Modeling reducibility on ground terms using constraints

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Abstract

In this note, we explain how to model (ir)reducibility of rewriting on ground terms using (dis)equational constraints. We show in particular that innermost (ir)reducibility can be modeled with a particular narrowing relation and that (dis)equational constraints are issued from the most general unifiers of this narrowing relation.

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1. Introduction

In [1], we introduced an induction based technique for specifically proving termination of innermost rewriting, which was then applied, again for the innermost strategy, to weak termination [2], termination of probabilistic rewriting [3] and termination of priority rewriting [4].

For a given rewrite system, the principle of the technique is to prove on the ground term algebra that any rewriting chain starting from any term terminates, provided that terms smaller than the starting term do, with an induction ordering satisfying ordering constraints built along the proof.

To carry out this principle, we develop proof trees representing the rewriting trees, using an abstraction mechanism, consisting in replacing subterms in a term by a variable representing their normal form and narrowing representing all possibilities of rewriting ground instances of the abstracted terms.

A special narrowing relation has been proposed to simulate the innermost rewriting relation on ground terms, lying on the description of sets of reducible and irreducible terms for a given rewrite rule [1, 5]. These sets are defined from narrowing substitutions expressed as cunjunctions of equalities, and negations of substitutions expressed as disjunctions of disequalities. In this note, we give a complete formalization of this simulation.

2. Some notations

$\mathcal{T}(\mathcal{F}, \mathcal{X})$ is the set of terms built from a finite set $\mathcal{F}$ of function symbols f with arity $n \in \mathbb{N}$ (denoted $f : n$), and a set $\mathcal{X}$ of variables denoted $x, y \ldots$.

$\mathcal{V}ar(t)$ is the set of variables of the term t. $\mathcal{T}(\mathcal{F})$ is the set of ground terms (without variables).

Symbols of arity 0 are called *constants*. Positions in a term are represented as sequences of integers. The empty sequence ϵ denotes the top position. Let p and p' be two positions. The position p' is a (strict) suffix of p if $p' = pq$, where q is a (non-empty) sequence of integers. The notation $t|_p$ stands for the subterm of t at position p. If p is a position in t, then $t[t']_p$ denotes the term obtained from t by replacing the subterm at position p by the term t'.

A substitution is an assignment from $\mathcal{X}$ to $\mathcal{T}(\mathcal{F},\mathcal{X})$, written $\sigma = (x \mapsto t)\ldots(y \mapsto u)$. It uniquely extends to an endomorphism of $\mathcal{T}(\mathcal{F},\mathcal{X})$. The result of applying σ to a term $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ is written $\sigma(t)$ or σt. The domain of σ, denoted $Dom(\sigma)$ is the finite subset of $\mathcal{X}$ such that $\sigma x \neq x$. The range of σ, denoted $\mathcal{R}an(\sigma)$, is defined by $\mathcal{R}an(\sigma) = \bigcup_{x \in Dom(\sigma)} Var(\sigma x)$. The composition of substitutions σ_1 followed by σ_2 is denoted $\sigma_2 \circ \sigma_1$ or simply $\sigma_2\sigma_1$. Given a subset $\mathcal{X}_1$ of $\mathcal{X}$, we write $\sigma_{\mathcal{X}_1}$ for the *restriction* of σ to the variables of $\mathcal{X}_1$, i.e. the substitution such that $Dom(\sigma_{\mathcal{X}_1}) \subseteq \mathcal{X}_1$ and $\forall x \in Dom(\sigma_{\mathcal{X}_1}) : \sigma_{\mathcal{X}_1}x = \sigma x$. A ground substitution is an assignment θ from $\mathcal{X}$ to $\mathcal{T}(\mathcal{F})$. The set of ground substitutions is denoted by Θ. For $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ and a ground substitution θ, θt is called ground instance of t.

A unifier of two terms s and t is a substitution μ such that $\mu s = \mu t$. The most general unifier of s and t, denoted $mgu(s,t)$, is the unifier σ (unique up to a variable renaming) such that for every unifier μ of s and t, there exists a substitution ν such that $\mu = \nu\sigma$.

A set $\mathcal{R}$ of rewrite rules is a set of pairs of terms of $\mathcal{T}(\mathcal{F},\mathcal{X})$, denoted $l \to r$, such that $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. In this note, we only consider finite sets of rewrite rules.

The rewriting relation induced by $\mathcal{R}$ is denoted by $\to^{\mathcal{R}}$ ($\to$ if there is no ambiguity on $\mathcal{R}$), and defined by $s \to t$ iff there is a substitution σ and a position p in s such that $s|_p = \sigma l$ for some rule $l \to r$ of $\mathcal{R}$, and $t = s[\sigma r]_p$. This is written $s \to^{\mathcal{R}}_{p,l\to r,\sigma} t$ where either p, $l \to r$, σ or $\mathcal{R}$ may be omitted; $s|_p$ is called a redex.

The *innermost* rewriting strategy consists in always reducing at lowest possible position. The *innermost* rewriting relation is denoted $\to^{Inn}$.

Given $\mathcal{R}$ a rewrite system on $\mathcal{T}(\mathcal{F},\mathcal{X})$ and $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$, if there is a rule $l \to r \in \mathcal{R}$ such that $s \to t$ for some $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$, then s is said to be reducible for $\mathcal{R}$. Conversely, if there is no such rule, then s is said to be irreducible (or in normal form) for $\mathcal{R}$.

Let $\mathcal{R}$ be a rewrite system on $\mathcal{T}(\mathcal{F},\mathcal{X})$. A term t is *narrowed* into t', at the non-variable position p, using the rewrite rule $l \to r$ of $\mathcal{R}$ and the substitution σ, when σ is a most general unifier (mgu) of $t|_p$ and l, and $t' = \sigma(t[r]_p)$. This is denoted $t \leadsto^{\mathcal{R}}_{p,l\to r,\sigma} t'$ where either p, $l \to r$, σ or $\mathcal{R}$ may be omitted. It is always assumed that there is no variable in common between the rule and the term i.e., that $\mathcal{V}ar(l) \cap \mathcal{V}ar(t) = \emptyset$.

3. Relation between (ir)reducible ground instances and solutions of (dis)equational constraints

Given a free term and a rewrite system, the goal here is to characterize the (ir)reducible ground instances of the term.

Let u be a term of $\mathcal{T}(\mathcal{F},\mathcal{X})$, l a left-hand side of rule on a disjoint variable set $\mathcal{Y}$, $\sigma = mgu(u,l)$ such that $Dom(\sigma)\cap\mathcal{R}an(\sigma)=\emptyset$.

The set of ground instances of the term u reducible by $l \rightarrow r$ at position ϵ is:

$$RED_l(u) = \{\beta \in \Theta \mid Dom(\beta) = \mathcal{V}ar(u), \exists\alpha, \beta(u) = \alpha(l)\}.$$

Let us first prove that this set is also the set of ground instances of $mgu(u,l)$.

Proposition 3.1.

$$RED_l(u) = \{\beta \in \Theta \mid Dom(\beta) = \mathcal{V}ar(u), \exists\mu, \beta = \mu\sigma[\mathcal{V}ar(u)]\}$$

where $\sigma = mgu(u,l)$.

PROOF. Since $\beta(u) = \alpha(l)$ for some ground substitution α, $\beta\cup\alpha$ is a ground unifier of u and l. So there exists a ground substitution μ defined on $\mathcal{R}an(\sigma)$ such that $\beta\cup\alpha = \mu\sigma[Var(u)\cup Var(l)]$. By restricting to variables of $\mathcal{X}$, we get $\beta = \mu\sigma[Var(u)]$.

Conversely, if $\exists\mu, \beta = \mu\sigma[\mathcal{V}ar(u)]$, $\beta(u) = \mu\sigma(u) = \mu\sigma(l)$ since $\sigma = mgu(u,l)$, so $\beta \in RED_l(u)$.

In the following, we identify a substitution $\sigma = (x_1 \mapsto t_1)\dots(x_n \mapsto t_n)$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$ with the finite conjunction of solved equations $(x_1 = t_1)\wedge\dots\wedge(x_n = t_n)$, where $x_i \notin \mathcal{V}ar(t_i)$ for $i = 1,\dots,n$. Since $\mathcal{R}an(\sigma)\cup Dom(\sigma) = \emptyset$, for $i = 1,\dots,n$, $\sigma(t_i) = t_i$ and thus σ is solution of $(x_1 = t_1)\wedge\dots\wedge(x_n = t_n)$. Any ground instance of σ is also a ground solution.

Let us consider the set of ground solutions of this equational constraint:

$$\begin{array}{ll} \Phi^{u,\sigma} = & \{\zeta \in \Theta \mid Dom(\zeta) = \mathcal{V}ar(u)\cup\mathcal{R}an(\sigma) \\ , & \zeta \textit{ solution of } \sigma = \bigwedge_i (x_i = t_i), x_i \in \mathcal{V}ar(u), t_i \in \mathcal{T}(\mathcal{F},\mathcal{X})\}. \end{array}$$

We can also write:

$$\begin{array}{ll} \Phi^{u,\sigma} = & \{\zeta \in \Theta \mid Dom(\zeta) = \mathcal{V}ar(u)\cup\mathcal{R}an(\sigma), \\ & \forall i, \zeta(x_i) = \zeta(t_i), \sigma = \bigwedge_i (x_i = t_i), x_i \in \mathcal{V}ar(u), t_i \in \mathcal{T}(\mathcal{F},\mathcal{X})\}. \end{array}$$

Let us denote $\Phi^{u,\sigma}_{\mathcal{V}ar(u)} = \{\zeta_{\mathcal{V}ar(u)} \mid \zeta \in \Phi^{u,\sigma}\}$.

Proposition 3.2. $RED_l(u) = \Phi^{u,\sigma}_{\mathcal{V}ar(u)}$.

PROOF. If β is in $RED_l(u)$, since by Proposition 3.1, $\beta(x) = \mu\sigma(x)$ for any variable x of $\mathcal{V}ar(u)$, $\forall i = 1,...,n,\ \beta(x_i) = \mu\sigma(x_i)$. Let us define $\zeta = \beta\cup\mu$. ζ is defined on $\mathcal{V}ar(u)\cup\mathcal{R}an(\sigma)$ and coincides with β on $\mathcal{V}ar(u)$ and with μ on $\mathcal{R}an(\sigma)$. Therefore, $\zeta(x_i) = \mu\sigma(x_i) = \mu(t_i) = \zeta(t_i)$. So ζ is solution of the constraint and $\zeta \in \Phi^{u,\sigma}$. Then $\beta = \zeta_{\mathcal{V}ar(u)} \in \Phi^{u,\sigma}_{\mathcal{V}ar(u)}$.

Conversely if $\zeta \in \Phi^{u,\sigma}$, ζ is a ground solution of the constraint, so it is an instance of the most general solution σ. There exists μ such that $\zeta = \mu\sigma[\mathcal{V}ar(u) \cup \mathcal{R}an(\sigma)]$ since $\bigcup_{i=1,\ldots n}\{x_i, \mathcal{V}ar(t_i)\} = \mathcal{V}ar(u) \cup \mathcal{R}an(\sigma)$. Then $\zeta_{\mathcal{V}ar(u)} = \mu\sigma[\mathcal{V}ar(u)]$ and according to Proposition 3.1, $\zeta_{\mathcal{V}ar(u)}$ belongs to $RED_l(u)$.

Now let us define $\Gamma^u = \{\beta \in \Theta \mid Dom(\beta) = \mathcal{V}ar(u)\}$ and consider its subset called $IRRED_l(u)$ which is the set of ground instances of the term u that are not reducible by $l \to r$ at position ϵ:

$$
\begin{aligned}
IRRED_l(u) = \quad & \{\beta \in \Theta \mid Dom(\beta) = \mathcal{V}ar(u), \\
& \beta(u) \textit{ not reducible by } l \to r \textit{ at position } \epsilon\}.
\end{aligned}
$$

Two cases may happen: either u and l are unifiable or not. Let us first consider the second case:

Proposition 3.3. *If u and l are not unifiable, $IRRED_l(u) = \Gamma^u$.*

Proof. According to the definitions, $IRRED_l(u) = \Gamma^u \setminus RED_l(u)$. Moreover, $RED_l(u)$ is empty. Otherwise there would exist ground substitutions α s.t. $\beta(u) = \alpha(l)$, which contradicts the fact that u and l are not unifiable.

In the first case, i.e. provided u and l are unifiable, we can prove:

Proposition 3.4. *If u and l are unifiable,*

$$
\begin{aligned}
IRRED_l(u) = \quad & \{\beta \in \Theta \mid Dom(\beta) = \mathcal{V}ar(u), \\
& \forall\mu, \exists x \in \mathcal{V}ar(u), \beta(x) \neq \mu\sigma(x)\}.
\end{aligned}
$$

where $\sigma = mgu(u, l)$.

Proof. We have:
$IRRED_l(u) = \{\beta \in \Theta \mid Dom(\beta) = \mathcal{V}ar(u), \beta \notin RED_l(u)\}$.
By proposition 3.1,
$RED_l(u) = \{\beta \in \Theta \mid Dom(\beta) = \mathcal{V}ar(u), \exists\mu, \beta = \mu\sigma[\mathcal{V}ar(u)]\}$,
Thus we get:
$IRRED_l(u) = \{\beta \in \Theta \mid Dom(\beta) = \mathcal{V}ar(u), \forall\mu, \exists x \in \mathcal{V}ar(u), \beta(x) \neq \mu\sigma(x)\}$.

If $\sigma = mgu(u, l)$, let us define the disequational formula: $\overline{\sigma} = (x_1 \neq t_1) \vee \ldots \vee (x_n \neq t_n) = \bigvee_i (x_i \neq t_i)$ where for $i = 1, \ldots, n$, $x_i \notin \mathcal{V}ar(t_i)$ and $\mathcal{V}ar(t_i) \subseteq \mathcal{R}an(\sigma)$.

Let us define $\Delta^{u,\sigma} = \{\beta \in \Theta \mid Dom(\beta) = \mathcal{V}ar(u) \cup \mathcal{R}an(\sigma)\}$. Let us consider the subset $\Psi^{u,\sigma}$ of $\Delta^{u,\sigma}$ which is the set of ground solutions of the constraint $\overline{\sigma}$:

$$
\begin{aligned}
\Psi^{u,\sigma} = \quad & \{\zeta \in \Theta \mid Dom(\zeta) = \mathcal{V}ar(u) \cup \mathcal{R}an(\sigma), \\
& \zeta \textit{ solution of } \overline{\sigma} = \textstyle\bigvee_i (x_i \neq t_i), x_i \in \mathcal{V}ar(u), t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})\}.
\end{aligned}
$$

We can also write:

$$\begin{aligned}\Psi^{u,\sigma} = \quad & \{\zeta \in \Theta \mid Dom(\zeta) = \mathcal{V}ar(u) \cup \mathcal{R}an(\sigma), \\ & \exists i, \zeta(x_i) \neq \zeta(t_i), \sigma = \bigwedge_i (x_i = t_i), x_i \in \mathcal{V}ar(u), t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})\}.\end{aligned}$$

Proposition 3.5. *Let $\sigma = mgu(u, l)$ and $\zeta \in \Delta^{u,\sigma}$. $\zeta \in \Phi^{u,\sigma}$ iff $\zeta \notin \Psi^{u,\sigma}$.*

Proof. $\zeta \in \Phi^{u,\sigma}$ iff ζ ground solution of $\bigwedge_i (x_i = t_i)$ iff $\zeta \notin \Psi^{u,\sigma}$.

Proposition 3.6. *We have $\beta \in \Phi^{u,\sigma}_{Var(u)}$ iff $\beta \notin \Psi^{u,\sigma}_{Var(u)}$.*

Proof. **Since the x_i are in** $Var(u)$, this is an obvious consequence of Proposition 3.5.

Proposition 3.7. *Let us assume u and l unifiable and let $\sigma = mgu(u, l)$. $IRRED_l(u) = \Psi^{u,\sigma}_{\mathcal{V}ar(u)}$.*

Proof. Let $\beta \in \Theta$ such that $Dom(\beta) = \mathcal{V}ar(u)$. We have $\beta \in IRRED_l(u)$ iff $\beta \notin RED_l(u)$ iff, by Proposition 3.2, $\beta \notin \Phi^{u,\sigma}_{Var(u)}$ iff, by Proposition 3.6, $\beta \in \Psi^{u,\sigma}_{Var(u)}$.

4. The Case of Innermost Rewriting

We now want to simulate the innermost rewriting relation on ground terms by narrowing. For that, an innermost narrowing redex in a term t must correspond to an innermost rewriting redex in a ground instance of t. This is the case only if, in the rewriting chain of the ground instance of t, there is no rewriting redex anymore in the part of the term brought by the instantiation. This condition is fulfilled by considering the variable of t as special variables, whose ground instances have already been reduced in normal form. These special variables are called abstraction variables.

Definition 4.1 (abstraction variable [5]). Let $\mathcal{X}_A$ be a set of variables disjoint from $\mathcal{X}$. Symbols of $\mathcal{X}_A$ are called *abstraction variables*. Ground substitutions are extended to $\mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$ in the following way: for any ground substitutions θ such that $Dom(\theta)$ contains a variable $X \in \mathcal{X}_A$, θX is normal form.

Definition 4.2 (constrained substitution). A *constrained substitution* σ is a formula $\sigma_0 \wedge c$, where σ_0 is a substitution and c is a conjunction of complement formulas.

Definition 4.3 (Innermost narrowing [5]). A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$ *innermost narrows* into a term $t' \in \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$ at the non-variable position p of t, using the rule $l \to r \in \mathcal{R}$ with the constrained substitution $\sigma = \sigma_0 \wedge \bigwedge_{j \in [1..k]} \overline{\sigma_j}$, which is written $t \leadsto^{Inn}_{p,l\to r,\sigma} t'$ iff

$$\sigma_0(l) = \sigma_0(t|_p) \text{ and } t' = \sigma_0(t[r]_p)$$

where σ_0 is the most general unifier of $t|_p$ and l and $\sigma_j, j \in [1..k]$ are all most general unifiers of $\sigma_0 t|_{p'}$ with a left-hand side l' of a rule of $\mathcal{R}$, for all suffix positions p' of p in t.

As said previously, it is always assumed that there is no variable in common between the rule and the term i.e., that $\mathcal{V}ar(l) \cap \mathcal{V}ar(t) = \emptyset$. This requirement of disjoint variables is easily fulfilled by an appropriate renaming of variables in the rules when narrowing is performed. Observe that for the most general unifier σ used in the above definition, $Dom(\sigma) \subseteq \mathcal{V}ar(l) \cup \mathcal{V}ar(t)$ and we can choose $\mathcal{R}an(\sigma) \cap (\mathcal{V}ar(l) \cup \mathcal{V}ar(t)) = \emptyset$, thus introducing in the range of σ only fresh variables.

Now, we are only interested on narrowing terms t of $\mathcal{T}(\mathcal{F}, \mathcal{X}_A)$. Then, from Definition 4.1 we infer that for the most general unifiers σ produced for narrowing, all variables of $\mathcal{R}an(\sigma)$ are abstraction variables.

The following lifting lemma generalizes to the innermost mechanism the one given in [6]. Note that a generic lifting lemma has been proposed in [5] to model rewriting on ground terms under the innermost, outermost and local strategies. The proof of the lemma given here is similar to the one of the generic lemma, but is particularized to the innermost strategy, and the irreducibility arguments are formalized with the set Ψ of ground solutions of a disequational formula defined above.

Lemma 4.4 (Innermost lifting Lemma). *Let $\mathcal{R}$ be a rewrite system. Let $s \in \mathcal{T}(\mathcal{F}, \mathcal{X}_A)$, α a (normalized) ground substitution and $\mathcal{Y} \subseteq \mathcal{X}$ a set of variables such that $\mathcal{V}ar(s) \cup Dom(\alpha) \subseteq \mathcal{Y}$. If $\alpha s \rightarrow^{Inn}_{p,l\rightarrow r} t'$, then there exist a term $s' \in \mathcal{T}(\mathcal{F}, \mathcal{X}_A)$ and substitutions $\beta, \sigma = \sigma_0 \wedge \bigwedge_{j\in[1..k]} \overline{\sigma_j}$ such that:*

1. $s \rightsquigarrow^{Inn}_{p,l\rightarrow r,\sigma} s'$,
2. $\beta s' = t'$,
3. $\beta \sigma_0 = \alpha[\mathcal{Y}]$
4. β satisfies $\bigwedge_{j\in[1..k]} \overline{\sigma_j}$

where σ_0 is the most general unifier of $s|_p$ and l, for $j \in [1..k]$ the σ_j are the most general unifiers of $\sigma_0 s|_{p'}$ with a left-hand side l' of a rule of $\mathcal{R}$, for all suffix positions p' of p in s.

The proof of the lemma needs the following two propositions (the first one is obvious).

Proposition 4.5. *Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and σ a substitution of $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Then $\mathcal{V}ar(\sigma t) = (\mathcal{V}ar(t) - Dom(\sigma)) \cup \mathcal{R}an(\sigma_{\mathcal{V}ar(t)})$.*

Proposition 4.6. *Suppose we have substitutions σ, μ, ν and sets A, B of variables such that $(B - Dom(\sigma)) \cup \mathcal{R}an(\sigma) \subseteq A$. If $\mu = \nu[A]$ then $\mu\sigma = \nu\sigma[B]$.*

PROOF. Let us consider $(\mu\sigma)_B$, which can be divided as follows: $(\mu\sigma)_B = (\mu\sigma)_{B\cap Dom(\sigma)} \cup (\mu\sigma)_{B-Dom(\sigma)}$.

For $x \in B \cap Dom(\sigma)$, we have $\mathcal{V}ar(\sigma x) \subseteq \mathcal{R}an(\sigma)$, and then $(\mu\sigma)x = \mu(\sigma x) = \mu_{\mathcal{R}an(\sigma)}(\sigma x) = (\mu_{\mathcal{R}an(\sigma)}\sigma)x$. Therefore $(\mu\sigma)_{B\cap Dom(\sigma)} = (\mu_{\mathcal{R}an(\sigma)}\sigma)_{B\cap Dom(\sigma)}$.
For $x \in B - Dom(\sigma)$, we have $\sigma x = x$, and then $(\mu\sigma)x = \mu(\sigma x) = \mu x$. Therefore we have $(\mu\sigma)_{B-Dom(\sigma)} = \mu_{B-Dom(\sigma)}$. Henceforth we get $(\mu\sigma)_B = (\mu_{\mathcal{R}an(\sigma)}\sigma)_{B\cap Dom(\sigma)} \cup \mu_{B-Dom(\sigma)}$.
By a similar reasoning, we get $(\nu\sigma)_B = (\nu_{\mathcal{R}an(\sigma)}\sigma)_{B\cap Dom(\sigma)} \cup \nu_{B-Dom(\sigma)}$.
By hypothesis, we have $\mathcal{R}an(\sigma) \subseteq A$ and $\mu = \nu[A]$. Then we can infer $\mu_{\mathcal{R}an(\sigma)} = \nu_{\mathcal{R}an(\sigma)}$. Likewise, since $B - Dom(\sigma) \subseteq A$, we have $\mu_{B-Dom(\sigma)} = \nu_{B-Dom(\sigma)}$.
Then we have $(\mu\sigma)_B = (\mu_{\mathcal{R}an(\sigma)}\sigma)_{B\cap Dom(\sigma)} \cup \mu_{B-Dom(\sigma)} = (\nu_{\mathcal{R}an(\sigma)}\sigma)_{B\cap Dom(\sigma)} \cup \nu_{B-Dom(\sigma)} = (\nu\sigma)_B$. Therefore $(\mu\sigma) = (\nu\sigma)[B]$.

We can now give the proof of the lifting lemma.

Proof. To show the point 1., we will need to fulfill the conditions of the innermost narrowing definition, given in Definition 4.3. In the following, we assume that $\mathcal{V}ar(\mathcal{Y}) \cap \mathcal{V}ar(l) = \emptyset$ for every $l \to r \in \mathcal{R}$.
If $\alpha s \to^{Inn}_{p,l\to r} t'$, then there exists a substitution τ such that $Dom(\tau) \subseteq \mathcal{V}ar(l)$ and $(\alpha s)|_p = \tau l$. Moreover, since α is normalized, p is a non variable position of s and we have $(\alpha s)|_p = \alpha(s|_p)$. Denoting $\mu = \alpha \wedge \tau$, we have:

$$\begin{array}{lll} \mu(s|_p) & = \alpha(s|_p) & \text{for } Dom(\tau) \subseteq \mathcal{V}ar(l) \text{ and } \mathcal{V}ar(l) \cap \mathcal{V}ar(s) = \emptyset \\ & = \tau l & \text{by definition of } \tau \\ & = \mu l & \text{for } Dom(\alpha) \subseteq \mathcal{Y} \text{ and } \mathcal{Y} \cap \mathcal{V}ar(l) = \emptyset, \end{array}$$

and therefore $s|_p$ and l are unifiable. Let us note σ_0 the most general unifier of $s|_p$ and l, and $s' = \sigma_0(s[r]_p)$.

If there exist most general unifiers σ_j of $\sigma_0 s$ and a left-hand side of rule of $\mathcal{R}$ at strict suffix positions of p, we can build the complement formula $\bigwedge_{j\in[1..k]} \overline{\sigma_j}$, that otherwise reduces to the identity constraint. Therefore, denoting $\sigma = \sigma_0 \wedge \bigwedge_{j\in[1..k]} \overline{\sigma_j}$, we get, by definition: $s \rightsquigarrow^{Inn}_{p,l\to r,\sigma} s'$, and then the point 1. of the current lemma holds.

Since σ_0 is more general than μ, there exists a substitution ρ such that $\rho\sigma_0 = \mu$. Let $\mathcal{Y}_1 = (\mathcal{Y} - Dom(\sigma_0)) \cup \mathcal{R}an(\sigma_0)$. We define $\beta = \rho_{\mathcal{Y}_1}$. Clearly $Dom(\beta) \subseteq \mathcal{Y}_1$.
We now show that $\mathcal{V}ar(s') \subseteq \mathcal{Y}_1$, by the following reasoning:

- since $s' = \sigma_0(s[r]_p)$, we have $\mathcal{V}ar(s') = \mathcal{V}ar(\sigma_0(s[r]_p))$;
- the rule $l \to r$ is such that $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$, therefore we have $\mathcal{V}ar(\sigma_0(s[r]_p)) \subseteq \mathcal{V}ar(\sigma_0(s[l]_p))$, and then, thanks to the previous point, $\mathcal{V}ar(s') \subseteq \mathcal{V}ar(\sigma_0(s[l]_p))$;
- since $\sigma_0(s[l]_p) = \sigma_0 s[\sigma_0 l]_p$ and since σ_0 unifies l and $s|_p$, we get $\sigma_0(s[l]_p) = \sigma_0 s[\sigma_0(s|_p)]_p = \sigma_0 s[s|_p]_p = \sigma_0 s$ and, thanks to the previous point: $\mathcal{V}ar(s') \subseteq \mathcal{V}ar(\sigma_0 s)$;
- according to Proposition 4.5, we have $\mathcal{V}ar(\sigma_0 s) = (\mathcal{V}ar(s) - Dom(\sigma_0)) \cup \mathcal{R}an(\sigma_{0\mathcal{V}ar(s)})$; by hypothesis, $\mathcal{V}ar(s) \subseteq \mathcal{Y}$. Moreover, since $\mathcal{R}an(\sigma_{0\mathcal{V}ar(s)}) \subseteq \mathcal{R}an(\sigma_0)$, we have $\mathcal{V}ar(\sigma_0(s)) \subseteq (\mathcal{Y} - Dom(\sigma_0)) \cup \mathcal{R}an(\sigma_0)$, that is $\mathcal{V}ar(\sigma_0 s) \subseteq \mathcal{Y}_1$. Therefore, with the previous point, we get $\mathcal{V}ar(s') \subseteq \mathcal{Y}_1$.

From $Dom(\beta) \subseteq \mathcal{Y}_1$ and $\mathcal{V}ar(s') \subseteq \mathcal{Y}_1$, we infer $Dom(\beta) \cup \mathcal{V}ar(s') \subseteq \mathcal{Y}_1$.

We are now going to demonstrate the point 2., that is $\beta s' = t'$.
Since $\beta = \rho_{\mathcal{Y}_1}$, we have $\beta = \rho[\mathcal{Y}_1]$. Since $\mathcal{V}ar(s') \subseteq \mathcal{Y}_1$, we get $\beta s' = \rho s'$. Since $s' = \sigma_0(s[r]_p)$, we have $\rho s' = \rho\sigma_0(s[r]_p) = \mu(s[r]_p) = \mu s[\mu r]_p$. Then $\beta s' = \mu s[\mu r]_p$.
We have $Dom(\tau) \subseteq \mathcal{V}ar(l)$ and $\mathcal{Y} \cap \mathcal{V}ar(l) = \emptyset$, then we have $\mathcal{Y} \cap Dom(\tau) = \emptyset$. Therefore, from $\mu = \alpha \cup \tau$, we get $\mu = \alpha[\mathcal{Y}]$. Since $\mathcal{V}ar(s) \subseteq \mathcal{Y}$, we get $\mu s = \alpha s$. Likewise, by hypothesis we have $Dom(\alpha) \subseteq \mathcal{Y}$, $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$ and $\mathcal{Y} \cap \mathcal{V}ar(l) = \emptyset$, then we get $\mathcal{V}ar(r) \cap Dom(\alpha) = \emptyset$, and then we have $\mu = \tau[\mathcal{V}ar(r)]$, and therefore $\mu r = \tau r$.
From $\mu s = \alpha s$ and $\mu r = \tau r$ we get $\mu s[\mu r]_p = \alpha s[\tau r]_p$. Since, by hypothesis, $\alpha s \rightarrow_p t'$, with $\tau l = (\alpha s)|_p$, then $\alpha s[\tau r]_p = t'$. Finally, we get $\beta s' = t'$ (2).

Next we show that $\beta\sigma_0 = \alpha[\mathcal{Y}]$ (point 3. of the current lemma). Reminding that $\mathcal{Y}_1 = (\mathcal{Y} - Dom(\sigma_0)) \cup \mathcal{R}an(\sigma_0)$, Proposition 4.6 (with the notations $\mathcal{Y}_1$ for A, $\mathcal{Y}$ for B, β for μ, ρ for ν and σ_0 for σ) yields $\beta\sigma_0 = \rho\sigma_0[\mathcal{Y}]$. We already noticed that $\mu = \alpha[\mathcal{Y}]$. Linking these two equalities via the equation $\rho\sigma_0 = \mu$ yields $\beta\sigma_0 = \alpha[\mathcal{Y}]$ (3).

Let us now prove that β satisfies the formula $\bigwedge_{j \in [1..k]} \overline{\sigma_j} = \bigwedge_j \bigvee_{i_j} (x_{i_j} \neq t_{i_j})$. For that, it must satisfy every conjunct $\overline{\sigma_j} = \bigvee_{i_j} (x_{i_j} \neq t_{i_j})$, where σ_j is the most general unifier of $\sigma_0 s|_{p_j}$ with a left-hand side of rule l_j. Let $u_j = \sigma_0 s|_{p_j}$ and

$$\begin{aligned} \Psi^{u_j,\sigma_j} = \quad & \{\zeta \in \Theta \mid Dom(\zeta) = \mathcal{V}ar(u_j) \cup \mathcal{R}an(\sigma_j), \\ & \zeta \textit{ solution of } \overline{\sigma_j} = \textstyle\bigvee_{i_j} (x_{i_j} \neq t_{i_j}), x_{i_j} \in \mathcal{V}ar(u_j), t_{i_j} \in \mathcal{T}(\mathcal{F}, \mathcal{X})\}. \end{aligned}$$

Then β verifies $\overline{\sigma_j}$ iff $\beta \in \Psi^{u_j,\sigma_j}$ iff, by definition of $\Psi^{u_j,\sigma_j}_{\mathcal{V}ar(u_j)}$, $\beta_{\mathcal{V}ar(u_j)} \in \Psi^{u_j,\sigma_j}_{\mathcal{V}ar(u_j)}$. According to Proposition 3.7, $\beta_{\mathcal{V}ar(u_j)} \in \Psi^{u_j,\sigma_j}_{\mathcal{V}ar(u_j)}$ iff $\beta_{\mathcal{V}ar(u_j)} \in IRRED_{l_j}\ (u_j)$, which is true since $\alpha s = \beta\sigma_0 s$ is irreducible at position p_j greater than p.

Finally, since β is defined on $\mathcal{Y}_1 \subseteq \mathcal{X}_A$, β is necessarily normalized.

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