Une définition générale de l'influence entre processus stochastiques

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Résumé: Nous présentons la notion de indépendance locale faible conditionnelle (WCLI) entre processus stochastiques dans une large classe de semi-martingales, que nous appelons $\mathcal{D}'$. La définition est fondée sur la mesurabilité des caractéristiques de la décomposition de la semi-martingale cible. Nous donnons également une définition liée à la notion de vraisemblance sur la base de certains processus, en utilisant le théorème de Girsanov. Sous certaines conditions, les deux définitions coïncident sur $\mathcal{D}'$. Nous présentons le modèle graphique associé à cette notion d'indépendance (ou d'influence) dans un ensemble de processus stochastiques. Ces résultats peuvent être utilisés dans des modèles de causalité. Nous pensons que cette définition permet de décrire l'influence d'une composante d'un processus stochastique sur une autre sans ambiguïté. De WCLI on peut construire un concept d'indépendance conditionnelle locale forte (SCLI). Lorsque WCLI n'est pas vérifié, il y a une influence directe ; lorsque SCLI n'est pas vérifié, il y a une influence directe ou indirecte. Nous étudions le lien entre WCLI et les conditions classiques d'indépendance.

Abstract: We present the notion of weak local conditional independence (WCLI) between stochastic processes in a large class of semi-martingales that we call $\mathcal{D}'$. The definition is based on measurability condition on the characteristics of the decomposition of the involved semi-martingales. We also give a definition related to the same concept based on certain likelihood processes, using the Girsanov theorem. Under certain conditions, the

two definitions coincide on $\boldsymbol{\mathcal{D}}'$. We present the graphical model associated to this notion of independence (or influence) in a set of stochastic processes. These results may be used in causal models. With these tools, influences of one component of a stochastic process on another can be described without ambiguity. From WCLI we can construct a concept of strong local conditional independence (SCLI). When WCLI does not hold, there is a direct influence while when SCLI does not hold there is direct or indirect influence. We investigate whether WCLI and SCLI can be defined via conventional independence conditions and find that this is the case for the latter but not for the former.

1 Context

The issue of causality has attracted more and more interest from statisticians in recent years. One approach is directly based on dynamical models and has been developed, starting with Granger (1969) and Schweder (1970), and more recently developed using the formalism of stochastic processes, by Aalen (1987), Florens and Fougère (1996) and Didelez (2007, 2008).

Recently we have given more development to the dynamical models approach (Commenges and Gégout-Petit, 2009) using the basic idea of the Doob-Meyer decomposition proposed in Aalen (1987). We have proposed a definition of weak local independence between processes (WCLI) for a certain class of special semi-martingales (called class $\mathcal{D}$) which involves the compensator of the Doob-Meyer decomposition of the studied semi-martingale. Although it can be used in discrete time, this definition is especially adapted to continuous-time processes for which definitions based on conventional conditional independence may fail. The aim of this paper is to give an even more general definition of WCLI, and conversely of direct influence. What we call direct influence of one component X_j on another component X_k of a multivariate stochastic process $\boldsymbol{X}$ (noted $X_j \longrightarrow_{\boldsymbol{X}} X_k$) is that X_k is not WCLI of X_j (we use WCLI both as the name of the condition and as an adjective, that is the "I" may mean "independence" or "independent" according to the context). This concept of influence is a good starting point for defining *causal* influence.

In the perspective of extending WCLI to a larger class of processes, we see two ways. The first one is to stay in the class of semi-martingales and try to be more general about the conditions. In particular we could use the triplet of the characteristics of a semi-martingales. For an exact definition of the characteristics of a semi-martingale, see Jacod and Shiryaev (2003). Roughly speaking, the characteristics of a semi-martingale are represented by the triplet (B, C, ν) where ν is the compensator of the jump part of the semi-martingale, B the finite variation part not included in ν, and C is the angle bracket process of the continuous martingale. The second way is to work with the likelihood of the process which is also tightly linked with the characteristics of the semi-martingale. In this paper we explore these two ways, extending the WCLI definition to a very large class of processes that we call $\boldsymbol{\mathcal{D}}'$, and showing that another definition of WCLI is possible by the use of likelihood processes. Another issue that we explore is the link between WCLI and analogous definitions based on conventional conditional independence; this angle of attack is closer to Granger (1969) proposal for time series.

2 Definitions and Results

In order to give the general definition of WCLI, using the characteristics of the semi-martingales, we present the notation

Consider a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and a multivariate stochastic process $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\geq 0}$; $\boldsymbol{X}_t$ takes values in $\Re^m$, and the whole process $\boldsymbol{X}$ takes values in $D(\Re^m)$, the Skorohod space of all cadlag functions: $\Re_+ \to \Re^m$. We suppose that all the filtrations satisfy "the usual conditions". We have $\boldsymbol{X} = (X_j, j = 1, \ldots, m)$ where $X_j = (X_{jt})_{t\geq 0}$. We denote by $\mathcal{X}_t$ the history of $\boldsymbol{X}$ up to time t, that is $\mathcal{X}_t$ is the σ-field $\sigma(\boldsymbol{X}_u, 0 \leq u \leq t)$, and by $(\mathcal{X}_t) = (\mathcal{X}_t)_{t\geq 0}$ the families of these histories, that is the filtration generated by $\boldsymbol{X}$. Similarly we shall denote by $\mathcal{X}_{jt}$ and $(\mathcal{X}_{jt})$ the histories and the filtration associated to X_j. Let $\mathcal{F}_t = \mathcal{H} \vee \mathcal{X}_t$; $\mathcal{H}$ may contain information known at $t = 0$, in addition to the initial value of $\boldsymbol{X}$. We shall consider the class of special semi-martingales in the filtration $(\mathcal{F}_t)$. We denote by (B, C, ν) the characteristics of the semi-martingale $\boldsymbol{X}$ under probability P, by M_j the martingale part of X_j, and by M_j^c the continuous part of this martingale. We denote by (B^k, C^k, ν^k) the characteristics of the semi-martingale X_k under probability P.

We shall assume two conditions on $\boldsymbol{X}$:

A1 M_j and M_k are square integrable orthogonal martingales, for all $j \neq k$.

Under assumption **A1**, the jumping parts of the martingales M_j and M_k are orthogonal. Moreover, the characteristic C of $\boldsymbol{X}$ (the angle bracket of the continuous part of the martingale) is a diagonal matrix. Indeed by definition of orthogonality of semi-martingales, $C_{ij} =< M_i^c, M_j^c >= 0$ for all $1 \leq i, j \leq m$; we note $C^k = C_{kk}$.

A2' C^j is deterministic for all j.

We call $\boldsymbol{\mathcal{D}}'$ the class of all special semi-martingales satisfying **A1** and **A2'**. In fact, **A1** and **A2'** coud be merged into a single compact assuption: the characteristic C of $\boldsymbol{X}$ is a deterministic diagonal matrix. $\boldsymbol{\mathcal{D}}'$ is stable by change of absolutely continuous probability (C does not change with the probability). $\boldsymbol{\mathcal{D}}'$ is a very large class of processes: it includes random measures, marked point processes, diffusions and diffusions with jumps.

Definition 1 (Weak conditional local independence (WCLI)) *Let $\boldsymbol{X}$ be in the class $\boldsymbol{\mathcal{D}}'$. X_k is WCLI of X_j in $\boldsymbol{X}$ on $[r, s]$ if and only if the characteristics B^k and ν^k are such that $B_{kt} - B_{kr}$ and $\nu_{kt} - \nu_{kr}$ are $(\mathcal{F}_{-jt})$-predictable on $[r, s]$. Equivalently we can say that X_k has the same characteristic triplet (B^k, C^k, ν^k) in $(\mathcal{F}_t)$ and in $(\mathcal{F}_{-jt})$ on the interval $[r, s]$.*

We give now the definition based on the likelihood process

Definition 2 *[Likelihood-based weak conditional local independence (LWCLI)]*
Let $\boldsymbol{X} = (X_j, j = 1, \ldots, m)$ be in the class $\boldsymbol{\mathcal{D}}'$.

1. *Suppose the existence of a probability P_0 such that (i) $P \ll P_0$, (ii) the characteristics of the semi-martingales X_i's with $i \neq k$ are the same under P and P_0 and (iii) the*

P_0-characteristics (B_0^k, C_0^k, ν_0^k) of the semi-martingale X_k are deterministic. We say that X_k is LWCLI of X_j in $\boldsymbol{X}$ on $[0,t]$ if and only if the likelihood ratio process $Z_t^{P/P_0} = \mathcal{L}_{\mathcal{F}_t}^{P/P_0}$ is $(\mathcal{F}_{-jt})$-measurable on $[0,t]$. We have denoted $\mathcal{F}_{-jt} = \mathcal{H} \vee \mathcal{X}_{-jt}$ and $\mathcal{X}_{-jt} = \vee_{l \neq j} \mathcal{X}_{-lt}$.

2. *X_k is LWCLI of X_j in $\boldsymbol{X}$ on $[r,s]$ if and only if the process $\frac{Z_t^{P/P_0}}{Z_r^{P/P_0}}$ is $(\mathcal{F}_{-jt})$-predictable for all $t \in [r,s]$ for all the probabilities P_0 as above.*

These definitions allow us to define a strong local conditional independence (SCLI) and to use graphical model. It describes a mathematical framework which is well suited for formalizing causality. However we prefer to speak of influence rather than causal influence. A causal interpretation needs an epistemological act to link the mathematical model to a physical reality.

3 References

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