Cl(E \dot{\Rightarrow} F)$  by induction on  $k$ .
    - \* If  $k = 0$ , then  $\rho$  is normal and  $(\Delta, \rho) \in F$ , therefore  $\lambda\alpha.\rho$  is normal and  $(\Delta, \lambda\alpha.\rho) \in E \dot{\Rightarrow} F$ , as we can choose  $\alpha$  such that  $\Delta \vdash \alpha : A_E$ , by  $\alpha$ -conversion. Finally,  $\pi = \lambda\alpha.\rho \in Cl^0(E \dot{\Rightarrow} F)$ .
    - \* If  $k > 0$ , then there exists  $\nu \in \mathcal{T}$ , and  $(\mu_i)_{i \leq n} \subseteq SN$ , each neutral not normal such that  $\rho = (\mu_i/\alpha_i)_{i \leq n} \nu$  and  $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$ , such that for all  $i \leq n$ ,  $\rho_i \in \mathcal{L}(\mu_i)$ , then  $(\rho_i/\alpha_i)_{i \leq n} \nu \in Cl^{k-1}(F)$ . Hence  $\pi = (\mu_i/\alpha_i)_{i \leq n} (\lambda\alpha.\nu) \in Cl(E \dot{\Rightarrow} F)$  by induction hypothesis and remark 9.
  - If  $\pi$  is neutral and normal, let  $\alpha \in \mathcal{X}$  such that  $\Delta \vdash \alpha : A_E$ , then  $\pi\alpha \in Cl(F)$ . Moreover  $\pi$  is neutral and normal, therefore  $\pi\alpha$  is normal, hence  $\pi\alpha \in Cl^0(F)$ . Then  $\Delta \vdash \pi\alpha : A_F$  and  $\Delta \vdash \pi : A_E \Rightarrow A_F$ . Let  $(\Gamma', \pi') \in E$ , then  $\Gamma' \vdash \pi' : A_E$ , by  $(P_{\equiv})$ , therefore  $\Delta\Gamma' \vdash \pi\pi' : A_F$ . Finally, as  $\pi$  is neutral and normal and  $\pi' \in WN$ , we have  $\pi\pi' \in WN$ , and  $\pi\pi'$  is isolated, therefore  $(\Delta\Gamma', \pi\pi') \in F$  as it satisfies  $(P_{3_{\equiv}})$ . Hence  $(\Delta, \pi) \in E \dot{\Rightarrow} F$  and  $\pi \in Cl^0(E \dot{\Rightarrow} F)$ , as it is normal.
  - Otherwise,  $\pi \in SN$ , is neutral and not normal. All its leaves are either neutral, either proof-abstractions and all these leaves are in  $Cl(E) \dot{\Rightarrow} Cl(F)$ , as it satisfies  $(CR_2)$ , therefore they also are in  $Cl(E \dot{\Rightarrow} F)$ , as we saw in the previous points. Finally,  $\pi \in Cl(E \dot{\Rightarrow} F)$ , by lemma 14.

### $\forall$ -morphism

We prove now that for all sorts  $T$  and  $f \in \mathring{A}_T$ ,  $Cl(\mathring{\forall}_T f) = \mathring{\forall}_T Cl \circ f$ . Notice that for all functions  $f \in \mathring{A}_T$ ,  $Cl \circ f \in \mathring{A}_T$ .

**Lemma 15.** *For all  $E \in \mathcal{D}_{\equiv}$ ,  $k \in \mathbb{N}$ , terms  $t$  and term-variables  $x$  of same sort, proof-terms  $\pi'$ , if  $(t/x)\pi \in Cl^k(E)$ , then  $(\lambda x.\pi)t \in Cl^k(E)$ .*

*Proof.* By induction on  $k$ .

- If  $k = 0$ , by  $(P_{3_{\equiv}})$ .
- If  $k > 0$ , by induction hypothesis.

**Proposition 5.** *For all sorts  $T$  and  $f \in \mathring{A}_T$ ,  $Cl(\mathring{\forall}_T f) = \mathring{\forall}_T Cl \circ f$ .*

*Proof.* ⊆ Let  $\pi \in Cl(\mathring{\forall}_T f)$ , then there exists (a minimal)  $k \in \mathbb{N}$  such that  $\pi \in Cl^k(\mathring{\forall}_T f)$ . By induction on  $k$ .

- If  $k = 0$ ,  $(\Delta, \pi) \in \mathring{\forall}_T f$  and  $\pi \in SN$ , then for all  $t \in \hat{T}$ ,  $(\Delta, \pi t) \in f(t)$ , therefore  $\pi t \in SN$  and its normal form is in  $Cl^0 \circ f(t)$ , hence  $\pi t \in Cl \circ f(t)$ , by lemma ???. Finally,  $\pi \in \mathring{\forall}_T Cl \circ f$ .

- If  $k > 0$ , then  $\pi = (\mu_i/\alpha_i)_i\nu$ , with each  $\mu_i$  neutral not normal and such that for all  $(\rho_i)_{i \leq n}$  each respectively a leaf of  $\mu_i$ , we have  $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1}(\check{\forall}_T f) \subseteq \check{\forall}_T Cl \circ f$ , by induction hypothesis. Let  $t \in \hat{T}$ , then if we write  $\nu' = \nu t$ , we have  $\pi t = (\mu_i/\alpha_i)_i\nu'$  and for all  $(\rho_i)_{i \leq n}$  each respectively a leaf of  $\mu_i$ ,  $(\rho_i/\alpha_i)_i\nu' = (\rho_i/\alpha_i)_i\nu t \in Cl \circ f(t)$ . Therefore  $\pi t \in Cl \circ f(t)$  by remark 9. Finally,  $\pi \in \check{\forall}_T Cl \circ f$ .
- $\supseteq$  Let  $\pi \in \check{\forall}_T Cl \circ f$ , then there exists a minimal  $k \in \mathbb{N}$  such that there exists  $t \in \hat{T}$ ,  $\pi t \in Cl^k \circ f(t)$ . By induction on  $k$ .
  - If  $k = 0$ , then there exists  $t \in \hat{T}$  such that  $\pi t \in Cl^0 \circ f(t)$ . Hence  $(\Delta, \pi t) \in f(t)$  and  $\pi t$  is normal. Hence  $\pi$  is normal and for all  $t' \in \hat{T}$ ,  $\pi t'$  is also normal, therefore, as  $\pi t' \in Cl \circ f(t)$ , we have, in particular,  $\pi t' \in Cl^0 \circ f(t)$ . Finally, for all  $t' \in \hat{T}$ ,  $(\Delta, \pi t') \in f(t)$ , therefore  $(\Delta, \pi) \in \check{\forall}_T f$ , and  $\pi \in Cl^0(\check{\forall}_T f)$ , as it is normal.
  - If  $k > 0$ , let  $t \in \hat{T}$  such that  $\pi t \in Cl^k \circ f(t)$ . Therefore  $\pi t = (\mu_i/\alpha_i)_i\nu$ , with each  $\mu_i$  neutral not normal and such that for all  $(\rho_i)_{i \leq n}$  each respectively a leaf of  $\mu_i$ , we have  $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1} \circ f(t)$ .
    - \* If  $\nu \neq \alpha_1$ , then  $\nu = \nu' t$ , with  $\pi = (\mu_i/\alpha_i)_i\nu'$ , and for all  $(\rho_i)_{i \leq n}$  each respectively a leaf of  $\mu_i$ , we have  $(\rho_i/\alpha_i)_i\nu' \in Cl(\check{\forall}_T f)$ , by induction hypothesis. We conclude by lemma 14.
    - \* Otherwise, every leaf of  $\pi t$  is in  $Cl^{k-1} \circ f(t)$ . If  $\pi$  is isolated, then all its leaves  $\rho$  are neutral and normal, hence  $\rho t$  is a leaf of  $\pi t$ , therefore  $\rho \in Cl(\check{\forall}_T f)$ , by induction hypothesis, and we conclude by lemma 14. If  $\pi$  reduces to  $\lambda x.\pi'$  then all leaves of  $(t/x)\pi'$  are in  $Cl^{k-1} \circ f(t)$ , therefore, for all leaves  $\rho$  of  $\pi'$ , we have  $(\lambda x.\rho)t \in Cl^{k-1} \circ f(t)$ , by lemma 15, hence  $\lambda x.\rho \in Cl(\check{\forall}_T f)$ , by induction hypothesis. And finally,  $\lambda x.\pi' \in Cl(\check{\forall}_T f)$ , and so does  $\pi$ .

We finally get the following (second) completeness result:

**Proposition 6.**

*If  $\mathcal{L}_{\equiv}$  is strongly normalizing, then  $Cl \circ [\cdot]$  is a  $\mathcal{C}'$ -valued model of  $\mathcal{L}_{\equiv}$  (and each element of the model contains an infinity of proof-variables).*

*Proof.* By lemma 1 and propositions 2, 3, 4, 5.

## 5 Soundness

We finally prove in this section, that having a  $\mathcal{C}'$ -valued model is also a sound condition of strongly normalizing theories  $\mathcal{L}_{\equiv}$ .

**Lemma 16.** *If  $[\cdot]$  is a  $\mathcal{C}'$ -valued model of a theory  $\mathcal{L}_{\equiv}$ , such that each element of the model contains an infinity of proof-variables, then for all  $A \in \mathcal{P}$ , contexts  $\Gamma$ ,  $\varphi \in \text{VAL}(A) \cap \text{VAL}(\Gamma)$ ,  $\pi \in \mathcal{T}$  and  $\sigma$  substitutions such that for all declarations  $\alpha : B$  in  $\Gamma$ ,  $\sigma\alpha \in \llbracket B \rrbracket_{\varphi}$ , we have:*

$$\text{if } \Gamma \vdash \pi : A \text{ then } \sigma\varphi\pi \in \llbracket A \rrbracket_{\varphi}.$$

*Proof.* By induction on the length of the derivation of  $\Gamma \vdash \pi : A$ . By case on the last rule used. If the last rule used is :

- axiom: in this case,  $\pi$  is a variable  $\alpha$ , and  $\Gamma$  contains a declaration  $\alpha : B$  with  $A \equiv B$  (therefore  $|A|_\varphi \equiv |B|_\varphi$ ). Then  $\sigma\varphi\pi = \sigma\alpha \in \llbracket B \rrbracket_\varphi = \llbracket A \rrbracket_\varphi$ .
- $\Rightarrow$ -intro: in this case,  $\pi$  is an abstraction  $\lambda\alpha.\tau$ , and we have  $\Gamma, \alpha : B \vdash \tau : C$  with  $A \equiv B \Rightarrow C$ . Hence, by induction hypothesis, if we choose  $\alpha$  such that  $\alpha \in \llbracket B \rrbracket_\varphi$ , by  $\alpha$ -conversion, we have  $\sigma(\alpha/\alpha)\varphi\tau = \sigma\varphi\tau \in \llbracket C \rrbracket_\varphi$ . Therefore  $\sigma\varphi(\lambda\alpha.\tau) = \lambda\alpha.\sigma\varphi\tau \in \llbracket B \rrbracket_\varphi \Rightarrow \llbracket C \rrbracket_\varphi = \llbracket B \Rightarrow C \rrbracket_\varphi$ .
- $\Rightarrow$ -elim: in this case,  $\pi$  is an application  $\rho\tau$ , and we have  $\Gamma \vdash \rho : C \equiv B \Rightarrow A$  and  $\Gamma \vdash \tau : B$ . Then  $\sigma\varphi\tau \in \llbracket B \rrbracket_\varphi$ , by induction hypothesis.
  - If  $\sigma\varphi\rho$  is a proof-abstraction then  $\rho$  is a proof-abstraction  $\lambda\alpha.\rho'$ , and we have  $\Gamma, \alpha : B \vdash \rho' : A$ , therefore  $(\sigma\varphi\tau/\alpha)\sigma\varphi\rho' \in \llbracket A \rrbracket_\varphi$ , by induction hypothesis, hence  $\sigma\varphi(\lambda\alpha.\rho' \tau) \in \llbracket A \rrbracket_\varphi$  as it satisfies (CR'<sub>3</sub>).
  - If  $\sigma\varphi\rho$  is neutral and normal, as  $\sigma\varphi\rho \in \llbracket A \Rightarrow B \rrbracket_\varphi = \llbracket A \rrbracket_\varphi \Rightarrow \llbracket B \rrbracket_\varphi$ , we have  $\sigma\varphi(\rho\tau) \in \llbracket A \rrbracket_\varphi$ .
  - Otherwise,  $\sigma\varphi\rho$  is neutral and not normal and all its leaves  $\mu$  satisfy  $\mu(\sigma\varphi\tau) \in \llbracket A \rrbracket_\varphi$  as we saw in the previous points.

Finally,  $\sigma\varphi(\rho\tau) \in \llbracket A \rrbracket_\varphi$  as it satisfies (CR'<sub>3</sub>).

- $\forall$ -intro: in this case,  $\pi$  is a term abstraction  $\lambda x.\pi'$  and we have  $\Gamma \vdash \pi' : B$  with  $A \equiv \forall x.B$ . Let  $t \in \hat{T}$  (with  $T$  the sort of  $x$ ), and  $\varphi' = \varphi + \langle x, t \rangle$ . Then  $\sigma\varphi'\pi' = \sigma\varphi(t/x)\pi' \in \llbracket B \rrbracket_{\varphi'}$ , by induction hypothesis. Therefore,  $\sigma\varphi(\lambda x.\pi') \in \check{\forall}_T(t \mapsto \llbracket B \rrbracket_{\varphi + \langle x, t \rangle}) = \llbracket A \rrbracket_\varphi$  (by induction on the maximal length of a reductions sequence from  $\pi t$ , with  $t \in \hat{T}$ , using the fact that for all  $t \in \hat{T}$ ,  $\llbracket B \rrbracket_{\varphi + \langle x, t \rangle}$  satisfies (CR<sub>2</sub>) and (CR<sub>3</sub>')).
- $\forall$ -elim: in this case,  $\pi$  is an application  $\rho t$ , and we have  $\Gamma \vdash \rho : \forall x.B$  with  $A = (t/x)B$  and  $x \notin FV(\Gamma)$ . By induction hypothesis, we have  $\sigma\varphi\rho \in \llbracket \forall x.B, \varphi \rrbracket = \check{\forall}_T(t \mapsto \llbracket B \rrbracket_\varphi + \langle x, t \rangle)$ . Therefore  $\sigma\varphi(\rho t) = \sigma\varphi\rho(\varphi t) \in \llbracket B \rrbracket_{\varphi + \langle x, \varphi t \rangle} = \llbracket (t/x)B \rrbracket_\varphi = \llbracket A \rrbracket_\varphi$

**Theorem 1.** *If  $\mathcal{L}_\equiv$  has a  $\mathcal{C}'$ -valued model (such that each element contains an infinite number of variables), then  $\mathcal{L}_\equiv$  is strongly normalizing.*

*Proof.* If  $\mathcal{F}$  is a  $\mathcal{C}'$ -valued model of  $\equiv$  then for all typing judgement  $\Gamma \vdash \pi : A$  and  $\sigma$  and  $\varphi$  as in the previous proposition, we have  $\sigma\varphi\pi \in \llbracket A \rrbracket_\varphi \neq \emptyset$  hence  $\sigma\varphi\pi \in SN$ , therefore  $\pi \in SN$ .

## 6 Rice Salad

**Theorem 2.** *If  $\mathcal{L}_\equiv$  is weakly normalizing then it is strongly normalizing.*



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