Estimation de la Fiabilité des Systèmes avec Redondance

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Interval Estimation of Redundant Systems Reliability

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Introduction.

Let us consider redundant systems with one main unit and $m-1$ stand-by units operating in "warm" conditions, i.e. under lower stress than the main one. We shall use notation $S(1, m-1)$ for such systems. The problem is to obtain confidence intervals for the cumulative distribution functions of redundant systems using failure data of two groups of units, the first group functioning in "hot" and second – in "warm" conditions.

We suppose that switching from "warm" to "hot" conditions does not do any damage to units. Bagdonavičius, Masiulaityte and Nikulin (2008) give mathematical formulation of "fluent switch on" and propose tests for verification of this hypothesis. The formulation is based on the "principle of Sedyakin" (Sedyakin (1966), Bagdonavičius and Nikoulina (1997), Bagdonavičius and Nikulin (2002)).

Denote by T_1, F_1 and f_1 the failure time, the c.d.f. and the probability density function of the main unit. The failure times of the stand-by units denote by $T_2, \ldots, T_m$. In "hot" conditions their distribution functions are also F_1. In "warm" conditions the c.d.f. of T_i is F_2 and the p.d.f. is f_2, $i = 2, \ldots, m$. If a stand-by unit is switched to "hot" conditions, its c.d.f. is different from F_1 and F_2. For $i = 1, 2$ denote by $S_i = 1 - F_i$, $\lambda_i = f_i/S_i$ and $\Lambda_i = -\ln S_i$ the survival function, hazard rate and cumulative hazard, respectively.

The failure time of the system $S(1, m-1)$ is $T^{(m)} = T_1 \vee T_2 \vee \ldots \vee T_m$. Denote by K_j and k_j the c.d.f. and the p.d.f. of $T^{(j)}$, respectively, $(j = 2, \ldots, m)$, $K_1 = F_1$, $k_1 = f_1$. The c.d.f K_j can be written in terms of the c.d.f. K_{j-1} and F_1:

$$K_j(t) = \mathbf{P}(T^{(j)} \le t) = \int_0^t \mathbf{P}(T_j \le t | T^{(j-1)} = y) dK_{j-1}(y). \tag{1}$$

The "fluent switch on" hypothesis H_0 formulated by Bagdonavičius, Masiulaityte & Nikulin (2008) states that

$$f_{T_j|T^{(j-1)}=y}(t) = \begin{cases} f_2(t) & \text{if} \quad t \le y, \\ f_1(t + g(y) - y) & \text{if} \quad t > y; \end{cases} , \quad g(y) = F_1^{-1}(F_2(y)). \tag{2}$$

This model implies that

$$K_j(t) = \int_0^t F_1(t + g(y) - y) dK_{j-1}(y). \tag{3}$$

So the distribution function K_m of the system with $m-1$ stand-by units is defined recurrently using the formula (3). In particular, if we suppose that the distribution of units functioning in "warm" and "hot" conditions differ only in scale, i.e.

$$F_2(t) = F_1(rt)$$

for all $t \geq 0$ and some $r > 0$, then $g(y) = ry$. Combining this assumption and the model (2) we have more strict hypothesis H_0^*. Goodness-of-fit tests for both models are given in Bagdonavičius, Masiulaityte & Nikulin (2008).

Point estimators of the c.d.f. of redundant systems

Suppose that the following data are available :

a) complete ordered sample $T_{11}, \ldots, T_{1n_1}$ of the failure times of units tested in "hot" conditions;

b) the time to obtain complete data in "warm" conditions may be long, so we suppose that n_2 units are tested time t_1 in "warm" conditions and the ordered first failure times $T_{21}, \ldots, T_{2m_2}$ are obtained.

Nonparametric estimation

Denote by

$$\hat{F}_j(t) = \frac{1}{n_i} \sum_{i=1}^{n_j} \mathbf{1}_{\{T_{ji} \leq t\}}, \quad \hat{F}_j^{-1}(y) = \inf\{s : \hat{F}_j(s) \geq y\}$$

the empirical distribution function and its inverse, respectively, for the jth sample.

The estimator of the function $g(t)$ is

$$\hat{g}(t) = \hat{F}_1^{-1}(\hat{F}_2(t)), \quad t \leq t_1.$$

Under H_0 for any $t \leq t_1$ the value $K_j(t)$ of the c.d.f is estimated recurrently:

$$\hat{K}_j(t) = \int_0^t \hat{F}_1(t + \hat{g}(y) - y) d\hat{K}_{j-1}(y), \quad \hat{K}_1(t) = \hat{F}_1(t). \tag{4}$$

If we suppose that the distribution of units functioning in "warm" and "hot" conditions differ only in scale, i.e. $g(y) = ry$ then the c.d.f. $K_j(t)$ can be estimated at any point $t \geq 0$ replacing $\hat{g}(y)$ by $\hat{r}y$ in (4), where $\hat{r}$ is a convenient estimator of r. Set

$$N_1(t) = \sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i} \leq t\}}, \quad N_2(t) = \sum_{i=1}^{n_2} \mathbf{1}_{\{T_{2i} \leq t, t \leq t_1\}},$$

$$Y_1(t) = \sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i} \geq t\}}, \quad Y_2(t) = \sum_{i=1}^{n_2} \mathbf{1}_{\{T_{2i} \geq t, t \leq t_1\}}.$$

Bagdonavičius, Masiulaityte & Nikulin (2008) give the following estimator of the parameter r:

$$\hat{r} = \tilde{U}^{-1}(0) = \sup\{r : \tilde{U}(r) > 0\};$$

here

$$\tilde{U}(r) = -\int_0^{rt_1} \frac{Y_2(v/r)dN_1(v)}{Y_1(v) + Y_2(v/r)} + \int_0^{t_1} \frac{Y_1(ru)dN_2(u)}{Y_1(ru) + Y_2(u)},$$

is ĉadlag stochastic process with trajectories which are non-increasing step functions satisfying the inequalities $\tilde{U}(0+) > 0$, $\tilde{U}(+\infty) < 0$.

Parametric estimation

Suppose that in hot conditions the c.d.f $F_1(t;\theta)$ is absolutely continuous and depends on finite dimensional parameter $\theta \in \Theta \subset \mathbf{R}^k$. Set $\gamma = (r, \theta^T)^T$.

The maximum likelihood estimator $\gamma^* = (r^*, (\theta^*)^T)^T$ of the parameter γ maximizes the loglikelihood function

$$\ell(\gamma) = \sum_{i=1}^{n_1} \ln f_1(T_{1i};\theta) + m_2 \ln r + \sum_{i=1}^{m_2} \ln f_1(rT_{2i};\theta) + (n_2 - m_2) \ln S_1(rt_1;\theta).$$

Under H_0^* for any $t \geq 0$ and $j \geq 2$ the c.d.f. $K_j(t)$ is estimated recurrently:

$$\hat{K}_j(t) = \int_0^t F_1(t + r^*y - y; \theta^*) d\hat{K}_{j-1}(y), \quad \hat{K}_1(t) = F_1(t;\theta^*). \tag{5}$$

Asymptotic distribution of $\hat{K}_j$ and confidence intervals for $K_j(t)$

Suppose that

$$\frac{n_i}{n} = l_i + O(\frac{1}{n}), \quad l_i \in (0,1), \quad \text{as} \quad n = n_1 + n_2 \to \infty.$$

Nonparametric case

The limit distribution of the empirical distribution functions is well known:

$$\sqrt{n}(\hat{F}_i - F_i) \xrightarrow{\mathcal{D}} U_i \tag{6}$$

on $D(A_i)$, where $\xrightarrow{\mathcal{D}}$ means weak convergence, $A_1 = [0, \infty)$, $A_2 = [0, t_1]$, U_1, U_2 are independent Gaussian martingales with $U_i(0) = 0$ and the covariances

$$\mathbf{cov}(U_i(u), U_i(v)) = \frac{1}{l_i} F_i(u \wedge v) S_i(u \vee v). \tag{7}$$

Let us find the asymptotic distribution of the estimator $\hat{r}$. Denote by $r_0 \in (0,1)$ the true value of r. Under the model H_0^* it is the ratio of the mean failure times μ_1 and μ_2 of units functioning in "hot" and "warm" conditions, respectively.

Lemma 1. *Suppose that the c.d.f. F_1 is absolutely continuous with positive p.d.f. f_1 on $(0,\infty)$ and the hypothesis $F_2(t) = F_1(r_0 t)$ is true. If*

$$A = -\frac{1}{r_0} \int_0^{r_0 t_1} u f_1(u) d\Lambda_1(u) - t_1 f_1(r_0 t_1) \neq 0, \tag{8}$$

then

$$\sqrt{n}(\hat{r} - r_0) \xrightarrow{d} Y = -\frac{W}{A}, \tag{9}$$

where

$$W = -\int_0^{t_1} [U_1(r_0 u) - U_2(u) d\Lambda_2(u) - U_1(r_0 t_1) + U_2(t_1). \tag{10}$$

Remark 1.

If samples are complete then

$$W = -\int_0^{\infty} [U_1(r_0 u) - U_2(u)] d\Lambda_2(u), \quad A = -\frac{1}{r_0}\int_0^{\infty} u f_1(u) d\Lambda_1(u), \tag{12}$$

and

$$\sqrt{n}(\hat{r} - r_0) \xrightarrow{\mathcal{D}} Y = -\frac{W}{A} \sim N(0, \frac{1}{l_1 l_2 A^2}). \tag{13}$$

Theorem 1.

If F_1 is continuously differentiable on $[0, \infty)$ then under H_0^* for any $t > 0$ and any natural $j \geq 2$

$$\sqrt{n}(\hat{K}_j(s) - K_j(s)) \xrightarrow{\mathcal{D}}$$

$$W_j(s) = \int_0^s U_1(s + r_0 y - y) dK_{j-1}(y) + \mu^{(j-1)}(s) Y + \int_0^s F_1(s + r_0 y - y) dW_{j-1}(y) \tag{13}$$

on $D[0, t]$, where $W_1(s) = U_1(s)$,

$$\mu^{(j-1)}(s) = \int_0^s y f_1(s + r_0 y - y) dK_{j-1}(y).$$

The asymptotic variance of $\sqrt{n}(\hat{K}_j(t) - K_j(t))$, $j \geq 2$ might be estimated recurrently, using the equation (13): the covariances $\mathbf{Cov}(W_j(s), W_j(t)) = \mathbf{E}(W_j(s), W_j(t))$ can be written in terms of the covariances

$$\mathbf{E}(W_{j-1}(u), W_{j-1}(v)), \quad \mathbf{E}(W_{j-1}(u), U_1(v)), \quad \mathbf{E}(W_{j-1}(u), U_2(v)),$$

$$\mathbf{E}(U_1(u), U_1(v)), \quad \mathbf{E}(U_2(u), U_2(v)). \tag{14}$$

Note that for $j = 2$ these covariances are

$$\mathbf{E}(W_1(u), W_1(v)) = \mathbf{E}(W_1(u), U_1(v)) = \mathbf{E}(U_1(u), U_1(v)) = \frac{1}{l_1} F_1(u \wedge v) S_1(u \vee v),$$

$$\mathbf{E}(W_1(u), V_2(v)) = \mathbf{E}(U_1(u), U_2(v)) = 0, \quad \mathbf{E}(U_2(u), U_2(v)) = \frac{1}{l_2} F_2(u \wedge v) S_2(u \vee v). \tag{16}$$

Let us find the asymptotic variance of $\sqrt{n}(\hat{K}_2(t) - K_2(t))$. Suppose first that samples are complete. In the following we skip the index in r_0 .

The formulas (11) and (12) imply that $\sqrt{n}(\hat{K}_2(s) - K_2(s)) \xrightarrow{\mathcal{D}} W_2(s)$, where

$$W_2(t) = F_2(t)U_1(t) + \int_0^t U_1(t+ry-y)dF_1(y) - \int_0^t U_1(y)dF_1(t+ry-y) +$$

$$\frac{\mu(t)}{A}\left(\int_0^\infty U_1(ry)d\Lambda_2(y) - \int_0^\infty U_2(y)d\Lambda_2(y)\right) = (V_1+V_2+V_3+V_4)(t). \quad (14)$$

$$\mu(t) = \mu^{(1)}(t) = \int_0^t y f_1(t+ry-y)dF_1(y), \quad A = -\frac{1}{r}\int_0^\infty u f_1(u)d\Lambda_1(u). \quad (15)$$

The random variable $W_2(t)$ has zero mean. Set

$$\nu(t) = \int_0^t F_1(t+ry-y)\,dF_1(y).$$

For any $t \geq 0$ the variances $\mathbf{Var}(W_2(t))$ is defined by the following formula:

$$l_1\mathbf{Var}(W_2(t)) = -4\nu^2(t) + \int_0^t F_1(t+ry-y)[F_1(t+ry-y) + 2F_1(y)]\,dF_1(y) +$$

$$2F_1(t)\nu(rt) + 2\int_{rt}^t F_1(t+ry-y)F_1((t-y)/(1-r))\,dF_1(y) + \frac{\mu^2(t)}{l_2A^2} +$$

$$\frac{2\mu(t)}{A}\left[\nu(t) + \int_0^t [F_1(t+ry-y)\ln S_1(y) - S_1(t+ry-y)\ln S_1(t+ry-y)]dF_1(y))\right].$$

Set

$$Z_{1i} = \hat{F}_1(t+(\hat{r}-1)T_{1i}-), \quad \hat{F}_1(t-) = \frac{1}{n_1}\sum_{i=1}^{n_1}\mathbf{1}_{\{T_{1i}<t\}}, \quad Z_{2i} = \hat{F}_1(\frac{t-T_{1i}}{1-\hat{r}}-),$$

$$Z_{3i} = \hat{F}_1(T_{1i}-), \quad Z_{4i} = \hat{f}_1(t+(\hat{r}-1)T_{1i}-), \quad \hat{\mu}(t) = \frac{1}{n_1}\sum_{T_{1i}\leq t} T_{1i}Z_{4i}, \quad Z_{5i} = \hat{f}_1(T_{1i}-).$$

The variance $\mathbf{Var}(W_2(t))$ is estimated using the statistic

$$\frac{n_1}{n}\hat{\mathbf{Var}}(W_2(t)) = -4\hat{\phi}_1^2(t) + \hat{\phi}_2(t) + \frac{n\hat{\mu}^2(t)}{n_2\hat{A}^2} + \frac{2\hat{\mu}(t)}{\hat{A}}\hat{\phi}_3(t);$$

here

$$\hat{\phi}_1(t) = \frac{1}{n_1}\sum_{T_{1i}\leq t} Z_{1i}, \quad \hat{A} = -\frac{1}{\hat{r}n_1}\sum_{i=1}^{n_1}\frac{T_{1i}Z_{5i}}{1-Z_{3i}},$$

$$\hat{\phi}_2(t) = \frac{1}{n_1}\sum_{T_{1i}\leq t} Z_{1i}[Z_{1i} + 2Z_{3i} + 2\hat{F}_1(t)\mathbf{1}_{\{T_{1i}\leq \hat{r}t\}} + 2Z_{2i}\mathbf{1}_{\{T_{1i}>\hat{r}t\}}],$$

$$\hat{\phi}_3(t) = \frac{1}{n_1} \sum_{T_{1i} \le t} [Z_{1i}(1 + \ln(1 - Z_{3i}) - (1 - Z_{1i}) \ln(1 - Z_{1i})].$$

So the variance $\sigma^2_{\hat{K}_2}$ of the estimator $\hat{K}_2(t)$ is estimated by

$$\hat{\sigma}^2_{\hat{K}_2} = \frac{1}{n} \hat{\mathbf{Var}}(W_2(t)).$$

The asymptotic $1 - \alpha$ confidence interval for $K_2(t)$ is $(\underline{K}_2(t), \overline{K}_2(t)$, where

$$\underline{K}_2(t) = \left(1 + \frac{1 - \hat{K}_2(t)}{\hat{K}_2(t)} \exp\left\{ \frac{\hat{\sigma}_{\hat{K}_2} z_{1-\alpha/2}}{\sqrt{\hat{K}_2(t)(1 - \hat{K}_2(t))}} \right\}\right)^{-1},$$

$$\overline{K}_2(t) = \left(1 + \frac{1 - \hat{K}_2(t)}{\hat{K}_2(t)} \exp\left\{ - \frac{\hat{\sigma}_{\hat{K}_2} z_{1-\alpha/2}}{\sqrt{\hat{K}_2(t)(1 - \hat{K}_2(t))}} \right\}\right)^{-1}.$$

Remark 2. *In the case of censoring the expression in parenthesis of the term* V_4 *in (17) is replaced by* $\int_0^{t_1} [U_1(ru) - U_2(u) d\Lambda_2(u) + U_1(rt_1) + U_2(t_1)$, *so only minor modifications are needed.*

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