# A semantic method to prove strong normalization from weak normalization

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## 1 Minimal deduction modulo

### 1.1 Syntax

We consider a set $(T)$ of *sorts*, an infinite set of *variables* of each sort, a set $(f)$ of function symbols, and a set $(P)$ of predicate symbols, that come with their *rank*. The formation rules for objects and propositions are the usual ones.

- Variables of sort $T$ are terms of sort $T$.
- If $f$ is a function symbol of rank $\langle T_1, \ldots, T_n, U\rangle$ and $t_1, \ldots, t_n$ are respectively objects of sort $T_1, \ldots, T_n$, then $f(t_1, \ldots, t_n)$ is a term of sort $U$.
- If $P$ is a predicate symbol of rank $\langle T_1, \ldots, T_n\rangle$ and $t_1, \ldots, t_n$ are respectively objects of sort $T_1, \ldots, T_n$, then $P(t_1, \ldots, t_n)$ is an *atomic proposition*.

Propositions are built-up from atomic propositions with the usual connective $\Rightarrow$ and quantifier $\forall$ . Remark that, implicitly, quantification in $\forall x.A$ is restricted over the sort of the variable $x$.

**Definition 1 (Terms and Propositions).**
*We call $\mathcal{O}$ (as* objects*), the set of terms:* $\quad t ::= x \mid f\ t \ldots t$
*We call $\mathcal{P}$, the set of propositions:* $\quad A ::= P\ t \ldots t \mid A \Rightarrow A \mid \forall x.A$

In this language, proof-terms can contain both term variables (written $x, y, \ldots$) and proof variables (written $\alpha, \beta, \ldots$). Terms are written $t, u, \ldots$ while proof-terms are written $\pi, \rho, \ldots$. We call $\mathcal{X}$ the set of proof variables and $\mathcal{Y}$ the set of term variables.

**Definition 2 (Proof-terms).**
*We call $\mathcal{T}$, the set of proof-terms:* $\quad \pi := \quad \alpha \mid \lambda\alpha.\pi \mid \pi\ \pi' \mid \lambda x.\pi \mid \pi\ t$

Notice that variables $\alpha$ and $x$ are bound in the constructions $\lambda\alpha.\pi$, and $\lambda x.\pi$. Alphabetic equivalence, free and bound variables are defined as usual.

Each proof-term construction corresponds to a natural deduction rule: terms of the form $\alpha$ express proofs built with the axiom rule, terms of the form $\lambda\alpha.\pi$ and $(\pi\ \pi')$ express proofs built respectively with the introduction and elimination rules of the implication and terms of the form $\lambda x.\pi$ and $(\pi\ t)$ express proofs built with the introduction and elimination rules of the universal quantifier.

We call *neutral* those proof-terms of $\mathcal{T}$ that are not abstractions *i.e.* of the form $\alpha$, $(\pi\pi')$ or $(\pi t)$. A proof-term is called *isolated* if it is neutral and only reduces on neutral terms.

### 1.2 Typing rules

We call *contexts*, lists of declarations $[\alpha : A]$ where $\alpha$ is a proof-variable and $A$ is a proposition, such that each variable in a declaration is different from all the other variables of the context (in this way, we only consider *well formed* contexts, therefore we have to deal with alphabetic equivalence, when concatening them).

Given a congruence relation on propositions $\equiv$ , we define typing rules as usual, in deduction modulo:

$$\frac{}{\Gamma, \alpha : A \vdash_{\equiv} \alpha : B}\ A \equiv B\ \ \text{(axiom)}$$

$$\frac{\Gamma, \alpha : A \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \lambda\alpha\ \pi : C}\ C \equiv A \Rightarrow B\ \ (\Rightarrow\text{-intro})$$

$$\frac{\Gamma \vdash_{\equiv} \pi : C \qquad \Gamma \vdash_{\equiv} \pi' : A}{\Gamma \vdash_{\equiv} (\pi\ \pi') : B}\ C \equiv A \Rightarrow B\ \ (\Rightarrow\text{-elim})$$

$$\frac{\Gamma \vdash_{\equiv} \pi : A}{\Gamma \vdash_{\equiv} \lambda x.\pi : B}\ B \equiv \forall x.A,\ \ x \notin FV(\Gamma)\ \ (\forall\text{-intro})$$

$$\frac{\Gamma \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \pi\ t : C}\ B \equiv \forall x.A,\ \ C \equiv (t/x)A,\ \ t \text{ has the sort of } x\ (\forall\text{-elim})$$

**Fig. 1.** Typing rules

### 1.3 Proof reduction rules and strong normalization

As usual in deduction modulo, the process of cut elimination is modeled by $\beta$-reduction. We consider the contextual closure of the reduction rules given figure 2. These rules correspond to proof reduction in natural deduction.

$$(\lambda\alpha.\pi\ \pi') \rightarrow_{\beta_\pi} (\pi'/\alpha)\pi$$
$$(\lambda x.\pi\ t) \rightarrow_{\beta_t} (t/x)\pi \qquad \text{(if } x \text{ and } t \text{ have the same sort)}$$

**Fig. 2.** Proof reduction rules

We write $(\pi'/\alpha)\pi$ (resp. $(t/x)\pi$) the substitution of $\alpha$ (resp. $x$) by $\pi'$ (resp. $t$) in $\pi$. We say that $\pi$ reduces to $\pi'$ if $\pi \to_{\beta_t} \pi'$ or $\pi \to_{\beta_\pi} \pi'$. We write $\pi \to \pi'$ if $\pi$ reduces in one step to $\pi'$, $\pi \to^+ \pi'$ if $\pi$ reduces in at least one step to $\pi'$, and $\pi \to^* \pi'$ if $\pi$ reduces in an arbitrary number of steps to $\pi'$.
A proof is said to be *normal* if it contains no redex. It is said to be weakly *normalizing* if it has a normal form and *strongly normalizing* if all reduction sequences issued from this proofs are finite. We write $SN$ for the set of strongly normalizing proofs.

### 1.4 Theories expressed in minimal deduction modulo

A theory expressed in minimal deduction modulo is defined by a many-sorted language in predicate logic $\mathcal{L} = \langle((T), (F), (P)\rangle$ and a congruence relation $\equiv$ on propositions of the associated many-sorted logic. We suppose $\equiv$ not *ambiguous*, *i.e.* there does not exist $x \in \mathcal{Y}$, $A, B, C \in \mathcal{P}$ such that $A \Rightarrow B \equiv \forall x.C$. We will call $\mathcal{L}_\equiv$ this theory.

*Remark 1.* Given a theory $\mathcal{L}_\equiv$, we will write $\vdash$ for $\vdash_\equiv$.

**Proposition 1 (confluence and subject-reduction).** *$\to$ is confluent. And for all contexts $\Gamma$, proof-terms $\pi, \pi'$ and propositions $A$, if $\Gamma \vdash \pi : A$ and $\pi \to \pi'$, then $\Gamma \vdash \pi' : A$.*

**Example** As mentioned above deduction modulo permits to express (intentional) simple type theory [1] without any axiom. We show in the following, how minimal deduction modulo permits to express minimal (intentional) simple type theory, without any axiom (see [6] for details).

The *sorts* are *simple types* inductively defined by:
– $\iota$ and $o$ are sorts,
– if $T$ and $U$ are sorts then $T \to U$ is a sort.

The language is composed of the individual symbols
- $S_{T,U,V}$ of sort $(T \to U \to V) \to (T \to U) \to T \to V$,
- $K_{T,U}$ of sort $T \to U \to T$,
- $\dot{\Rightarrow}$, of sort $o$,
- $\dot{\forall}_T$ of sort $(T \to o) \to o$,

the function symbols $\alpha_{T,U}$ of rank $\langle T \to U, T, U\rangle$,
and the predicate symbol $\varepsilon$ of rank $\langle o\rangle$.

The combinators $S_{T,U,V}$ and $K_{T,U}$ are used to express functions. The objects $\dot{\Rightarrow}$, and $\dot{\forall}_T$ allow to represent propositions as objects of sort $o$. Finally, the predicate $\varepsilon$ allows to transform such an object $t$ of type $o$ into the actual corresponding proposition $\varepsilon(t)$.

$$\begin{aligned}
\alpha(\alpha(\alpha(S_{T,U,V}, x), y), z) &\to \alpha(\alpha(x, z), \alpha(y, z)) \\
\alpha(\alpha(K_{T,U}, x), y) &\to x \\
\varepsilon(\alpha(\alpha(\dot{\Rightarrow}, x), y)) &\to \varepsilon(x) \Rightarrow \varepsilon(y) \\
\varepsilon(\alpha(\dot{\forall}, x)) &\to \forall y\ \varepsilon(\alpha(x, y))
\end{aligned}$$

## 2 Language-dependent truth values algebras

### 2.1 Definition

For all sets $E$, we call $\mathbb{P}(E)$ the set of subsets of $E$.
For all sorts $T$ of a language $\mathcal{L}$, we write $\hat{T}$, the set of closed terms of sort $T$.

**Definition 3 (language-dependent tvas).**
*Let $\mathcal{L} = \langle (T), (f), (P) \rangle$ be a many-sorted language in predicate logic.*
*$\langle \mathcal{B}, \hat{\Rightarrow}, (\hat{\mathcal{A}}_T), (\hat{\forall}_T) \rangle$ is a* LDTVA *for $\mathcal{L}$ if and only if:*

- *$\mathcal{B}$ is a set,*
- *$\hat{\Rightarrow}$ is a function from $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$,*
- *for all sorts $T$,*
  - *$\hat{\mathcal{A}}_T$ is a set of functions from $\hat{T}$ to $\mathcal{B}$: $\hat{\mathcal{A}}_T \subseteq \hat{T} \mapsto \mathcal{B}$*
  - *$\hat{\forall}_T$ is a function from $\hat{\mathcal{A}}_T$ to $\mathcal{B}$.*

**Definition 4 (Morphism).**
*Let $\mathcal{B}^1 = \langle \mathcal{B}^1, \hat{\Rightarrow}^1, (\hat{\mathcal{A}}^1_T), (\hat{\forall}^1_T) \rangle\rangle$ and $\mathcal{B}^2 = \langle \mathcal{B}^2, \hat{\Rightarrow}^2, (\hat{\mathcal{A}}^2_T), (\hat{\forall}^2_T) \rangle$ be two* LDTVA*s.*
*A* morphism *from $\mathcal{B}^1$ to $\mathcal{B}^2$ is a function $F$ from $\mathcal{B}^1$ to $\mathcal{B}^2$ such that:*
- *for all $E, G \in \mathcal{B}^1$, $F(E \hat{\Rightarrow}^1 G) = F(E) \hat{\Rightarrow}^2 F(G)$,*
- *for all sorts $T$, $x \in \hat{T}$ and $f \in \mathring{\mathcal{A}}_T$, $F(\hat{\forall}^1_T\ f) = \hat{\forall}^2_T\ F \circ f$.*

**Definition 5 (Valuation).**
*Given a* LDTVA *for $\mathcal{L} = \langle (T), (f), (P) \rangle$, a* valuation *$\varphi$ is a substitution mapping term-variables of a sort to closed terms of the same sort. For all propositions $A$ (resp. terms $t$), we call $\textsc{Val}(A)$ (resp. $\textsc{Val}(t)$) the set of valuations whose domain contains the set of free variables of $A$ (resp. $t$).*
*We write $x \notin \varphi$ for expressing the fact that $\varphi(x)$ is not defined.*

**Definition 6.** *For all $A \in \mathcal{P}$, terms $t$ and $\varphi \in \textsc{Val}(A)$, we write $|A|_\varphi$ the result of the substitution $\varphi$ on $A$.*

**Definition 7 (Models).**
*Let $\mathcal{L} = \langle (T), (f), (P) \rangle$ be a many-sorted language in predicate logic,*
*let $\equiv$ a congruence relation on propositions of minimal deduction based on $\mathcal{L}$,*
*let $\mathcal{B} = \langle \mathcal{B}, \hat{\Rightarrow}, (\hat{\mathcal{A}}_T), (\hat{\forall}_T) \rangle$ be a* LDTVA *for $\mathcal{L}$.*

1. *We call $\mathcal{B}$-*valued interpretations *those functions which map every ordered pair of a proposition $A$ and a valuation in $\textsc{Val}(A)$ to an element of $\mathcal{B}$.*
2. *A $\mathcal{B}$-valued interpretation $[\![.]\!]_.$ is a $\mathcal{B}$-*valued model *if and only if:*
   - *for all $A, B \in \mathcal{P}$ and $\varphi \in \textsc{Val}(A \Rightarrow B)$, $[\![A \Rightarrow B]\!]_\varphi = [\![A]\!]_\varphi \hat{\Rightarrow} [\![B]\!]_\varphi$*
   - *for all $A \in \mathcal{P}$, $x$ of sort $T$ and $\varphi \in \textsc{Val}(\forall x.A)$ such that $x \notin \varphi$,* $[\![\forall x.A]\!]_\varphi = \hat{\forall}_T(t \mapsto [\![A]\!]_{\varphi + \langle x,t \rangle})$
   - *for all $A \in \mathcal{P}$, $x$ of sort $T$, $t \in \hat{T}$ and $\varphi \in \textsc{Val}(\forall x.A)$ such that $x \notin \varphi$,* $[\![(t/x)A]\!]_\varphi = [\![A]\!]_{\varphi + \langle x,t \rangle}$.

3. *A $\mathcal{B}$-valued model $[\![.]\!]_.$ is a* model of the theory $\mathcal{L}_\equiv$ *if and only if: for all $A, A' \in \mathcal{P}$, $\varphi \in$ $\mathrm{Val}(A)$ and $\psi \in$ $\mathrm{Val}(A')$, if $|A|_\varphi \equiv |A'|_\psi$, then $[\![A]\!]_\varphi = [\![A']\!]_\psi$*

*Remark 2.* The previous conditions can be reformulated as: 2. Interpretations of propositions have to be adapted to the connectives to be a model. 3. Models have to be adapted to the congruence to be a model of the associated theory.

The following lemma explains that our definition of morphism is correct for the property of being a model of a theory $\mathcal{L}_\equiv$.

**Lemma 1.** *For all* LDTVA*s $\mathcal{B}_1$ and $\mathcal{B}_2$ and morphisms $F$ from $\mathcal{B}_1$ to $\mathcal{B}_2$, if $[\![.]\!]_.$ is a $\mathcal{B}_1$-valued model of a theory $\mathcal{L}_\equiv$, then $F \circ [\![.]\!]_.$ is a $\mathcal{B}_2$-valued model of $\mathcal{L}_\equiv$.*

## 3 About *WN*-reducibility candidates and typing

### 3.1 $\mathcal{D}_\equiv$, a ldtva of ($\equiv$) well-typed *WN*-reducibility candidates

**Definition 8 ($\mathcal{U}$).**
*$\mathcal{U} = \{(\Gamma, \pi)$ such that $\Gamma$ is a context and $\pi$ is a proof-term $\}$.*

**Definition 9.**
*For all $E \subseteq \mathcal{U}$, we define the following properties :*

*($P_\equiv$) There exists $A_E$ such that $\forall(\Gamma, \pi) \in E$, $\Gamma \vdash \pi : A_E$*
*($P_{1_\equiv}$) For all $(\Gamma, \pi) \in E$, $\pi \in WN$*
*($P_{2_\equiv}$) For all $(\Gamma, \pi) \in E$ and $\pi' \in WN$ such that $\pi \to \pi'$, $(\Gamma, \pi') \in E$*
*($P_{3_\equiv}$) For all $(\Gamma, \pi) \in \mathcal{U}$,*
- *If $\pi \in WN$, $\pi$ is isolated and $\Gamma \vdash \pi : A_E$, then $(\Gamma, \pi) \in E$*
- *If $(\Gamma, \pi) \in E$ and $\pi' \to_{\beta_t} \pi$, then $(\Gamma, \pi') \in E$.*

*Remark 3.* For all $E \subseteq \mathcal{U}$, if $E$ satisfies ($P_\equiv$) and ($P_{3_\equiv}$), then for all proof-variables $\alpha$, $(\alpha : A_E, \alpha) \in E$, as $\alpha$ is isolated and in *WN*.

**Definition 10 (domain $\mathcal{D}_\equiv$).** *We call $\mathcal{D}_\equiv$ the set of subsets of $\mathcal{U}$ which satisfy ($P_\equiv$), ($P_{1_\equiv}$), ($P_{2_\equiv}$) and ($P_{3_\equiv}$).*

**Definition 11 (leaves).**
*The* leaves *of a proof-term $\pi$ are its first reducts which are normal or not neutral. ($\rho$ is a leaf of $\pi$ if and only if it is normal or not neutral and there exists $n \geq 0$ and $\pi_1 \dots \pi_{n-1}$ neutral not normal terms such that $\pi = \pi_1 \to \dots \to \pi_{n-1} \to \rho$). We call $\mathcal{L}(\pi)$ the set of leaves of $\pi$, $\mathcal{L}_\downarrow(\pi)$ the set of neutral normal leaves of $\pi$, and $\mathcal{L}_\lambda$ the set of not neutral leaves of $\pi$.*

*Remark 4.*
- The only leaf of a normal or not neutral proof-term is itself.
- If $\pi$ is a neutral non-normal proof-term, then $\rho \in \mathcal{L}(\pi)$ if and only if there exists a one-step reduct $\pi'$ of $\pi$ such that $\rho \in \mathcal{L}(\pi')$.
- If $\pi \in WN$, then $\mathcal{L}(\pi) \neq \emptyset$.

**Definition 12** **($\mathring{\Rightarrow}$).** *For all $E, F \in \mathcal{D}_{\equiv}$,*
$E \mathring{\Rightarrow} F = \{(\Gamma, \pi) \in \mathcal{U}$ *such that* $\pi \in WN$, $\Gamma \vdash \pi : A_E \Rightarrow A_F$ *and*
- $\forall \rho \in \mathcal{L}_{\lambda}(\pi)$, $\rho = \lambda\alpha.\rho'$ *with* $(\Gamma, \alpha : A_E,\ \rho) \in F$
- $\forall \rho \in \mathcal{L}_{\downarrow}(\pi)$ *and* $(\Gamma', \pi') \in E$, $(\Gamma\Gamma',\ \rho\pi') \in F\}$

*Remark 5.* We recall the fact that we only consider well-formed contexts, therefore the only variables $\Gamma$ and $\Gamma'$ can share have to be declared proofs of equivalent propositions, otherwise we have to deal with $\alpha$-conversion when concatening $\Gamma$ and $\Gamma'$.

**Lemma 2.** *$\mathring{\Rightarrow}$ is a function from $\mathcal{D}_{\equiv} \times \mathcal{D}_{\equiv}$ to $\mathcal{D}_{\equiv}$.*

*Proof.* Let $E, F \in \mathcal{D}_{\equiv}$, and $(\Gamma, \pi) \in E \mathring{\Rightarrow} F$,

($\mathrm{P}_{\equiv}$) By definition, $A_{E \mathring{\Rightarrow} F} \equiv A_E \Rightarrow A_F$.

($\mathrm{P}_{1_{\equiv}}$) By definition.

($\mathrm{P}_{2_{\equiv}}$) By subject-reduction, the fact that $F$ satisfies ($\mathrm{P}_{2_{\equiv}}$) and the fact that all leaves of a reduct of a proof term $\pi$ are also leaves or reducts of leaves of $\pi$.

($\mathrm{P}_{3_{\equiv}}$)
- By the fact that an isolated term has only neutral leaves, and that if $\pi$ is a neutral normal term, and $\pi'$ is a term in $WN$, then $\pi\pi'$ is isolated and in $WN$.
- By the fact that if $\pi' \rightarrow_{\beta_t} \pi$, then in a given context, $\pi$ and $\pi'$ have the same type, if $\pi \in WN$ then so does $\pi'$ and all leaves of $\pi'$ are either leaves of $\pi$, either "$\beta_t$-expansions" of leaves of $\pi$.

**Definition 13** **($\mathring{\mathcal{A}}_T$).** *For all sorts $T$,*
$\mathring{\mathcal{A}}_T = \{f : \hat{T} \mapsto \mathcal{D}_{\equiv}$, *such that there exists* $A_f \in \mathcal{P}$ *and* $x_f \in \mathcal{X}$ *such that* *for all* $t \in \hat{T}$ *and* $(\Gamma, \pi) \in f(t)$, $\Gamma \vdash \pi : (t/x_f)A_f\}$

**Definition 14** **($\mathring{\forall}_T$).** *For all sorts $T$ and functions $f \in \mathring{\mathcal{A}}_T$,*
$\mathring{\forall}_T.f = \{(\Gamma, \pi) \in \mathcal{U}$ *such that for all* $t \in \hat{T}$, $(\Gamma,\ \pi\, t) \in f(t)\}$

**Lemma 3.** *For all sorts $T$, $\mathring{\forall}_T$ is a function from $\mathring{\mathcal{A}}_T$ to $\mathcal{D}_{\equiv}$.*

*Proof.* Let $f \in \mathring{\mathcal{A}}_T$, and $(\Gamma, \pi) \in \mathring{\forall}_T.f$

($\mathrm{P}_{\equiv}$) Let $t \in \hat{T}$ ($\neq \emptyset$). Then $(\Gamma, \pi t) \in f(t)$. As $f \in \mathring{\forall}_T$, we have $\Gamma \vdash \pi t : (t/x_f)A_f$. Therefore $\Gamma \vdash \pi : \forall x_f.A_f$, by case on the last rule used in the derivation of $\Gamma \vdash \pi t : (t/x_f)A_f$. Finally, $A_{\mathring{\forall}_T f} \equiv \forall x_f.A_f$.

($\mathrm{P}_{1_{\equiv}}$) Let $t \in \hat{T}$ ($\neq \emptyset$). Then $(\Gamma, \pi t) \in f(t) \in \mathcal{D}_{\equiv}$ therefore $\pi t \in WN$ and so does $\pi$.

($\mathrm{P}_{2_{\equiv}}$) Let $\pi'$ such that $\pi \rightarrow \pi'$. Then, for all $t \in \hat{T}$, $\pi t \rightarrow \pi' t$, therefore $\pi' t \in f(t) \in \mathcal{D}_{\equiv}$.

($\mathrm{P}_{3_{\equiv}}$)
- Let $(\Gamma, \tau) \in \mathcal{U}$ such that $\tau \in WN$, $\tau$ is isolated and $\Gamma \vdash \tau : \forall x_f.A_f$. Let $t \in \hat{T}$ then $\Gamma \vdash \tau t : (t/x_f)A_f$, $\tau t$ is isolated as $\tau$ is, and $\tau t \in WN$, as $\tau \in WN$. Finally, $\tau \in \mathring{\forall}_T.f$, as $f(t)$ satisfies ($\mathrm{P}_{3_{\equiv}}$), for all $t \in \hat{T}$.

- Let $\pi'$ such that $\pi' \to_{\beta_t} \pi$, then for all $t \in \hat{T}$, $\pi' t \to_{\beta_t} \pi t$ therefore $(\Gamma, \pi' t) \in f(t)$ as it satisfies $(\mathrm{P}_{3_\equiv})$. Hence $(\Gamma, \pi') \in \mathring{\forall}_T f$.

**Definition 15** $(\mathcal{D}_\equiv)$**.** *$\mathcal{D}_\equiv$ is the* LDTVA $\langle \mathcal{D}_\equiv, \mathring{\Rightarrow}, (\mathring{\mathcal{A}}_T), (\mathring{\forall}_T)\rangle$.

### 3.2 Building a $\mathcal{D}_\equiv$-valued interpretation of *WN* theories $\mathcal{L}_\equiv$

Let us now define a first $\mathcal{D}_\equiv$-valued model, by using directly definitions of $\mathring{\Rightarrow}$ and $\mathring{\forall}_T$, and well-chosen interpretations of atomic propositions.

**Definition 16.** *Let $A$ be a proposition and $\varphi \in \mathrm{Val}(A)$.*
*We define the subset of $\mathcal{U}$, $[A]_\varphi$ by induction over the structure of $A$.*

- $[P\ t_1 \dots t_n]_\varphi = \{(\Gamma, \pi) \in \mathcal{U}$ *such that* $\pi \in WN$ *and* $\Gamma \vdash \pi : P\ \varphi(t_1) \dots \varphi(t_n)\}$
- $[B \Rightarrow C]_\varphi = [B]_\varphi \mathring{\Rightarrow} [C]_\varphi$
- $[\forall x.B]_\varphi = \mathring{\forall}_T\ (t \mapsto [B]_{\varphi + \langle x,t\rangle})$

**Lemma 4.** *For all $A \in \mathcal{P}$, $x$ of sort $T$, $t \in \hat{T}$ and $\varphi \in \mathrm{Val}(\forall x.A)$ such that $x \notin \varphi$, we have $[(t/x)A]_\varphi = [A]_{\varphi+\langle x,t\rangle}$.*

*Proof.* By induction on $A$.

**Lemma 5.** *For all $A \in \mathcal{P}$, and $\varphi \in \mathrm{Val}(A)$,*
$[A]_\varphi \in \mathcal{D}_\equiv$ *with* $A_{[A]_\varphi} = A_\varphi$ *(i.e,* $\forall (\Gamma, \pi) \in [A]_\varphi$*,* $\Gamma \vdash \pi : A_\varphi$*).*

*Proof.* By induction on $A$.

- If $A$ is an atomic proposition $P\ t_1 \dots t_n$,
  $(\mathrm{P}_\equiv)$ By definition. (with $A_{[P\ t_1 \dots t_n]_\varphi} \equiv P\ \varphi(t_1) \dots \varphi(t_n)$).
  $(\mathrm{P}_{1_\equiv})$ By definition.
  $(\mathrm{P}_{2_\equiv})$ By subject-reduction.
  $(\mathrm{P}_{3_\equiv})$ By definition.
- If $A = B \Rightarrow C$, as $\mathring{\Rightarrow} : \mathcal{D}_\equiv \times \mathcal{D}_\equiv \mapsto \mathcal{D}_\equiv$, we conclude by induction hypothesis (with $A_{[B \Rightarrow C]_\varphi} = A_{[B]_\varphi \mathring{\Rightarrow} [C]_\varphi} \equiv A_{[B]_\varphi} \Rightarrow A_{[C]_\varphi} \equiv B_\varphi \Rightarrow C_\varphi = (B \Rightarrow C)_\varphi$).
- If $A = \forall x.B$, let $T$ be the sort of $x$ and $f = t \mapsto [B]_{\varphi+\langle x,t\rangle}$.
  Then $f$ is a function from $\hat{T}$ to $\mathcal{D}_\equiv$, by induction hypothesis. Moreover, for all $t \in \hat{T}$, $A_{f(t)} = B_{\varphi+\langle x,t\rangle} = (t/x)B_\varphi$, by induction hypothesis. Therefore $f \in \mathring{\mathcal{A}}_T$ and $\mathring{\forall}_T f \in \mathcal{D}_\equiv$ (with $A_{[\forall x.B]_\varphi} = \forall x.A_f = \forall x.B_\varphi$).

At this point, we have $\mathcal{D}_\equiv$-valued model which is adapted to typing but not necessarily $\equiv$-adapted. Indeed, in a theory where we have two atomic proposition symbols $P$ and $Q$ such that $P \equiv (Q \Rightarrow Q)$ (notice that such a theory can be weakly normalizing), then for all valuations $\varphi \in \mathrm{Val}(P) \cap \mathrm{Val}(Q)$, $[P]_\varphi \neq [Q]_\varphi \mathring{\Rightarrow} [Q]_\varphi$. We have then to modify this interpretation to make it a $\mathcal{D}_\equiv$-valued model of $\mathcal{L}_\equiv$.

### 3.3 Adapting this interpretation to the congruence

**Definition 17.** *We define a second interpretation* $\lfloor . \rfloor_.$*, as follows :*
*for all* $A \in \mathcal{P}$ *and* $\varphi \in \textsc{Val}(A)$*,*

$$\lfloor A \rfloor_\varphi = \bigcap_{A_\varphi \equiv A'_\psi} [A']_\psi$$

*Remark 6.* For all $A, A' \in \mathcal{P}$, $\varphi \in \textsc{Val}(A)$ and $\psi \in \textsc{Val}(A')$ such that $A_\varphi \equiv A'_\psi$, we have $\lfloor A \rfloor_\varphi = \lfloor A' \rfloor_\psi$, by definition.

Then we prove that $\lfloor . \rfloor_.$ is also a $\mathcal{D}_\equiv$-valued interpretation adapted to typing.

**Lemma 6.** *For all* $A \in \mathcal{P}$*, and* $\varphi \in \textsc{Val}(A)$*,*
$\lfloor A \rfloor_\varphi \in \mathcal{D}_\equiv$ *with* $A_{\lfloor A \rfloor_\varphi} = A_\varphi$ *(i.e,* $\forall (\Gamma, \pi) \in \lfloor A \rfloor_\varphi$*,* $\Gamma \vdash \pi : A_\varphi$*).*

*Proof.* Let $A \in \mathcal{P}$, and $\varphi \in \textsc{Val}(A)$,
By lemma 5 and the fact that $\lfloor A \rfloor_\varphi \subseteq [A']_\psi$, for all $A'_\psi \equiv A_\varphi$.

**Lemma 7.** *For all* $A \in \mathcal{P}$*,* $x$ *of sort* $T$*,* $t \in \hat{T}$ *and* $\varphi \in \textsc{Val}(\forall x.A)$ *such that* $x \notin \varphi$*, we have* $\lfloor (t/x)A \rfloor_\varphi = \lfloor A \rfloor_{\varphi + \langle x, t \rangle}$*.*

*Proof.* By lemma 7.

Finally, we proved, that $\lfloor . \rfloor_.$ is a $\mathcal{D}_\equiv$-valued interpretation of propositions adapted to typing and to the congruence relation $\equiv$. Let us now show that $\lfloor . \rfloor_.$ is also a $\mathcal{D}_\equiv$-valued model of weakly normalizing theories $\mathcal{L}_\equiv$, *i.e.* it is also adapted to connectives, when $\mathcal{L}_\equiv$ is weakly normalizing.

### 3.4 $\lfloor . \rfloor_.$ is a $\mathcal{D}_\equiv$-valued model of weakly normalizing theories $\mathcal{L}_\equiv$

In order to prove that $\lfloor . \rfloor_.$ is a $\mathcal{D}_\equiv$-valued model of $\mathcal{L}_\equiv$, if it is weakly normalizing, we proceed by *reductio ad absurdum*, showing that if $\lfloor . \rfloor_.$ is not connectives-adapted, then we can exhibit a typing judgement $\Gamma \vdash \pi : A$ such that $\pi \notin \mathit{WN}$.

**Lemma 8.**
*If there exists* $A, B \in \mathcal{P}$ *and* $\varphi \in \textsc{Val}(A \Rightarrow B)$*, such that* $\lfloor A \Rightarrow B \rfloor_\varphi \neq \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi$ *then there exists* $\pi \in \mathcal{T}$*,* $C \in \mathcal{P}$*,* $\psi \in \textsc{Val}(C)$ *such that* $\Gamma \vdash \pi : C_\psi$ *and* $(\Gamma, \pi) \notin \lfloor C \rfloor_\psi$*.*

*Proof.*
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin \lfloor A \Rightarrow B \rfloor_\varphi$ and $(\Gamma, \pi) \in \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi$. Then $\Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi = (A \Rightarrow B)_\varphi$.
  We take $C = A \Rightarrow B$ and $\psi = \varphi$.
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \in \lfloor A \Rightarrow B \rfloor_\varphi$ and $(\Gamma, \pi) \notin \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi$. Notice that as $\Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi$, $\pi$ cannot reduce to a term-abstraction, by subject-reduction. Then, as $\pi \in \mathit{WN}$ and $\Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi$, either there exists $\lambda\alpha.\rho \in \mathcal{L}_\lambda(\pi)$ such that $(\Gamma, \alpha : A_\varphi,\ \rho) \notin \lfloor B \rfloor_\varphi$, with $\Gamma, \alpha : A_\varphi \vdash \rho : B_\varphi$ by subject-reduction. Either there exists $\rho \in \mathcal{L}_\downarrow(\pi)$ and $(\Gamma', \pi') \in \lfloor A \rfloor_\varphi$ such that $(\Gamma\Gamma', \rho\pi') \notin \lfloor B \rfloor_\varphi$, with $\Gamma\Gamma' \vdash \rho\pi' : B_\varphi$ by subject-reduction. We take $C = B$ and $\psi = \varphi$

**Lemma 9.**
*If there exists $A \in \mathcal{P}$, $\varphi \in \mathrm{VAL}(A)$, and $x$ of sort $T$ such that $x \notin \varphi$, and*
$$\lfloor \forall x.A \rfloor_\varphi \neq \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi+\langle x,t\rangle})$$
*then there exists $\pi \in \mathcal{T}$, $C \in \mathcal{P}$, $\psi \in \mathrm{VAL}(C)$ such that $\Gamma \vdash \pi : C_\psi$ and $(\Gamma, \pi) \notin \lfloor C \rfloor_\psi$.*

*Proof.*
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin \lfloor \forall x.A \rfloor_\varphi$ and $(\Gamma, \pi) \in \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi+\langle x,t\rangle})$. Then $\Gamma \vdash \pi : (\forall x.A)_\varphi$.
  We take $C = \forall x.A$ and $\psi = \varphi$.
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \in \lfloor \forall x.A \rfloor_\varphi$ and $(\Gamma, \pi) \notin \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi+\langle x,t\rangle})$. Then there exists $t \in \hat{T}$ such that $(\Gamma, \pi t) \notin \lfloor A \rfloor_{\varphi+\langle x,t\rangle}$. As $\Gamma \vdash \pi : (\forall x.A)_\varphi$, we have $\Gamma \vdash \pi t : (t/x)A_\varphi = A_{\varphi+\langle x,t\rangle}$. We take $C = A$ and $\psi = \varphi + \langle x, t\rangle$

**Lemma 10.**
*If there exists $A, B \in \mathcal{P}$, $\varphi \in \mathrm{VAL}(A \Rightarrow B)$ or $\varphi' \in \mathrm{VAL}(\forall x.A)$ with $x$ of sort $T$, $x \notin \varphi'$ and*
$$\lfloor A \Rightarrow B \rfloor_\varphi \neq \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi \quad \text{or} \quad \lfloor \forall x.A \rfloor_{\varphi'} \neq \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi'+\langle x,t\rangle})$$
*then there exists $D \in \mathcal{P}$, $\pi \in \mathcal{T}$, $\psi \in \mathrm{VAL}(D)$ such that $\Gamma \vdash \pi : D_\psi$ and $(\Gamma, \pi) \notin [D]_\psi$.*

*Proof.* By lemmas 8 and 9, there exists $C$, $\Gamma$, $\pi$ and $\psi$ such that $\Gamma \vdash \pi : C_\psi$ and $(\Gamma, \pi) \notin \lfloor C \rfloor_\psi$. Therefore, there exists a proposition $D$ and $\psi' \in \mathrm{VAL}(D)$ such that $D_{\psi'} \equiv C_\psi$ and $(\Gamma, \pi) \notin [D]_{\psi'}$. And $\Gamma \vdash \pi : D_{\psi'}$, by equivalence of $C_\psi$ and $D_{\psi'}$.

**Lemma 11.**
*If there exists $A, B \in \mathcal{P}$, $\varphi \in \mathrm{VAL}(A \Rightarrow B)$ or $\varphi' \in \mathrm{VAL}(\forall x.A)$ with $x$ of sort $T$, $x \notin \varphi'$*
$$\text{and} \quad \lfloor A \Rightarrow B \rfloor_\varphi \neq \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi \quad \text{or} \quad \lfloor \forall x.A \rfloor_{\varphi'} \neq \mathring{\forall}_T(t \mapsto \lfloor A \rfloor_{\varphi'+\langle x,t\rangle})$$
*then there exists a (term-closed) proposition $E$, $\pi \in \mathcal{T}$ and a context $\Gamma$ such that*
$$\Gamma \vdash \pi : E \text{ and } \pi \notin WN.$$

*Proof.* By lemma 10, there exists a proposition $D$, a context $\Gamma$, a proof $\pi$ and $\varphi \in \mathcal{V}(D)$ such that $\Gamma \vdash \pi : D_\varphi$ and $(\Gamma, \pi) \notin [D]_\varphi$. By induction on $D$.
- if $D$ is atomic, then as $\Gamma \vdash \pi : D_\varphi$, we have $\pi \notin WN$.
- if $D = (F \Rightarrow G)_\varphi$,
  then $\Gamma \vdash \pi : (F \Rightarrow G)_\varphi$ and $(\Gamma, \pi) \notin [F \Rightarrow G]_\varphi = [F]_\varphi \mathring{\Rightarrow} [G]_\varphi$. If $\pi \in WN$, either there exists $\lambda\alpha.\rho \in \mathcal{L}_\lambda(\pi)$ such that $(\Gamma, \alpha : F_\varphi,\ \rho) \notin [G]_\varphi$. Either there exists $\rho \in \mathcal{L}_\downarrow(\pi)$ and $(\Gamma', \pi') \in [F]_\varphi$ such that $(\Gamma\Gamma', \rho\pi') \notin [G]_\varphi$, with $\Gamma\Gamma' \vdash \rho\pi' : G_\varphi$. We conclude by induction hypothesis.
- if $D = \forall x.F$,
  then $\Gamma \vdash \pi : (\forall x.F)_\varphi$ and $(\Gamma, \pi) \notin [\forall x.F]_\varphi$. Therefore there exists $t \in \hat{T}$ such that $(\Gamma, \pi t) \notin [F]_{\varphi+\langle x,t\rangle}$, with $\Gamma \vdash \pi t : A_{\varphi+\langle x,t\rangle}$. We conclude by induction hypothesis.

**Proposition 2 (Completeness).** *If the theory $\mathcal{L}_\equiv$ is weakly normalizing, then $\lfloor . \rfloor_. = \langle A, \varphi\rangle \mapsto \lfloor A \rfloor_\varphi$ is a $\mathcal{D}_\equiv$-model of this theory.*

*Proof.* By remark 6 and lemmas 6 and 11.

## 4 From $\mathcal{D}_{\equiv}$ to $\mathcal{C}'$

### 4.1 $\mathcal{C}'$, yet another algebra of candidates.

**Definition 18.**
*For all sets $E$ of proof-terms, we define the following properties :*

*($CR_1$)* For all $\pi \in E$, $\pi \in SN$.

*($CR_2$)* For all $\pi \in E$, for all $\pi' \in \mathcal{T}$ such that $\pi \to \pi'$, then $\pi' \in E$.

*($CR'_3$)* for all $n \in \mathbb{N}$, for all $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$, if
- for all $i \leq n$, $\mu_i$ is neutral and not normal,
- $\forall \rho_1, \dots, \rho_n \in \mathcal{T}$ such that for all $i \leq n$, $\mu_{i\leq n} \to \rho_i$, $(\rho_i/\alpha_i)_{i\leq n}\nu \in E$

*then $(\mu_i/\alpha_i)\nu \in E$.*

*Remark 7.* If $E$ satisfies ($\mathrm{CR}'_3$) then, in particular, all neutral *not normal* terms whose all one-steps reducts are in $E$, is in $E$. That is slightly different from the usual ($\mathrm{CR}_3$) of reducibility candidates, where the neutral term can be normal, therefore all neutral normal terms are in all reducibility candidates.

**Definition 19 ($\tilde{\Rightarrow}$).**
*For all $E, F \subseteq \mathcal{T}$, $E \tilde{\Rightarrow} F = \{\pi \in SN$ such that*
*- $\forall \rho \in \mathcal{L}_\lambda(\pi)$, $\rho = \lambda\alpha.\rho'$ with $\rho \in F$*
*- $\forall \rho \in \mathcal{L}_\downarrow(\pi)$ and $\pi' \in E$, $\rho\pi' \in F\}$*

**Lemma 12.** *$\tilde{\Rightarrow}$ is a function from $\mathcal{C}' \times \mathcal{C}'$ to $\mathcal{C}'$.*

*Proof.* Let $E, F \in \mathcal{C}'$ and $\pi \in E \tilde{\Rightarrow} F$,

($\mathrm{CR}_1$) $\pi \in SN$, by definition.
($\mathrm{CR}_2$) If $\pi'$ is a one-step reduct of $\pi$, then for all $\pi' \in SN$ and all its leaves are leaves of $\pi$, or reducts of leaves of $\pi$.
($\mathrm{CR}'_3$) Let $\pi = (\mu_i/\alpha_i)\nu$ with each $\mu_i$ neutral not normal and such that for all $(\rho_i)$ each respectively a one-step reduct of $\mu_i$, $(\rho_i/\alpha_i)\nu \in E \tilde{\Rightarrow} F$. We can first notice that $\pi$ cannot reduce to a term-abstraction, by confluence. Let us prove that $\pi \in E \tilde{\Rightarrow} F$, by induction on the length $l$ of the maximal length of a reductions sequence from $\pi$ to one of its leaves.

- If $l = 0$, then $\pi$ is either normal and neutral, either a proof-abstraction.
  - If $\pi$ is neutral and normal then none of the $\mu_i$ appears in $\nu$, hence $\pi \in E \tilde{\Rightarrow} F$.
  - If $\pi = \lambda\alpha.\pi'$ then, as each $\mu_i$ is neutral, $\nu = \lambda\alpha.\nu'$, with $\pi' = (\mu_i/\alpha_i)\nu'$. And for all $(\rho_i)$ each respectively a one-step reduct of $\mu_i$, $(\rho_i/\alpha_i)\nu = \lambda\alpha.(\rho_i/\alpha_i)\nu' \in E \tilde{\Rightarrow} F$, therefore $(\rho_i/\alpha_i)\nu' \in F$. Finally, $\pi' \in F$ as it satisfies ($\mathrm{CR}'_3$), and $\pi = \lambda\alpha.\pi' \in E \tilde{\Rightarrow} F$.
- If $l > 0$, then all its leaves are leaves of a one-step reduct of $\pi$, wich is in $E \tilde{\Rightarrow} F$, by induction hypothesis.

**Definition 20 ($\tilde{\mathcal{A}}_T$).**
*For all sorts $T$, $\tilde{\mathcal{A}}_T = \hat{T} \mapsto \mathcal{C}'$.*

**Definition 21** ($\tilde{\forall}_T$)**.** *For all sorts $T$ and function $f \in \tilde{\mathcal{A}}_T$,*
$\tilde{\forall}_T.f = \{\pi \in \mathcal{T}$ *such that for all* $t \in \hat{T}$, $\pi t \in f(t)\}$

**Lemma 13.** *For all sorts $T$, $\tilde{\forall}_T$ is a function from $\tilde{\mathcal{A}}_T$ to $\mathcal{C}'$.*

*Proof.* Let $T$ be a sort, $f \in \tilde{\mathcal{A}}_T$ and $\pi \in \tilde{\forall}_T.f$.

($\text{CR}_1$) Let $t \in \hat{T}$ ($\neq \emptyset$), then $\pi t \in f(t) \in \mathcal{C}'$, therefore $\pi t \in SN$ and so does $\pi$.
($\text{CR}_2$) Let $\pi'$ such that $\pi \to \pi'$. Then for all $t \in \hat{T}$, $\pi' t$ is a one-step reduct of $\pi t$.
($\text{CR}_3'$) If there exists $\nu, \mu_1, \ldots, \mu_n \in \mathcal{T}$, such that each $\mu_i$ is neutral not normal, $\tau = (\mu_i/\alpha_i)_{i\leq n}\ \nu$ and for all $(\rho_i)_{i\leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_{i\leq n} \to \rho_i$, then $(\rho_i/\alpha_i)_{i\leq n}\ \nu \in \tilde{\forall}_T.f$. Then, for all $t \in \hat{T}$, $\tau t = (\mu_i/\alpha_i)_{i\leq n}\ \nu t = (\mu_i/\alpha_i)_{i\leq n}\ \nu'$ with $\nu' = \nu t$. And for all $(\rho_i)_{i\leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_{i\leq n} \to \rho_i$, we have $(\rho_i/\alpha_i)_{i\leq n}\ \nu' = (\rho_i/\alpha_i)_{i\leq n}\ \nu\ t \in f(t)$ by hypothesis, therefore $\tau t \in f(t)$ as it satifies ($\text{CR}_3'$). And finally, $\tau \in \tilde{\forall}_T.f$.

**Definition 22** ($\mathcal{C}'$)**.** *$\mathcal{C}'$ is the* LDTVA $\langle \mathcal{C}', \tilde{\Rightarrow}, (\tilde{\mathcal{A}}_T), (\tilde{\forall}_T)\rangle$.

### 4.2 Building a function from $\mathcal{D}_{\equiv}$ to $\mathcal{C}'$

**Definition 23** ($\Delta$)**.** *We consider a context which contains an infinite number of variables for each proposition. $\Delta = (\beta_i^A : A)_{A\in\mathcal{P}, i\in\mathbb{N}}$.*

**Definition 24** (Cl)**.** *For all $E \subseteq \mathcal{U}$, we define $Cl(E)$ as follows :*
*for all $k \in \mathbb{N}$,*

- $Cl^0(E) = \{\pi \in \mathcal{T}$ *such that* $(\Delta, \pi) \in E$ *and* $\pi$ *is normal* $\}$
- $Cl^{k+1}(E) = \{\pi \in \mathcal{T}$*, such that there exists* $n \in \mathbb{N}$*:*
  $\exists \nu_\pi \in \mathcal{T}, \exists (\mu_i)_{i\leq n} \subseteq SN$*, each neutral not normal s.t.*
  $\pi = (\mu_i/\alpha_i)_{i\leq n}\ \nu_\pi$ *and* $\forall (\rho_i)_{i\leq n} \subseteq \mathcal{T}$*, s.t.* $\forall i \leq n, \rho_i \in \mathcal{L}(\mu_i)$,
  *we have* $(\rho_i/\alpha_i)_{i\leq n}\ \nu_\pi \in Cl^k(E)\}$
- $Cl(E) = \cup_{n\in\mathbb{N}} Cl^n(E)$

*Remark 8.* For all $E \in \mathcal{D}_{\equiv}$,

1. for all $k \in \mathbb{N}$, $Cl^k(E) \subseteq Cl^{k+1}(E)$,
2. $Cl(E) \neq \emptyset$ as $Cl^0(E)$ contains all variables $\alpha$ such that $\Delta \vdash \alpha : A_E$.
3. if $\pi \in Cl(E)$ and $\pi$ is normal, then $\pi \in Cl^0(E)$.

**Proposition 3.**
*For all $E \in \mathcal{D}_{\equiv}$, $Cl(E) \in \mathcal{C}'$.*

*Proof.* See [2]

### 4.3 Proving that $Cl(.)$ is a morphism

**$\Rightarrow$-morphism**
We prove now that for all $E, F \in \mathcal{D}_{\equiv}$, we have $Cl(E \stackrel{\circ}{\Rightarrow} F) = Cl(E) \stackrel{\sim}{\Rightarrow} Cl(F)$.

**Lemma 14.** *For all $E \subseteq \mathcal{T}$ and $\pi \in \mathcal{T}$,*
*If $\pi \in SN$, $\pi$ is neutral not normal and $\forall \rho \in \mathcal{L}(\pi)$, $\rho \in Cl(E)$, then $\pi \in Cl(E)$*

*Proof.* As $\pi \in SN$, $\mathcal{L}(\pi)$ is defined and finite.
And, if we call $k_m = \max\{\min\{k, \rho \in Cl^k(E)\},\ \rho \in \mathcal{L}(\pi)\}$,
then $\pi \in Cl^{k_m+1}(E) \subseteq Cl(E)$.

*Remark 9.* In the same way, if there exists $\nu_\pi \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\pi = (\mu_i/\alpha_i)_{i \leq n}\ \nu_\pi$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n, \rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n}\ \nu_\pi \in Cl(E)$, we have $\pi \in Cl(E)$.

*Remark 10.* If $\pi \in Cl(E \stackrel{\circ}{\Rightarrow} F)$, then its normal form $\rho$ is in $Cl^0(E \stackrel{\circ}{\Rightarrow} F)$, hence $\Delta \vdash \rho : A_E \Rightarrow A_F$ and therefore, $\pi$ cannot reduce to a term abstraction, by confluence.

**Proposition 4.** *For all $E, F \in \mathcal{D}_{\equiv}$, then $Cl(E \stackrel{\circ}{\Rightarrow} F) = Cl(E) \stackrel{\sim}{\Rightarrow} Cl(F)$.*

*Proof.* $\subseteq$ Let $\pi \in Cl(E \stackrel{\circ}{\Rightarrow} F)$,
then $\pi \in SN$ as $Cl(E \stackrel{\circ}{\Rightarrow} F)$ satisfies (CR$_1$).

- Let $\rho \in \mathcal{L}_{\downarrow}(\pi)$, then $\rho \in Cl(E \stackrel{\circ}{\Rightarrow} F)$ by (CR$_2$), and as it is normal, it is, in particular, in $Cl^0(E \stackrel{\circ}{\Rightarrow} F)$, hence $(\Delta, \rho) \in E \stackrel{\circ}{\Rightarrow} F$. Let $\pi' \in Cl(E)$, then there exists (a minimal) $j \in \mathbb{N}$, such that $\pi' \in Cl^j(E)$. Let us show that $\rho\pi' \in Cl(F)$ by induction on $j$.
  * If $j = 0$, then $\pi'$ is normal and $(\Delta, \pi') \in E$, therefore $(\Delta, \rho\pi') \in F$, as $(\Delta, \rho) \in E \stackrel{\circ}{\Rightarrow} F$. Moreover, $\rho\pi'$ is normal as $\pi'$ is normal and $\rho$ is neutral and normal. Finally $\rho\pi' \in Cl^0(F)$.
  * If $j > 0$, then there exists $\nu_{\pi'} \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\pi' = (\mu_i/\alpha_i)_{i \leq n}\ \nu_{\pi'}$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n}\ \nu_{\pi'} \in Cl^{j-1}(E)$, therefore $\rho\ (\rho_i/\alpha_i)_{i \leq n}\ \nu_{\pi'} \in Cl(F)$, by induction hypothesis. Finally, $\rho\pi' = (\mu_i/\alpha_i)_{i \leq n}\ (\rho\nu_{\pi'}) \in Cl(F)$ by remark 9.
- Let $\lambda\alpha.\rho \in \mathcal{L}_{\lambda}(\pi)$, then $\lambda\alpha.\rho \in Cl(E \stackrel{\circ}{\Rightarrow} F)$ by (CR$_2$), and there exists (a minimal) $k \in \mathbb{N}$, such that $\lambda\alpha.\rho \in Cl^k(E \stackrel{\circ}{\Rightarrow} F)$. Let us prove that $\rho \in Cl(F)$ by induction on $k$.
  * If $k = 0$, then $\lambda\alpha.\rho \in SN$ and $(\Delta, \lambda\alpha.\rho) \in E \stackrel{\circ}{\Rightarrow} F$, therefore $\rho \in SN$ and $(\Delta, \rho) \in F$, as we can choose $\alpha$ such that $\Delta \vdash \alpha : A_E$, by $\alpha$-conversion. Finally, $\rho \in Cl^0(F)$.
  * If $k > 0$, then there exists $\nu \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\lambda\alpha.\rho = (\mu_i/\alpha_i)_{i \leq n}\ \nu_{\pi'}$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n}\ \nu \in Cl^{k-1}(E \stackrel{\circ}{\Rightarrow} F)$. As each $\mu_i$ is neutral, there exists $\nu'$ such that $\nu = \lambda\alpha.\nu'$, therefore $(\rho_i/\alpha_i)_{i \leq n}\ \nu' \in Cl(F)$, by induction hypothesis. Finally, $\rho = (\mu_i/\alpha_i)_{i \leq n}\ (\nu') \in Cl(F)$ by remark 9.

Finally, $\pi \in Cl(E)\tilde{\Rightarrow}Cl(F)$.

$\supseteq$ Let $\pi \in Cl(E)\tilde{\Rightarrow}Cl(F)$. then $\pi \in SN$ and $\pi$ cannot reduce to a term-abstraction, by definition of $\tilde{\Rightarrow}$.

- If $\pi$ is a proof-abstraction $\lambda\alpha.\rho$, then $\rho \in Cl(F)$ and there exists (a minimal) $k \in \mathbb{N}$, such that $\rho \in Cl^k(F)$. Let us prove that $\pi \in Cl(E\mathring{\Rightarrow}F)$ by induction on $k$.
  * If $k = 0$, then $\rho$ is normal and $(\Delta, \rho) \in F$, therefore $\lambda\alpha.\rho$ is normal and $(\Delta, \lambda\alpha.\rho) \in E\mathring{\Rightarrow}F$, as we can choose $\alpha$ such that $\Delta \vdash \alpha : A_E$, by $\alpha$-conversion. Finally, $\pi = \lambda\alpha.\rho \in Cl^0(E\mathring{\Rightarrow}F)$.
  * If $k > 0$, then there exists $\nu \in \mathcal{T}$, and $(\mu_i)_{i\leq n} \subseteq SN$, each neutral not normal such that $\rho = (\mu_i/\alpha_i)_{i\leq n}\ \nu$ and $\forall(\rho_i)_{i\leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i\leq n}\ \nu \in Cl^{k-1}(F)$. Hence $\pi = (\mu_i/\alpha_i)_{i\leq n}\ (\lambda\alpha.\nu) \in Cl(E\mathring{\Rightarrow}F)$ by induction hypothesis and remark 9.
- If $\pi$ is neutral and normal, let $\alpha \in \mathcal{X}$ such that $\Delta \vdash \alpha : A_E$, then $\pi\alpha \in Cl(F)$. Moreover $\pi$ is neutral and normal, therefore $\pi\alpha$ is normal, hence $\pi\alpha \in Cl^0(F)$. Then $\Delta \vdash \pi\alpha : A_F$ and $\Delta \vdash \pi : A_E \Rightarrow A_F$. Let $(\Gamma', \pi') \in E$, then $\Gamma' \vdash \pi' : A_E$, by ($\mathrm{P}_\equiv$), therefore $\Delta\Gamma' \vdash \pi\pi' : A_F$. Finally, as $\pi$ is neutral and normal and $\pi' \in WN$, we have $\pi\pi' \in WN$, and $\pi\pi'$ is isolated, therefore $(\Delta\Gamma', \pi\pi') \in F$ as it satisfies ($\mathrm{P}3_\equiv$). Hence $(\Delta, \pi) \in E\mathring{\Rightarrow}F$ and $\pi \in Cl^0(E\mathring{\Rightarrow}F)$, as it is normal.
- Otherwise, $\pi \in SN$, is neutral and not normal. All its leaves are either neutral, either proof-abstractions and all these leaves are in $Cl(E)\tilde{\Rightarrow}Cl(F)$, as it satisfies ($\mathrm{CR}_2$), therefore they also are in $Cl(E\mathring{\Rightarrow}F)$, as we saw in the previous points. Finally, $\pi \in Cl(E\mathring{\Rightarrow}F)$, by lemma 14.

**$\forall$-morphism**

We prove now that for all sorts $T$ and $f \in \mathring{\mathcal{A}}_T$, $Cl(\mathring{\forall}_T f) = \tilde{\forall}_T\ Cl \circ f$. Notice that for all functions $f \in \mathring{\mathcal{A}}_T$, $Cl \circ f \in \tilde{\mathcal{A}}_T$.

**Lemma 15.** *For all $E \in \mathcal{D}_\equiv$, $k \in \mathbb{N}$, terms $t$ and term-variables $x$ of same sort, proof-terms $\pi'$,*
*if $(t/x)\pi \in Cl^k(E)$, then $(\lambda x.\pi)t \in Cl^k(E)$.*

*Proof.* By induction on $k$.

- If $k = 0$, by ($\mathrm{P}3_\equiv$).
- If $k > 0$, by induction hypothesis.

**Proposition 5.** *For all sorts $T$ and $f \in \mathring{\mathcal{A}}_T$, $Cl(\mathring{\forall}_T f) = \tilde{\forall}_T\ Cl \circ f$.*

*Proof.* $\subseteq$ Let $\pi \in Cl(\mathring{\forall}_T f)$, then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi \in Cl^k(\mathring{\forall}_T f)$. By induction on $k$.

- If $k = 0$, $(\Delta, \pi) \in \mathring{\forall}_T f$ and $\pi \in SN$, then for all $t \in \hat{T}$, $(\Delta, \pi t) \in f(t)$, therefore $\pi t \in SN$ and its normal form is in $Cl^0 \circ f(t)$ , hence $\pi t \in Cl \circ f(t)$, by lemma **??**. Finally, $\pi \in \tilde{\forall}_T\ Cl \circ f$.

- If $k > 0$, then $\pi = (\mu_i/\alpha_i)_i\nu$, with each $\mu_i$ neutral not normal and such that for all $(\rho_i)_{i\leq n}$ each respectively a leaf of $\mu_i$, we have $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1}(\mathring{\forall}_T f) \subseteq \tilde{\forall}_T\ Cl \circ f$, by induction hypothesis. Let $t \in \hat{T}$, then if we write $\nu' = \nu t$, we have $\pi t = (\mu_i/\alpha_i)_i\nu'$ and for all $(\rho_i)_{i\leq n}$ each respectively a leaf of $\mu_i$, $(\rho_i/\alpha_i)_i\nu' = (\rho_i/\alpha_i)_i\nu\ t \in Cl \circ f(t)$. Therefore $\pi t \in Cl \circ f(t)$ by remark 9. Finally, $\pi \in \tilde{\forall}_T\ Cl \circ f$.

$\supseteq$ Let $\pi \in \tilde{\forall}_T\ Cl \circ f$, then there exists a minimal $k \in \mathbb{N}$ such that there exists $t \in \hat{T}$, $\pi t \in Cl^k \circ f(t)$. By induction on $k$.
  - If $k = 0$, then there exists $t \in \hat{T}$ such that $\pi t \in Cl^0 \circ f(t)$. Hence $(\Delta, \pi t) \in f(t)$ and $\pi t$ is normal. Hence $\pi$ is normal and for all $t' \in \hat{T}$, $\pi t'$ is also normal, therefore, as $\pi t' \in Cl \circ f(t)$, we have, in particular, $\pi t' \in Cl^0 \circ f(t)$. Finally, for all $t' \in \hat{T}$, $(\Delta, \pi t') \in f(t)$, therefore $(\Delta, \pi) \in \mathring{\forall}_T f$, and $\pi \in Cl^0(\mathring{\forall}_T f)$, as it is normal.
  - If $k > 0$, let $t \in \hat{T}$ such that $\pi t \in Cl^k \circ f(t)$. Therefore $\pi t = (\mu_i/\alpha_i)_i\nu$, with each $\mu_i$ neutral not normal and such that for all $(\rho_i)_{i\leq n}$ each respectively a leaf of $\mu_i$, we have $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1} \circ f(t)$.
    * If $\nu \neq \alpha_1$, then $\nu = \nu' t$, with $\pi = (\mu_i/\alpha_i)_i\nu'$, and for all $(\rho_i)_{i\leq n}$ each respectively a leaf of $\mu_i$, we have $(\rho_i/\alpha_i)_i\nu' \in Cl(\mathring{\forall}_T f)$, by induction hypothesis. We conclude by lemma 14.
    * Otherwise, every leaf of $\pi t$ is in $Cl^{k-1} \circ f(t)$. If $\pi$ is isolated, then all its leaves $\rho$ are neutral and normal, hence $\rho t$ is a leaf of $\pi t$, therefore $\rho \in Cl(\mathring{\forall}_T f)$, by induction hypothesis, and we conclude by lemma 14. If $\pi$ reduces to $\lambda x.\pi'$ then all leaves of $(t/x)\pi'$ are in $Cl^{k-1} \circ f(t)$, therefore, for all leaves $\rho$ of $\pi'$, we have $(\lambda x.\rho)t \in Cl^{k-1} \circ f(t)$, by lemma 15, hence $\lambda x.\rho \in Cl(\mathring{\forall}_T f)$, by induction hypothesis. And finally, $\lambda x.\pi' \in Cl(\mathring{\forall}_T f)$, and so does $\pi$.

We finally get the following (second) completeness result:

**Proposition 6.**
*If $\mathcal{L}_\equiv$ is strongly normalizing, then $Cl \circ \lfloor . \rfloor_.$ is a $\mathcal{C}'$-valued model of $\mathcal{L}_\equiv$ (and each element of the model contains an infinity of proof-variables).*

*Proof.* By lemma 1 and propositions 2, 3, 4, 5.

## 5 Soundness

We finally prove in this section, that having a $\mathcal{C}'$-valued model is also a sound condition of strongly normalizing theories $\mathcal{L}_\equiv$.

**Lemma 16.** *If $[\![ . ]\!]_.$ is a $\mathcal{C}'$-valued model of a theory $\mathcal{L}_\equiv$, such that each element of the model contains an infinity of proof-variables,*
*then for all $A \in \mathcal{P}$, contexts $\Gamma$, $\varphi \in \mathrm{VAL}(A) \cap \mathrm{VAL}(\Gamma)$, $\pi \in \mathcal{T}$ and $\sigma$ substitutions such that for all declarations $\alpha : B$ in $\Gamma$, $\sigma\alpha \in [\![B]\!]_\varphi$, we have:*

$$\text{if } \Gamma \vdash \pi : A \text{ then } \sigma\varphi\pi \in [\![A]\!]_\varphi .$$

*Proof.* By induction on the length of the derivation of $\Gamma \vdash \pi : A$. By case on the last rule used. If the last rule used is :

– axiom: in this case, $\pi$ is a variable $\alpha$, and $\Gamma$ contains a declaration $\alpha : B$ with $A \equiv B$ (therefore $|A|_\varphi \equiv |B|_\varphi$). Then $\sigma\varphi\pi = \sigma\alpha \in [\![B]\!]_\varphi = [\![A]\!]_\varphi$.

– $\Rightarrow$-intro: in this case, $\pi$ is an abstraction $\lambda\alpha.\tau$, and we have $\Gamma, \alpha : B \vdash \tau : C$ with $A \equiv B \Rightarrow C$. Hence, by induction hypothesis, if we choose $\alpha$ such that $\alpha \in [\![B]\!]_\varphi$, by $\alpha$-conversion, we have $\sigma(\alpha/\alpha)\varphi\tau = \sigma\varphi\tau \in [\![C]\!]_\varphi$. Therefore $\sigma\varphi(\lambda\alpha.\tau) = \lambda\alpha.\sigma\varphi\tau \in [\![B]\!]_\varphi \tilde{\Rightarrow} [\![C]\!]_\varphi = [\![B \Rightarrow C]\!]_\varphi$.

– $\Rightarrow$-elim: in this case, $\pi$ is an application $\rho\tau$, and we have $\Gamma \vdash \rho : C \equiv B \Rightarrow A$ and $\Gamma \vdash \tau : B$. Then $\sigma\varphi\tau \in [\![B]\!]_\varphi$, by induction hypothesis.
  - If $\sigma\varphi\rho$ is a proof-abstraction then $\rho$ is a proof-abstraction $\lambda\alpha.\rho'$, and we have $\Gamma, \alpha : B \vdash \rho' : A$, therefore $(\sigma\varphi\tau/\alpha)\sigma\varphi\rho' \in [\![A]\!]_\varphi$, by induction hypothesis, hence $\sigma\varphi(\lambda\alpha.\rho'\ \tau) \in [\![A]\!]_\varphi$ as it satisfies ($CR_3'$).
  - If $\sigma\varphi\rho$ is neutral and normal, as $\sigma\varphi\rho \in [\![A \Rightarrow B]\!]_\varphi = [\![A]\!]_\varphi \tilde{\Rightarrow} [\![B]\!]_\varphi$, we have $\sigma\varphi(\rho\tau) \in [\![A]\!]_\varphi$.
  - Otherwise, $\sigma\varphi\rho$ is neutral and not normal and all its leaves $\mu$ satisfy $\mu\ (\sigma\varphi\tau) \in [\![A]\!]_\varphi$ as we saw in the previous points.

  Finally, $\sigma\varphi(\rho\tau) \in [\![A]\!]_\varphi$ as it satisfies ($CR_3'$).

– $\forall$-intro: in this case, $\pi$ is a term abstraction $\lambda x.\pi'$ and we have $\Gamma \vdash \pi' : B$ with $A \equiv \forall x.B$. Let $t \in \hat{T}$ (with $T$ the sort of $x$), and $\varphi' = \varphi + \langle x, t\rangle$. Then $\sigma\varphi'\pi' = \sigma\varphi(t/x)\pi' \in [\![B]\!]_{\varphi'}$, by induction hypothesis. Therefore, $\sigma\varphi(\lambda x.\pi') \in \tilde{\forall}_T(t \mapsto [\![B]\!]_{\varphi+\langle x,t\rangle} = [\![A]\!]_\varphi$ (by induction on the maximal length of a reductions sequence from $\pi t$, with $t \in \hat{T}$, using the fact that for all $t \in \hat{T}$, $[\![B]\!]_{\varphi+\langle x,t\rangle}$ satisfies ($CR_2$) and ($CR_3$')).

– $\forall$-elim: in this case, $\pi$ is an application $\rho t$, and we have $\Gamma \vdash \rho : \forall x.B$ with $A = (t/x)B$ and $x \notin FV(\Gamma)$. By induction hypothesis, we have $\sigma\varphi\rho \in [\![\forall x.B, \varphi]\!] = \tilde{\forall}_T(t \mapsto [\![B]\!]\varphi + \langle x, t\rangle)$. Therefore $\sigma\varphi(\rho t) = \sigma\varphi\rho\ (\varphi t) \in [\![B]\!]_{\varphi+\langle x,\varphi t\rangle} = [\![(t/x)B]\!]_\varphi = [\![A]\!]_\varphi$

**Theorem 1.** *If $\mathcal{L}_\equiv$ has a $\mathcal{C}'$-valued model (such that each element contains an infinite number of variables), then $\mathcal{L}_\equiv$ is strongly normalizing.*

*Proof.* If $\mathcal{F}$ is a $\mathcal{C}'$-valued model of $\equiv$ then for all typing judgement $\Gamma \vdash \pi : A$ and $\sigma$ and $\varphi$ as in the previous proposition, we have $\sigma\varphi\pi \in [\![A]\!]_\varphi \neq \emptyset$ hence $\sigma\varphi\pi \in SN$, therefore $\pi \in SN$.

## 6 Rice Salad

**Theorem 2.** *If $\mathcal{L}_\equiv$ is weakly normalizing then it is strongly normalizing.*

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