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A semantic method to prove strong normalization from weak normalization (draft)

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1 Minimal deduction modulo

1.1 Syntax

We consider a set (T) of *sorts*, an infinite set of *variables* of each sort, a set (f) of function symbols, and a set (P) of predicate symbols, that come with their *rank*. The formation rules for objects and propositions are the usual ones.

- Variables of sort T are terms of sort T .
- If f is a function symbol of rank $\langle T_1, \dots, T_n, U \rangle$ and t_1, \dots, t_n are respectively objects of sort T_1, \dots, T_n , then $f(t_1, \dots, t_n)$ is a term of sort U .
- If P is a predicate symbol of rank $\langle T_1, \dots, T_n \rangle$ and t_1, \dots, t_n are respectively objects of sort T_1, \dots, T_n , then $P(t_1, \dots, t_n)$ is an *atomic proposition*.

Propositions are built-up from atomic propositions with the usual connective \Rightarrow and quantifier \forall . Remark that, implicitly, quantification in $\forall x.A$ is restricted over the sort of the variable x .

Definition 1 (Terms and Propositions).

We call \mathcal{O} (as objects), the set of terms: $t ::= x \mid f t \dots t$
We call \mathcal{P} , the set of propositions: $A ::= P t \dots t \mid A \Rightarrow A \mid \forall x.A$

In this language, proof-terms can contain both term variables (written x, y, \dots) and proof variables (written α, β, \dots). Terms are written t, u, \dots while proof-terms are written π, ρ, \dots . We call \mathcal{X} the set of proof variables and \mathcal{Y} the set of term variables.

Definition 2 (Proof-terms).

We call \mathcal{T} , the set of proof-terms: $\pi ::= \alpha \mid \lambda \alpha. \pi \mid \pi \pi' \mid \lambda x. \pi \mid \pi t$

Notice that variables α and x are bound in the constructions $\lambda \alpha. \pi$, and $\lambda x. \pi$. Alphabetic equivalence, free and bound variables are defined as usual.

Each proof-term construction corresponds to a natural deduction rule: terms of the form α express proofs built with the axiom rule, terms of the form $\lambda \alpha. \pi$ and $(\pi \pi')$ express proofs built respectively with the introduction and elimination

rules of the implication and terms of the form $\lambda x.\pi$ and (πt) express proofs built with the introduction and elimination rules of the universal quantifier.

We call *neutral* those proof-terms of \mathcal{T} that are not abstractions *i.e.* of the form α , $(\pi\pi')$ or (πt) . A proof-term is called *isolated* if it is neutral and only reduces on neutral terms.

1.2 Typing rules

We call *contexts*, lists of declarations $[\alpha : A]$ where α is a proof-variable and A is a proposition, such that each variable in a declaration is different from all the other variables of the context (in this way, we only consider *well formed* contexts, therefore we have to deal with alphabetic equivalence, when concatening them).

Given a congruence relation on propositions \equiv , we define typing rules as usual, in deduction modulo:

$$\begin{array}{c}
\frac{}{\Gamma, \alpha : A \vdash_{\equiv} \alpha : B} \quad A \equiv B \quad (\text{axiom}) \\
\frac{\Gamma, \alpha : A \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \lambda \alpha. \pi : C} \quad C \equiv A \Rightarrow B \quad (\Rightarrow\text{-intro}) \\
\frac{\Gamma \vdash_{\equiv} \pi : C \quad \Gamma \vdash_{\equiv} \pi' : A}{\Gamma \vdash_{\equiv} (\pi \pi') : B} \quad C \equiv A \Rightarrow B \quad (\Rightarrow\text{-elim}) \\
\frac{\Gamma \vdash_{\equiv} \pi : A}{\Gamma \vdash_{\equiv} \lambda x. \pi : B} \quad B \equiv \forall x. A, \quad x \notin FV(\Gamma) \quad (\forall\text{-intro}) \\
\frac{\Gamma \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \pi t : C} \quad B \equiv \forall x. A, \quad C \equiv (t/x)A, \quad t \text{ has the sort of } x \quad (\forall\text{-elim})
\end{array}$$

Fig. 1. Typing rules

1.3 Proof reduction rules and strong normalization

As usual in deduction modulo, the process of cut elimination is modeled by β -reduction. We consider the contextual closure of the reduction rules given figure 2. These rules correspond to proof reduction in natural deduction.

$$\begin{array}{c}
(\lambda \alpha. \pi \pi') \rightarrow_{\beta_{\pi}} (\pi' / \alpha) \pi \\
(\lambda x. \pi t) \rightarrow_{\beta_t} (t/x) \pi \quad (\text{if } x \text{ and } t \text{ have the same sort})
\end{array}$$

Fig. 2. Proof reduction rules

We write $(\pi'/\alpha)\pi$ (resp. $(t/x)\pi$) the substitution of α (resp. x) by π' (resp. t) in π . We say that π reduces to π' if $\pi \rightarrow_{\beta_i} \pi'$ or $\pi \rightarrow_{\beta_\pi} \pi'$. We write $\pi \rightarrow \pi'$ if π reduces in one step to π' , $\pi \rightarrow^+ \pi'$ if π reduces in at least one step to π' , and $\pi \rightarrow^* \pi'$ if π reduces in an arbitrary number of steps to π' .

A proof is said to be *normal* if it contains no redex. It is said to be weakly *normalizing* if it has a normal form and *strongly normalizing* if all reduction sequences issued from this proofs are finite. We write SN for the set of strongly normalizing proofs.

1.4 Theories expressed in minimal deduction modulo

A theory expressed in minimal deduction modulo is defined by a many-sorted language in predicate logic $\mathcal{L} = \langle (T), (F), (P) \rangle$ and a congruence relation \equiv on propositions of the associated many-sorted logic. We suppose \equiv not *ambiguous*, i.e. there does not exist $x \in \mathcal{Y}$, $A, B, C \in \mathcal{P}$ such that $A \Rightarrow B \equiv \forall x.C$. We will call \mathcal{L}_{\equiv} this theory.

Remark 1. Given a theory \mathcal{L}_{\equiv} , we will write \vdash for \vdash_{\equiv} .

Proposition 1 (confluence and subject-reduction). *\rightarrow is confluent.*

And for all contexts Γ , proof-terms π, π' and propositions A , if $\Gamma \vdash \pi : A$ and $\pi \rightarrow \pi'$, then $\Gamma \vdash \pi' : A$.

Example As mentioned above deduction modulo permits to express (intentional) simple type theory [1] without any axiom. We show in the following, how minimal deduction modulo permits to express minimal (intentional) simple type theory, without any axiom (see [6] for details).

The *sorts* are *simple types* inductively defined by:

- ι and o are sorts,
- if T and U are sorts then $T \rightarrow U$ is a sort.

The language is composed of the individual symbols

- $S_{T,U,V}$ of sort $(T \rightarrow U \rightarrow V) \rightarrow (T \rightarrow U) \rightarrow T \rightarrow V$,
- $K_{T,U}$ of sort $T \rightarrow U \rightarrow T$,
- $\dot{\Rightarrow}$, of sort o ,
- $\dot{\forall}_T$ of sort $(T \rightarrow o) \rightarrow o$,

the function symbols $\alpha_{T,U}$ of rank $\langle T \rightarrow U, T, U \rangle$, and the predicate symbol ε of rank $\langle o \rangle$.

The combinators $S_{T,U,V}$ and $K_{T,U}$ are used to express functions. The objects $\dot{\Rightarrow}$, and $\dot{\forall}_T$ allow to represent propositions as objects of sort o . Finally, the predicate ε allows to transform such an object t of type o into the actual corresponding proposition $\varepsilon(t)$.

$$\begin{aligned} \alpha(\alpha(\alpha(S_{T,U,V}, x), y), z) &\rightarrow \alpha(\alpha(x, z), \alpha(y, z)) \\ \alpha(\alpha(K_{T,U}, x), y) &\rightarrow x \\ \varepsilon(\alpha(\alpha(\dot{\Rightarrow}, x), y)) &\rightarrow \varepsilon(x) \dot{\Rightarrow} \varepsilon(y) \\ \varepsilon(\alpha(\dot{\forall}_T, x)) &\rightarrow \forall y \varepsilon(\alpha(x, y)) \end{aligned}$$

2 Language-dependent truth values algebras

2.1 Definition

For all sets E , we call $\mathbb{P}(E)$ the set of subsets of E .

For all sorts T of a language \mathcal{L} , we write \hat{T} , the set of closed terms of sort T .

Definition 3 (language-dependent tvas).

Let $\mathcal{L} = \langle (T), (f), (P) \rangle$ be a many-sorted language in predicate logic.

$\langle \mathcal{B}, \Rightarrow, (\hat{A}_T), (\hat{V}_T) \rangle$ is a LDTVA for \mathcal{L} if and only if:

- \mathcal{B} is a set,
- \Rightarrow is a function from $\mathcal{B} \times \mathcal{B}$ to \mathcal{B} ,
- for all sorts T ,
 - \hat{A}_T is a set of functions from \hat{T} to \mathcal{B} : $\hat{A}_T \subseteq \hat{T} \mapsto \mathcal{B}$
 - \hat{V}_T is a function from \hat{A}_T to \mathcal{B} .

Definition 4 (Morphism).

Let $\mathcal{B}^1 = \langle \mathcal{B}^1, \Rightarrow^1, (\hat{A}_T^1), (\hat{V}_T^1) \rangle$ and $\mathcal{B}^2 = \langle \mathcal{B}^2, \Rightarrow^2, (\hat{A}_T^2), (\hat{V}_T^2) \rangle$ be two LDTVAs. A morphism from \mathcal{B}^1 to \mathcal{B}^2 is a function F from \mathcal{B}^1 to \mathcal{B}^2 such that:

- for all $E, G \in \mathcal{B}^1$, $F(E \Rightarrow^1 G) = F(E) \Rightarrow^2 F(G)$,
- for all sorts T , $x \in \hat{T}$ and $f \in \hat{A}_T^1$, $F(\hat{V}_T^1 f) = \hat{V}_T^2 F \circ f$.

Definition 5 (Valuation).

Given a LDTVA for $\mathcal{L} = \langle (T), (f), (P) \rangle$, a valuation φ is a substitution mapping term-variables of a sort to closed terms of the same sort. For all propositions A (resp. terms t), we call $\text{VAL}(A)$ (resp. $\text{VAL}(t)$) the set of valuations whose domain contains the set of free variables of A (resp. t).

We write $x \notin \varphi$ for expressing the fact that $\varphi(x)$ is not defined.

Definition 6. For all $A \in \mathcal{P}$, terms t and $\varphi \in \text{VAL}(A)$, we write $|A|_\varphi$ the result of the substitution φ on A .

Definition 7 (Models).

Let $\mathcal{L} = \langle (T), (f), (P) \rangle$ be a many-sorted language in predicate logic,

let \equiv be a congruence relation on propositions of minimal deduction based on \mathcal{L} ,

let $\mathcal{B} = \langle \mathcal{B}, \Rightarrow, (\hat{A}_T), (\hat{V}_T) \rangle$ be a LDTVA for \mathcal{L} .

1. We call \mathcal{B} -valued interpretations those functions which map every ordered pair of a proposition A and a valuation in $\text{VAL}(A)$ to an element of \mathcal{B} .
2. A \mathcal{B} -valued interpretation $\llbracket \cdot \rrbracket$ is a \mathcal{B} -valued model if and only if:
 - for all $A, B \in \mathcal{P}$ and $\varphi \in \text{VAL}(A \Rightarrow B)$, $\llbracket A \Rightarrow B \rrbracket_\varphi = \llbracket A \rrbracket_\varphi \Rightarrow \llbracket B \rrbracket_\varphi$
 - for all $A \in \mathcal{P}$, x of sort T and $\varphi \in \text{VAL}(\forall x.A)$ such that $x \notin \varphi$,
 $\llbracket \forall x.A \rrbracket_\varphi = \hat{V}_T(t \mapsto \llbracket A \rrbracket_{\varphi + \langle x, t \rangle})$
 - for all $A \in \mathcal{P}$, x of sort T , $t \in \hat{T}$ and $\varphi \in \text{VAL}(\forall x.A)$ such that $x \notin \varphi$,
 $\llbracket (t/x)A \rrbracket_\varphi = \llbracket A \rrbracket_{\varphi + \langle x, t \rangle}$.

3. A \mathcal{B} -valued model $\llbracket \cdot \rrbracket$ is a model of the theory \mathcal{L}_{\equiv} if and only if:
for all $A, A' \in \mathcal{P}$, $\varphi \in \text{VAL}(A)$ and $\psi \in \text{VAL}(A')$, if $|A|_{\varphi} \equiv |A'|_{\psi}$,
then $\llbracket A \rrbracket_{\varphi} = \llbracket A' \rrbracket_{\psi}$

Remark 2. The previous conditions can be reformulated as: 2. Interpretations of propositions have to be adapted to the connectives to be a model. 3. Models have to be adapted to the congruence to be a model of the associated theory.

The following lemma explains that our definition of morphism is correct for the property of being a model of a theory \mathcal{L}_{\equiv} .

Lemma 1. For all LDTVAs \mathcal{B}_1 and \mathcal{B}_2 and morphisms F from \mathcal{B}_1 to \mathcal{B}_2 , if $\llbracket \cdot \rrbracket$ is a \mathcal{B}_1 -valued model of a theory \mathcal{L}_{\equiv} , then $F \circ \llbracket \cdot \rrbracket$ is a \mathcal{B}_2 -valued model of \mathcal{L}_{\equiv} .

3 About WN-reducibility candidates and typing

3.1 \mathcal{D}_{\equiv} , a ldtva of (\equiv) well-typed WN-reducibility candidates

Definition 8 (\mathcal{U}).

$\mathcal{U} = \{(\Gamma, \pi) \text{ such that } \Gamma \text{ is a context and } \pi \text{ is a proof-term}\}$.

Definition 9.

For all $E \subseteq \mathcal{U}$, we define the following properties :

- (P_{\equiv}) There exists A_E such that $\forall (\Gamma, \pi) \in E, \Gamma \vdash \pi : A_E$
- ($P_{1_{\equiv}}$) For all $(\Gamma, \pi) \in E, \pi \in \text{WN}$
- ($P_{2_{\equiv}}$) For all $(\Gamma, \pi) \in E$ and $\pi' \in \text{WN}$ such that $\pi \rightarrow \pi', (\Gamma, \pi') \in E$
- ($P_{3_{\equiv}}$) For all $(\Gamma, \pi) \in \mathcal{U}$,
 - If $\pi \in \text{WN}$, π is isolated and $\Gamma \vdash \pi : A_E$, then $(\Gamma, \pi) \in E$
 - If $(\Gamma, \pi) \in E$ and $\pi' \rightarrow_{\beta_t} \pi$, then $(\Gamma, \pi') \in E$.

Remark 3. For all $E \subseteq \mathcal{U}$, if E satisfies (P_{\equiv}) and ($P_{3_{\equiv}}$), then for all proof-variables α , $(\alpha : A_E, \alpha) \in E$, as α is isolated and in WN .

Definition 10 (domain \mathcal{D}_{\equiv}). We call \mathcal{D}_{\equiv} the set of subsets of \mathcal{U} which satisfy (P_{\equiv}), ($P_{1_{\equiv}}$), ($P_{2_{\equiv}}$) and ($P_{3_{\equiv}}$).

Definition 11 (leaves).

The leaves of a proof-term π are its first reducts which are normal or not neutral. (ρ is a leaf of π if and only if it is normal or not neutral and there exists $n \geq 0$ and $\pi_1 \dots \pi_{n-1}$ neutral not normal terms such that $\pi = \pi_1 \rightarrow \dots \rightarrow \pi_{n-1} \rightarrow \rho$). We call $\mathcal{L}(\pi)$ the set of leaves of π , $\mathcal{L}_1(\pi)$ the set of neutral normal leaves of π , and \mathcal{L}_{λ} the set of not neutral leaves of π .

Remark 4. - The only leaf of a normal or not neutral proof-term is itself.
- If π is a neutral non-normal proof-term, then $\rho \in \mathcal{L}(\pi)$ if and only if there exists a one-step reduct π' of π such that $\rho \in \mathcal{L}(\pi')$.
- If $\pi \in \text{WN}$, then $\mathcal{L}(\pi) \neq \emptyset$.

Definition 12 (\Rightarrow). For all $E, F \in \mathcal{D}_{\equiv}$,
 $E \Rightarrow F = \{(\Gamma, \pi) \in \mathcal{U} \text{ such that } \pi \in WN, \Gamma \vdash \pi : A_E \Rightarrow A_F \text{ and}$
 $\quad - \forall \rho \in \mathcal{L}_\lambda(\pi), \rho = \lambda\alpha.\rho' \text{ with } (\Gamma, \alpha : A_E, \rho) \in F$
 $\quad - \forall \rho \in \mathcal{L}_\downarrow(\pi) \text{ and } (\Gamma', \pi') \in E, (\Gamma\Gamma', \rho\pi') \in F\}$

Remark 5. We recall the fact that we only consider well-formed contexts, therefore the only variables Γ and Γ' can share have to be declared proofs of equivalent propositions, otherwise we have to deal with α -conversion when concatenating Γ and Γ' .

Lemma 2. \Rightarrow is a function from $\mathcal{D}_{\equiv} \times \mathcal{D}_{\equiv}$ to \mathcal{D}_{\equiv} .

Proof. Let $E, F \in \mathcal{D}_{\equiv}$, and $(\Gamma, \pi) \in E \Rightarrow F$,

(P _{\equiv}) By definition, $A_{E \Rightarrow F} \equiv A_E \Rightarrow A_F$.

(P_{1 \equiv}) By definition.

(P_{2 \equiv}) By subject-reduction, the fact that F satisfies (P_{2 \equiv}) and the fact that all leaves of a reduct of a proof term π are also leaves or reducts of leaves of π .

(P_{3 \equiv})

- By the fact that an isolated term has only neutral leaves, and that if π is a neutral normal term, and π' is a term in WN , then $\pi\pi'$ is isolated and in WN .
- By the fact that if $\pi' \rightarrow_{\beta_t} \pi$, then in a given context, π and π' have the same type, if $\pi \in WN$ then so does π' and all leaves of π' are either leaves of π , either " β_t -expansions" of leaves of π .

Definition 13 (\mathring{A}_T). For all sorts T ,

$\mathring{A}_T = \{f : \hat{T} \mapsto \mathcal{D}_{\equiv}, \text{ such that there exists } A_f \in \mathcal{P} \text{ and } x_f \in \mathcal{X} \text{ such that}$
 $\quad \text{for all } t \in \hat{T} \text{ and } (\Gamma, \pi) \in f(t), \Gamma \vdash \pi : (t/x_f)A_f\}$

Definition 14 ($\mathring{\forall}_T$). For all sorts T and functions $f \in \mathring{A}_T$,

$\mathring{\forall}_T.f = \{(\Gamma, \pi) \in \mathcal{U} \text{ such that for all } t \in \hat{T}, (\Gamma, \pi t) \in f(t)\}$

Lemma 3. For all sorts T , $\mathring{\forall}_T$ is a function from \mathring{A}_T to \mathcal{D}_{\equiv} .

Proof. Let $f \in \mathring{A}_T$, and $(\Gamma, \pi) \in \mathring{\forall}_T.f$

(P _{\equiv}) Let $t \in \hat{T} (\neq \emptyset)$. Then $(\Gamma, \pi t) \in f(t)$. As $f \in \mathring{\forall}_T$, we have $\Gamma \vdash \pi t : (t/x_f)A_f$. Therefore $\Gamma \vdash \pi : \forall x_f.A_f$, by case on the last rule used in the derivation of $\Gamma \vdash \pi t : (t/x_f)A_f$. Finally, $A_{\mathring{\forall}_T.f} \equiv \forall x_f.A_f$.

(P_{1 \equiv}) Let $t \in \hat{T} (\neq \emptyset)$. Then $(\Gamma, \pi t) \in f(t) \in \mathcal{D}_{\equiv}$ therefore $\pi t \in WN$ and so does π .

(P_{2 \equiv}) Let π' such that $\pi \rightarrow \pi'$. Then, for all $t \in \hat{T}$, $\pi t \rightarrow \pi' t$, therefore $\pi' t \in f(t) \in \mathcal{D}_{\equiv}$.

(P_{3 \equiv})

- Let $(\Gamma, \tau) \in \mathcal{U}$ such that $\tau \in WN$, τ is isolated and $\Gamma \vdash \tau : \forall x_f.A_f$. Let $t \in \hat{T}$ then $\Gamma \vdash \tau t : (t/x_f)A_f$, τt is isolated as τ is, and $\tau t \in WN$, as $\tau \in WN$. Finally, $\tau \in \mathring{\forall}_T.f$, as $f(t)$ satisfies (P_{3 \equiv}), for all $t \in \hat{T}$.

- Let π' such that $\pi' \rightarrow_{\beta_t} \pi$, then for all $t \in \hat{T}$, $\pi't \rightarrow_{\beta_t} \pi t$ therefore $(\Gamma, \pi't) \in f(t)$ as it satisfies $(P_{3_{\equiv}})$. Hence $(\Gamma, \pi') \in \check{\forall}_T f$.

Definition 15 (\mathcal{D}_{\equiv}). \mathcal{D}_{\equiv} is the LDTVA $\langle \mathcal{D}_{\equiv}, \overset{\circ}{\Rightarrow}, (\overset{\circ}{\mathcal{A}}_T), (\overset{\circ}{\forall}_T) \rangle$.

3.2 Building a \mathcal{D}_{\equiv} -valued interpretation of WN theories \mathcal{L}_{\equiv}

Let us now define a first \mathcal{D}_{\equiv} -valued model, by using directly definitions of $\overset{\circ}{\Rightarrow}$ and $\overset{\circ}{\forall}_T$, and well-chosen interpretations of atomic propositions.

Definition 16. Let A be a proposition and $\varphi \in \text{VAL}(A)$.

We define the subset of \mathcal{U} , $[A]_{\varphi}$ by induction over the structure of A .

- $[P t_1 \dots t_n]_{\varphi} = \{(\Gamma, \pi) \in \mathcal{U} \text{ such that } \pi \in \text{WN and } \Gamma \vdash \pi : P \varphi(t_1) \dots \varphi(t_n)\}$
- $[B \Rightarrow C]_{\varphi} = [B]_{\varphi} \overset{\circ}{\Rightarrow} [C]_{\varphi}$
- $[\forall x.B]_{\varphi} = \overset{\circ}{\forall}_T (t \mapsto [B]_{\varphi+\langle x,t \rangle})$

Lemma 4. For all $A \in \mathcal{P}$, x of sort T , $t \in \hat{T}$ and $\varphi \in \text{VAL}(\forall x.A)$ such that $x \notin \varphi$, we have $[(t/x)A]_{\varphi} = [A]_{\varphi+\langle x,t \rangle}$.

Proof. By induction on A .

Lemma 5. For all $A \in \mathcal{P}$, and $\varphi \in \text{VAL}(A)$,

$[A]_{\varphi} \in \mathcal{D}_{\equiv}$ with $A_{[A]_{\varphi}} = A_{\varphi}$ (i.e., $\forall(\Gamma, \pi) \in [A]_{\varphi}, \Gamma \vdash \pi : A_{\varphi}$).

Proof. By induction on A .

- If A is an atomic proposition $P t_1 \dots t_n$,
 - (P_{\equiv}) By definition. (with $A_{[P t_1 \dots t_n]_{\varphi}} \equiv P \varphi(t_1) \dots \varphi(t_n)$).
 - ($P_{1_{\equiv}}$) By definition.
 - ($P_{2_{\equiv}}$) By subject-reduction.
 - ($P_{3_{\equiv}}$) By definition.
- If $A = B \Rightarrow C$, as $\overset{\circ}{\Rightarrow} : \mathcal{D}_{\equiv} \times \mathcal{D}_{\equiv} \mapsto \mathcal{D}_{\equiv}$, we conclude by induction hypothesis (with $A_{[B \Rightarrow C]_{\varphi}} = A_{[B]_{\varphi} \overset{\circ}{\Rightarrow} [C]_{\varphi}} \equiv A_{[B]_{\varphi}} \Rightarrow A_{[C]_{\varphi}} \equiv B_{\varphi} \Rightarrow C_{\varphi} = (B \Rightarrow C)_{\varphi}$).
- If $A = \forall x.B$, let T be the sort of x and $f = t \mapsto [B]_{\varphi+\langle x,t \rangle}$.
Then f is a function from \hat{T} to \mathcal{D}_{\equiv} , by induction hypothesis. Moreover, for all $t \in \hat{T}$, $A_{f(t)} = B_{\varphi+\langle x,t \rangle} = (t/x)B_{\varphi}$, by induction hypothesis. Therefore $f \in \overset{\circ}{\mathcal{A}}_T$ and $\overset{\circ}{\forall}_T f \in \mathcal{D}_{\equiv}$ (with $A_{[\forall x.B]_{\varphi}} = \forall x.A_f = \forall x.B_{\varphi}$).

At this point, we have \mathcal{D}_{\equiv} -valued model which is adapted to typing but not necessarily \equiv -adapted. Indeed, in a theory where we have two atomic proposition symbols P and Q such that $P \equiv (Q \Rightarrow Q)$ (notice that such a theory can be weakly normalizing), then for all valuations $\varphi \in \text{VAL}(P) \cap \text{VAL}(Q)$, $[P]_{\varphi} \neq [Q]_{\varphi} \overset{\circ}{\Rightarrow} [Q]_{\varphi}$. We have then to modify this interpretation to make it a \mathcal{D}_{\equiv} -valued model of \mathcal{L}_{\equiv} .

3.3 Adapting this interpretation to the congruence

Definition 17. We define a second interpretation $[\cdot]$, as follows :
for all $A \in \mathcal{P}$ and $\varphi \in \text{VAL}(A)$,

$$[A]_\varphi = \bigcap_{A_\varphi \equiv A'_\psi} [A']_\psi$$

Remark 6. For all $A, A' \in \mathcal{P}$, $\varphi \in \text{VAL}(A)$ and $\psi \in \text{VAL}(A')$ such that $A_\varphi \equiv A'_\psi$, we have $[A]_\varphi = [A']_\psi$, by definition.

Then we prove that $[\cdot]$ is also a \mathcal{D}_\equiv -valued interpretation adapted to typing.

Lemma 6. For all $A \in \mathcal{P}$, and $\varphi \in \text{VAL}(A)$,

$$[A]_\varphi \in \mathcal{D}_\equiv \quad \text{with } A_{[A]_\varphi} = A_\varphi \quad (\text{i.e.}, \forall (\Gamma, \pi) \in [A]_\varphi, \Gamma \vdash \pi : A_\varphi).$$

Proof. Let $A \in \mathcal{P}$, and $\varphi \in \text{VAL}(A)$,

By lemma 5 and the fact that $[A]_\varphi \subseteq [A']_\psi$, for all $A'_\psi \equiv A_\varphi$.

Lemma 7. For all $A \in \mathcal{P}$, x of sort T , $t \in \hat{T}$ and $\varphi \in \text{VAL}(\forall x.A)$ such that $x \notin \varphi$, we have $[(t/x)A]_\varphi = [A]_{\varphi+(x,t)}$.

Proof. By lemma 7.

Finally, we proved, that $[\cdot]$ is a \mathcal{D}_\equiv -valued interpretation of propositions adapted to typing and to the congruence relation \equiv . Let us now show that $[\cdot]$ is also a \mathcal{D}_\equiv -valued model of weakly normalizing theories \mathcal{L}_\equiv , i.e. it is also adapted to connectives, when \mathcal{L}_\equiv is weakly normalizing.

3.4 $[\cdot]$ is a \mathcal{D}_\equiv -valued model of weakly normalizing theories \mathcal{L}_\equiv

In order to prove that $[\cdot]$ is a \mathcal{D}_\equiv -valued model of \mathcal{L}_\equiv , if it is weakly normalizing, we proceed by *reductio ad absurdum*, showing that if $[\cdot]$ is not connectives-adapted, then we can exhibit a typing judgement $\Gamma \vdash \pi : A$ such that $\pi \notin \text{WN}$.

Lemma 8.

If there exists $A, B \in \mathcal{P}$ and $\varphi \in \text{VAL}(A \Rightarrow B)$, such that $[A \Rightarrow B]_\varphi \neq [A]_\varphi \overset{\Rightarrow}{\Rightarrow} [B]_\varphi$ then there exists $\pi \in \mathcal{T}$, $C \in \mathcal{P}$, $\psi \in \text{VAL}(C)$ such that $\Gamma \vdash \pi : C_\psi$ and $(\Gamma, \pi) \notin [C]_\psi$.

Proof. – If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin [A \Rightarrow B]_\varphi$ and

$$(\Gamma, \pi) \in [A]_\varphi \overset{\Rightarrow}{\Rightarrow} [B]_\varphi. \quad \text{Then } \Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi = (A \Rightarrow B)_\varphi.$$

We take $C = A \Rightarrow B$ and $\psi = \varphi$.

– If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \in [A \Rightarrow B]_\varphi$ and $(\Gamma, \pi) \notin [A]_\varphi \overset{\Rightarrow}{\Rightarrow} [B]_\varphi$.

Notice that as $\Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi$, π cannot reduce to a term-abstraction, by subject-reduction. Then, as $\pi \in \text{WN}$ and $\Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi$, either there exists $\lambda\alpha.\rho \in \mathcal{L}_\lambda(\pi)$ such that $(\Gamma, \alpha : A_\varphi, \rho) \notin [B]_\varphi$, with $\Gamma, \alpha : A_\varphi \vdash \rho : B_\varphi$ by subject-reduction. Either there exists $\rho \in \mathcal{L}_\perp(\pi)$ and $(\Gamma', \pi') \in [A]_\varphi$ such that $(\Gamma\Gamma', \rho\pi') \notin [B]_\varphi$, with $\Gamma\Gamma' \vdash \rho\pi' : B_\varphi$ by subject-reduction. We take $C = B$ and $\psi = \varphi$

Lemma 9.

If there exists $A \in \mathcal{P}$, $\varphi \in \text{VAL}(A)$, and x of sort T such that $x \notin \varphi$, and

$$[\forall x.A]_{\varphi} \neq \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi+\langle x,t \rangle})$$

then there exists $\pi \in \mathcal{T}$, $C \in \mathcal{P}$, $\psi \in \text{VAL}(C)$ such that $\Gamma \vdash \pi : C_{\psi}$ and $(\Gamma, \pi) \notin [C]_{\psi}$.

Proof. – If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin [\forall x.A]_{\varphi}$ and

$$(\Gamma, \pi) \in \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi+\langle x,t \rangle}). \text{ Then } \Gamma \vdash \pi : (\forall x.A)_{\varphi}.$$

We take $C = \forall x.A$ and $\psi = \varphi$.

– If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \in [\forall x.A]_{\varphi}$ and $(\Gamma, \pi) \notin \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi+\langle x,t \rangle})$.

Then there exists $t \in \hat{T}$ such that $(\Gamma, \pi t) \notin [A]_{\varphi+\langle x,t \rangle}$. As $\Gamma \vdash \pi : (\forall x.A)_{\varphi}$,

we have $\Gamma \vdash \pi t : (t/x)A_{\varphi} = A_{\varphi+\langle x,t \rangle}$. We take $C = A$ and $\psi = \varphi + \langle x, t \rangle$

Lemma 10.

If there exists $A, B \in \mathcal{P}$, $\varphi \in \text{VAL}(A \Rightarrow B)$ or $\varphi' \in \text{VAL}(\forall x.A)$ with x of sort T , $x \notin \varphi'$ and

$$[A \Rightarrow B]_{\varphi} \neq [A]_{\varphi} \overset{\circ}{\Rightarrow} [B]_{\varphi} \quad \text{or} \quad [\forall x.A]_{\varphi'} \neq \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi'+\langle x,t \rangle})$$

then there exists $D \in \mathcal{P}$, $\pi \in \mathcal{T}$, $\psi \in \text{VAL}(D)$ such that $\Gamma \vdash \pi : D_{\psi}$ and $(\Gamma, \pi) \notin [D]_{\psi}$.

Proof. By lemmas 8 and 9, there exists C, Γ, π and ψ such that $\Gamma \vdash \pi : C_{\psi}$ and $(\Gamma, \pi) \notin [C]_{\psi}$. Therefore, there exists a proposition D and $\psi' \in \text{VAL}(D)$ such that $D_{\psi'} \equiv C_{\psi}$ and $(\Gamma, \pi) \notin [D]_{\psi'}$. And $\Gamma \vdash \pi : D_{\psi'}$, by equivalence of C_{ψ} and $D_{\psi'}$.

Lemma 11.

If there exists $A, B \in \mathcal{P}$, $\varphi \in \text{VAL}(A \Rightarrow B)$ or $\varphi' \in \text{VAL}(\forall x.A)$ with x of sort T , $x \notin \varphi'$

$$\text{and} \quad [A \Rightarrow B]_{\varphi} \neq [A]_{\varphi} \overset{\circ}{\Rightarrow} [B]_{\varphi} \quad \text{or} \quad [\forall x.A]_{\varphi'} \neq \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi'+\langle x,t \rangle})$$

then there exists a (term-closed) proposition E , $\pi \in \mathcal{T}$ and a context Γ such that

$$\Gamma \vdash \pi : E \text{ and } \pi \notin \text{WN}.$$

Proof. By lemma 10, there exists a proposition D , a context Γ , a proof π and $\varphi \in \mathcal{V}(D)$ such that $\Gamma \vdash \pi : D_{\varphi}$ and $(\Gamma, \pi) \notin [D]_{\varphi}$. By induction on D .

– if D is atomic, then as $\Gamma \vdash \pi : D_{\varphi}$, we have $\pi \notin \text{WN}$.

– if $D = (F \Rightarrow G)_{\varphi}$,

then $\Gamma \vdash \pi : (F \Rightarrow G)_{\varphi}$ and $(\Gamma, \pi) \notin [F \Rightarrow G]_{\varphi} = [F]_{\varphi} \overset{\circ}{\Rightarrow} [G]_{\varphi}$. If $\pi \in \text{WN}$, either there exists $\lambda\alpha.\rho \in \mathcal{L}_{\lambda}(\pi)$ such that $(\Gamma, \alpha : F_{\varphi}, \rho) \notin [G]_{\varphi}$. Either there exists $\rho \in \mathcal{L}_{\downarrow}(\pi)$ and $(\Gamma', \pi') \in [F]_{\varphi}$ such that $(\Gamma\Gamma', \rho\pi') \notin [G]_{\varphi}$, with $\Gamma\Gamma' \vdash \rho\pi' : G_{\varphi}$. We conclude by induction hypothesis.

– if $D = \forall x.F$,

then $\Gamma \vdash \pi : (\forall x.F)_{\varphi}$ and $(\Gamma, \pi) \notin [\forall x.F]_{\varphi}$. Therefore there exists $t \in \hat{T}$ such that $(\Gamma, \pi t) \notin [F]_{\varphi+\langle x,t \rangle}$, with $\Gamma \vdash \pi t : A_{\varphi+\langle x,t \rangle}$. We conclude by induction hypothesis.

Proposition 2 (Completeness). *If the theory \mathcal{L}_{\equiv} is weakly normalizing, then $[\cdot]_{\cdot} = \langle A, \varphi \rangle \mapsto [A]_{\varphi}$ is a \mathcal{D}_{\equiv} -model of this theory.*

Proof. By remark 6 and lemmas 6 and 11.

4 From \mathcal{D}_{\equiv} to \mathcal{C}'

4.1 \mathcal{C}' , yet another algebra of candidates.

Definition 18.

For all sets E of proof-terms, we define the following properties :

- (CR₁) For all $\pi \in E$, $\pi \in SN$.
- (CR₂) For all $\pi \in E$, for all $\pi' \in \mathcal{T}$ such that $\pi \rightarrow \pi'$, then $\pi' \in E$.
- (CR'₃) for all $n \in \mathbb{N}$, for all $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$, if
 - for all $i \leq n$, μ_i is neutral and not normal,
 - $\forall \rho_1, \dots, \rho_n \in \mathcal{T}$ such that for all $i \leq n$, $\mu_i \leq_n \rightarrow \rho_i$, $(\rho_i/\alpha_i)_{i \leq n} \nu \in E$
 then $(\mu_i/\alpha_i)\nu \in E$.

Remark 7. If E satisfies (CR'₃) then, in particular, all neutral *not normal* terms whose all one-steps reducts are in E , is in E . That is slightly different from the usual (CR₃) of reducibility candidates, where the neutral term can be normal, therefore all neutral normal terms are in all reducibility candidates.

Definition 19 (\rightleftharpoons).

For all $E, F \subseteq \mathcal{T}$, $E \rightleftharpoons F = \{\pi \in SN \text{ such that}$

- $\forall \rho \in \mathcal{L}_\lambda(\pi)$, $\rho = \lambda\alpha.\rho'$ with $\rho \in F$
- $\forall \rho \in \mathcal{L}_\perp(\pi)$ and $\pi' \in E$, $\rho\pi' \in F\}$

Lemma 12. \rightleftharpoons is a function from $\mathcal{C}' \times \mathcal{C}'$ to \mathcal{C}' .

Proof. Let $E, F \in \mathcal{C}'$ and $\pi \in E \rightleftharpoons F$,

- (CR₁) $\pi \in SN$, by definition.
- (CR₂) If π' is a one-step reduct of π , then for all $\pi' \in SN$ and all its leaves are leaves of π , or reducts of leaves of π .
- (CR'₃) Let $\pi = (\mu_i/\alpha_i)\nu$ with each μ_i neutral not normal and such that for all (ρ_i) each respectively a one-step reduct of μ_i , $(\rho_i/\alpha_i)\nu \in E \rightleftharpoons F$. We can first notice that π cannot reduce to a term-abstraction, by confluence. Let us prove that $\pi \in E \rightleftharpoons F$, by induction on the length l of the maximal length of a reductions sequence from π to one of its leaves.
 - If $l = 0$, then π is either normal and neutral, either a proof-abstraction.
 - * If π is neutral and normal then none of the μ_i appears in ν , hence $\pi \in E \rightleftharpoons F$.
 - * If $\pi = \lambda\alpha.\pi'$ then, as each μ_i is neutral, $\nu = \lambda\alpha.\nu'$, with $\pi' = (\mu_i/\alpha_i)\nu'$. And for all (ρ_i) each respectively a one-step reduct of μ_i , $(\rho_i/\alpha_i)\nu = \lambda\alpha.(\rho_i/\alpha_i)\nu' \in E \rightleftharpoons F$, therefore $(\rho_i/\alpha_i)\nu' \in F$. Finally, $\pi' \in F$ as it satisfies (CR'₃), and $\pi = \lambda\alpha.\pi' \in E \rightleftharpoons F$.
 - If $l > 0$, then all its leaves are leaves of a one-step reduct of π , wich is in $E \rightleftharpoons F$, by induction hypothesis.

Definition 20 ($\tilde{\mathcal{A}}_T$).

For all sorts T , $\tilde{\mathcal{A}}_T = \hat{T} \mapsto \mathcal{C}'$.

Definition 21 ($\tilde{\forall}_T$). For all sorts T and function $f \in \tilde{\mathcal{A}}_T$,
 $\tilde{\forall}_T.f = \{\pi \in \mathcal{T} \text{ such that for all } t \in \hat{T}, \pi t \in f(t)\}$

Lemma 13. For all sorts T , $\tilde{\forall}_T$ is a function from $\tilde{\mathcal{A}}_T$ to \mathcal{C}' .

Proof. Let T be a sort, $f \in \tilde{\mathcal{A}}_T$ and $\pi \in \tilde{\forall}_T.f$.

(CR₁) Let $t \in \hat{T}$ ($\neq \emptyset$), then $\pi t \in f(t) \in \mathcal{C}'$, therefore $\pi t \in SN$ and so does π .

(CR₂) Let π' such that $\pi \rightarrow \pi'$. Then for all $t \in \hat{T}$, $\pi' t$ is a one-step reduct of πt .

(CR'₃) If there exists $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$, such that each μ_i is neutral not normal, $\tau = (\mu_i/\alpha_i)_{i \leq n} \nu$ and for all $(\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \leq n \rightarrow \rho_i$, then $(\rho_i/\alpha_i)_{i \leq n} \nu \in \tilde{\forall}_T.f$. Then, for all $t \in \hat{T}$, $\tau t = (\mu_i/\alpha_i)_{i \leq n} \nu t = (\mu_i/\alpha_i)_{i \leq n} \nu'$ with $\nu' = \nu t$. And for all $(\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \leq n \rightarrow \rho_i$, we have $(\rho_i/\alpha_i)_{i \leq n} \nu' = (\rho_i/\alpha_i)_{i \leq n} \nu t \in f(t)$ by hypothesis, therefore $\tau t \in f(t)$ as it satisfies (CR'₃). And finally, $\tau \in \tilde{\forall}_T.f$.

Definition 22 (\mathcal{C}'). \mathcal{C}' is the LDTVVA $\langle \mathcal{C}', \Rightarrow, (\tilde{\mathcal{A}}_T), (\tilde{\forall}_T) \rangle$.

4.2 Building a function from \mathcal{D}_{\equiv} to \mathcal{C}'

Definition 23 (Δ). We consider a context which contains an infinite number of variables for each proposition. $\Delta = (\beta_i^A : A)_{A \in \mathcal{P}, i \in \mathbb{N}}$.

Definition 24 (Cl). For all $E \subseteq \mathcal{U}$, we define $Cl(E)$ as follows :
for all $k \in \mathbb{N}$,

- $Cl^0(E) = \{\pi \in \mathcal{T} \text{ such that } (\Delta, \pi) \in E \text{ and } \pi \text{ is normal} \}$
- $Cl^{k+1}(E) = \{\pi \in \mathcal{T}, \text{ such that there exists } n \in \mathbb{N}:$
 $\exists \nu_\pi \in \mathcal{T}, \exists (\mu_i)_{i \leq n} \subseteq SN, \text{ each neutral not normal s.t.}$
 $\pi = (\mu_i/\alpha_i)_{i \leq n} \nu_\pi \text{ and } \forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}, \text{ s.t. } \forall i \leq n, \rho_i \in \mathcal{L}(\mu_i),$
 $\text{we have } (\rho_i/\alpha_i)_{i \leq n} \nu_\pi \in Cl^k(E)\}$
- $Cl(E) = \cup_{n \in \mathbb{N}} Cl^n(E)$

Remark 8. For all $E \in \mathcal{D}_{\equiv}$,

1. for all $k \in \mathbb{N}$, $Cl^k(E) \subseteq Cl^{k+1}(E)$,
2. $Cl(E) \neq \emptyset$ as $Cl^0(E)$ contains all variables α such that $\Delta \vdash \alpha : A_E$.
3. if $\pi \in Cl(E)$ and π is normal, then $\pi \in Cl^0(E)$.

Proposition 3.

For all $E \in \mathcal{D}_{\equiv}$, $Cl(E) \in \mathcal{C}'$.

Proof. See [2]

4.3 Proving that $Cl(\cdot)$ is a morphism

\Rightarrow -morphism

We prove now that for all $E, F \in \mathcal{D}_{\equiv}$, we have $Cl(E \Rightarrow F) = Cl(E) \Rightarrow Cl(F)$.

Lemma 14. *For all $E \subseteq \mathcal{T}$ and $\pi \in \mathcal{T}$,*

If $\pi \in SN$, π is neutral not normal and $\forall \rho \in \mathcal{L}(\pi)$, $\rho \in Cl(E)$, then $\pi \in Cl(E)$

Proof. As $\pi \in SN$, $\mathcal{L}(\pi)$ is defined and finite.

And, if we call $k_m = \max\{\min\{k, \rho \in Cl^k(E)\}, \rho \in \mathcal{L}(\pi)\}$,

then $\pi \in Cl^{k_m+1}(E) \subseteq Cl(E)$.

Remark 9. In the same way, if there exists $\nu_\pi \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\pi = (\mu_i/\alpha_i)_{i \leq n} \nu_\pi$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n} \nu_\pi \in Cl(E)$, we have $\pi \in Cl(E)$.

Remark 10. If $\pi \in Cl(E \Rightarrow F)$, then its normal form ρ is in $Cl^0(E \Rightarrow F)$, hence $\Delta \vdash \rho : A_E \Rightarrow A_F$ and therefore, π cannot reduce to a term abstraction, by confluence.

Proposition 4. *For all $E, F \in \mathcal{D}_{\equiv}$, then $Cl(E \Rightarrow F) = Cl(E) \Rightarrow Cl(F)$.*

Proof. \subseteq Let $\pi \in Cl(E \Rightarrow F)$,

then $\pi \in SN$ as $Cl(E \Rightarrow F)$ satisfies (CR₁).

- Let $\rho \in \mathcal{L}_\downarrow(\pi)$, then $\rho \in Cl(E \Rightarrow F)$ by (CR₂), and as it is normal, it is, in particular, in $Cl^0(E \Rightarrow F)$, hence $(\Delta, \rho) \in E \Rightarrow F$. Let $\pi' \in Cl(E)$, then there exists (a minimal) $j \in \mathbb{N}$, such that $\pi' \in Cl^j(E)$. Let us show that $\rho\pi' \in Cl(F)$ by induction on j .
 - * If $j = 0$, then π' is normal and $(\Delta, \pi') \in E$, therefore $(\Delta, \rho\pi') \in F$, as $(\Delta, \rho) \in E \Rightarrow F$. Moreover, $\rho\pi'$ is normal as π' is normal and ρ is neutral and normal. Finally $\rho\pi' \in Cl^0(F)$.
 - * If $j > 0$, then there exists $\nu_{\pi'} \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\pi' = (\mu_i/\alpha_i)_{i \leq n} \nu_{\pi'}$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n} \nu_{\pi'} \in Cl^{j-1}(E)$, therefore $\rho(\rho_i/\alpha_i)_{i \leq n} \nu_{\pi'} \in Cl(F)$, by induction hypothesis. Finally, $\rho\pi' = (\mu_i/\alpha_i)_{i \leq n} (\rho\nu_{\pi'}) \in Cl(F)$ by remark 9.
- Let $\lambda\alpha.\rho \in \mathcal{L}_\lambda(\pi)$, then $\lambda\alpha.\rho \in Cl(E \Rightarrow F)$ by (CR₂), and there exists (a minimal) $k \in \mathbb{N}$, such that $\lambda\alpha.\rho \in Cl^k(E \Rightarrow F)$. Let us prove that $\rho \in Cl(F)$ by induction on k .
 - * If $k = 0$, then $\lambda\alpha.\rho \in SN$ and $(\Delta, \lambda\alpha.\rho) \in E \Rightarrow F$, therefore $\rho \in SN$ and $(\Delta, \rho) \in F$, as we can choose α such that $\Delta \vdash \alpha : A_E$, by α -conversion. Finally, $\rho \in Cl^0(F)$.
 - * If $k > 0$, then there exists $\nu \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\lambda\alpha.\rho = (\mu_i/\alpha_i)_{i \leq n} \nu$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n} \nu \in Cl^{k-1}(E \Rightarrow F)$. As each μ_i is neutral, there exists ν' such that $\nu = \lambda\alpha.\nu'$, therefore $(\rho_i/\alpha_i)_{i \leq n} \nu' \in Cl(F)$, by induction hypothesis. Finally, $\rho = (\mu_i/\alpha_i)_{i \leq n} (\nu') \in Cl(F)$ by remark 9.

Finally, $\pi \in Cl(E) \dot{\Rightarrow} Cl(F)$.

⊇ Let $\pi \in Cl(E) \dot{\Rightarrow} Cl(F)$. then $\pi \in SN$ and π cannot reduce to a term-abstraction, by definition of $\dot{\Rightarrow}$.

- If π is a proof-abstraction $\lambda\alpha.\rho$, then $\rho \in Cl(F)$ and there exists (a minimal) $k \in \mathbb{N}$, such that $\rho \in Cl^k(F)$. Let us prove that $\pi \in Cl(E \dot{\Rightarrow} F)$ by induction on k .
 - * If $k = 0$, then ρ is normal and $(\Delta, \rho) \in F$, therefore $\lambda\alpha.\rho$ is normal and $(\Delta, \lambda\alpha.\rho) \in E \dot{\Rightarrow} F$, as we can choose α such that $\Delta \vdash \alpha : A_E$, by α -conversion. Finally, $\pi = \lambda\alpha.\rho \in Cl^0(E \dot{\Rightarrow} F)$.
 - * If $k > 0$, then there exists $\nu \in \mathcal{T}$, and $(\mu_i)_{i \leq n} \subseteq SN$, each neutral not normal such that $\rho = (\mu_i/\alpha_i)_{i \leq n} \nu$ and $\forall (\rho_i)_{i \leq n} \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_{i \leq n} \nu \in Cl^{k-1}(F)$. Hence $\pi = (\mu_i/\alpha_i)_{i \leq n} (\lambda\alpha.\nu) \in Cl(E \dot{\Rightarrow} F)$ by induction hypothesis and remark 9.
- If π is neutral and normal, let $\alpha \in \mathcal{X}$ such that $\Delta \vdash \alpha : A_E$, then $\pi\alpha \in Cl(F)$. Moreover π is neutral and normal, therefore $\pi\alpha$ is normal, hence $\pi\alpha \in Cl^0(F)$. Then $\Delta \vdash \pi\alpha : A_F$ and $\Delta \vdash \pi : A_E \Rightarrow A_F$. Let $(\Gamma', \pi') \in E$, then $\Gamma' \vdash \pi' : A_E$, by (P_{\equiv}) , therefore $\Delta\Gamma' \vdash \pi\pi' : A_F$. Finally, as π is neutral and normal and $\pi' \in WN$, we have $\pi\pi' \in WN$, and $\pi\pi'$ is isolated, therefore $(\Delta\Gamma', \pi\pi') \in F$ as it satisfies $(P_{3_{\equiv}})$. Hence $(\Delta, \pi) \in E \dot{\Rightarrow} F$ and $\pi \in Cl^0(E \dot{\Rightarrow} F)$, as it is normal.
- Otherwise, $\pi \in SN$, is neutral and not normal. All its leaves are either neutral, either proof-abstractions and all these leaves are in $Cl(E) \dot{\Rightarrow} Cl(F)$, as it satisfies (CR_2) , therefore they also are in $Cl(E \dot{\Rightarrow} F)$, as we saw in the previous points. Finally, $\pi \in Cl(E \dot{\Rightarrow} F)$, by lemma 14.

\forall -morphism

We prove now that for all sorts T and $f \in \mathring{A}_T$, $Cl(\mathring{\forall}_T f) = \mathring{\forall}_T Cl \circ f$. Notice that for all functions $f \in \mathring{A}_T$, $Cl \circ f \in \mathring{A}_T$.

Lemma 15. *For all $E \in \mathcal{D}_{\equiv}$, $k \in \mathbb{N}$, terms t and term-variables x of same sort, proof-terms π' , if $(t/x)\pi \in Cl^k(E)$, then $(\lambda x.\pi)t \in Cl^k(E)$.*

Proof. By induction on k .

- If $k = 0$, by $(P_{3_{\equiv}})$.
- If $k > 0$, by induction hypothesis.

Proposition 5. *For all sorts T and $f \in \mathring{A}_T$, $Cl(\mathring{\forall}_T f) = \mathring{\forall}_T Cl \circ f$.*

Proof. ⊆ Let $\pi \in Cl(\mathring{\forall}_T f)$, then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi \in Cl^k(\mathring{\forall}_T f)$. By induction on k .

- If $k = 0$, $(\Delta, \pi) \in \mathring{\forall}_T f$ and $\pi \in SN$, then for all $t \in \hat{T}$, $(\Delta, \pi t) \in f(t)$, therefore $\pi t \in SN$ and its normal form is in $Cl^0 \circ f(t)$, hence $\pi t \in Cl \circ f(t)$, by lemma ???. Finally, $\pi \in \mathring{\forall}_T Cl \circ f$.

- If $k > 0$, then $\pi = (\mu_i/\alpha_i)_i\nu$, with each μ_i neutral not normal and such that for all $(\rho_i)_{i \leq n}$ each respectively a leaf of μ_i , we have $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1}(\check{\forall}_T f) \subseteq \check{\forall}_T Cl \circ f$, by induction hypothesis. Let $t \in \hat{T}$, then if we write $\nu' = \nu t$, we have $\pi t = (\mu_i/\alpha_i)_i\nu'$ and for all $(\rho_i)_{i \leq n}$ each respectively a leaf of μ_i , $(\rho_i/\alpha_i)_i\nu' = (\rho_i/\alpha_i)_i\nu t \in Cl \circ f(t)$. Therefore $\pi t \in Cl \circ f(t)$ by remark 9. Finally, $\pi \in \check{\forall}_T Cl \circ f$.
- \supseteq Let $\pi \in \check{\forall}_T Cl \circ f$, then there exists a minimal $k \in \mathbb{N}$ such that there exists $t \in \hat{T}$, $\pi t \in Cl^k \circ f(t)$. By induction on k .
 - If $k = 0$, then there exists $t \in \hat{T}$ such that $\pi t \in Cl^0 \circ f(t)$. Hence $(\Delta, \pi t) \in f(t)$ and πt is normal. Hence π is normal and for all $t' \in \hat{T}$, $\pi t'$ is also normal, therefore, as $\pi t' \in Cl \circ f(t)$, we have, in particular, $\pi t' \in Cl^0 \circ f(t)$. Finally, for all $t' \in \hat{T}$, $(\Delta, \pi t') \in f(t)$, therefore $(\Delta, \pi) \in \check{\forall}_T f$, and $\pi \in Cl^0(\check{\forall}_T f)$, as it is normal.
 - If $k > 0$, let $t \in \hat{T}$ such that $\pi t \in Cl^k \circ f(t)$. Therefore $\pi t = (\mu_i/\alpha_i)_i\nu$, with each μ_i neutral not normal and such that for all $(\rho_i)_{i \leq n}$ each respectively a leaf of μ_i , we have $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1} \circ f(t)$.
 - * If $\nu \neq \alpha_1$, then $\nu = \nu' t$, with $\pi = (\mu_i/\alpha_i)_i\nu'$, and for all $(\rho_i)_{i \leq n}$ each respectively a leaf of μ_i , we have $(\rho_i/\alpha_i)_i\nu' \in Cl(\check{\forall}_T f)$, by induction hypothesis. We conclude by lemma 14.
 - * Otherwise, every leaf of πt is in $Cl^{k-1} \circ f(t)$. If π is isolated, then all its leaves ρ are neutral and normal, hence ρt is a leaf of πt , therefore $\rho \in Cl(\check{\forall}_T f)$, by induction hypothesis, and we conclude by lemma 14. If π reduces to $\lambda x.\pi'$ then all leaves of $(t/x)\pi'$ are in $Cl^{k-1} \circ f(t)$, therefore, for all leaves ρ of π' , we have $(\lambda x.\rho)t \in Cl^{k-1} \circ f(t)$, by lemma 15, hence $\lambda x.\rho \in Cl(\check{\forall}_T f)$, by induction hypothesis. And finally, $\lambda x.\pi' \in Cl(\check{\forall}_T f)$, and so does π .

We finally get the following (second) completeness result:

Proposition 6.

If \mathcal{L}_{\equiv} is strongly normalizing, then $Cl \circ [\cdot]$ is a \mathcal{C}' -valued model of \mathcal{L}_{\equiv} (and each element of the model contains an infinity of proof-variables).

Proof. By lemma 1 and propositions 2, 3, 4, 5.

5 Soundness

We finally prove in this section, that having a \mathcal{C}' -valued model is also a sound condition of strongly normalizing theories \mathcal{L}_{\equiv} .

Lemma 16. *If $[\cdot]$ is a \mathcal{C}' -valued model of a theory \mathcal{L}_{\equiv} , such that each element of the model contains an infinity of proof-variables, then for all $A \in \mathcal{P}$, contexts Γ , $\varphi \in \text{VAL}(A) \cap \text{VAL}(\Gamma)$, $\pi \in \mathcal{T}$ and σ substitutions such that for all declarations $\alpha : B$ in Γ , $\sigma\alpha \in \llbracket B \rrbracket_{\varphi}$, we have:*

$$\text{if } \Gamma \vdash \pi : A \text{ then } \sigma\varphi\pi \in \llbracket A \rrbracket_{\varphi}.$$

Proof. By induction on the length of the derivation of $\Gamma \vdash \pi : A$. By case on the last rule used. If the last rule used is :

- axiom: in this case, π is a variable α , and Γ contains a declaration $\alpha : B$ with $A \equiv B$ (therefore $|A|_\varphi \equiv |B|_\varphi$). Then $\sigma\varphi\pi = \sigma\alpha \in \llbracket B \rrbracket_\varphi = \llbracket A \rrbracket_\varphi$.
- \Rightarrow -intro: in this case, π is an abstraction $\lambda\alpha.\tau$, and we have $\Gamma, \alpha : B \vdash \tau : C$ with $A \equiv B \Rightarrow C$. Hence, by induction hypothesis, if we choose α such that $\alpha \in \llbracket B \rrbracket_\varphi$, by α -conversion, we have $\sigma(\alpha/\alpha)\varphi\tau = \sigma\varphi\tau \in \llbracket C \rrbracket_\varphi$. Therefore $\sigma\varphi(\lambda\alpha.\tau) = \lambda\alpha.\sigma\varphi\tau \in \llbracket B \rrbracket_\varphi \Rightarrow \llbracket C \rrbracket_\varphi = \llbracket B \Rightarrow C \rrbracket_\varphi$.
- \Rightarrow -elim: in this case, π is an application $\rho\tau$, and we have $\Gamma \vdash \rho : C \equiv B \Rightarrow A$ and $\Gamma \vdash \tau : B$. Then $\sigma\varphi\tau \in \llbracket B \rrbracket_\varphi$, by induction hypothesis.
 - If $\sigma\varphi\rho$ is a proof-abstraction then ρ is a proof-abstraction $\lambda\alpha.\rho'$, and we have $\Gamma, \alpha : B \vdash \rho' : A$, therefore $(\sigma\varphi\tau/\alpha)\sigma\varphi\rho' \in \llbracket A \rrbracket_\varphi$, by induction hypothesis, hence $\sigma\varphi(\lambda\alpha.\rho' \tau) \in \llbracket A \rrbracket_\varphi$ as it satisfies (CR'₃).
 - If $\sigma\varphi\rho$ is neutral and normal, as $\sigma\varphi\rho \in \llbracket A \Rightarrow B \rrbracket_\varphi = \llbracket A \rrbracket_\varphi \Rightarrow \llbracket B \rrbracket_\varphi$, we have $\sigma\varphi(\rho\tau) \in \llbracket A \rrbracket_\varphi$.
 - Otherwise, $\sigma\varphi\rho$ is neutral and not normal and all its leaves μ satisfy $\mu(\sigma\varphi\tau) \in \llbracket A \rrbracket_\varphi$ as we saw in the previous points.

Finally, $\sigma\varphi(\rho\tau) \in \llbracket A \rrbracket_\varphi$ as it satisfies (CR'₃).

- \forall -intro: in this case, π is a term abstraction $\lambda x.\pi'$ and we have $\Gamma \vdash \pi' : B$ with $A \equiv \forall x.B$. Let $t \in \hat{T}$ (with T the sort of x), and $\varphi' = \varphi + \langle x, t \rangle$. Then $\sigma\varphi'\pi' = \sigma\varphi(t/x)\pi' \in \llbracket B \rrbracket_{\varphi'}$, by induction hypothesis. Therefore, $\sigma\varphi(\lambda x.\pi') \in \tilde{\forall}_T(t \mapsto \llbracket B \rrbracket_{\varphi + \langle x, t \rangle}) = \llbracket A \rrbracket_\varphi$ (by induction on the maximal length of a reductions sequence from πt , with $t \in \hat{T}$, using the fact that for all $t \in \hat{T}$, $\llbracket B \rrbracket_{\varphi + \langle x, t \rangle}$ satisfies (CR₂) and (CR₃')).
- \forall -elim: in this case, π is an application ρt , and we have $\Gamma \vdash \rho : \forall x.B$ with $A = (t/x)B$ and $x \notin FV(\Gamma)$. By induction hypothesis, we have $\sigma\varphi\rho \in \llbracket \forall x.B, \varphi \rrbracket = \tilde{\forall}_T(t \mapsto \llbracket B \rrbracket_\varphi + \langle x, t \rangle)$. Therefore $\sigma\varphi(\rho t) = \sigma\varphi\rho(\varphi t) \in \llbracket B \rrbracket_{\varphi + \langle x, \varphi t \rangle} = \llbracket (t/x)B \rrbracket_\varphi = \llbracket A \rrbracket_\varphi$

Theorem 1. *If \mathcal{L}_\equiv has a \mathcal{C}' -valued model (such that each element contains an infinite number of variables), then \mathcal{L}_\equiv is strongly normalizing.*

Proof. If \mathcal{F} is a \mathcal{C}' -valued model of \equiv then for all typing judgement $\Gamma \vdash \pi : A$ and σ and φ as in the previous proposition, we have $\sigma\varphi\pi \in \llbracket A \rrbracket_\varphi \neq \emptyset$ hence $\sigma\varphi\pi \in SN$, therefore $\pi \in SN$.

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Theorem 2. *If \mathcal{L}_\equiv is weakly normalizing then it is strongly normalizing.*

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