An Exponential Extension

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Résumé

Nous présentons ici une généralization de la loi exponentielle. La densité de cette nouvelle loi est toujour décroissant mais son taux de hasard peut être constant, croissant et décroissant. Cette famille peut utilisée pour étudier la fiabilité et l'analyse de survie. Comme exemple, nous appliquons cette loi aux données de tremblement de terre en Iran.

Abstract

A generalization of the exponential distribution is presented. The generalization always has a decreasing probability density function and yet allows for increasing, decreasing and constant hazard rates. It can be used for modeling lifetime data from reliability and survival studies. One example is discussed to illustrate its applicability.

Keywords: Estimation; Exponential distribution; Hazard rate function; Weibull distribution.

1 Introduction

In lifetime data analysis, monotone hazard rate occurs commonly in practice. Such situations are currently modeled using the families of the Weibull or the Gamma distributions. Among them, the Weibull distribution is more popular. Because the survival function of the Gamma distribution cannot be expressed in a closed form and one needs to obtain the survival function or the failure rate by numerical integration, while the Weibull distribution has a nice survival and hazard function. One can refer to Murthy et al. (2004) for details about Weibull models.
Gupta and Kundu (2001), presented the Exponentiated Exponential (Generalized Exponential) distribution. This family has lots of properties which are quite similar to those of a Gamma distribution but it has an explicit expression of the survival function like a Weiull distribution. Gupta and Kundu (2007) provided a detailed review and some developments on the Exponentiated Exponential distribution.

In this paper, we consider a two-parameter extension of Exponential distribution and study some of its properties. The two parameters of the new distribution represent the shape and scale parameter. It is observed that the new family always has a decreasing probability function like an Exponential distribution but it allows for increasing, decreasing and constant hazard rates like a Weibull distribution or an Exponentiated Exponential distribution. The new distribution has an explicit expression of survival function and failure rate hazard function. Due to convenient form of the distribution function, the new distribution random variable can be easily generated for simulation purposes.
The two-parameter extension of Exponential distribution is a particular member of the three-parameter Generalized Power Weibull distribution, introduced by Nikulin and Haghighi (2006). Moreover, the new distribution is a special case of Gurvich (1977) class as $F(t) = 1 - exp\{-aG(t)\}$, where $G(t)$ is a monotonically increasing function of t with the only limitation $G(t) \geq 0$.
The main aim of this paper is to introduce a new family of distribution which may have a better fit compared to a Weibull family or Exponentiated Exponential family in certain situations.

2 The Distribution

The family is most conveniently specified in terms of survival function,

$$S(t) = \exp\{1 - (1 + \lambda t)^{\alpha}\} \tag{1}$$

for $\alpha > 0$, $\lambda > 0$ and $t > 0$. The corresponding cumulative distribution function (cdf), probability density function (pdf) and the quantile function are

$$F(t) = 1 - \exp\{1 - (1 + \lambda t)^{\alpha}\}, \tag{2}$$

$$f(t) = \alpha\lambda(1 + \lambda t)^{\alpha-1} \exp\{1 - (1 + \lambda t)^{\alpha}\} \tag{3}$$

and

$$Q(p) = \frac{1}{\lambda}\{(1 - \log(1 - p))^{\frac{1}{\alpha}} - 1\}, \quad 0 < p < 1.$$

The median is

$$Median(T) = \frac{1}{\lambda}\{(1 - \log(0.5))^{\frac{1}{\alpha}} - 1\}.$$

The hazard function is

$$h(t) = \alpha\lambda(1 + \lambda t)^{\alpha-1}. \tag{4}$$

For $\alpha = 1$, the family reduce to the exponential distribution.

3 Shapes

It follows from (3) that

$$\frac{d\log f(t)}{dt} = \frac{\lambda(\alpha-1)}{1+\lambda t} - \alpha\lambda(1+\lambda t)^{\alpha-1}$$

and

$$\frac{d^2\log f(t)}{dt^2} = -\frac{\lambda^2(\alpha-1)}{(1+\lambda t)^2} - \alpha(\alpha-1)\lambda^2(1+\lambda t)^{\alpha-2}.$$

So, if $\alpha \leq 1$ then $d\log f(t)/dt < 0$ for all t. If $\alpha > 1$ then $d\log f(t_0)/dt_0 = 0$ and $d^2\log f(t_0)/dt_0^2 < 0$, where $t_0 = (1/\lambda)\{(1-1/\alpha)^{1/\alpha} - 1\}$. But $t_0 < 0$. Hence, the only shape possible is that $f(t)$ monotonically decreases with $f(0) = \alpha\lambda$ and $f(t) \to 0$ as $t \to \infty$. Figure 1 illustrates this shape for selected parameter values.

The hazard rate function in (4) exhibits the following shapes:

- if $\alpha < 1$ then $h(t)$ monotonically decreases with $h(0) = \alpha\lambda$ and $h(t) \to 0$ as $t \to \infty$.
- if $\alpha > 1$ then $h(t)$ monotonically increases with $h(0) = \alpha\lambda$ and $h(t) \to \infty$ as $t \to \infty$.
- if $\alpha = 1$ then $h(t) = \alpha\lambda$ for all t.

4 Estimation and Fitting

Now consider estimation by the method of maximum likelihood. The log-likelihood function of the two parameters is:

$$\log L(\alpha,\lambda) = n\log(\alpha\lambda) + (\alpha-1)\sum_{i=1}^{n}\log(1+\lambda t_i) + n - \sum_{i=1}^{n}(1+\lambda t_i)^{\alpha}. \quad (5)$$

It follows that the maximum likelihood estimators are the simultaneous solutions of the equations:

$$\frac{n}{\alpha} + \sum_{i=1}^{n}\log(1+\lambda t_i) - \sum_{i=1}^{n}(1+\lambda t_i)^{\alpha}\log(1+\lambda t_i) = 0$$

and

$$\frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n} t_i(1+\lambda t_i)^{-1} - \sum_{i=1}^{n}\alpha t_i(1+\lambda t_i)^{\alpha-1} = 0.$$

We consider the earthquakes in the last century in Iran. In Table (1) the dates of successive earthquakes with magnitudes greater or equal to 6.5 are recorded with their exact times, latitudes, longitudes, depths, magnitudes and references between the years 1989 and 2008. The data set given in Table (2) includes the time intervals (in days) of the successive earthquakes mentioned above. The data are taken from International Institute of Earthquake Engineering and Seismology (IIEES, web address: http://www.iiees.ac.ir).

Table 1. The powerful earthquakes in Iran

Dates	Time	Latitude	Longitude	Depth	Magnitude	Reference
1989/09/16	02:05:09	40.34	51.56	40	Ms:6.5	EHB
1990/06/20	21:00:10	36.99	49.35	19	Ms:7.7	ISC
1990/11/06	18:45:53	28.24	55.4	11	Ms:6.6	EHB
1997/02/04	10:37:49	37.73	57.31	15	Ms:6.8	EHB
1997/05/10	07:57:32	33.85	59.81	13	Ms:7.3	EHB
1998/03/14	19:40:28	30.14	57.59	9	Ms:6.9	EHB
1999/03/04	05:38:27	28.27	57.21	28	Mw:6.6	EHB
2000/11/25	18:09:13	40.20	49.93	50	Mw:6.8	EHB
2000/12/06	17:11:08	39.53	54.80	33	Ms:7.4	EHB
2002/06/22	02:58:23	35;60	49.02	11	Mw:6.5	EHB
2003/12/26	01:56:56	29.08	58.38	13	Ms:6.5	IIEES

Table 2. The time intervals of successive earthquakes

284	246	139	2280	95	308	355	607	11	563	553

The MLE's, Log-likelihood, Chi-squared and Kolmogrov-Smirnov statistics under the new distribution, Weibull and Exponentiated Exponential models are given in Table (3). It is observed that the statistics take the smallest values under new model with regard to the other models. The new model offers an alternative to Weibull and Exponentiated Exponential models as in analyzed real data sets.

Table 3. Parameter estimation, Log-likelihood, Chi-squared and Kolmogrov-Smirnov statistics for the data set

					K-S		
Data set	Distribution	$\hat{\theta} = (\hat{\alpha}, \hat{\lambda})$	L-L	χ^2_1	D^+	D^-	D
(n=11)	New distribution	$\hat{\theta} = (0.8, 0.003)$	-79.12	0.6747	0.1836	0.1538	0.1836
	Weibull	$\hat{\theta} = (0.93, 0.002)$	-79.20	0.9018	0.2096	0.1266	0.2096
	EE	$\hat{\theta} = (0.95, 0.002)$	- 79.23	0.7240	0.1885	0.1425	0.1885

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