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Résumé

On s'intéresse à l'estimation d'une fonction d'autorégression en un point fixé dans un modèle autorégressif avec un bruit Gaussien. Un estimateur à noyau adaptatif qui atteint la vitesse minimax est construit pour le risk minimax.

Abstract

This paper deals with the estimation of a autoregression function at a given point in nonparametric autoregression models with Gaussian noise. An adaptive kernel estimator which attains the minimax rate is constructed for the minimax risk.

Key words: Adaptive estimation, kernel estimator, minimax, nonparametric autoregression.

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1 Introduction

We consider the following nonparametric autoregressive model

$$y_k = S(x_k) y_{k-1} + \xi_k \,, \quad 1 \le k \le n \,, \tag{1.1}$$

where $S(\cdot)$ is an unknown $\mathbb{R} \to \mathbb{R}$ function, $x_k = k/n$, y_0 is a constant and the noise random variables $(\xi_k)_{1\le k\le n}$ are i.i.d. standard Gaussian.

We are interested in the estimation of the autoregression coefficient function S at a given point $z_0 \in]0;1[$. We assume that the autoregression function belongs to a Hölder class but its smoothness parameter β remains unknown. The paper is organized as follows. In the next section we give the description of the problem with all assumptions needed and all definitions of necessary mathematical object. The lower bound of the minimax risk is given in section 3. In Section 4 we construct an adaptive estimator for which an upper bound of the risk is found.

2 Statement of the problem

Consider model (1.1) where $S \in \mathbf{C}_1([0,1],\mathbb{R})$ is unknown function. We want to estimate the autoregression function S at a fixed point z_0. To obtain a stable (uniformly with respect to the function S) model (1.1) we assume (see [1]) that for some fixed $0 < \varepsilon < 1$ the unknown function S belongs to the stability set

$$\Gamma_\varepsilon = \{S \in \mathbf{C}_1[0,1] : \|S\| \le 1 - \varepsilon\}, \tag{2.1}$$

where $\|S\| = \sup_{0 \le x \le 1} |S(x)|$. Here is the Banah space of continuously differentiable $[0,1] \to \mathbb{R}$ functions. For fixed constants $K > 0$, $0 \le \alpha < 1$ and $\beta \in [\beta_*;\beta^*] \subset]1;2[$ we define the corresponding *stable local Hölder class* at the point z_0 as

$$\mathcal{H}^{(\beta)}(z_0,K,\varepsilon) = \left\{ S \in \Gamma_\varepsilon \,:\, \|\dot{S}\| \le K \quad \text{and} \quad \Omega^*(z_0,S) \le K \right\} \tag{2.2}$$

with $\beta = 1 + \alpha$ and

$$\Omega^*(z_0,S) = \sup_{x \in [0,1]} \frac{|\dot{S}(x) - \dot{S}(z_0)|}{|x - z_0|^\alpha}\,.$$

The smoothness parameter β is supposed to be unknown whereas the interval $[\beta_*;\beta^*]$ is considered as known.

The risk of an estimator $\hat{S}$ of $S(z_0)$ is defined over the neighborhood $\mathcal{H}^{(\beta)}(z_0,K,\varepsilon)$ by

$$\mathcal{R}_n(\hat{S}_n,S) = \sup_{\beta \in [\beta_*;\beta^*]} \; \sup_{S \in \mathcal{H}^{(\beta)}(z_0,K,\varepsilon)} N(\beta)\,\mathbf{E}_S|\hat{S}_n(z_0) - S(z_0)|\,, \tag{2.3}$$

where $N(\beta) = \left(\frac{n}{\ln n}\right)^{\beta/(2\beta+1)}$ and $\mathbf{E}_S$ is the expectation taken with respect to the distribution $\mathbf{P}_S$ of the vector $(y_1,...,y_n)$ in (1.1) corresponding to the function S.

3 Lower bounds

In this section we give the lower bound for the minimax risk. We denote $d_n = c\frac{n}{\ln n}$, $c > 0$, and $\tilde{N}(\beta) = d_n^{\beta/(2\beta+1)}$, $\tilde{h}(\beta) = d_n^{-1/(2\beta+1)}$. We show that with this convergence rate $\tilde{N}(\beta)$, the lower bound of minimax risk is positive, which implies that it is also the case for the rate $N(\beta) = \left(\frac{n}{\ln n}\right)^{\beta/(2\beta+1)}$.

Theorem 3.1. *For any constant K, great enough, one have*

$$\underline{\lim}_{n\to\infty} \inf_{\tilde{S}_n} \mathcal{R}_n(\tilde{S}_n) > 0.$$

where the infimum is taken over all estimators.

4 Upper bound

The paper [1] handles the autoregression non adaptive case, considering the kernel estimator

$$S_h^*(z_0) = \frac{1}{A_n(h)} \sum_{k=1}^{n} Q(u_k)\, y_{k-1}\, y_k\, \mathbf{1}_{(A_n(h) \geq d)}, \tag{4.1}$$

where $Q(\cdot)$ is a kernel function,

$$A_n = \sum_{k=1}^{n} Q(u_k) y_{k-1}^2 \quad \text{with} \quad u_k = \frac{x_k - z_0}{h}\,;$$

d and h are some positive parameters. Taking into account the fact that β is unknown we can not use such an estimator because the bandwidths h depends on β. That is the reason why we create a partition of the interval $[\beta_*;\beta^*]$ in the following way :

$$\beta_l = \beta_* + l\frac{\beta^* - \beta_*}{\ln n}, \quad l = 0, \cdots, [\ln n],$$

where [a] denotes the integral part of a number a, and we define the corresponding bandwidths $h_l = h(\beta_l) = \left(\frac{n}{\ln n}\right)^{-1/(2\beta_l+1)}$. Then we set

$$\hat{l} = \max\left\{ 0 \leq l \leq [\ln n] : \max_{0 \leq j \leq l} \left(|S_{h_l}^*(z_0) - S_{h_j}^*(z_0)| - \frac{\lambda}{N_j} \right) \leq 0 \right\},$$

where $N_j = N(\beta_j)$ and $\lambda >$.
Notice that $\hat{l}$ really exists because the set above contains the index 0.

The adaptive estimator is now defined as $\hat{S}_n = S_{h_{\hat{l}}}^*(z_0)$. Furthermore we associate with the unknown parameter β the unique integer $l(\beta) \in \{0, \cdots, [\ln n] - 1\}$ such that $\beta_{l(\beta)} \leq \beta < \beta_{l(\beta)+1}$.

The following result gives an upper bound for the risk of the adaptive estimator.

Theorem 4.1. *For any* $0 < \varepsilon < 1$, *one has*

$$\overline{\lim}_{n\to\infty} \mathcal{R}_n(\hat{S}_n, S) < \infty,$$

then $\hat{S}_n$ is adaptive estimator on convergence rate.

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