A semantic method to prove strong normalization from weak normalization

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1 Minimal deduction modulo

1.1 Syntax

We consider a set (T) of *sorts*, an infinite set of *variables* of each sort, a set (f) of function symbols, and a set (P) of predicate symbols, that come with their *rank*. The formation rules for objects and propositions are the usual ones.

- Variables of sort T are terms of sort T.
- If f is a function symbol of rank $\langle T_1, \ldots, T_n, U\rangle$ and $t_1, \ldots, t_n$ are respectively objects of sort $T_1, \ldots, T_n$, then $f(t_1, \ldots, t_n)$ is a term of sort U.
- If P is a predicate symbol of rank $\langle T_1, \ldots, T_n\rangle$ and $t_1, \ldots, t_n$ are respectively objects of sort $T_1, \ldots, T_n$, then $P(t_1, \ldots, t_n)$ is an *atomic proposition*.

Propositions are built-up from atomic propositions with the usual connective $\Rightarrow$ and quantifier $\forall$. Remark that, implicitly, quantification in $\forall x.A$ is restricted over the sort of the variable x.

Definition 1 (Terms and Propositions).
We call $\mathcal{O}$ (as objects*), the set of terms:* $t ::= x \mid f\ t \ldots t$
We call $\mathcal{P}$, the set of propositions: $A ::= P\ t \ldots t \mid A \Rightarrow A \mid \forall x.A$

In this language, proof-terms can contain both term variables (written $x, y, \ldots$) and proof variables (written $\alpha, \beta, \ldots$). Terms are written $t, u, \ldots$ while proof-terms are written $\pi, \rho, \ldots$. We call $\mathcal{X}$ the set of proof variables and $\mathcal{Y}$ the set of term variables.

Definition 2 (Proof-terms).
We call $\mathcal{T}$, the set of proof-terms: $\pi := \alpha \mid \lambda\alpha.\pi \mid \pi\ \pi' \mid \lambda x.\pi \mid \pi\ t$

Notice that variables α and x are bound in the constructions $\lambda\alpha.\pi$, and $\lambda x.\pi$. Alphabetic equivalence, free and bound variables are defined as usual.

Each proof-term construction corresponds to a natural deduction rule: terms of the form α express proofs built with the axiom rule, terms of the form $\lambda\alpha.\pi$ and $(\pi\ \pi')$ express proofs built respectively with the introduction and elimination rules of the implication and terms of the form $\lambda x.\pi$ and $(\pi\ t)$ express proofs built with the introduction and elimination rules of the universal quantifier.

We call *neutral* those proof-terms of $\mathcal{T}$ that are not abstractions *i.e.* of the form α, $(\pi\pi')$ or (πt). A proof-term is called *isolated* if it is neutral and only reduces on neutral terms.

1.2 Typing rules

We call *contexts*, lists of declarations $[\alpha : A]$ where α is a proof-variable and A is a proposition, such that each variable in a declaration is different from all the other variables of the context (in this way, we only consider *well formed* contexts, therefore we have to deal with alphabetic equivalence, when concatening them).

Given a congruence relation on propositions $\equiv$, we define typing rules as usual, in deduction modulo:

$$\frac{}{\Gamma, \alpha : A \vdash_{\equiv} \alpha : B}\ A \equiv B\ \ \text{(axiom)}$$

$$\frac{\Gamma, \alpha : A \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \lambda\alpha\ \pi : C}\ C \equiv A \Rightarrow B\ \ (\Rightarrow\text{-intro})$$

$$\frac{\Gamma \vdash_{\equiv} \pi : C \qquad \Gamma \vdash_{\equiv} \pi' : A}{\Gamma \vdash_{\equiv} (\pi\ \pi') : B}\ C \equiv A \Rightarrow B\ \ (\Rightarrow\text{-elim})$$

$$\frac{\Gamma \vdash_{\equiv} \pi : A}{\Gamma \vdash_{\equiv} \lambda x.\pi : B}\ B \equiv \forall x.A,\ \ x \notin FV(\Gamma)\ \ (\forall\text{-intro})$$

$$\frac{\Gamma \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \pi\ t : C}\ B \equiv \forall x.A,\ \ C \equiv (t/x)A,\ \ t \text{ has the sort of } x\ (\forall\text{-elim})$$

Fig. 1. Typing rules

1.3 Proof reduction rules and strong normalization

As usual in deduction modulo, the process of cut elimination is modeled by β-reduction. We consider the contextual closure of the reduction rules given figure 2. These rules correspond to proof reduction in natural deduction.

$$(\lambda\alpha.\pi\ \pi') \to_{\beta_\pi} (\pi'/\alpha)\pi$$
$$(\lambda x.\pi\ t) \to_{\beta_t} (t/x)\pi \qquad \text{(if } x \text{ and } t \text{ have the same sort)}$$

Fig. 2. Proof reduction rules

We write $(\pi'/\alpha)\pi$ (resp. $(t/x)\pi$) the substitution of α (resp. x) by π' (resp. t) in π. We say that π reduces to π' if $\pi \rightarrow_{\beta_t} \pi'$ or $\pi \rightarrow_{\beta_\pi} \pi'$. We write $\pi \rightarrow \pi'$ if π reduces in one step to π', $\pi \rightarrow^+ \pi'$ if π reduces in at least one step to π', and $\pi \rightarrow^* \pi'$ if π reduces in an arbitrary number of steps to π'.
A proof is said to be *normal* if it contains no redex. It is said to be weakly *normalizing* if it has a normal form and *strongly normalizing* if all reduction sequences issued from this proofs are finite. We write SN for the set of strongly normalizing proofs.

1.4 Theories expressed in minimal deduction modulo

A theory expressed in minimal deduction modulo is defined by a many-sorted language in predicate logic $\mathcal{L} = \langle((T),(F),(P)\rangle$ and a congruence relation $\equiv$ on propositions of the associated many-sorted logic. We suppose $\equiv$ not *ambiguous*, *i.e.* there does not exist $x \in \mathcal{Y}$, $A, B, C \in \mathcal{P}$ such that $A \Rightarrow B \equiv \forall x.C$. We will call $\mathcal{L}_\equiv$ this theory.

Remark 1. Given a theory $\mathcal{L}_\equiv$, we will write $\vdash$ for $\vdash_\equiv$.

Proposition 1 (confluence and subject-reduction). *$\rightarrow$ is confluent. And for all contexts Γ, proof-terms π, π' and propositions A, if $\Gamma \vdash \pi : A$ and $\pi \rightarrow \pi'$, then $\Gamma \vdash \pi' : A$.*

Example As mentioned above deduction modulo permits to express (intentional) simple type theory [1] without any axiom. We show in the following, how minimal deduction modulo permits to express minimal (intentional) simple type theory, without any axiom (see [6] for details).

The *sorts* are *simple types* inductively defined by:
- ι and o are sorts,
- if T and U are sorts then $T \rightarrow U$ is a sort.

The language is composed of the individual symbols
- $S_{T,U,V}$ of sort $(T \rightarrow U \rightarrow V) \rightarrow (T \rightarrow U) \rightarrow T \rightarrow V$,
- $K_{T,U}$ of sort $T \rightarrow U \rightarrow T$,
- $\dot{\Rightarrow}$, of sort o,
- $\dot{\forall}_T$ of sort $(T \rightarrow o) \rightarrow o$,

the function symbols $\alpha_{T,U}$ of rank $\langle T \rightarrow U, T, U\rangle$,
and the predicate symbol ε of rank $\langle o\rangle$.

The combinators $S_{T,U,V}$ and $K_{T,U}$ are used to express functions. The objects $\dot{\Rightarrow}$, and $\dot{\forall}_T$ allow to represent propositions as objects of sort o. Finally, the predicate ε allows to transform such an object t of type o into the actual corresponding proposition $\varepsilon(t)$.

$$\begin{aligned}
\alpha(\alpha(\alpha(S_{T,U,V}, x), y), z) &\rightarrow \alpha(\alpha(x, z), \alpha(y, z)) \\
\alpha(\alpha(K_{T,U}, x), y) &\rightarrow x \\
\varepsilon(\alpha(\alpha(\dot{\Rightarrow}, x), y)) &\rightarrow \varepsilon(x) \Rightarrow \varepsilon(y) \\
\varepsilon(\alpha(\dot{\forall}, x)) &\rightarrow \forall y\ \varepsilon(\alpha(x, y))
\end{aligned}$$

2 Language-dependent truth values algebras

2.1 Definition

For all sets E, we call $\mathbb{P}(E)$ the set of subsets of E.
For all sorts T of a language $\mathcal{L}$, we write $\hat{T}$, the set of closed terms of sort T.

Definition 3 (language-dependent tvas).
Let $\mathcal{L} = \langle (T), (f), (P) \rangle$ be a many-sorted language in predicate logic.
$\langle \mathcal{B}, \hat{\Rightarrow}, (\hat{\mathcal{A}}_T), (\hat{\forall}_T) \rangle$ is a LDTVA *for $\mathcal{L}$ if and only if:*

- *$\mathcal{B}$ is a set,*
- *$\hat{\Rightarrow}$ is a function from $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$,*
- *for all sorts T,*
  - *$\hat{\mathcal{A}}_T$ is a set of functions from $\hat{T}$ to $\mathcal{B}$: $\hat{\mathcal{A}}_T \subseteq \hat{T} \mapsto \mathcal{B}$*
  - *$\hat{\forall}_T$ is a function from $\hat{\mathcal{A}}_T$ to $\mathcal{B}$.*

Definition 4 (Morphism).
Let $\mathcal{B}^1 = \langle \mathcal{B}^1, \hat{\Rightarrow}^1, (\hat{\mathcal{A}}^1_T), (\hat{\forall}^1_T)) \rangle$ and $\mathcal{B}^2 = \langle \mathcal{B}^2, \hat{\Rightarrow}^2, (\hat{\mathcal{A}}^2_T), (\hat{\forall}^2_T)) \rangle$ be two LDTVA*s.*
A morphism *from $\mathcal{B}^1$ to $\mathcal{B}^2$ is a function F from $\mathcal{B}^1$ to $\mathcal{B}^2$ such that:*
- *for all $E, G \in \mathcal{B}^1$, $F(E \hat{\Rightarrow}^1 G) = F(E) \hat{\Rightarrow}^2 F(G)$,*
- *for all sorts T, $x \in \hat{T}$ and $f \in \mathring{\mathcal{A}}_T$, $F(\hat{\forall}^1_T\ f) = \hat{\forall}^2_T\ F \circ f$.*

Definition 5 (Valuation).
Given a LDTVA *for $\mathcal{L} = \langle (T), (f), (P) \rangle$, a* valuation *$\varphi$ is a substitution mapping term-variables of a sort to closed terms of the same sort. For all propositions A (resp. terms t), we call $\textsc{Val}(A)$ (resp. $\textsc{Val}(t)$) the set of valuations whose domain contains the set of free variables of A (resp. t).*
We write $x \notin \varphi$ for expressing the fact that $\varphi(x)$ is not defined.

Definition 6. *For all $A \in \mathcal{P}$, terms t and $\varphi \in \textsc{Val}(A)$, we write $|A|_\varphi$ the result of the substitution φ on A.*

Definition 7 (Models).
Let $\mathcal{L} = \langle (T), (f), (P) \rangle$ be a many-sorted language in predicate logic,
let $\equiv$ a congruence relation on propositions of minimal deduction based on $\mathcal{L}$,
let $\mathcal{B} = \langle \mathcal{B}, \hat{\Rightarrow}, (\hat{\mathcal{A}}_T), (\hat{\forall}_T) \rangle$ be a LDTVA *for $\mathcal{L}$.*

1. *We call $\mathcal{B}$-*valued interpretations *those functions which map every ordered pair of a proposition A and a valuation in $\textsc{Val}(A)$ to an element of $\mathcal{B}$.*
2. *A $\mathcal{B}$-valued interpretation $[\![.]\!]_.$ is a $\mathcal{B}$-*valued model *if and only if:*
   - *for all $A, B \in \mathcal{P}$ and $\varphi \in \textsc{Val}(A \Rightarrow B)$, $[\![A \Rightarrow B]\!]_\varphi = [\![A]\!]_\varphi \hat{\Rightarrow} [\![B]\!]_\varphi$*
   - *for all $A \in \mathcal{P}$, x of sort T and $\varphi \in \textsc{Val}(\forall x.A)$ such that $x \notin \varphi$,*
     $[\![\forall x.A]\!]_\varphi = \hat{\forall}_T(t \mapsto [\![A]\!]_{\varphi + \langle x,t \rangle})$
   - *for all $A \in \mathcal{P}$, x of sort T, $t \in \hat{T}$ and $\varphi \in \textsc{Val}(\forall x.A)$ such that $x \notin \varphi$,*
     $[\![(t/x)A]\!]_\varphi = [\![A]\!]_{\varphi + \langle x,t \rangle}$.

3. *A $\mathcal{B}$-valued model $[\![.]\!]_.$ is a* model of the theory $\mathcal{L}_\equiv$ *if and only if:* *for all $A, A' \in \mathcal{P}$, $\varphi \in \text{VAL}(A)$ and $\psi \in \text{VAL}(A')$, if $|A|_\varphi \equiv |A'|_\psi$, then $[\![A]\!]_\varphi = [\![A']\!]_\psi$*

Remark 2. The previous conditions can be reformulated as: 2. Interpretations of propositions have to be adapted to the connectives to be a model. 3. Models have to be adapted to the congruence to be a model of the associated theory.

The following lemma explains that our definition of morphism is correct for the property of being a model of a theory $\mathcal{L}_\equiv$.

Lemma 1. *For all* LDTVA*s $\mathcal{B}_1$ and $\mathcal{B}_2$ and morphisms F from $\mathcal{B}_1$ to $\mathcal{B}_2$, if $[\![.]\!]_.$ is a $\mathcal{B}_1$-valued model of a theory $\mathcal{L}_\equiv$, then $F \circ [\![.]\!]_.$ is a $\mathcal{B}_2$-valued model of $\mathcal{L}_\equiv$.*

3 About *WN*-reducibility candidates and typing

3.1 $\mathcal{D}_\equiv$, a ldtva of ($\equiv$) well-typed *WN*-reducibility candidates

Definition 8 ($\mathcal{U}$).
$\mathcal{U} = \{(\Gamma, \pi)$ *such that Γ is a context and π is a proof-term* $\}$.

Definition 9.
For all $E \subseteq \mathcal{U}$, we define the following properties :

($P_\equiv$) There exists A_E such that $\forall(\Gamma, \pi) \in E,\ \Gamma \vdash \pi : A_E$
($P_{1_\equiv}$) For all $(\Gamma, \pi) \in E$, $\pi \in WN$
($P_{2_\equiv}$) For all $(\Gamma, \pi) \in E$ and $\pi' \in WN$ such that $\pi \rightarrow \pi'$, $(\Gamma, \pi') \in E$
($P_{3_\equiv}$) For all $(\Gamma, \pi) \in \mathcal{U}$,
- *If $\pi \in WN$, π is isolated and $\Gamma \vdash \pi : A_E$, then $(\Gamma, \pi) \in E$*
- *If $(\Gamma, \pi) \in E$ and $\pi' \rightarrow_{\beta_t} \pi$, then $(\Gamma, \pi') \in E$.*

Remark 3. For all $E \subseteq \mathcal{U}$, if E satisfies ($P_\equiv$) and ($P_{3_\equiv}$),
then for all proof-variables α, $(\alpha : A_E, \alpha) \in E$, as α is isolated and in *WN*.

Definition 10 (domain $\mathcal{D}_\equiv$). *We call $\mathcal{D}_\equiv$ the set of subsets of $\mathcal{U}$ which satisfy ($P_\equiv$), ($P_{1_\equiv}$), ($P_{2_\equiv}$) and ($P_{3_\equiv}$).*

Definition 11 (leaves).
The leaves *of a proof-term π are its first reducts which are normal or not neutral. (ρ is a leaf of π if and only if it is normal or not neutral and there exists $n \geq 0$ and $\pi_1 \dots \pi_{n-1}$ neutral not normal terms such that $\pi = \pi_1 \rightarrow \dots \rightarrow \pi_{n-1} \rightarrow \rho$). We call $\mathcal{L}(\pi)$ the set of leaves of π, $\mathcal{L}_\downarrow(\pi)$ the set of neutral normal leaves of π, and $\mathcal{L}_\lambda$ the set of not neutral leaves of π.*

Remark 4.
- The only leaf of a normal or not neutral proof-term is itself.
- If π is a neutral non-normal proof-term, then $\rho \in \mathcal{L}(\pi)$ if and only if there exists a one-step reduct π' of π such that $\rho \in \mathcal{L}(\pi')$.
- If $\pi \in WN$, then $\mathcal{L}(\pi) \neq \emptyset$.

Definition 12 **($\mathring{\Rightarrow}$).** *For all $E, F \in \mathcal{D}_{\equiv}$,*
$E \mathring{\Rightarrow} F = \{(\Gamma, \pi) \in \mathcal{U}$ *such that* $\pi \in WN$, $\Gamma \vdash \pi : A_E \Rightarrow A_F$ *and*
- $\forall \rho \in \mathcal{L}_\lambda(\pi)$, $\rho = \lambda\alpha.\rho'$ *with* $(\Gamma, \alpha : A_E,\ \rho) \in F$
- $\forall \rho \in \mathcal{L}_\downarrow(\pi)$ *and* $(\Gamma', \pi') \in E$, $(\Gamma\Gamma',\ \rho\pi') \in F\}$

Remark 5. We recall the fact that we only consider well-formed contexts, therefore the only variables Γ and Γ' can share have to be declared proofs of equivalent propositions, otherwise we have to deal with α-conversion when concatening Γ and Γ'.

Lemma 2. *$\mathring{\Rightarrow}$ is a function from $\mathcal{D}_{\equiv} \times \mathcal{D}_{\equiv}$ to $\mathcal{D}_{\equiv}$.*

Proof. Let $E, F \in \mathcal{D}_{\equiv}$, and $(\Gamma, \pi) \in E \mathring{\Rightarrow} F$,

($P_{\equiv}$) By definition, $A_{E \mathring{\Rightarrow} F} \equiv A_E \Rightarrow A_F$.

($P_{1_{\equiv}}$) By definition.

($P_{2_{\equiv}}$) By subject-reduction, the fact that F satisfies ($P_{2_{\equiv}}$) and the fact that all leaves of a reduct of a proof term π are also leaves or reducts of leaves of π.

($P_{3_{\equiv}}$)
- By the fact that an isolated term has only neutral leaves, and that if π is a neutral normal term, and π' is a term in WN, then $\pi\pi'$ is isolated and in WN.
- By the fact that if $\pi' \rightarrow_{\beta_t} \pi$, then in a given context, π and π' have the same type, if $\pi \in WN$ then so does π' and all leaves of π' are either leaves of π, either "β_t-expansions" of leaves of π.

Definition 13 **($\mathring{\mathcal{A}}_T$).** *For all sorts T,*
$\mathring{\mathcal{A}}_T = \{f : \hat{T} \mapsto \mathcal{D}_{\equiv}$, *such that there exists* $A_f \in \mathcal{P}$ *and* $x_f \in \mathcal{X}$ *such that*
for all $t \in \hat{T}$ *and* $(\Gamma, \pi) \in f(t)$, $\Gamma \vdash \pi : (t/x_f)A_f\}$

Definition 14 **($\mathring{\forall}_T$).** *For all sorts T and functions $f \in \mathring{\mathcal{A}}_T$,*
$\mathring{\forall}_T.f = \{(\Gamma, \pi) \in \mathcal{U}$ *such that for all* $t \in \hat{T}$, $(\Gamma,\ \pi\, t) \in f(t)\}$

Lemma 3. *For all sorts T, $\mathring{\forall}_T$ is a function from $\mathring{\mathcal{A}}_T$ to $\mathcal{D}_{\equiv}$.*

Proof. Let $f \in \mathring{\mathcal{A}}_T$, and $(\Gamma, \pi) \in \mathring{\forall}_T.f$

($P_{\equiv}$) Let $t \in \hat{T}$ ($\neq \emptyset$). Then $(\Gamma, \pi t) \in f(t)$. As $f \in \mathring{\forall}_T$, we have $\Gamma \vdash \pi t : (t/x_f)A_f$. Therefore $\Gamma \vdash \pi : \forall x_f.A_f$, by case on the last rule used in the derivation of $\Gamma \vdash \pi t : (t/x_f)A_f$. Finally, $A_{\mathring{\forall}_T f} \equiv \forall x_f.A_f$.

($P_{1_{\equiv}}$) Let $t \in \hat{T}$ ($\neq \emptyset$). Then $(\Gamma, \pi t) \in f(t) \in \mathcal{D}_{\equiv}$ therefore $\pi t \in WN$ and so does π.

($P_{2_{\equiv}}$) Let π' such that $\pi \rightarrow \pi'$. Then, for all $t \in \hat{T}$, $\pi t \rightarrow \pi' t$, therefore $\pi' t \in f(t) \in \mathcal{D}_{\equiv}$.

($P_{3_{\equiv}}$)
- Let $(\Gamma, \tau) \in \mathcal{U}$ such that $\tau \in WN$, τ is isolated and $\Gamma \vdash \tau : \forall x_f.A_f$. Let $t \in \hat{T}$ then $\Gamma \vdash \tau t : (t/x_f)A_f$, τt is isolated as τ is, and $\tau t \in WN$, as $\tau \in WN$. Finally, $\tau \in \mathring{\forall}_T.f$, as $f(t)$ satisfies ($P_{3_{\equiv}}$), for all $t \in \hat{T}$.

- Let π' such that $\pi' \rightarrow_{\beta_t} \pi$, then for all $t \in \hat{T}$, $\pi' t \rightarrow_{\beta_t} \pi t$ therefore $(\Gamma, \pi' t) \in f(t)$ as it satisfies $(\mathrm{P}_{3_\equiv})$. Hence $(\Gamma, \pi') \in \mathring{\forall}_T f$.

Definition 15 $(\mathcal{D}_\equiv)$**.** $\mathcal{D}_\equiv$ *is the* LDTVA $\langle \mathcal{D}_\equiv, \mathring{\Rightarrow}, (\mathring{\mathcal{A}}_T), (\mathring{\forall}_T)\rangle$.

3.2 Building a $\mathcal{D}_\equiv$-valued interpretation of *WN* theories $\mathcal{L}_\equiv$

Let us now define a first $\mathcal{D}_\equiv$-valued model, by using directly definitions of $\mathring{\Rightarrow}$ and $\mathring{\forall}_T$, and well-chosen interpretations of atomic propositions.

Definition 16. *Let A be a proposition and $\varphi \in \mathrm{VAL}(A)$.*
We define the subset of $\mathcal{U}$, $[A]_\varphi$ by induction over the structure of A.

- $[P\ t_1 \ldots t_n]_\varphi = \{(\Gamma, \pi) \in \mathcal{U}$ *such that* $\pi \in \mathit{WN}$ *and* $\Gamma \vdash \pi : P\ \varphi(t_1) \ldots \varphi(t_n)\}$
- $[B \Rightarrow C]_\varphi = [B]_\varphi \mathring{\Rightarrow} [C]_\varphi$
- $[\forall x.B]_\varphi = \mathring{\forall}_T\ (t \mapsto [B]_{\varphi + \langle x, t\rangle})$

Lemma 4. *For all $A \in \mathcal{P}$, x of sort T, $t \in \hat{T}$ and $\varphi \in \mathrm{VAL}(\forall x.A)$ such that $x \notin \varphi$, we have $[(t/x)A]_\varphi = [A]_{\varphi + \langle x, t\rangle}$.*

Proof. By induction on A.

Lemma 5. *For all $A \in \mathcal{P}$, and $\varphi \in \mathrm{VAL}(A)$,*
$[A]_\varphi \in \mathcal{D}_\equiv$ *with* $A_{[A]_\varphi} = A_\varphi$ *(i.e,* $\forall (\Gamma, \pi) \in [A]_\varphi$, $\Gamma \vdash \pi : A_\varphi$*).*

Proof. By induction on A.

- If A is an atomic proposition $P\ t_1 \ldots t_n$,
  - $(\mathrm{P}_\equiv)$ By definition. (with $A_{[P\ t_1 \ldots t_n]_\varphi} \equiv P\ \varphi(t_1) \ldots \varphi(t_n)$).
  - $(\mathrm{P}_{1_\equiv})$ By definition.
  - $(\mathrm{P}_{2_\equiv})$ By subject-reduction.
  - $(\mathrm{P}_{3_\equiv})$ By definition.
- If $A = B \Rightarrow C$, as $\mathring{\Rightarrow} : \mathcal{D}_\equiv \times \mathcal{D}_\equiv \mapsto \mathcal{D}_\equiv$, we conclude by induction hypothesis (with $A_{[B \Rightarrow C]_\varphi} = A_{[B]_\varphi \mathring{\Rightarrow} [C]_\varphi} \equiv A_{[B]_\varphi} \Rightarrow A_{[C]_\varphi} \equiv B_\varphi \Rightarrow C_\varphi = (B \Rightarrow C)_\varphi$).
- If $A = \forall x.B$, let T be the sort of x and $f = t \mapsto [B]_{\varphi + \langle x, t\rangle}$.
  Then f is a function from $\hat{T}$ to $\mathcal{D}_\equiv$, by induction hypothesis. Moreover, for all $t \in \hat{T}$, $A_{f(t)} = B_{\varphi + \langle x, t\rangle} = (t/x)B_\varphi$, by induction hypothesis. Therefore $f \in \mathring{\mathcal{A}}_T$ and $\mathring{\forall}_T f \in \mathcal{D}_\equiv$ (with $A_{[\forall x.B]_\varphi} = \forall x.A_f = \forall x.B_\varphi$).

At this point, we have $\mathcal{D}_\equiv$-valued model which is adapted to typing but not necessarily $\equiv$-adapted. Indeed, in a theory where we have two atomic proposition symbols P and Q such that $P \equiv (Q \Rightarrow Q)$ (notice that such a theory can be weakly normalizing), then for all valuations $\varphi \in \mathrm{VAL}(P) \cap \mathrm{VAL}(Q)$, $[P]_\varphi \neq [Q]_\varphi \mathring{\Rightarrow} [Q]_\varphi$. We have then to modify this interpretation to make it a $\mathcal{D}_\equiv$-valued model of $\mathcal{L}_\equiv$.

3.3 Adapting this interpretation to the congruence

Definition 17. *We define a second interpretation $\lfloor . \rfloor_.$, as follows :*
for all $A \in \mathcal{P}$ and $\varphi \in \textsc{Val}(A)$,

$$\lfloor A \rfloor_\varphi = \bigcap_{A_\varphi \equiv A'_\psi} [A']_\psi$$

Remark 6. For all $A, A' \in \mathcal{P}$, $\varphi \in \textsc{Val}(A)$ and $\psi \in \textsc{Val}(A')$ such that $A_\varphi \equiv A'_\psi$, we have $\lfloor A \rfloor_\varphi = \lfloor A' \rfloor_\psi$, by definition.

Then we prove that $\lfloor . \rfloor_.$ is also a $\mathcal{D}_\equiv$-valued interpretation adapted to typing.

Lemma 6. *For all $A \in \mathcal{P}$, and $\varphi \in \textsc{Val}(A)$,*
$\lfloor A \rfloor_\varphi \in \mathcal{D}_\equiv$ *with* $A_{\lfloor A \rfloor_\varphi} = A_\varphi$ *(i.e, $\forall (\Gamma, \pi) \in \lfloor A \rfloor_\varphi$, $\Gamma \vdash \pi : A_\varphi$).*

Proof. Let $A \in \mathcal{P}$, and $\varphi \in \textsc{Val}(A)$,
By lemma 5 and the fact that $\lfloor A \rfloor_\varphi \subseteq [A']_\psi$, for all $A'_\psi \equiv A_\varphi$.

Lemma 7. *For all $A \in \mathcal{P}$, x of sort T, $t \in \hat{T}$ and $\varphi \in \textsc{Val}(\forall x.A)$ such that $x \notin \varphi$, we have $\lfloor (t/x)A \rfloor_\varphi = \lfloor A \rfloor_{\varphi + \langle x,t \rangle}$.*

Proof. By lemma 7.

Finally, we proved, that $\lfloor . \rfloor_.$ is a $\mathcal{D}_\equiv$-valued interpretation of propositions adapted to typing and to the congruence relation $\equiv$. Let us now show that $\lfloor . \rfloor_.$ is also a $\mathcal{D}_\equiv$-valued model of weakly normalizing theories $\mathcal{L}_\equiv$, *i.e.* it is also adapted to connectives, when $\mathcal{L}_\equiv$ is weakly normalizing.

3.4 $\lfloor . \rfloor_.$ is a $\mathcal{D}_\equiv$-valued model of weakly normalizing theories $\mathcal{L}_\equiv$

In order to prove that $\lfloor . \rfloor_.$ is a $\mathcal{D}_\equiv$-valued model of $\mathcal{L}_\equiv$, if it is weakly normalizing, we proceed by *reductio ad absurdum*, showing that if $\lfloor . \rfloor_.$ is not connectives-adapted, then we can exhibit a typing judgement $\Gamma \vdash \pi : A$ such that $\pi \notin \mathit{WN}$.

Lemma 8.
If there exists $A, B \in \mathcal{P}$ and $\varphi \in \textsc{Val}(A \Rightarrow B)$, such that $\lfloor A \Rightarrow B \rfloor_\varphi \neq \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi$ then there exists $\pi \in \mathcal{T}$, $C \in \mathcal{P}$, $\psi \in \textsc{Val}(C)$ such that $\Gamma \vdash \pi : C_\psi$ and $(\Gamma, \pi) \notin \lfloor C \rfloor_\psi$.

Proof.
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin \lfloor A \Rightarrow B \rfloor_\varphi$ and $(\Gamma, \pi) \in \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi$. Then $\Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi = (A \Rightarrow B)_\varphi$.
  We take $C = A \Rightarrow B$ and $\psi = \varphi$.
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \in \lfloor A \Rightarrow B \rfloor_\varphi$ and $(\Gamma, \pi) \notin \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi$. Notice that as $\Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi$, π cannot reduce to a term-abstraction, by subject-reduction. Then, as $\pi \in \mathit{WN}$ and $\Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi$, either there exists $\lambda\alpha.\rho \in \mathcal{L}_\lambda(\pi)$ such that $(\Gamma, \alpha : A_\varphi,\ \rho) \notin \lfloor B \rfloor_\varphi$, with $\Gamma, \alpha : A_\varphi \vdash \rho : B_\varphi$ by subject-reduction. Either there exists $\rho \in \mathcal{L}_\downarrow(\pi)$ and $(\Gamma', \pi') \in \lfloor A \rfloor_\varphi$ such that $(\Gamma\Gamma', \rho\pi') \notin \lfloor B \rfloor_\varphi$, with $\Gamma\Gamma' \vdash \rho\pi' : B_\varphi$ by subject-reduction. We take $C = B$ and $\psi = \varphi$

Lemma 9.
If there exists $A \in \mathcal{P}$, $\varphi \in \textsc{Val}(A)$, and x of sort T such that $x \notin \varphi$, and
$$\lfloor \forall x.A \rfloor_\varphi \neq \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi+\langle x,t\rangle})$$
then there exists $\pi \in \mathcal{T}$, $C \in \mathcal{P}$, $\psi \in \textsc{Val}(C)$ such that $\Gamma \vdash \pi : C_\psi$ and $(\Gamma, \pi) \notin \lfloor C \rfloor_\psi$.

Proof.
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin \lfloor \forall x.A \rfloor_\varphi$ and $(\Gamma, \pi) \in \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi+\langle x,t\rangle})$. Then $\Gamma \vdash \pi : (\forall x.A)_\varphi$.
  We take $C = \forall x.A$ and $\psi = \varphi$.
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \in \lfloor \forall x.A \rfloor_\varphi$ and $(\Gamma, \pi) \notin \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi+\langle x,t\rangle})$.
  Then there exists $t \in \hat{T}$ such that $(\Gamma, \pi t) \notin \lfloor A \rfloor_{\varphi+\langle x,t\rangle}$. As $\Gamma \vdash \pi : (\forall x.A)_\varphi$, we have $\Gamma \vdash \pi t : (t/x)A_\varphi = A_{\varphi+\langle x,t\rangle}$. We take $C = A$ and $\psi = \varphi + \langle x, t\rangle$

Lemma 10.
If there exists $A, B \in \mathcal{P}$, $\varphi \in \textsc{Val}(A \Rightarrow B)$ or $\varphi' \in \textsc{Val}(\forall x.A)$ with x of sort T, $x \notin \varphi'$ and
$$\lfloor A \Rightarrow B \rfloor_\varphi \neq \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi \quad \text{or} \quad \lfloor \forall x.A \rfloor_{\varphi'} \neq \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi'+\langle x,t\rangle})$$
then there exists $D \in \mathcal{P}$, $\pi \in \mathcal{T}$, $\psi \in \textsc{Val}(D)$ such that $\Gamma \vdash \pi : D_\psi$ and $(\Gamma, \pi) \notin [D]_\psi$.

Proof. By lemmas 8 and 9, there exists C, Γ, π and ψ such that $\Gamma \vdash \pi : C_\psi$ and $(\Gamma, \pi) \notin \lfloor C \rfloor_\psi$. Therefore, there exists a proposition D and $\psi' \in \textsc{Val}(D)$ such that $D_{\psi'} \equiv C_\psi$ and $(\Gamma, \pi) \notin [D]_{\psi'}$. And $\Gamma \vdash \pi : D_{\psi'}$, by equivalence of C_ψ and $D_{\psi'}$.

Lemma 11.
If there exists $A, B \in \mathcal{P}$, $\varphi \in \textsc{Val}(A \Rightarrow B)$ or $\varphi' \in \textsc{Val}(\forall x.A)$ with x of sort T, $x \notin \varphi'$
$$\text{and} \qquad \lfloor A \Rightarrow B \rfloor_\varphi \neq \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi \quad \text{or} \quad \lfloor \forall x.A \rfloor_{\varphi'} \neq \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi'+\langle x,t\rangle})$$
then there exists a (term-closed) proposition E, $\pi \in \mathcal{T}$ and a context Γ such that
$$\Gamma \vdash \pi : E \text{ and } \pi \notin WN.$$

Proof. By lemma 10, there exists a proposition D, a context Γ, a proof π and $\varphi \in \mathcal{V}(D)$ such that $\Gamma \vdash \pi : D_\varphi$ and $(\Gamma, \pi) \notin [D]_\varphi$. By induction on D.
- if D is atomic, then as $\Gamma \vdash \pi : D_\varphi$, we have $\pi \notin WN$.
- if $D = (F \Rightarrow G)_\varphi$,
  then $\Gamma \vdash \pi : (F \Rightarrow G)_\varphi$ and $(\Gamma, \pi) \notin [F \Rightarrow G]_\varphi = [F]_\varphi \mathring{\Rightarrow} [G]_\varphi$. If $\pi \in WN$, either there exists $\lambda\alpha.\rho \in \mathcal{L}_\lambda(\pi)$ such that $(\Gamma, \alpha : F_\varphi, \ \rho) \notin [G]_\varphi$. Either there exists $\rho \in \mathcal{L}_\downarrow(\pi)$ and $(\Gamma', \pi') \in [F]_\varphi$ such that $(\Gamma\Gamma', \rho\pi') \notin [G]_\varphi$, with $\Gamma\Gamma' \vdash \rho\pi' : G_\varphi$. We conclude by induction hypothesis.
- if $D = \forall x.F$,
  then $\Gamma \vdash \pi : (\forall x.F)_\varphi$ and $(\Gamma, \pi) \notin [\forall x.F]_\varphi$. Therefore there exists $t \in \hat{T}$ such that $(\Gamma, \pi t) \notin [F]_{\varphi+\langle x,t\rangle}$, with $\Gamma \vdash \pi t : A_{\varphi+\langle x,t\rangle}$. We conclude by induction hypothesis.

Proposition 2 (Completeness). *If the theory $\mathcal{L}_\equiv$ is weakly normalizing, then $\lfloor . \rfloor_. = \langle A, \varphi\rangle \mapsto \lfloor A \rfloor_\varphi$ is a $\mathcal{D}_\equiv$-model of this theory.*

Proof. By remark 6 and lemmas 6 and 11.

4 From $\mathcal{D}_{\equiv}$ to $\mathcal{C}'$

4.1 $\mathcal{C}'$, yet another algebra of candidates.

Definition 18.
For all sets E of proof-terms, we define the following properties :

(CR_1) *For all $\pi \in E$, $\pi \in SN$.*
(CR_2) *For all $\pi \in E$, for all $\pi' \in \mathcal{T}$ such that $\pi \rightarrow \pi'$, then $\pi' \in E$.*
(CR'_3) *for all $n \in \mathbb{N}$, for all $\nu, \mu_1, \ldots, \mu_n \in \mathcal{T}$, if*
- for all $i \leq n$, μ_i is neutral and not normal,
- $\forall \rho_1, \ldots, \rho_n \in \mathcal{T}$ such that for all $i \leq n$, $\mu_{i \leq n} \rightarrow \rho_i$, $(\rho_i/\alpha_i)_{i \leq n}\nu \in E$
then $(\mu_i/\alpha_i)\nu \in E$.

Remark 7. If E satisfies (CR'_3) then, in particular, all neutral *not normal* terms whose all one-steps reducts are in E, is in E. That is slightly different from the usual (CR_3) of reducibility candidates, where the neutral term can be normal, therefore all neutral normal terms are in all reducibility candidates.

Definition 19 ($\tilde{\Rightarrow}$).
For all $E, F \subseteq \mathcal{T}$, $E \tilde{\Rightarrow} F = \{\pi \in SN$ such that
- $\forall \rho \in \mathcal{L}_{\lambda}(\pi)$, $\rho = \lambda\alpha.\rho'$ with $\rho \in F$
- $\forall \rho \in \mathcal{L}_{\downarrow}(\pi)$ and $\pi' \in E$, $\rho\pi' \in F\}$

Lemma 12. *$\tilde{\Rightarrow}$ is a function from $\mathcal{C}' \times \mathcal{C}'$ to $\mathcal{C}'$.*

Proof. Let $E, F \in \mathcal{C}'$ and $\pi \in E \tilde{\Rightarrow} F$,

(CR_1) $\pi \in SN$, by definition.
(CR_2) If π' is a one-step reduct of π, then for all $\pi' \in SN$ and all its leaves are leaves of π, or reducts of leaves of π.
(CR'_3) Let $\pi = (\mu_i/\alpha_i)\nu$ with each μ_i neutral not normal and such that for all (ρ_i) each respectively a one-step reduct of μ_i, $(\rho_i/\alpha_i)\nu \in E \tilde{\Rightarrow} F$. We can first notice that π cannot reduce to a term-abstraction, by confluence. Let us prove that $\pi \in E \tilde{\Rightarrow} F$, by induction on the length l of the maximal length of a reductions sequence from π to one of its leaves.

- If $l = 0$, then π is either normal and neutral, either a proof-abstraction.
  - If π is neutral and normal then none of the μ_i appears in ν, hence $\pi \in E \tilde{\Rightarrow} F$.
  - If $\pi = \lambda\alpha.\pi'$ then, as each μ_i is neutral, $\nu = \lambda\alpha.\nu'$, with $\pi' = (\mu_i/\alpha_i)\nu'$. And for all (ρ_i) each respectively a one-step reduct of μ_i, $(\rho_i/\alpha_i)\nu = \lambda\alpha.(\rho_i/\alpha_i)\nu' \in E \tilde{\Rightarrow} F$, therefore $(\rho_i/\alpha_i)\nu' \in F$. Finally, $\pi' \in F$ as it satisfies (CR'_3), and $\pi = \lambda\alpha.\pi' \in E \tilde{\Rightarrow} F$.
- If $l > 0$, then all its leaves are leaves of a one-step reduct of π, wich is in $E \tilde{\Rightarrow} F$, by induction hypothesis.

Definition 20 ($\tilde{\mathcal{A}}_T$).
For all sorts T, $\tilde{\mathcal{A}}_T = \hat{T} \mapsto \mathcal{C}'$.

Definition 21 **($\tilde{\forall}_T$).** *For all sorts T and function $f \in \tilde{\mathcal{A}}_T$,*
$\tilde{\forall}_T.f = \{\pi \in \mathcal{T}$ *such that for all* $t \in \hat{T}$, $\pi t \in f(t)\}$

Lemma 13. *For all sorts T, $\tilde{\forall}_T$ is a function from $\tilde{\mathcal{A}}_T$ to $\mathcal{C}'$.*

Proof. Let T be a sort, $f \in \tilde{\mathcal{A}}_T$ and $\pi \in \tilde{\forall}_T.f$.

(CR$_1$) Let $t \in \hat{T}$ ($\neq \emptyset$), then $\pi t \in f(t) \in \mathcal{C}'$, therefore $\pi t \in SN$ and so does π.
(CR$_2$) Let π' such that $\pi \to \pi'$. Then for all $t \in \hat{T}$, $\pi' t$ is a one-step reduct of πt.
(CR$'_3$) If there exists $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$, such that each μ_i is neutral not normal, $\tau = (\mu_i/\alpha_i)_{i \leq n}\ \nu$ and for all $(\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_{i \leq n} \to \rho_i$, then $(\rho_i/\alpha_i)_{i \leq n}\ \nu \in \tilde{\forall}_T.f$. Then, for all $t \in \hat{T}$, $\tau t = (\mu_i/\alpha_i)_{i \leq n}\ \nu t = (\mu_i/\alpha_i)_{i \leq n}\ \nu'$ with $\nu' = \nu t$. And for all $(\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_{i \leq n} \to \rho_i$, we have $(\rho_i/\alpha_i)_{i \leq n}\ \nu' = (\rho_i/\alpha_i)_{i \leq n}\ \nu\ t \in f(t)$ by hypothesis, therefore $\tau t \in f(t)$ as it satifies (CR$'_3$). And finally, $\tau \in \tilde{\forall}_T.f$.

Definition 22 **($\mathcal{C}'$).** *$\mathcal{C}'$ is the* LDTVA $\langle \mathcal{C}', \tilde{\Rightarrow}, (\tilde{\mathcal{A}}_T), (\tilde{\forall}_T) \rangle$.

4.2 Building a function from $\mathcal{D}_{\equiv}$ to $\mathcal{C}'$

Definition 23 **(Δ).** *We consider a context which contains an infinite number of variables for each proposition.* $\Delta = (\beta_i^A : A)_{A \in \mathcal{P}, i \in \mathbb{N}}$.

Definition 24 **(Cl).** *For all $E \subseteq \mathcal{U}$, we define $Cl(E)$ as follows :*
for all $k \in \mathbb{N}$,

- $Cl^0(E) = \{\pi \in \mathcal{T}$ *such that* $(\Delta, \pi) \in E$ *and* π *is normal* $\}$
- $Cl^{k+1}(E) = \{\pi \in \mathcal{T}$, *such that there exists* $n \in \mathbb{N}$:
  $\exists \nu_\pi \in \mathcal{T}, \exists (\mu_i)_{i \leq n} \subseteq SN$, *each neutral not normal s.t.*
  $\pi = (\mu_i/\alpha_i)_{i \leq n}\ \nu_\pi$ *and* $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, *s.t.* $\forall i \leq n, \rho_i \in \mathcal{L}(\mu_i)$,
  we have $(\rho_i/\alpha_i)_{i \leq n}\ \nu_\pi \in Cl^k(E)\}$
- $Cl(E) = \cup_{n \in \mathbb{N}} Cl^n(E)$

Remark 8. For all $E \in \mathcal{D}_{\equiv}$,
1. for all $k \in \mathbb{N}$, $Cl^k(E) \subseteq Cl^{k+1}(E)$,
2. $Cl(E) \neq \emptyset$ as $Cl^0(E)$ contains all variables α such that $\Delta \vdash \alpha : A_E$.
3. if $\pi \in Cl(E)$ and π is normal, then $\pi \in Cl^0(E)$.

Proposition 3.
For all $E \in \mathcal{D}_{\equiv}$, $Cl(E) \in \mathcal{C}'$.

Proof. See [2]

4.3 Proving that $Cl(.)$ is a morphism

$\Rightarrow$-morphism
We prove now that for all $E, F \in \mathcal{D}_{\equiv}$, we have $Cl(E \overset{\circ}{\Rightarrow} F) = Cl(E) \tilde{\Rightarrow} Cl(F)$.

Lemma 14. *For all $E \subseteq \mathcal{T}$ and $\pi \in \mathcal{T}$,*
If $\pi \in SN$, π is neutral not normal and $\forall \rho \in \mathcal{L}(\pi)$, $\rho \in Cl(E)$, then $\pi \in Cl(E)$

Proof. As $\pi \in SN$, $\mathcal{L}(\pi)$ is defined and finite.
And, if we call $k_m = \max\{\min\{k, \rho \in Cl^k(E)\},\ \rho \in \mathcal{L}(\pi)\}$,
then $\pi \in Cl^{k_m+1}(E) \subseteq Cl(E)$.

Remark 9. In the same way, if there exists $\nu_\pi \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\pi = (\mu_i/\alpha_i)_{i \leq n}\ \nu_\pi$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n, \rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n}\ \nu_\pi \in Cl(E)$, we have $\pi \in Cl(E)$.

Remark 10. If $\pi \in Cl(E \overset{\circ}{\Rightarrow} F)$, then its normal form ρ is in $Cl^0(E \overset{\circ}{\Rightarrow} F)$, hence $\Delta \vdash \rho : A_E \Rightarrow A_F$ and therefore, π cannot reduce to a term abstraction, by confluence.

Proposition 4. *For all $E, F \in \mathcal{D}_{\equiv}$, then $Cl(E \overset{\circ}{\Rightarrow} F) = Cl(E) \tilde{\Rightarrow} Cl(F)$.*

Proof. $\subseteq$ Let $\pi \in Cl(E \overset{\circ}{\Rightarrow} F)$,
then $\pi \in SN$ as $Cl(E \overset{\circ}{\Rightarrow} F)$ satisfies (CR_1).

- Let $\rho \in \mathcal{L}_\downarrow(\pi)$, then $\rho \in Cl(E \overset{\circ}{\Rightarrow} F)$ by (CR_2), and as it is normal, it is, in particular, in $Cl^0(E \overset{\circ}{\Rightarrow} F)$, hence $(\Delta, \rho) \in E \overset{\circ}{\Rightarrow} F$. Let $\pi' \in Cl(E)$, then there exists (a minimal) $j \in \mathbb{N}$, such that $\pi' \in Cl^j(E)$. Let us show that $\rho\pi' \in Cl(F)$ by induction on j.
  - $*$ If $j = 0$, then π' is normal and $(\Delta, \pi') \in E$, therefore $(\Delta, \rho\pi') \in F$, as $(\Delta, \rho) \in E \overset{\circ}{\Rightarrow} F$. Moreover, $\rho\pi'$ is normal as π' is normal and ρ is neutral and normal. Finally $\rho\pi' \in Cl^0(F)$.
  - $*$ If $j > 0$, then there exists $\nu_{\pi'} \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\pi' = (\mu_i/\alpha_i)_{i \leq n}\ \nu_{\pi'}$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n}\ \nu_{\pi'} \in Cl^{j-1}(E)$, therefore $\rho\ (\rho_i/\alpha_i)_{i \leq n}\ \nu_{\pi'} \in Cl(F)$, by induction hypothesis. Finally, $\rho\pi' = (\mu_i/\alpha_i)_{i \leq n}\ (\rho\nu_{\pi'}) \in Cl(F)$ by remark 9.
- Let $\lambda\alpha.\rho \in \mathcal{L}_\lambda(\pi)$, then $\lambda\alpha.\rho \in Cl(E \overset{\circ}{\Rightarrow} F)$ by (CR_2), and there exists (a minimal) $k \in \mathbb{N}$, such that $\lambda\alpha.\rho \in Cl^k(E \overset{\circ}{\Rightarrow} F)$. Let us prove that $\rho \in Cl(F)$ by induction on k.
  - $*$ If $k = 0$, then $\lambda\alpha.\rho \in SN$ and $(\Delta, \lambda\alpha.\rho) \in E \overset{\circ}{\Rightarrow} F$, therefore $\rho \in SN$ and $(\Delta, \rho) \in F$, as we can choose α such that $\Delta \vdash \alpha : A_E$, by α-conversion. Finally, $\rho \in Cl^0(F)$.
  - $*$ If $k > 0$, then there exists $\nu \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\lambda\alpha.\rho = (\mu_i/\alpha_i)_{i \leq n}\ \nu_{\pi'}$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n}\ \nu \in Cl^{k-1}(E \overset{\circ}{\Rightarrow} F)$. As each μ_i is neutral, there exists ν' such that $\nu = \lambda\alpha.\nu'$, therefore $(\rho_i/\alpha_i)_{i \leq n}\ \nu' \in Cl(F)$, by induction hypothesis. Finally, $\rho = (\mu_i/\alpha_i)_{i \leq n}\ (\nu') \in Cl(F)$ by remark 9.

Finally, $\pi \in Cl(E)\tilde{\Rightarrow}Cl(F)$.

$\supseteq$ Let $\pi \in Cl(E)\tilde{\Rightarrow}Cl(F)$. then $\pi \in SN$ and π cannot reduce to a term-abstraction, by definition of $\tilde{\Rightarrow}$.

- If π is a proof-abstraction $\lambda\alpha.\rho$, then $\rho \in Cl(F)$ and there exists (a minimal) $k \in \mathbb{N}$, such that $\rho \in Cl^k(F)$. Let us prove that $\pi \in Cl(E\mathring{\Rightarrow}F)$ by induction on k.
  - If $k = 0$, then ρ is normal and $(\Delta, \rho) \in F$, therefore $\lambda\alpha.\rho$ is normal and $(\Delta, \lambda\alpha.\rho) \in E\mathring{\Rightarrow}F$, as we can choose α such that $\Delta \vdash \alpha : A_E$, by α-conversion. Finally, $\pi = \lambda\alpha.\rho \in Cl^0(E\mathring{\Rightarrow}F)$.
  - If $k > 0$, then there exists $\nu \in \mathcal{T}$, and $(\mu_i)_{i\leq n} \subseteq SN$, each neutral not normal such that $\rho = (\mu_i/\alpha_i)_{i\leq n}\ \nu$ and $\forall(\rho_i)_{i\leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i\leq n}\ \nu \in Cl^{k-1}(F)$. Hence $\pi = (\mu_i/\alpha_i)_{i\leq n}\ (\lambda\alpha.\nu) \in Cl(E\mathring{\Rightarrow}F)$ by induction hypothesis and remark 9.
- If π is neutral and normal, let $\alpha \in \mathcal{X}$ such that $\Delta \vdash \alpha : A_E$, then $\pi\alpha \in Cl(F)$. Moreover π is neutral and normal, therefore $\pi\alpha$ is normal, hence $\pi\alpha \in Cl^0(F)$. Then $\Delta \vdash \pi\alpha : A_F$ and $\Delta \vdash \pi : A_E \Rightarrow A_F$. Let $(\Gamma', \pi') \in E$, then $\Gamma' \vdash \pi' : A_E$, by $(\mathrm{P}_\equiv)$, therefore $\Delta\Gamma' \vdash \pi\pi' : A_F$. Finally, as π is neutral and normal and $\pi' \in WN$, we have $\pi\pi' \in WN$, and $\pi\pi'$ is isolated, therefore $(\Delta\Gamma', \pi\pi') \in F$ as it satisfies $(\mathrm{P}_{3_\equiv})$. Hence $(\Delta, \pi) \in E\mathring{\Rightarrow}F$ and $\pi \in Cl^0(E\mathring{\Rightarrow}F)$, as it is normal.
- Otherwise, $\pi \in SN$, is neutral and not normal. All its leaves are either neutral, either proof-abstractions and all these leaves are in $Cl(E)\tilde{\Rightarrow}Cl(F)$, as it satisfies (CR_2), therefore they also are in $Cl(E\mathring{\Rightarrow}F)$, as we saw in the previous points. Finally, $\pi \in Cl(E\mathring{\Rightarrow}F)$, by lemma 14.

$\forall$-morphism

We prove now that for all sorts T and $f \in \mathring{\mathcal{A}}_T$, $Cl(\mathring{\forall}_T f) = \tilde{\forall}_T\ Cl \circ f$. Notice that for all functions $f \in \mathring{\mathcal{A}}_T$, $Cl \circ f \in \tilde{\mathcal{A}}_T$.

Lemma 15. *For all $E \in \mathcal{D}_\equiv$, $k \in \mathbb{N}$, terms t and term-variables x of same sort, proof-terms π',
if $(t/x)\pi \in Cl^k(E)$, then $(\lambda x.\pi)t \in Cl^k(E)$.*

Proof. By induction on k.

- If $k = 0$, by $(\mathrm{P}_{3_\equiv})$.
- If $k > 0$, by induction hypothesis.

Proposition 5. *For all sorts T and $f \in \mathring{\mathcal{A}}_T$, $Cl(\mathring{\forall}_T f) = \tilde{\forall}_T\ Cl \circ f$.*

Proof. $\subseteq$ Let $\pi \in Cl(\mathring{\forall}_T f)$, then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi \in Cl^k(\mathring{\forall}_T f)$. By induction on k.

- If $k = 0$, $(\Delta, \pi) \in \mathring{\forall}_T f$ and $\pi \in SN$, then for all $t \in \hat{T}$, $(\Delta, \pi t) \in f(t)$, therefore $\pi t \in SN$ and its normal form is in $Cl^0 \circ f(t)$, hence $\pi t \in Cl \circ f(t)$, by lemma **??**. Finally, $\pi \in \tilde{\forall}_T\ Cl \circ f$.

- If $k > 0$, then $\pi = (\mu_i/\alpha_i)_i\nu$, with each μ_i neutral not normal and such that for all $(\rho_i)_{i\leq n}$ each respectively a leaf of μ_i, we have $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1}(\mathring{\forall}_T f) \subseteq \tilde{\forall}_T\ Cl \circ f$, by induction hypothesis. Let $t \in \hat{T}$, then if we write $\nu' = \nu t$, we have $\pi t = (\mu_i/\alpha_i)_i\nu'$ and for all $(\rho_i)_{i\leq n}$ each respectively a leaf of μ_i, $(\rho_i/\alpha_i)_i\nu' = (\rho_i/\alpha_i)_i\nu\ t \in Cl \circ f(t)$. Therefore $\pi t \in Cl \circ f(t)$ by remark 9. Finally, $\pi \in \tilde{\forall}_T\ Cl \circ f$.

$\supseteq$ Let $\pi \in \tilde{\forall}_T\ Cl \circ f$, then there exists a minimal $k \in \mathbb{N}$ such that there exists $t \in \hat{T}$, $\pi t \in Cl^k \circ f(t)$. By induction on k.
  - If $k = 0$, then there exists $t \in \hat{T}$ such that $\pi t \in Cl^0 \circ f(t)$. Hence $(\Delta, \pi t) \in f(t)$ and πt is normal. Hence π is normal and for all $t' \in \hat{T}$, $\pi t'$ is also normal, therefore, as $\pi t' \in Cl \circ f(t)$, we have, in particular, $\pi t' \in Cl^0 \circ f(t)$. Finally, for all $t' \in \hat{T}$, $(\Delta, \pi t') \in f(t)$, therefore $(\Delta, \pi) \in \mathring{\forall}_T f$, and $\pi \in Cl^0(\mathring{\forall}_T f)$, as it is normal.
  - If $k > 0$, let $t \in \hat{T}$ such that $\pi t \in Cl^k \circ f(t)$. Therefore $\pi t = (\mu_i/\alpha_i)_i\nu$, with each μ_i neutral not normal and such that for all $(\rho_i)_{i\leq n}$ each respectively a leaf of μ_i, we have $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1} \circ f(t)$.
    * If $\nu \neq \alpha_1$, then $\nu = \nu' t$, with $\pi = (\mu_i/\alpha_i)_i\nu'$, and for all $(\rho_i)_{i\leq n}$ each respectively a leaf of μ_i, we have $(\rho_i/\alpha_i)_i\nu' \in Cl(\mathring{\forall}_T f)$, by induction hypothesis. We conclude by lemma 14.
    * Otherwise, every leaf of πt is in $Cl^{k-1} \circ f(t)$. If π is isolated, then all its leaves ρ are neutral and normal, hence ρt is a leaf of πt, therefore $\rho \in Cl(\mathring{\forall}_T f)$, by induction hypothesis, and we conclude by lemma 14. If π reduces to $\lambda x.\pi'$ then all leaves of $(t/x)\pi'$ are in $Cl^{k-1} \circ f(t)$, therefore, for all leaves ρ of π', we have $(\lambda x.\rho)t \in Cl^{k-1} \circ f(t)$, by lemma 15, hence $\lambda x.\rho \in Cl(\mathring{\forall}_T f)$, by induction hypothesis. And finally, $\lambda x.\pi' \in Cl(\mathring{\forall}_T f)$, and so does π.

We finally get the following (second) completeness result:

Proposition 6.
If $\mathcal{L}_\equiv$ is strongly normalizing, then $Cl \circ \lfloor . \rfloor_.$ is a $\mathcal{C}'$-valued model of $\mathcal{L}_\equiv$ (and each element of the model contains an infinity of proof-variables).

Proof. By lemma 1 and propositions 2, 3, 4, 5.

5 Soundness

We finally prove in this section, that having a $\mathcal{C}'$-valued model is also a sound condition of strongly normalizing theories $\mathcal{L}_\equiv$.

Lemma 16. *If $[\![.]\!]_.$ is a $\mathcal{C}'$-valued model of a theory $\mathcal{L}_\equiv$, such that each element of the model contains an infinity of proof-variables, then for all $A \in \mathcal{P}$, contexts Γ, $\varphi \in \textsc{Val}(A) \cap \textsc{Val}(\Gamma)$, $\pi \in \mathcal{T}$ and σ substitutions such that for all declarations $\alpha : B$ in Γ, $\sigma\alpha \in [\![B]\!]_\varphi$, we have:*

$$\textit{if } \Gamma \vdash \pi : A \textit{ then } \sigma\varphi\pi \in [\![A]\!]_\varphi.$$

Proof. By induction on the length of the derivation of $\Gamma \vdash \pi : A$. By case on the last rule used. If the last rule used is :

– axiom: in this case, π is a variable α, and Γ contains a declaration $\alpha : B$ with $A \equiv B$ (therefore $|A|_\varphi \equiv |B|_\varphi$). Then $\sigma\varphi\pi = \sigma\alpha \in [\![B]\!]_\varphi = [\![A]\!]_\varphi$.

– $\Rightarrow$-intro: in this case, π is an abstraction $\lambda\alpha.\tau$, and we have $\Gamma, \alpha : B \vdash \tau : C$ with $A \equiv B \Rightarrow C$. Hence, by induction hypothesis, if we choose α such that $\alpha \in [\![B]\!]_\varphi$, by α-conversion, we have $\sigma(\alpha/\alpha)\varphi\tau = \sigma\varphi\tau \in [\![C]\!]_\varphi$. Therefore $\sigma\varphi(\lambda\alpha.\tau) = \lambda\alpha.\sigma\varphi\tau \in [\![B]\!]_\varphi \tilde{\Rightarrow} [\![C]\!]_\varphi = [\![B \Rightarrow C]\!]_\varphi$.

– $\Rightarrow$-elim: in this case, π is an application $\rho\tau$, and we have $\Gamma \vdash \rho : C \equiv B \Rightarrow A$ and $\Gamma \vdash \tau : B$. Then $\sigma\varphi\tau \in [\![B]\!]_\varphi$, by induction hypothesis.
  - If $\sigma\varphi\rho$ is a proof-abstraction then ρ is a proof-abstraction $\lambda\alpha.\rho'$, and we have $\Gamma, \alpha : B \vdash \rho' : A$, therefore $(\sigma\varphi\tau/\alpha)\sigma\varphi\rho' \in [\![A]\!]_\varphi$, by induction hypothesis, hence $\sigma\varphi(\lambda\alpha.\rho'\ \tau) \in [\![A]\!]_\varphi$ as it satisfies (CR_3').
  - If $\sigma\varphi\rho$ is neutral and normal, as $\sigma\varphi\rho \in [\![A \Rightarrow B]\!]_\varphi = [\![A]\!]_\varphi \tilde{\Rightarrow} [\![B]\!]_\varphi$, we have $\sigma\varphi(\rho\tau) \in [\![A]\!]_\varphi$.
  - Otherwise, $\sigma\varphi\rho$ is neutral and not normal and all its leaves μ satisfy $\mu\ (\sigma\varphi\tau) \in [\![A]\!]_\varphi$ as we saw in the previous points.

  Finally, $\sigma\varphi(\rho\tau) \in [\![A]\!]_\varphi$ as it satisfies (CR_3').

– $\forall$-intro: in this case, π is a term abstraction $\lambda x.\pi'$ and we have $\Gamma \vdash \pi' : B$ with $A \equiv \forall x.B$. Let $t \in \hat{T}$ (with T the sort of x), and $\varphi' = \varphi + \langle x, t\rangle$. Then $\sigma\varphi'\pi' = \sigma\varphi(t/x)\pi' \in [\![B]\!]_{\varphi'}$, by induction hypothesis. Therefore, $\sigma\varphi(\lambda x.\pi') \in \tilde{\forall}_T(t \mapsto [\![B]\!]_{\varphi+\langle x,t\rangle} = [\![A]\!]_\varphi$ (by induction on the maximal length of a reductions sequence from πt, with $t \in \hat{T}$, using the fact that for all $t \in \hat{T}$, $[\![B]\!]_{\varphi+\langle x,t\rangle}$ satisfies (CR_2) and (CR_3')).

– $\forall$-elim: in this case, π is an application ρt, and we have $\Gamma \vdash \rho : \forall x.B$ with $A = (t/x)B$ and $x \notin FV(\Gamma)$. By induction hypothesis, we have $\sigma\varphi\rho \in [\![\forall x.B, \varphi]\!] = \tilde{\forall}_T(t \mapsto [\![B]\!]\varphi + \langle x, t\rangle)$. Therefore $\sigma\varphi(\rho t) = \sigma\varphi\rho\ (\varphi t) \in [\![B]\!]_{\varphi+\langle x,\varphi t\rangle} = [\![(t/x)B]\!]_\varphi = [\![A]\!]_\varphi$

Theorem 1. *If $\mathcal{L}_\equiv$ has a $\mathcal{C}'$-valued model (such that each element contains an infinite number of variables), then $\mathcal{L}_\equiv$ is strongly normalizing.*

Proof. If $\mathcal{F}$ is a $\mathcal{C}'$-valued model of $\equiv$ then for all typing judgement $\Gamma \vdash \pi : A$ and σ and φ as in the previous proposition, we have $\sigma\varphi\pi \in [\![A]\!]_\varphi \neq \emptyset$ hence $\sigma\varphi\pi \in SN$, therefore $\pi \in SN$.

6 Rice Salad

Theorem 2. *If $\mathcal{L}_\equiv$ is weakly normalizing then it is strongly normalizing.*

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