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# INTERVAL ESTIMATION OF REDUNDANT SYSTEMS RELIABILITY

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## Introduction.

Let us consider redundant systems with one main unit and  $m - 1$  stand-by units operating in "warm" conditions, i.e. under lower stress than the main one. We shall use notation  $S(1, m - 1)$  for such systems. The problem is to obtain confidence intervals for the cumulative distribution functions of redundant systems using failure data of two groups of units, the first group functioning in "hot" and second – in "warm" conditions.

We suppose that switching from "warm" to "hot" conditions does not do any damage to units. Bagdonavičius, Masiulaityte and Nikulin (2008) give mathematical formulation of "fluent switch on" and propose tests for verification of this hypothesis. The formulation is based on the "principle of Sedyakin" (Sedyakin (1966), Bagdonavičius and Nikoulina (1997), Bagdonavičius and Nikulin (2002)).

Denote by  $T_1, F_1$  and  $f_1$  the failure time, the c.d.f. and the probability density function of the main unit. The failure times of the stand-by units denote by  $T_2, \dots, T_m$ . In "hot" conditions their distribution functions are also  $F_1$ . In "warm" conditions the c.d.f. of  $T_i$  is  $F_2$  and the p.d.f. is  $f_2, i = 2, \dots, m$ . If a stand-by unit is switched to "hot" conditions, its c.d.f. is different from  $F_1$  and  $F_2$ . For  $i = 1, 2$  denote by  $S_i = 1 - F_i, \lambda_i = f_i/S_i$  and  $\Lambda_i = -\ln S_i$  the survival function, hazard rate and cumulative hazard, respectively.

The failure time of the system  $S(1, m - 1)$  is  $T^{(m)} = T_1 \vee T_2 \vee \dots \vee T_m$ . Denote by  $K_j$  and  $k_j$  the c.d.f. and the p.d.f. of  $T^{(j)}$ , respectively, ( $j = 2, \dots, m$ ),  $K_1 = F_1, k_1 = f_1$ . The c.d.f  $K_j$  can be written in terms of the c.d.f.  $K_{j-1}$  and  $F_1$ :

$$K_j(t) = \mathbf{P}(T^{(j)} \leq t) = \int_0^t \mathbf{P}(T_j \leq t | T^{(j-1)} = y) dK_{j-1}(y). \quad (1)$$

The "fluent switch on" hypothesis  $H_0$  formulated by Bagdonavičius, Masiulaityte & Nikulin (2008) states that

$$f_{T_j | T^{(j-1)}=y}(t) = \begin{cases} f_2(t) & \text{if } t \leq y, \\ f_1(t + g(y) - y) & \text{if } t > y; \end{cases}, \quad g(y) = F_1^{-1}(F_2(y)). \quad (2)$$

This model implies that

$$K_j(t) = \int_0^t F_1(t + g(y) - y) dK_{j-1}(y). \quad (3)$$

So the distribution function  $K_m$  of the system with  $m - 1$  stand-by units is defined recurrently using the formula (3). In particular, if we suppose that the distribution of units functioning in "warm" and "hot" conditions differ only in scale, i.e.

$$F_2(t) = F_1(rt)$$

for all  $t \geq 0$  and some  $r > 0$ , then  $g(y) = ry$ . Combining this assumption and the model (2) we have more strict hypothesis  $H_0^*$ . Goodness-of-fit tests for both models are given in Bagdonavičius, Masiulaityte & Nikulin (2008).

### Point estimators of the c.d.f. of redundant systems

Suppose that the following data are available :

a) complete ordered sample  $T_{11}, \dots, T_{1n_1}$  of the failure times of units tested in "hot" conditions;

b) the time to obtain complete data in "warm" conditions may be long, so we suppose that  $n_2$  units are tested time  $t_1$  in "warm" conditions and the ordered first failure times  $T_{21}, \dots, T_{2m_2}$  are obtained.

### Nonparametric estimation

Denote by

$$\hat{F}_j(t) = \frac{1}{n_j} \sum_{i=1}^{n_j} \mathbf{1}_{\{T_{ji} \leq t\}}, \quad \hat{F}_j^{-1}(y) = \inf\{s : \hat{F}_j(s) \geq y\}$$

the empirical distribution function and its inverse, respectively, for the  $j$ th sample.

The estimator of the function  $g(t)$  is

$$\hat{g}(t) = \hat{F}_1^{-1}(\hat{F}_2(t)), \quad t \leq t_1.$$

Under  $H_0$  for any  $t \leq t_1$  the value  $K_j(t)$  of the c.d.f is estimated recurrently:

$$\hat{K}_j(t) = \int_0^t \hat{F}_1(t + \hat{g}(y) - y) d\hat{K}_{j-1}(y), \quad \hat{K}_1(t) = \hat{F}_1(t). \quad (4)$$

If we suppose that the distribution of units functioning in "warm" and "hot" conditions differ only in scale, i.e.  $g(y) = ry$  then the c.d.f.  $K_j(t)$  can be estimated at any point  $t \geq 0$  replacing  $\hat{g}(y)$  by  $\hat{r}y$  in (4), where  $\hat{r}$  is a convenient estimator of  $r$ . Set

$$N_1(t) = \sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i} \leq t\}}, \quad N_2(t) = \sum_{i=1}^{n_2} \mathbf{1}_{\{T_{2i} \leq t, t \leq t_1\}},$$

$$Y_1(t) = \sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i} \geq t\}}, \quad Y_2(t) = \sum_{i=1}^{n_2} \mathbf{1}_{\{T_{2i} \geq t, t \leq t_1\}}.$$

Bagdonavičius, Masiulaityte & Nikulin (2008) give the following estimator of the parameter  $r$ :

$$\hat{r} = \tilde{U}^{-1}(0) = \sup\{r : \tilde{U}(r) > 0\};$$

here

$$\tilde{U}(r) = - \int_0^{rt_1} \frac{Y_2(v/r)dN_1(v)}{Y_1(v) + Y_2(v/r)} + \int_0^{t_1} \frac{Y_1(ru)dN_2(u)}{Y_1(ru) + Y_2(u)},$$

is càdlag stochastic process with trajectories which are non-increasing step functions satisfying the inequalities  $\tilde{U}(0+) > 0$ ,  $\tilde{U}(+\infty) < 0$ .

### Parametric estimation

Suppose that in hot conditions the c.d.f  $F_1(t; \theta)$  is absolutely continuous and depends on finite dimensional parameter  $\theta \in \Theta \subset \mathbf{R}^k$ . Set  $\gamma = (r, \theta^T)^T$ .

The maximum likelihood estimator  $\gamma^* = (r^*, (\theta^*)^T)^T$  of the parameter  $\gamma$  maximizes the loglikelihood function

$$\ell(\gamma) = \sum_{i=1}^{n_1} \ln f_1(T_{1i}; \theta) + m_2 \ln r + \sum_{i=1}^{m_2} \ln f_1(rT_{2i}; \theta) + (n_2 - m_2) \ln S_1(rt_1; \theta).$$

Under  $H_0^*$  for any  $t \geq 0$  and  $j \geq 2$  the c.d.f.  $K_j(t)$  is estimated recurrently:

$$\hat{K}_j(t) = \int_0^t F_1(t + r^*y - y; \theta^*) d\hat{K}_{j-1}(y), \quad \hat{K}_1(t) = F_1(t; \theta^*). \quad (5)$$

### Asymptotic distribution of $\hat{K}_j$ and confidence intervals for $K_j(t)$

Suppose that

$$\frac{n_i}{n} = l_i + O\left(\frac{1}{n}\right), \quad l_i \in (0, 1), \quad \text{as } n = n_1 + n_2 \rightarrow \infty.$$

### Nonparametric case

The limit distribution of the empirical distribution functions is well known:

$$\sqrt{n}(\hat{F}_i - F_i) \xrightarrow{\mathcal{D}} U_i \quad (6)$$

on  $D(A_i)$ , where  $\xrightarrow{\mathcal{D}}$  means weak convergence,  $A_1 = [0, \infty)$ ,  $A_2 = [0, t_1]$ ,  $U_1, U_2$  are independent Gaussian martingales with  $U_i(0) = 0$  and the covariances

$$\mathbf{cov}(U_i(u), U_i(v)) = \frac{1}{l_i} F_i(u \wedge v) S_i(u \vee v). \quad (7)$$

Let us find the asymptotic distribution of the estimator  $\hat{r}$ . Denote by  $r_0 \in (0, 1)$  the true value of  $r$ . Under the model  $H_0^*$  it is the ratio of the mean failure times  $\mu_1$  and  $\mu_2$  of units functioning in "hot" and "warm" conditions, respectively.

**Lemma 1.** *Suppose that the c.d.f.  $F_1$  is absolutely continuous with positive p.d.f.  $f_1$  on  $(0, \infty)$  and the hypothesis  $F_2(t) = F_1(r_0 t)$  is true. If*

$$A = -\frac{1}{r_0} \int_0^{r_0 t_1} u f_1(u) d\Lambda_1(u) - t_1 f_1(r_0 t_1) \neq 0, \quad (8)$$

then

$$\sqrt{n}(\hat{r} - r_0) \xrightarrow{d} Y = -\frac{W}{A}, \quad (9)$$

where

$$W = -\int_0^{t_1} [U_1(r_0 u) - U_2(u)] d\Lambda_2(u) - U_1(r_0 t_1) + U_2(t_1). \quad (10)$$

**Remark 1.**

If samples are complete then

$$W = -\int_0^\infty [U_1(r_0 u) - U_2(u)] d\Lambda_2(u), \quad A = -\frac{1}{r_0} \int_0^\infty u f_1(u) d\Lambda_1(u), \quad (12)$$

and

$$\sqrt{n}(\hat{r} - r_0) \xrightarrow{\mathcal{D}} Y = -\frac{W}{A} \sim N\left(0, \frac{1}{l_1 l_2 A^2}\right). \quad (13)$$

**Theorem 1.**

If  $F_1$  is continuously differentiable on  $[0, \infty)$  then under  $H_0^*$  for any  $t > 0$  and any natural  $j \geq 2$

$$\sqrt{n}(\hat{K}_j(s) - K_j(s)) \xrightarrow{\mathcal{D}} W_j(s) = \int_0^s U_1(s + r_0 y - y) dK_{j-1}(y) + \mu^{(j-1)}(s) Y + \int_0^s F_1(s + r_0 y - y) dW_{j-1}(y) \quad (13)$$

on  $D[0, t]$ , where  $W_1(s) = U_1(s)$ ,

$$\mu^{(j-1)}(s) = \int_0^s y f_1(s + r_0 y - y) dK_{j-1}(y).$$

The asymptotic variance of  $\sqrt{n}(\hat{K}_j(t) - K_j(t))$ ,  $j \geq 2$  might be estimated recurrently, using the equation (13): the covariances  $\mathbf{Cov}(W_j(s), W_j(t)) = \mathbf{E}(W_j(s), W_j(t))$  can be written in terms of the covariances

$$\begin{aligned} & \mathbf{E}(W_{j-1}(u), W_{j-1}(v)), \quad \mathbf{E}(W_{j-1}(u), U_1(v)), \quad \mathbf{E}(W_{j-1}(u), U_2(v)), \\ & \mathbf{E}(U_1(u), U_1(v)), \quad \mathbf{E}(U_2(u), U_2(v)). \end{aligned} \quad (14)$$

Note that for  $j = 2$  these covariances are

$$\begin{aligned} \mathbf{E}(W_1(u), W_1(v)) &= \mathbf{E}(W_1(u), U_1(v)) = \mathbf{E}(U_1(u), U_1(v)) = \frac{1}{l_1} F_1(u \wedge v) S_1(u \vee v), \\ \mathbf{E}(W_1(u), U_2(v)) &= \mathbf{E}(U_1(u), U_2(v)) = 0, \quad \mathbf{E}(U_2(u), U_2(v)) = \frac{1}{l_2} F_2(u \wedge v) S_2(u \vee v). \end{aligned} \quad (16)$$

Let us find the asymptotic variance of  $\sqrt{n}(\hat{K}_2(t) - K_2(t))$ . Suppose first that samples are complete. In the following we skip the index in  $r_0$ .

The formulas (11) and (12) imply that  $\sqrt{n}(\hat{K}_2(s) - K_2(s)) \xrightarrow{\mathcal{D}} W_2(s)$ , where

$$W_2(t) = F_2(t)U_1(t) + \int_0^t U_1(t+ry-y)dF_1(y) - \int_0^t U_1(y)dF_1(t+ry-y) + \frac{\mu(t)}{A} \left( \int_0^\infty U_1(ry)d\Lambda_2(y) - \int_0^\infty U_2(y)d\Lambda_2(y) \right) = (V_1 + V_2 + V_3 + V_4)(t). \quad (14)$$

$$\mu(t) = \mu^{(1)}(t) = \int_0^t yf_1(t+ry-y)dF_1(y), \quad A = -\frac{1}{r} \int_0^\infty uf_1(u)d\Lambda_1(u). \quad (15)$$

The random variable  $W_2(t)$  has zero mean. Set

$$\nu(t) = \int_0^t F_1(t+ry-y) dF_1(y).$$

For any  $t \geq 0$  the variances  $\mathbf{Var}(W_2(t))$  is defined by the following formula:

$$\begin{aligned} l_1 \mathbf{Var}(W_2(t)) &= -4\nu^2(t) + \int_0^t F_1(t+ry-y)[F_1(t+ry-y) + 2F_1(y)] dF_1(y) + \\ &2F_1(t)\nu(rt) + 2 \int_{rt}^t F_1(t+ry-y)F_1((t-y)/(1-r)) dF_1(y) + \frac{\mu^2(t)}{l_2 A^2} + \\ &\frac{2\mu(t)}{A} \left[ \nu(t) + \int_0^t [F_1(t+ry-y) \ln S_1(y) - S_1(t+ry-y) \ln S_1(t+ry-y)] dF_1(y) \right]. \end{aligned}$$

Set

$$\begin{aligned} Z_{1i} &= \hat{F}_1(t + (\hat{r} - 1)T_{1i}-), \quad \hat{F}_1(t-) = \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i} < t\}}, \quad Z_{2i} = \hat{F}_1\left(\frac{t - T_{1i}}{1 - \hat{r}} -\right), \\ Z_{3i} &= \hat{F}_1(T_{1i}-), \quad Z_{4i} = \hat{f}_1(t + (\hat{r} - 1)T_{1i}-), \quad \hat{\mu}(t) = \frac{1}{n_1} \sum_{T_{1i} \leq t} T_{1i} Z_{4i}, \quad Z_{5i} = \hat{f}_1(T_{1i}-). \end{aligned}$$

The variance  $\mathbf{Var}(W_2(t))$  is estimated using the statistic

$$\frac{n_1}{n} \hat{\mathbf{Var}}(W_2(t)) = -4\hat{\phi}_1^2(t) + \hat{\phi}_2(t) + \frac{n\hat{\mu}^2(t)}{n_2 \hat{A}^2} + \frac{2\hat{\mu}(t)}{\hat{A}} \hat{\phi}_3(t);$$

here

$$\begin{aligned} \hat{\phi}_1(t) &= \frac{1}{n_1} \sum_{T_{1i} \leq t} Z_{1i}, \quad \hat{A} = -\frac{1}{\hat{r} n_1} \sum_{i=1}^{n_1} \frac{T_{1i} Z_{5i}}{1 - Z_{3i}}, \\ \hat{\phi}_2(t) &= \frac{1}{n_1} \sum_{T_{1i} \leq t} Z_{1i} [Z_{1i} + 2Z_{3i} + 2\hat{F}_1(t) \mathbf{1}_{\{T_{1i} \leq \hat{r}t\}} + 2Z_{2i} \mathbf{1}_{\{T_{1i} > \hat{r}t\}}], \end{aligned}$$

$$\hat{\phi}_3(t) = \frac{1}{n_1} \sum_{T_{1i} \leq t} [Z_{1i}(1 + \ln(1 - Z_{3i}) - (1 - Z_{1i}) \ln(1 - Z_{1i}))].$$

So the variance  $\sigma_{\hat{K}_2}^2$  of the estimator  $\hat{K}_2(t)$  is estimated by

$$\hat{\sigma}_{\hat{K}_2}^2 = \frac{1}{n} \mathbf{Var}(W_2(t)).$$

The asymptotic  $1 - \alpha$  confidence interval for  $K_2(t)$  is  $(\underline{K}_2(t), \overline{K}_2(t))$ , where

$$\underline{K}_2(t) = \left( 1 + \frac{1 - \hat{K}_2(t)}{\hat{K}_2(t)} \exp \left\{ \frac{\hat{\sigma}_{\hat{K}_2} z_{1-\alpha/2}}{\sqrt{\hat{K}_2(t)(1 - \hat{K}_2(t))}} \right\} \right)^{-1},$$

$$\overline{K}_2(t) = \left( 1 + \frac{1 - \hat{K}_2(t)}{\hat{K}_2(t)} \exp \left\{ -\frac{\hat{\sigma}_{\hat{K}_2} z_{1-\alpha/2}}{\sqrt{\hat{K}_2(t)(1 - \hat{K}_2(t))}} \right\} \right)^{-1}.$$

**Remark 2.** In the case of censoring the expression in parenthesis of the term  $V_4$  in (17) is replaced by  $\int_0^{t_1} [U_1(ru) - U_2(u)] d\Lambda_2(u) + U_1(rt_1) + U_2(t_1)$ , so only minor modifications are needed.

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