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Abstract

In quasi-periodic or nonlinear periodic homogenization, the corrector problem must be in general set on the whole space. Numerically computing the homogenization coefficient therefore implies a truncation error, due to the fact that the problem is approximated on a bounded, large domain. We present here an approach that improves the rate of convergence of this approximation.

1 Introduction

Many problems in the engineering sciences and in the life sciences involve several scales. When these scales are well separated, homogenization theory (see for instance [3, 20, 21, 25, 29]) gives an appropriate answer to the problem of defining an effective macroscopic equation. The coefficients of this equation may in general be computed using a *corrector problem*. In the linear periodic case, see [3], the corrector problem is set on a bounded, periodic cell, and is thus easy to solve using *e.g.* a standard finite element method. The question of computing effective coefficients is thus well understood and documented, both theoretically and numerically. In sharp contrast, when the original coefficients are not periodic, or when the equation is not linear, the corrector problem is, in principle, posed over the whole space. Even though the coefficients are periodic it might be wrong that the corrector problem amounts to a problem set on the periodic cell. Solving it numerically, and computing the effective coefficients, is then a challenging practical question. A large body of literature discusses the appropriate choice of a representative volume element along with adequate boundary conditions to be used in order to reach accuracy at a limited computational cost.

Our purpose here is to present a numerical approach that efficiently computes such homogenized coefficients in some difficult cases. Under consideration is a specific periodic setting, the quasi-periodic setting, and a nonlinear non-convex example. We admit the examples we treat are somewhat academic in nature but we believe they are representative of some generality of the problems actually met in practice. The approach is thus likely to be extended to other difficult cases. It is based on a filtering technique. Such techniques are well known in signal analysis. In [12, 13], a particular version of filtering was employed to exploit long range correlations in periodic and quasi-periodic signals in order to accelerate long-time averages in molecular dynamics simulations. That very same idea of *filters* can be useful in the homogenization context, and this is the purpose of the present contribution to investigate its capabilities.

In short, the filtering technique is based on the following simple remark. Consider a regular, quasi-periodic[1] function b on $\mathbb{R}^d$. Its average is defined by

$$\langle b \rangle = \lim_{R\to\infty} \frac{1}{|Q_R|} \int_{Q_R} b(x)dx,$$

where Q_R is the cube of size R. The rate of this convergence is only of order $1/R$. It is however possible to compute the average in a more efficient way. Fix a compactly supported, nonnegative function $\varphi \in C^k$, which sums to one, that is $\displaystyle\int_{\mathbb{R}^d} \varphi = 1$, and whose derivatives (up to order k) vanish on the boundary of the unit cube Q. Then,

$$\langle b \rangle = \lim_{R\to\infty} \int_{Q_R} b(x)\varphi\left(\frac{x}{R}\right)\frac{1}{R^d}\,dx,$$

but, provided b is sufficiently regular, the rate of this convergence is now of order $1/R^k$ (see [12, 13] or the proof of Proposition 3.1 below for a rigorous argument).

This elementary property, originally pointed out for accelerating the convergence of averages of functions, forms the bottom line of the approach we now present, in the context of homogenization theory.

Some filtering techniques have already been introduced in the context of homogenization (see [14, 27]). In sharp contrast to these prior works, the present approach introduces a filter *already* in the definition of the corrector problem (see (1.11)-(1.12) below), and not only at the level of the calculation of the average defining the homogenized coefficient. The speed-up obtained is considerably better. The acceleration is not limited to a prefactor but also improves the *rate* of convergence. In addition, the present contribution provides an analysis of the filtering approach, confirming the superiority of the numerical strategy chosen. We shall comment upon such issues later on in the present contribution.

[1]The definition of quasi-periodic functions is recalled in Definition 4.1 below

We first recall for consistency some basic elements of homogenization theory. For simplicity, we first restrict ourselves to the consideration of a *linear scalar second order elliptic equation*. This is the prototypical setting for homogenization theory. The case of nonlinear hyperelastic models is also considered in Section 5. As will be clear from the sequel, our approach may however carry over some other cases and apply to more general situations. Consider the problem

$$\left\{ \begin{array}{ll} -\operatorname{div}(A_\varepsilon \nabla u) = f & \text{in } \mathcal{D}, \\ u = 0 & \text{on } \partial\mathcal{D}, \end{array} \right. \tag{1.1}$$

with ε a small parameter, a right-hand side $f \in L^2(\mathcal{D})$ that is independent of ε, and $\mathcal{D}$ a fixed bounded connected open subset of $\mathbb{R}^d$. The family of matrices $A_\varepsilon = A_\varepsilon(x)$ is assumed to be uniformly elliptic and bounded, that is,

$$\exists \gamma > 0 \quad / \quad \forall \varepsilon > 0, \quad \forall \xi \in \mathbb{R}^d, \quad \xi^T A_\varepsilon(x) \xi \geq \gamma |\xi|^2, \tag{1.2}$$

almost everywhere in $x \in \mathcal{D}$, and

$$\exists M > 0 \quad / \quad \forall \varepsilon > 0, \quad \|A_\varepsilon\|_{L^\infty(\mathcal{D})} \leq M. \tag{1.3}$$

Under these hypotheses, Murat and Tartar (see [20, 21, 25]) have proved the following:

Theorem 1.1 (Murat, Tartar, [21]) *Let $\mathcal{D}$ be an open bounded subset of $\mathbb{R}^d$, where d is a positive integer. Consider a set of matrices A_ε satisfying (1.2) and (1.3). Let u_ε be the unique solution to* (1.1)*. Then, there exists a sequence $\varepsilon_n \longrightarrow 0$, a matrix A^* satisfying* (1.2) *and* (1.3)*, and $u^* \in H^1(\mathcal{D})$ such that*

$$\left\{ \begin{array}{ll} -\operatorname{div}(A^* \nabla u^*) = f & \text{in } \mathcal{D} \\ u = 0 & \text{on } \partial\mathcal{D}, \end{array} \right. \tag{1.4}$$

and

$$u_{\varepsilon_n} \rightharpoonup u^* \text{ in } H^1(\mathcal{D}), \quad A_{\varepsilon_n} \nabla u_{\varepsilon_n} \rightharpoonup A^* \nabla u^* \text{ in } L^2(\mathcal{D}).$$

This theorem states the existence of a limit problem, defined by the homogenized matrix A^* under extremely general hypotheses, namely (1.2)-(1.3). It does not provide, however, any hint on how to explicitly compute this matrix. In some specific situations, it is possible to derive an "explicit" expression for A^*. This is the case for instance if A_ε in (1.1) is a rescaled *periodic* function:

$$A_\varepsilon(x) = A_{\text{per}}\left(\frac{x}{\varepsilon}\right),$$

where A_{per} is a periodic matrix-valued function. Without loss of generality, we may then assume that the periodic cell of A_{per} is the unit cube Q.

In order to compute the homogenized matrix A^*, for problem (1.1) with $A_\varepsilon(x) = A_{\text{per}}\left(\frac{x}{\varepsilon}\right)$ we define the *corrector problem*

$$\left\{ \begin{array}{l} -\operatorname{div}\left(A_{\text{per}}(y)\,(p + \nabla w_p)\right) = 0, \\ \\ w_p \text{ is } \mathbb{Z}^d\text{-periodic.} \end{array} \right. \tag{1.5}$$

This problem admits a unique solution, up to the addition of a constant (see [3]). Then, the homogenized coefficients read

$$A^*_{ij} = \int_Q (e_i + \nabla w_{e_i}(y))^T A_{\rm per}(y) \left(e_j + \nabla w_{e_j}(y)\right) dy$$
$$= \int_Q e_i^T A_{\rm per}(y) \left(e_j + \nabla w_{e_j}(y)\right) dy, \quad (1.6)$$

where Q is the unit cube, and $(e_i)_{1\leq i\leq d}$ is the canonical basis of $\mathbb{R}^d$. Hence, for $A_{\rm per}$ given, (1.5)-(1.6) allows to compute A^*. In practice, (1.5) is solved using for instance a finite element method, and A^* is next evaluated using (1.6) and an appropriate quadrature formula. The computation can be made arbitrarily accurate. The accuracy only depends on the accuracy of the discretization approach used to solve (1.5) and that of the quadrature method used to approximate (1.6).

The periodic situation is somehow a serendipitous case. The general case is much more delicate to address. To convey the idea, consider the quasi-periodic (or almost-periodic)[2] setting. Then problem (1.5) is posed on the whole space $\mathbb{R}^d$. If

$$A_\varepsilon(x) = A_{\rm q-per}\left(\frac{x}{\varepsilon}\right),$$

where $A_{\rm q-per}$ is quasi-periodic, then (1.5) and (1.6) indeed read

$$\begin{cases} -\operatorname{div}\left(A_{\rm q-per}(y)\,(p + \nabla w_p)\right) = 0, \\ \\ w_p \text{ is almost periodic,} \end{cases} \quad (1.7)$$

$$A^*_{ij} = \left\langle (e_i + \nabla w_{e_i}(y))^T A_{q-per}(y) \left(e_j + \nabla w_{e_j}(y)\right) \right\rangle$$
$$= \left\langle e_i^T A_{q-per}(y) \left(e_j + \nabla w_{e_j}(y)\right)\right\rangle, \quad (1.8)$$

where $\langle g\rangle$ denotes the mean value of an almost-periodic function[3] g:

$$\langle g\rangle = \lim_{R\to+\infty} \frac{1}{|B_R|}\int_{B_R} g(y)dy.$$

In terms of accuracy, we are now in position to understand the striking difference between the computation of the homogenized coefficient in the periodic case and, say, in the quasi-periodic case. In the latter, we have to account for a truncation error. Its origin is twofold. First, (1.7), while theoretically posed on the whole space $\mathbb{R}^d$, is indeed in practice solved on a bounded domain. Second, the average (1.8) is defined as a limit $R \to +\infty$. Only *approximate* values will

[2]The definition of almost-periodic functions is recalled in Definition 4.2 below

[3]Since the space of quasi-periodic functions is not closed for uniform norms (see Section 4 and the references therein), the corrector problem is naturally set in the set of almost-periodic functions.

thus be computable. Both sources of error add up to the standard numerical error related to the actual solution procedure for (1.7). As will be seen below, and as confirmed by numerical experiments, a simple truncation on a box of size R induces an error of order $1/R$. In order to diminish this error, we introduce, in the spirit of the approach presented in [12, 13], a *filtered problem.* Define a filtering function φ such that

$$\left\{ \begin{array}{l} \varphi \in C^k(\overline{Q}), \quad \varphi \geq 0, \quad \int_Q \varphi = 1, \\ \forall j \leq k-1, \quad D^j \varphi_{|\partial Q} = 0. \end{array} \right. \tag{1.9}$$

Define likewise a *rescaled filtering function* φ_R by:

$$\varphi_R(y) = \frac{1}{R^d} \varphi\left(\frac{y}{R}\right), \tag{1.10}$$

which satisfies (1.9) in $Q_R = RQ$ instead of Q. We consider the following *filtered corrector problem*:

$$\left\{ \begin{array}{l} -\operatorname{div}\left[\varphi_R\left(y\right)\left(A_{\mathrm{q-per}}(y)\left(p+\nabla w_p\right)+\lambda\right)\right]=0, \quad \text{in} \quad Q_R, \\ \\ \displaystyle\int_{Q_R} \nabla w_p(y) \varphi_R\left(y\right) dy = 0, \end{array} \right. \tag{1.11}$$

where Q_R is the cube of size R, and $\lambda \in \mathbb{R}^d$ is the vector of the Lagrange multipliers associated with the constraint (second line of (1.11)). If the matrix $A_{\mathrm{q-per}}$ is symmetric, equation (1.11) is in fact the Euler-Lagrange equation of the following minimization problem:

$$\inf \left\{ \int_{Q_R} (p + \nabla w_p(y))^T A_{\mathrm{q-per}}(y) (p + \nabla w_p(y)) \varphi_R(y) dy, \right. \\ \left. \int_{Q_R} \nabla w_p(y) \varphi_R(y) = 0 \right\}.$$

The corresponding, approximated homogenized matrix A^*_R is defined by

$$\begin{aligned} (A^*_R)_{ij} &= \int_{Q_R} \left(e_i + \nabla w_{e_i}(y)\right)^T A_{\mathrm{q-per}}(y) \left(e_j + \nabla w_{e_j}(y)\right) \varphi_R\left(y\right) dy \\ &= \int_{Q_R} e_i^T A_{\mathrm{q-per}}(y) \left(e_j + \nabla w_{e_j}(y)\right) \varphi_R\left(y\right) dy. \end{aligned} \tag{1.12}$$

Note that (1.11)-(1.12) coincide with (1.7)-(1.8) when $\varphi = 1$. The introduction of the filter at the level of the corrector problem seems to be new, to the best of the authors' knowledge. For the sake of illustration, let us momentarily consider the one-dimensional quasi-periodic situation and motivate our strategy on this simple case. A natural approach consists in considering first the standard corrector problem

$$\left\{ \begin{array}{l} -\frac{d}{dy}\left[A_{\mathrm{q-per}}(y)\left(p + \frac{dw_p}{dy}(y)\right)\right] = 0 \quad \text{in} \quad Q_R = \left(-\frac{R}{2}, \frac{R}{2}\right), \\ w_p \quad \text{is} \quad R-\text{periodic}, \end{array} \right. \tag{1.13}$$

and next computing a *filtered* homogenized coefficient. The solution to (1.13) is known analytically and reads

$$\frac{dw_p}{dy}(y) = \left(\frac{1}{R}\int_{-R/2}^{R/2} A_{\mathrm{q-per}}^{-1}(y)dy\right)^{-1} \frac{p}{A_{\mathrm{q-per}}(y)} - p.$$

Therefore the filtered homogenized coefficient is

$$\begin{aligned} \tilde{A}_R^* &= \frac{1}{R}\int_{-R/2}^{R/2} A_{\mathrm{q-per}}(y)\left(1+\frac{dw_1}{dy}(y)\right)^2 \varphi_R(y)dy \\ &= \left(\frac{1}{R}\int_{-R/2}^{R/2} A_{\mathrm{q-per}}^{-1}(y)dy\right)^{-2} \frac{1}{R}\int_{-R/2}^{R/2} A_{\mathrm{q-per}}^{-1}(y)\varphi_R(y)dy. \end{aligned} \quad (1.14)$$

The rate of convergence when R, the size of the interval, goes to infinity is only 1/R, similarly to the non-filtered case. Only, perhaps, the prefactor has been improved. In contrast, consider now the *filtered* corrector problem (1.11) and insert its solution, again analytically known in this one-dimensional situation:

$$\frac{dw_p}{dy}(y) = \left(\frac{1}{R}\int_{-R/2}^{R/2} A_{\mathrm{q-per}}^{-1}(y)\varphi_R(y)dy\right)^{-1} \frac{p}{A_{\mathrm{q-per}}(y)} - p,$$

in the filtered coefficient (1.14) above. One obtains:

$$\begin{aligned} A_R^* &= \frac{1}{R}\int_{-R/2}^{R/2} A_{\mathrm{q-per}}(y)\left(1+\frac{dw_1}{dy}(y)\right)^2 \varphi_R(y)dy \\ &= \left(\frac{1}{R}\int_{-R/2}^{R/2} A_{\mathrm{q-per}}^{-1}(y)\varphi_R(y)dy\right)^{-1}. \end{aligned}$$

This time, the rate of convergence is $1/R^k$, if φ satisfies (1.9) [12, 13] and is better than in the previous "partially" filtered approach. A two dimensional numerical experiment would yield the same qualitative comparison, although of course analytic formulae are not accessible.

In dimension one, where both the solution to (1.7) and (1.11) may be computed explicitly, we shall prove below that

$$\exists C > 0, \quad |A_R^* - A^*| \leq \frac{C}{R^k}. \quad (1.15)$$

In higher dimensions, we shall develop a formal argument, based on a two-scale expansion, that shows that, for any $k \geq 2$,

$$\exists C > 0, \quad |A_R^* - A^*| \leq \frac{C}{R^2}. \quad (1.16)$$

We shall also report on some numerical experiments that confirm estimate (1.16) and the interest of the filtering approach.

Besides the linear (periodic or quasi-periodic) case, another interesting setting is the nonlinear (periodic) case. Indeed, as briefly mentioned above, it may happen in such a case that the corrector problem needs to be solved in the entire space instead of in the unit cell only. Let us introduce the corresponding notions: we assume that (here, $\mathbb{R}^{d\times d}$ denotes the space of square matrices of size d)

$$\begin{array}{rccc} W: & \mathbb{R}^d \times \mathbb{R}^{d\times d} & \longrightarrow & \mathbb{R} \\ & (y, A) & \longmapsto & W(y, A), \end{array}$$

is $\mathbb{Z}^d$-periodic in its first variable, quasiconvex (see [2, 18]) in its second variable, and satisfies a growth property of order $p > 1$ with respect to its second variable, that is,

$$\exists C_2 \geq C_1 > 0, \quad \forall y \in \mathbb{R}^d, \quad \forall A \in \mathbb{R}^{d\times d},$$
$$C_1 \left(|A|^p - 1\right) \leq W(y, A) \leq C_2 \left(|A|^p + 1\right).$$

We refer for instance to [8] for the details. The corresponding homogenization problem reads

$$I_\varepsilon = \inf\left\{ \int_{\mathcal{D}} W\left(\frac{x}{\varepsilon}, \nabla(u + \overline{u})\right), \quad u \in W_0^{1,p}(\mathcal{D}), \right\},$$

where $\overline{u}$ is a suitable boundary condition. Then, the same kind of result as for the linear case is valid [8], giving the following homogenized problem:

$$I^{\text{homog}} = \inf\left\{ \int_{\mathcal{D}} W^{\text{homog}}\left(x, \nabla(u + \overline{u})\right), \quad u \in W_0^{1,p}(\mathcal{D}), \right\},$$

where W^{homog} is defined by

$$W^{\text{homog}}(A) = \lim_{N\to\infty} \inf\left\{ \frac{1}{N^d} \int_{(0,N)^d} W(x, \nabla v(x) + A) dx \quad v \in W^{1,p}_{\text{per}}((0,N)^d; \mathbb{R}^d) \right\}. \quad (1.17)$$

If the energy density W is convex, then the limit $N \to \infty$ in (1.17) is not necessary, and one can simply use $N = 1$ to compute W^{homog}. In contrast, if W is quasiconvex but *not* convex, then the limit in (1.17) is necessary [19], and the value of (1.17) for $N < +\infty$ may be strictly larger than its limit $N \to \infty$. In particular,

$$W^{\text{homog}}(A) \leq W^{\text{homog,per}}(A), \quad (1.18)$$

where

$$W^{\text{homog,per}}(A) = \inf\left\{ \int_{(0,1)^d} W(x, \nabla v(x) + A) dx v \in W^{1,p}_{\text{per}}((0,1)^d; \mathbb{R}^d) \right\}. \quad (1.19)$$

The inequality in (1.18) may be a strict inequality. Hence, the filtering method we use here can in principle be applied to this case, hopefully speeding up the convergence as $N \to \infty$. We give in Section 5 an example of such an energy density W, together with a detailed numerical study of the corresponding filtered problem.

The article is organized as follows. We first prove, in Section 2, that the filtered corrector problem (1.11) for R fixed, is well posed. Then, in Section 3, we consider periodic homogenization. We perform a formal argument showing the rectitude of the approach and then perform some numerical tests that confirm the efficiency. In these numerical tests, we deliberately put ourselves in a difficult situation, *pretending* not to know the period and trying to nevertheless complete the computation. This case is seen as a mathematical test bed for other more relevant situations. We are in position to analyze the situation mathematically, notably proving error estimates, see (1.15) and (1.16). Section 4 is devoted to the quasi-periodic case. We show the efficiency of the filtering approach, even though we are not able to perform any mathematical analysis in this setting. Section 5 is devoted to a special example of a hyperelastic non-convex energy. In Section 6, we show a limitation of the approach: it cannot be applied in its present state to the stochastic setting. Further developments are needed to address this latter case.

To conclude this introduction, let us emphasize that we address here the question of the explicit computation of the homogenized coefficient A^*, with a view to solving (1.4) and thus determine the approximation u^* of u_ε. This is typically the situation of interest when ε is very small, and/or if the fine-scale structure of the solution u_ε is not needed. To the best of our knowledge, very few papers have addressed the question examined here of the practical computation of the homogenized coefficient. Yurinskiĭ [28] studied the effect of truncation (a different type of truncation, actually) on the computation of homogenized coefficients in stochastic problems. Bourgeat and Piatnitskii [6, 7] studied a situation that, although still stochastic in nature, is closer to that considered in the present article. In [15], Gloria and Otto studied a similar situation, but with discrete operators. Byström, Dasht and Wall provided in [11] a numerical study of the truncation error. We will return to the analysis of such contributions and to the stochastic setting in a future publication.

2 Study of the filtered corrector problem

We study here, for R fixed, the filtered corrector problem:

$$\left\{ \begin{aligned} & -\operatorname{div}\left[\varphi_R\left(y\right)\left(A(y)\left(p+\nabla w_p\right)+\lambda\right)\right]=0, \quad \text{in} \quad Q_R, \\ & \int_{Q_R} \nabla w_R(y)\varphi_R\left(y\right)dy=0. \end{aligned} \right. \tag{2.1}$$

The matrix A is assumed to satisfy (1.2)-(1.3), and the filtering function φ and its rescaled version φ_R are defined by (1.9) and (1.10), respectively. In addition, we assume that φ satisfies:

$$\exists \delta > 0, \quad \begin{cases} \forall x \in \partial Q, \quad t \mapsto \varphi(xt) \text{ is decreasing on } (1-\delta, 1), \\ \inf_{x \in Q_{1-\delta}} \varphi(x) > 0. \end{cases} \tag{2.2}$$

Remark 2.1 *Assumption* (2.2) *is satisfied by any function of the product form*

$$\varphi(x) = \prod_{i=1}^{d} \varphi_0(x_i),$$

where φ_0 *is even, and decreasing on* $(0, 1/2)$. *The filtering functions we have used in the numerical tests of Subsections 3.3 and 4.2 are of this form.*

In order to study problem (2.1), we first introduce an appropriate functional setting. For any φ_R satisfying (1.9)-(1.10), we define the spaces

$$\mathcal{H}_R = \left\{ u : Q_R \longrightarrow \mathbb{R}, \quad u \text{ measurable}, \quad \int_{Q_R} u(y)^2 \varphi_R\,(y)\, dy < +\infty \right\},$$

$$\mathcal{H}_R^1 = \left\{ u \in \mathcal{H}_R, \quad \nabla u \in (\mathcal{H}_R)^d \right\}.$$

Elementary tools of analysis allow to prove:

Proposition 2.2 *Let* φ *satisfy* (1.9), *and let* φ_R *be defined by* (1.10). *Then,*

(i) the space $\mathcal{H}_R$ *is a Hilbert space for the scalar product*

$$\langle u \mid v \rangle = \int_{Q_R} u(y)\, v(y)\, \varphi_R\,(y)\, dy;$$

(ii) the space $\mathcal{H}_R^1$ *is a Hilbert space of the scalar product*

$$\langle\langle u \mid v \rangle\rangle = \int_{Q_R} u(y)\, v(y)\, \varphi_R\,(y)\, dy + \int_{Q_R} \nabla u(y) \cdot \nabla v(y)\, \varphi_R\,(y)\, dy.$$

Next, we have the following density property:

Lemma 2.3 *The space* $C^\infty(Q_R)$ *is dense in* $\mathcal{H}_R$ *and in* $\mathcal{H}_R^1$.

Proof: Consider $u \in \mathcal{H}_R$. Then, $v = u\sqrt{\varphi_R} \in L^2(Q_R)$. Hence, one can find $v_n \in \mathcal{D}(Q_R)$ such that

$$v_n \longrightarrow v \quad \text{in} \quad L^2(Q_R). \tag{2.3}$$

We then define $u_n = v_n/\sqrt{\varphi}_R$, which is in $C^\infty(Q_R)$. The convergence (2.3) is exactly equivalent to $u_n \longrightarrow u$ in $\mathcal{H}_R$.

Next, consider $u \in \mathcal{H}_R^1$. Here, the above strategy does not apply since $\nabla v \notin L^2(Q_R)$ *a priori*. Therefore, fixing an integer $n > 0$, we note that $u \in H^1(Q_{R-\frac{1}{n}})$. One can thus find $u_n \in C^\infty(Q_{R-\frac{1}{n}})$ such that

$$\|u - u_n\|_{H^1(Q_{R-\frac{1}{n}})} \leq \frac{1}{n}.$$

Next, we consider $\overline{u}_n$, an extension of u_n to Q_R such that $\|\overline{u}_n\|_{H^1(Q_R)} \leq C\|u_n\|_{H^1(Q_{R-\frac{1}{n}})}$, with C independent of n (see for instance [10] for the existence of such an extension). Then, we have:

$$
\begin{aligned}
\|\overline{u}_n - u\|^2_{\mathcal{H}^1_R} \leq\ & \|\varphi_R\|_{L^\infty}\|\overline{u}_n - u\|^2_{H^1(Q_{R-\frac{1}{n}})} \\
& +2\int_{Q_R\setminus Q_{R-\frac{1}{n}}} \varphi_R\left(u^2 + |\nabla u|^2\right) + 2\int_{Q_R\setminus Q_{R-\frac{1}{n}}} \varphi_R\left(\overline{u}_n^2 + |\nabla \overline{u}_n|^2\right) \\
\leq\ & \frac{\|\varphi_R\|_{L^\infty}}{n^2} + 2\int_{Q_R\setminus Q_{R-\frac{1}{n}}} \varphi_R\left(u^2 + |\nabla u|^2\right) \\
& +2C^2\|\varphi_R\|_{L^\infty(Q_R\setminus Q_{R-\frac{1}{n}})}\|u_n\|^2_{H^1(Q_{R-\frac{1}{n}})}
\end{aligned}
$$

Each term of the right hand side goes to zero as $n \to \infty$, hence $\overline{u}_n$ converges to u in $\mathcal{H}^1_R$. $\square$

In the sequel, we will also need the following Poincaré-type inequality:

Lemma 2.4 *Consider φ satisfying* (1.9)*, and φ_R defined by* (1.10)*. Assume in addition that φ satisfies* (2.2)*. Then, there exists a constant $C_{\varphi,R}$ depending only on φ and R such that*

$$
\forall u \in \mathcal{H}^1_R, \quad \int_{Q_R}\left(u - \int_{Q_R} u\,\varphi_R\right)^2 \varphi_R \leq C_{\varphi,R}\int_{Q_R}|\nabla u|^2\varphi_R. \tag{2.4}
$$

We postpone the proof of this result to the Appendix. A simple corollary of this inequality is then:

Lemma 2.5 *The space $\mathcal{H}^1_R/\mathbb{R}$ is a Hilbert space for the scalar product*

$$
(u \mid v) = \int_{Q_R} \nabla u(y)\cdot\nabla v(y)\,\varphi_R(y)\,dy.
$$

Next, we introduce the following variational formulation of (2.1):

Definition 2.6 *For any $p \in \mathbb{R}^d$, we say that $(w,\lambda) \in \left(\mathcal{H}^1_R/\mathbb{R}\right)\times\mathbb{R}^d$ is a weak solution to* (2.1) *if*

$$
\begin{aligned}
&\forall (v,\mu) \in \left(\mathcal{H}^1_R/\mathbb{R}\right)\times\mathbb{R}^d, \\
&\qquad \int_{Q_R}\nabla v^T A\nabla w\,\varphi_R + \int_{Q_R}\lambda^T\nabla v\,\varphi_R - \int_{Q_R}\mu^T\nabla w\,\varphi_R = 0. \qquad (2.5)
\end{aligned}
$$

Then, we have

Lemma 2.7 *The pair $(w,\lambda) \in \left(\mathcal{H}^1_R/\mathbb{R}\right)\times\mathbb{R}^d$ is a solution to* (2.1) *in the sense of distributions if and only if it satisfies* (2.5)*.*

Proof: Using Lemma 2.3, the proof is straightforward. $\square$

The well-posedness of (2.1) is now a direct consequence of Lemma 2.7 and of the following:

Proposition 2.8 *Let A satisfy* (1.2)-(1.3)*, and let φ_R be defined by* (1.10)*, with φ satisfying* (1.9)*. Then, problem* (2.5) *has a unique solution $(w, \lambda) \in \left(\mathcal{H}^1_R/\mathbb{R}\right) \times \mathbb{R}^d$.*

Proof: Directly applying the Lax-Milgram Lemma to

$$\mathcal{B}\big[(w,\lambda),(v,\mu)\big] = \int_{Q_R} \nabla v^T A \nabla w \, \varphi_R + \int_{Q_R} \lambda^T \nabla v \varphi_R - \int_{Q_R} \mu^T \nabla w \, \varphi_R$$

is not possible because the bilinear form $\mathcal{B}$ is not coercive in $\left(\mathcal{H}^1_R/\mathbb{R}\right) \times \mathbb{R}^d$. Indeed, $\mathcal{B}\left[(0,\lambda),(0,\lambda)\right] = 0$ for any $\lambda \in \mathbb{R}^d$. We thus proceed by approximation and define, for $\delta > 0$, $\mathcal{B}_\delta\big[(w,\lambda),(v,\mu)\big] = \mathcal{B}\big[(w,\lambda),(v,\mu)\big] + \delta\lambda^T\mu$. It is coercive in $\left(\mathcal{H}^1_R/\mathbb{R}\right) \times \mathbb{R}^d$. Applying the Lax-Milgram Lemma, we know there exists $(w_\delta, \lambda_\delta) \in \left(\mathcal{H}^1_R/\mathbb{R}\right) \times \mathbb{R}^d$ such that

$$\forall (v,\mu) \in \left(\mathcal{H}^1_R/\mathbb{R}\right) \times \mathbb{R}^d, \quad \mathcal{B}_\delta\big[(w_\delta,\lambda_\delta),(v,\mu)\big] = 0,$$

namely

$$\begin{aligned}&\forall (v,\mu) \in \left(\mathcal{H}^1_R/\mathbb{R}\right) \times \mathbb{R}^d,\\ &\quad \int_{Q_R} \nabla v^T A \nabla w_\delta \, \varphi_R + \int_{Q_R} \lambda_\delta^T \nabla v \, \varphi_R - \int_{Q_R} \mu^T \nabla w_\delta \, \varphi_R + \delta\lambda_\delta^T \mu = 0. \end{aligned} \tag{2.6}$$

We are now going to bound $(w_\delta, \lambda_\delta)$ independently of δ, which will eventually allow us to let δ vanish and recover (2.5). For this purpose, we use $u = w_\delta$ and $\mu = \lambda_\delta$ in (2.6), and obtain

$$\int \nabla w_\delta^T A \nabla w_\delta \, \varphi_R + \delta|\lambda_\delta|^2 = \int \nabla w_\delta^T A p \, \varphi_R.$$

Hence, we have $\|\nabla w_\delta\|_{(\mathcal{H}_R)^d} \leq C, \quad |\lambda_\delta| \leq \dfrac{C}{\sqrt{\delta}}$, where C is a constant independent of δ. Thus, extracting a subsequence if necessary, we have

$$w_\delta \rightharpoonup w \text{ in } \mathcal{H}^1_R/\mathbb{R}, \quad \delta\lambda_\delta \longrightarrow 0,$$

as $\delta \to 0$. Thus, all terms in (2.6) are bounded except $\lambda_\delta^T \int \nabla v \, \varphi_R$. Hence, this term is also bounded independently of δ, for any $v \in \mathcal{H}^1_R/\mathbb{R}$. As a consequence, λ_δ is bounded. We may thus pass to the limit in all terms of (2.6), up to extracting a subsequence, getting.

$$\lambda_\delta \longrightarrow \lambda,$$

as $\delta \to 0$. We may now pass to the limit in (2.6), and obtain a solution to (2.5).

The uniqueness is easily proved by using (w, λ) as a test function in (2.5): this implies that $\nabla w = 0$. Hence, $\lambda^T \int \nabla v \, \varphi_R = 0$, for any $v \in \mathcal{H}^\infty{}_R$, which implies $\lambda = 0$. $\square$

Remark 2.9 *As pointed out in the introduction, in the case of a symmetric matrix A, the filtered corrector problem may be written as a minimization problem: for any $p \in \mathbb{R}^d$,*

$$A_R^* \, p = \inf \left\{ \int_{Q_R} \varphi_R (p + \nabla w)^T A(p + \nabla w), \quad w \in \mathcal{H}_R^1, \quad \int_{Q_R} \varphi_R \nabla w = 0 \right\}.$$

Using this formulation, it is also possible, using the same functional spaces as above and standard techniques of the calculus of variations, to prove existence and uniqueness (up to the addition of a constant) of the corrector w.

3 The periodic case

We study in this Section the case of a periodic matrix A_ε. We first address the one-dimensional case, which is fully explicit, hence allows for a very simple proof. Subsection 3.2 gives a (formal) proof for higher dimensions, and Subsection 3.3 reports on some numerical computations. We emphasize that the simple periodic setting is seen as a preliminary step toward the more relevant quasi-periodic setting and other settings.

3.1 One-dimensional case

As is well known, problem (1.5) may be solved explicitly in dimension one. Indeed:

$$w_p'(y) = \frac{p}{\left(\int_Q A_{\mathrm{per}}^{-1}\right) A_{\mathrm{per}}(y)} - p = \frac{p}{\langle A_{\mathrm{per}}^{-1} \rangle A_{\mathrm{per}}(y)} - p.$$

Thus, we have

$$A^* = \left(\int_Q A_{\mathrm{per}}^{-1}(y) dy \right)^{-1}. \tag{3.1}$$

Likewise, for the filtered problem,

$$w_{p,R}'(y) = \frac{p}{\left(\int_{Q_R} \varphi_R\,(y)\, A_{\mathrm{per}}^{-1}(y) dy\right) A_{\mathrm{per}}(y)} - p,$$

and

$$A_R^* = \left(\int_{Q_R} A_{\mathrm{per}}^{-1}(y) \varphi_R\,(y)\, dy \right)^{-1}. \tag{3.2}$$

Hence, the approach of [12, 13] implies:

Proposition 3.1 *Assume that $d = 1$, and that $A_\varepsilon = A_{\mathrm{per}}(x/\varepsilon)$ satisfies* (1.2) *and* (1.3)*, for a 1-periodic (scalar) matrix A_{per}. Assume that φ satisfies* (1.9) *with $k \geq 1$. Then, A^* and A_R^* are given by* (3.1) *and* (3.2) *respectively, and satisfy* (1.15)*, namely:*

$$\exists C > 0, \quad |A_R^* - A^*| \leq \frac{C}{R^k}.$$

Proof: We already know that (3.1) and (3.2) hold. Then, for the sake of consistency, we reproduce the proof of [12, 13]: we denote by $\widehat{\varphi}$ the Fourier transform of φ, and compute, setting $b(y) = A_{\mathrm{per}}^{-1}(y)$,

$$\begin{aligned} \frac{1}{A_R^*} &= \int_{\mathbb{R}} b(y)\, \varphi_R\,(y)\, dy = \frac{1}{R} \left\langle \widehat{b}, \widehat{\varphi_R} \right\rangle \\ &= \left\langle \widehat{b}, \widehat{\varphi}(R\xi) \right\rangle = \sum_{j\in\mathbb{Z}} c_j(b) \widehat{\varphi}(jR), \end{aligned}$$

where $c_j(b) = \int_{-1/2}^{1/2} b(y) \exp(-2ij\pi y) dy$ is the j^{th} Fourier coefficient of b. Since $\varphi \in C^k$ and has compact support, we have

$$\forall \xi \in \mathbb{R} \setminus \{0\}, \quad |\widehat{\varphi}(\xi)| \leq \frac{\|\varphi\|_{C^k}}{(2\pi|\xi|)^k}. \tag{3.3}$$

Hence,

$$\begin{aligned} \left| \sum_{j\neq 0} c_j(b) \widehat{\varphi}(2j\pi R) \right| &\leq \frac{\|\varphi\|_{C^k}}{(2\pi R)^k} \sum_{j\neq 0} \frac{|c_j(b)|}{|j|^k} \\ &\leq \frac{\|\varphi\|_{C^k}}{(2\pi R)^k} \|b\|_{L^2(-\frac{1}{2},\frac{1}{2})} \left(\sum_{j\neq 0} \frac{1}{|j|^{2k}} \right)^{1/2}. \end{aligned}$$

Since $c_0(b) = 1/A^*$, we have

$$\left| \frac{1}{A_R^*} - \frac{1}{A^*} \right| \leq \frac{C}{R^k},$$

for some constant C depending on φ and A_{per}. Actually, one can use $C = \left(\|\varphi\|_{C^k} \|A_{\mathrm{per}}^{-1}\|_{L^2(-1/2,1/2)}\right) / \left(2^k \pi^{k-1} \sqrt{6}\right)$. Since both A_R^* and A^* are bounded away from 0, this concludes the proof. $\square$

3.2 Formal analysis in higher dimensions

From Subsection 3.1, we know that, at least in dimension 1, the filtered problem provides a more accurate approximation of A^* than a simple truncation, that is, the filtered problem with $\varphi = \mathbf{1}_Q$. We give in this section a formal calculation that extends the argument to higher dimensions. The argument is based on the two-scale expansion method [1, 17, 22]. It remains formal, and does not constitute a mathematical proof. We only check that the expansion we postulate (see (3.5)-(3.6) below) fits with the appropriate asymptotics and allows for an efficient approximation strategy. We do not actually prove the expansion holds true. We believe it is only a matter of technicality to complete such a proof, given the usual tools of two-scale convergence [1, 17, 22]. We do not pursue in this direction, and prefer to concentrate on more practical issues in the present article.

We write the filtered corrector problem as

$$\begin{cases} -\operatorname{div}\left[\varphi_R(y)\left(A_{\rm per}(y)\left(p+\nabla w_p\right)+\lambda\right)\right]=0, \quad \text{in} \quad Q_R, \\ \displaystyle\int_{Q_R} \nabla w_p(y)\,\varphi_R(y)dy = 0, \end{cases} \tag{3.4}$$

(where we recall $\varphi_R(y) = R^{-d}\varphi\left(\dfrac{y}{R}\right)$.) We assume that w_p satisfies the following Ansatz:

$$w_p(y) = w_p^0\left(\frac{y}{R},y\right) + \frac{1}{R}w_p^1\left(\frac{y}{R},y\right) + \frac{1}{R^2}w_p^2\left(\frac{y}{R},y\right) + \dots, \tag{3.5}$$

$$\lambda = \lambda^0 + \frac{1}{R}\lambda^1 + \frac{1}{R^2}\lambda^2 + \dots, \tag{3.6}$$

where for each $j \in \mathbb{N}$, $w_p^j = w_p^j(x,y)$ is such that $\nabla_y w_p^j$ is periodic with respect to y. We further assume that the variables $x = y/R$ and y are independent, as is standard in this type of arguments.

Using (3.5), we have

$$\begin{aligned} A_{\rm per}(y)(\nabla w_p + p) &= A_{\rm per}(y)(\nabla_y w_p^0 + p) + \frac{1}{R}A_{\rm per}(y)\left(\nabla_x w_p^0 + \nabla_y w_p^1\right) \\ &\quad + \frac{1}{R^2}A_{\rm per}(y)\left(\nabla_x w_p^0 + \nabla_y w_p^1\right) + \dots \end{aligned}$$

We insert this expansion and (3.6) in the first line of (3.4), and get:

$$\begin{aligned} 0 = &\operatorname{div}_y\left[\varphi(x)\Big(A_{\rm per}(y)\left(\nabla_y w_p^0 + p\right) - \lambda^0\Big)\right] \\ &+\frac{1}{R}\Bigg[\operatorname{div}_x\left[\varphi(x)\Big(A_{\rm per}(y)\left(\nabla_y w_p^0 + p\right) - \lambda^0\Big)\right] \\ &\quad + \frac{1}{R}\operatorname{div}_y\left[\varphi(x)\Big(A_{\rm per}(y)\left(\nabla_x w_p^0 + \nabla_y w_p^1\right) - \lambda^1\Big)\right]\Bigg] \\ &+\frac{1}{R^2}\Bigg[\operatorname{div}_x\left[\varphi(x)\Big(A_{\rm per}(y)\left(\nabla_x w_p^0 + \nabla_y w_p^1\right) - \lambda^1\Big)\right] \\ &\quad + \frac{1}{R^2}\operatorname{div}_y\left[\varphi(x)\Big(A_{\rm per}(y)\left(\nabla_x w_p^1 + \nabla_y w_p^2\right) - \lambda^2\Big)\right]\Bigg] \\ &+ \dots \end{aligned} \tag{3.7}$$

We now successively equate all the coefficients of expansion (3.7) to zero. To begin with, we study the term of order 1:

$$0 = \operatorname{div}_y\left[A_{\rm per}(y)\left(\nabla_y w_p^0 + p\right) - \lambda^0\right] = \operatorname{div}_y\left[A_{\rm per}(y)\left(\nabla_y w_p^0 + p\right)\right].$$

The periodic corrector problem being uniquely solvable (up to the addition of a constant), we obtain

$$w_p^0(x,y) = w_p^{\rm per}(y) + v^0(x), \tag{3.8}$$

where $w_p^{\rm per}$ is the solution to (1.5) and v^0 is a function depending only on x.

We next study the term of order $1/R$ in (3.7). Integrating it with respect to y on Q yields

$$\operatorname{div}_x \left[\varphi(x) \int_Q A_{\rm per}(y)(\nabla_y w_p^0 + p) - \lambda^0 \, \varphi(x)\right] = 0.$$

Using (3.8) and the definition of A^*, we obtain $\operatorname{div}_x \left[\varphi(x) A^* p - \lambda^0 \, \varphi(x)\right] = 0$, whence $\nabla\varphi(x)^T (A^* p - \lambda_0) = 0$. Since this is valid for any $x \in Q$,

$$\lambda^0 = A^* p.$$

Inserting this equality in the term of order $1/R$ of (3.7), we obtain

$$\begin{aligned} -\operatorname{div}_y &\left[A_{\rm per}(y)\left(\nabla_y w_p^1 + \nabla_x v^0(x)\right)\right] \\ &= \frac{1}{\varphi(x)} \nabla\varphi(x)^T \left[A_{\rm per}(y)\left(\nabla_y w_p^{\rm per}(y) + p\right) - A^* p\right]. \end{aligned}$$

Therefore,

$$w_p^1(x,y) = w_{\nabla v^0(x)}^{\rm per}(y) + v^1(x,y), \tag{3.9}$$

where $w_{\nabla v^0(x)}^{\rm per}(y)$ is the solution to the periodic corrector problem (1.5) with $p = \nabla v^0(x)$, and v^1 is the unique solution (up to the addition of a function depending only on x) to

$$-\operatorname{div}_y \left[A_{\rm per}(y)\nabla_y v^1\right] = \frac{1}{\varphi(x)} \nabla\varphi(x)^T \left[A_{\rm per}(y)\left(\nabla_y w_p^{\rm per}(y) + p\right) - A^* p\right], \tag{3.10}$$

with periodic boundary conditions on $\nabla_y v^1$. In view of (3.10), its solution v^1 is of the form

$$v^1(x,y) = \frac{1}{\varphi(x)} \nabla\varphi(x)^T B(y) p + v^2(x), \tag{3.11}$$

where v^2 depends only on x, and B is a square matrix and does not depend on x. Indeed, the right-hand side of (3.10) is of the form $(\varphi(x))^{-1}\nabla\varphi(x) C(y) p$, where C is a matrix that does not depend on x (recall that ∇w_p is a linear function of p).

We now turn to the term of order $1/R^2$ in (3.7), and here again integrate on Q with respect to y. We find:

$$\operatorname{div}_x \left[\varphi(x) \int_Q A_{\rm per}(y)\left(\nabla_y w_p^1(x,y) + \nabla_x v^0(x)\right) dy - \lambda^1 \, \varphi(x)\right] = 0.$$

Since w_p^1 satisfies (3.9), we use here again the definition of A^*, and derive

$$\operatorname{div}_x \left[\varphi(x) A^* \nabla v^0(x) + \varphi(x) \int_Q A_{\mathrm{per}}(y) \nabla_y v^1(x,y) dy - \lambda^1 \varphi(x)\right] = 0, \quad (3.12)$$

where $\nabla_y v^1$ is uniquely defined by (3.10). The constant λ^1 is a Lagrange multiplier associated to the constraint satisfied by v^0. We now determine both v^0 and λ^1. To this end, we insert expansion (3.5) into the second line of (3.4), and get

$$\begin{aligned} 0 &= \int \nabla w_p(y)\, \varphi_R(y) dy \\ &= \int \nabla_y w_p^0\left(\frac{y}{R}, y\right) \varphi_R(y) dy \\ &\quad + \frac{1}{R} \int \left[\nabla_x w_p^0\left(\frac{y}{R}, y\right) + \nabla_y w_p^1\left(\frac{y}{R}, y\right)\right] \varphi_R(y) dy \\ &\quad + \frac{1}{R^2} \int \left[\nabla_x w_p^1\left(\frac{y}{R}, y\right) + \nabla_y w_p^2\left(\frac{y}{R}, y\right)\right] \varphi_R(y) dy + \dots \end{aligned}$$

Using (3.8) and (3.9), we thus have

$$\begin{aligned} 0 &= \int \nabla_y w_p^{\mathrm{per}}(y)\, \varphi_R(y) dy \\ &\quad + \frac{1}{R} \int \left[\nabla v^0\left(\frac{y}{R}\right) + \nabla w^{\mathrm{per}}_{\nabla v^0\left(\frac{y}{R}\right)}(y) + \nabla_y v^1\left(\frac{y}{R}, y\right)\right] \varphi_R(y) dy \\ &\quad + O\left(\frac{1}{R^2}\right). \end{aligned} \quad (3.13)$$

According to the results of Subsection 3.1, we know that

$$\int \nabla w_p^{\mathrm{per}}(y)\, \varphi_R(y) dy = \int_Q \nabla w_p^{\mathrm{per}}(y) dy + O\left(\frac{1}{R^k}\right) = O\left(\frac{1}{R^k}\right).$$

For the second term of (3.13), we remark

$$\int \nabla v^0\left(\frac{y}{R}\right) \varphi_R(y) dy = \int \nabla v^0(x)\, \varphi(x) dx,$$

for all R, while

$$\begin{aligned} \int \nabla_y w^{\mathrm{per}}_{\nabla v^0\left(\frac{y}{R}\right)}(y)\, \varphi_R(y) dy &= \int \nabla w^{\mathrm{per}}_{\nabla v^0(x)}(Rx)\, \varphi(x) dx, \\ &\overset{R \longrightarrow +\infty}{\longrightarrow} \int \left(\int_Q \nabla w^{\mathrm{per}}_{\nabla v^0(x)}(y) dy\right) \varphi(x) dx = 0. \end{aligned}$$

Similarly,

$$\frac{1}{R} \int \nabla_y v^1\left(\frac{y}{R}, y\right) \varphi_R(y) dy \overset{R \longrightarrow +\infty}{\longrightarrow} 0.$$

Collecting the three terms and inserting the information in (3.13), we obtain

$$\int \nabla v^0(x)\,\varphi(x)dx = 0. \tag{3.14}$$

The function v^0 therefore solves (3.12)-(3.14), namely:

$$\begin{cases} \operatorname{div}_x \left[\varphi(x)A^*\nabla v^0(x) + \varphi(x)\displaystyle\int_Q A_{\mathrm{per}}(y)\nabla_y v^1(x,y)dy - \lambda^1\,\varphi(x)\right] = 0, \\ \displaystyle\int \nabla v^0(x)\,\varphi(x)dx = 0, \end{cases} \tag{3.15}$$

for the Lagrange multiplier λ^1 (still to be determined at this stage).

The expansion (3.8) of the corrector now allows us to correspondingly expand the homogenized coefficient:

$$\begin{aligned} A_R^* p &= \int A_{\mathrm{per}}(y)\,(\nabla w_p(y) + p)\,\varphi_R(y)dy \\ &= \int A_{\mathrm{per}}(y)\,\left(\nabla w_p^{\mathrm{per}}(y) + p\right)\varphi_R(y)dy \\ &\quad + \frac{1}{R}\int A_{\mathrm{per}}(y)\left[\nabla v^0\left(\frac{y}{R}\right) + \nabla w^{\mathrm{per}}_{\nabla v^0\left(\frac{y}{R}\right)}(y) + \nabla_y v^1\left(\frac{y}{R}, y\right)\right]\varphi_R(y)dy \\ &\quad + O\left(\frac{1}{R^2}\right) \end{aligned} \tag{3.16}$$

Here again, the results of Subsection 3.1 imply

$$\begin{aligned} \int A_{\mathrm{per}}(y)\,(\nabla w_p(y) + p)\,\varphi_R(y)dy &= \int_Q A_{\mathrm{per}}(y)\left(\nabla w_p^{\mathrm{per}}(y) + p\right)dy \\ &\quad + O\left(\frac{1}{R^k}\right) \\ &= A^* p + O\left(\frac{1}{R^k}\right). \end{aligned} \tag{3.17}$$

In addition, we have

$$\begin{aligned} &\int A_{\mathrm{per}}(y)\left[\nabla v^0\left(\frac{y}{R}\right) + \nabla w^{\mathrm{per}}_{\nabla v^0\left(\frac{y}{R}\right)}(y)\right]\varphi_R(y)dy = \\ &\qquad \int\int_Q A_{\mathrm{per}}(y)\left[\nabla v^0(x) + \nabla w^{\mathrm{per}}_{\nabla v^0(x)}(y)\right]dy\,\varphi(x)dx + O\left(\frac{1}{R}\right) \\ &\qquad\qquad = \int A^*\nabla v^0(x)\,\varphi(x)dx + O\left(\frac{1}{R}\right) = O\left(\frac{1}{R}\right). \end{aligned} \tag{3.18}$$

Considering (3.11), we obtain

$$\int A_{\mathrm{per}}(y)\nabla_y v^1\left(\frac{y}{R},y\right)\varphi_R(y)dy$$
$$= \int\int_Q A_{\mathrm{per}}(y)\nabla_y v^1(x,y)\,dy\,\varphi(x)dx + O\left(\frac{1}{R}\right)$$
$$= \int\int_Q A_{\mathrm{per}}(y)\nabla_y\left[\nabla\varphi(x)^T B(y)p\right]dydx + O\left(\frac{1}{R}\right) = O\left(\frac{1}{R}\right). \quad (3.19)$$

Collecting (3.16), (3.17), (3.18) and (3.19), we obtain (1.16), that is, if $k \geq 2$,

$$A_R^* p = A^* p + O\left(\frac{1}{R^2}\right).$$

As said above, although the above manipulations formally allow for the determination of the terms of lowest order, they do not *prove* that the expansion (3.5)-(3.6) is actually correct. It is however immediate to see that this expansion is correct in one dimension. Indeed, recall that in 1D, the solution w_p^{per} of (1.5) reads

$$\frac{dw_p^{\mathrm{per}}}{dy}(y) = \frac{1}{A_{\mathrm{per}}(y)}A^*p - p. \quad (3.20)$$

On the other hand, equations (3.4), (3.10) (with periodic boundary conditions on $\nabla_y v^1$) and (3.15) respectively write

$$\begin{cases} -\dfrac{d}{dy}\left[\varphi_R(y)\left(A_{\mathrm{per}}(y)\left(p+w_p'\right)+\lambda\right)\right] = 0, \quad \text{in} \quad Q_R, \\ \displaystyle\int_{Q_R} w_p'(y)\,\varphi_R(y)dy = 0, \end{cases} \quad (3.21)$$

$$-\frac{\partial}{\partial y}\left[A_{\mathrm{per}}(y)\frac{\partial v^1}{\partial y}\right] = \frac{\varphi'(x)}{\varphi(x)}\left[A_{\mathrm{per}}(y)\left(\frac{dw_p^{\mathrm{per}}}{dy}(y)+p\right) - A^*p\right], \quad (3.22)$$

$$\begin{cases} \dfrac{d}{dx}\left[\varphi(x)A^*\dfrac{dv^0}{dx}(x) + \varphi(x)\displaystyle\int_Q A_{\mathrm{per}}(y)\frac{\partial v^1}{\partial y}(x,y)dy - \lambda^1\,\varphi(x)\right] = 0, \\ \displaystyle\int \frac{dv^0}{dx}(x)\,\varphi(x)dx = 0. \end{cases} \quad (3.23)$$

The solution of (3.21) writes

$$w_p'(y) = \frac{1}{A_{\mathrm{per}}(y)}\left(\int_Q \varphi_R\left(A_{\mathrm{per}}(z)\right)^{-1}dz\right)^{-1} p - p,$$

$$\lambda = -\left(\int_Q \varphi_R\left(A_{\mathrm{per}}(z)\right)^{-1}dz\right)^{-1} p.$$

Next, because of the explicit value (3.20) of $w_p^{\rm per}$, (3.22) becomes

$$-\frac{\partial}{\partial y}\left[A_{\rm per}(y)\frac{\partial v^1}{\partial y}\right] = 0. \tag{3.24}$$

The solution of (3.24) is

$$\frac{\partial v^1}{\partial y}(x,y) = 0.$$

We then insert this into (3.23), finding

$$\frac{dv^0}{dx}(x) = 0, \quad \lambda^1 = 0.$$

Hence,

$$w_p^0(x,y) = w_p^{\rm per}(y), \quad \lambda^0 = A^* p,$$
$$w_p^1(x,y) = 0, \quad \lambda^1 = 0,$$

up to the addition of a constant to w_p^0 and w_p^1. Thus, we have

$$w_p - w_p^0 - \frac{1}{R}w_p^1 = O\left(\frac{1}{R^k}\right) = O\left(\frac{1}{R^2}\right),$$

$$\lambda - \lambda^0 - \frac{1}{R}\lambda^1 = O\left(\frac{1}{R^k}\right) = O\left(\frac{1}{R^2}\right).$$

We conclude that the expansion (3.5)-(3.6) holds, at least up to order $1/R^2$, with $\frac{\partial w_p^j}{\partial y}$ periodic.

Remark 3.2 *With a view to addressing more general situations, it is interesting to note that the above computation shows the rate of convergence* (1.16) *provided:*

- *the filtering function is such that, for any periodic function g, we have $\int g(y)\,\varphi_R(y)dy = \langle g\rangle + O\left(\frac{1}{R^2}\right)$, and*
- *the problems considered within the calculations are well-posed (like the corrector problem itself and* (3.12)*).*

Remark 3.3 *In contrast to the one-dimensional situation, increasing the order of the filter does not seem to bring any additional speed-up of the convergence in higher dimensions. The rate of convergence remains of order $\dfrac{1}{R^2}$. We believe this owes to the presence of a residual error in the two-scale expansion. This error is absent in the one-dimensional setting where all calculations can be made explicit, and where, in fact, the value of the corrector itself factors out of the computation of the homogenized coefficient. In dimensions higher than or equal to two, the corrector problem is to be solved with only approximate boundary conditions, and this impacts the final result, irrespective of the order of the filter used.*

3.3 Numerical experiments

We present in this subsection some numerical experiments that confirm estimate (1.16).

Define

$$A_{\mathrm{per}}(y_1, y_2) = \left(\frac{2 + 1.8\sin(2\pi y_1)}{2 + 1.8\cos(2\pi y_2)} + \frac{2 + sin(2\pi y_2)}{2 + 1.8\cos(2\pi y_1)} \right) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We first compute a reference approximation of A^* using the periodic cell problem (1.5) and a finite element discretization. For this purpose, we use a mesh that is sufficiently fine to ensure that the computation has accuracy of the order 10^{-4}. This accuracy is typically guaranteed by standard *a priori* error analysis on Galerkin methods, since the matrix A_{per} is C^∞. The value obtained with this fine mesh calculation is used as a reference for the other computations. All relative errors mentioned throughout the section are computed with this value as a reference.

We now assume we know that A_{per} is periodic but we *pretend* not to know what its actual period is. This is of course a surrogate for a more complex problem, not necessarily periodic, where indeed we would not know what size of domain is appropriate to truncate the corrector problem. Next, we compute several approximations of A^* following three different numerical approaches. For R varying in the range $[1, 8]$,

- [i] we first compute the truncated corrector problem on a box of size R and compute the homogenized matrix as an average over this box of size R. This corresponds to the filtered computation (2.1) with a filtering function $\varphi \equiv 1$.
- [ii] we next apply an *oversampling* technique: we first solve the corrector problem in a box of size R, and then compute an approximation of A^* with an average over the box of size $R/2$.
- [iii] we finally use the filtering strategy, with a C^2 filter φ, namely $\varphi(x_1, x_2) = \varphi_0(x_1)\,\varphi_0(x_2)$, where

$$\varphi_0(t) = \begin{cases} t^2 & \text{if } 0 < t < 1/3, \\ -1/3 + 2t - 2t^2 & \text{if } 1/3 < t < 2/3, \\ (t-1)^2 & \text{if } 2/3 < t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

We plot on Figure 1 the relative numerical error for Approaches [i] and [iii] (relative to the periodic, fine mesh computation, we recall). The abscissa is the size of the simulation box $R \in [1, 8]$. The simplest computation (no filtering, that is, Approach [i] above, dashed curve of the figure) is exact when R is a multiple of the period of A_{per} (here, an integer). In such a case the problem solved is in fact equivalent to the cell problem (1.5). In contrast, when R is *not*

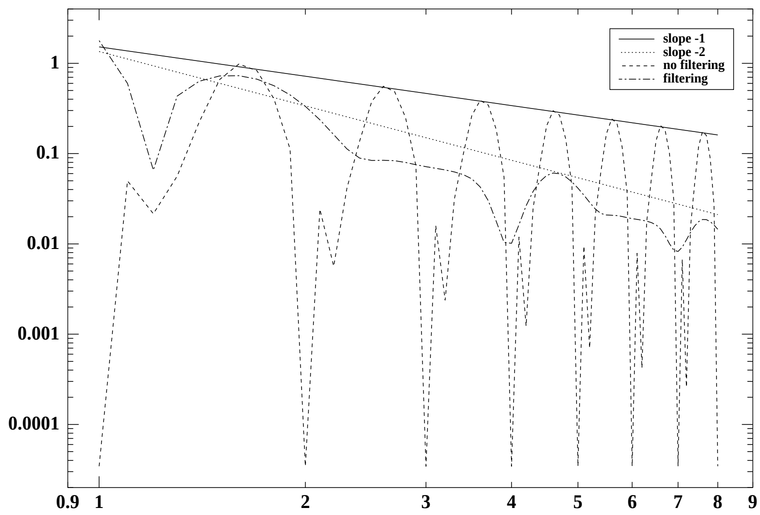

Figure 1: Numerical relative error (in log-log scale) on the homogenized coefficient in function of the box size, depending on the numerical approach: comparison of the unfiltered, direct approach [i] (dashed curve) and the filtering approach [iii] (dot-dashed curve). See the body of text for more details.

a multiple of the period, the accuracy of Approach [i] is poor, and scales, in a very oscillatory fashion, like $\frac{1}{R}$. On the other hand, the accuracy of the filtered approach [iii], that is the light dot-dashed line on the figure, is much better. It exhibits a convergence of higher order (although of course the calculation is not exact, as the unfiltered approach is, for integer values of R). The two straight lines of slope -1 and -2 represent in log scale multiples of the functions $1/R$ and $1/R^2$ respectively. This shows that the filtered strategy has, overall, an order of precision one order higher than the unfiltered strategy. In addition, the fact that the result is much less oscillatory as R varies makes easier extrapolation techniques (although those must be used cautiously).

For the sake of comparison, we plot on Figure 2 the numerical error for Approaches [i] and [ii]. The oversampling approach does not seemingly bring much of a better rate of convergence, at least in the present case.

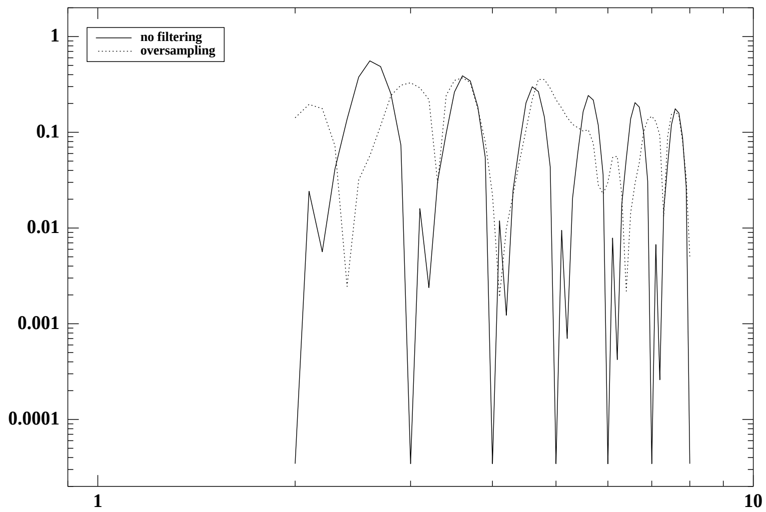

Figure 2: Numerical error (in log-log scale) on the homogenized coefficient as a function of the box size, depending on the numerical approach: comparison of the unfiltered approach [i] (solid curve) and the oversampling approach [ii] (dashed curve)

4 The quasi-periodic case

We now turn to the quasi-periodic case. For consistency, we first recall the definitions and basic properties of *quasi-periodic* and *almost periodic* functions. They are borrowed from the two classical monographs [4, 5] and from the references [9, 29], where many other details may be found. The reader familiar with these notion may directly proceed to Section 4.1.

Definition 4.1 *A function* $f \in C^0(\mathbb{R}^d)$ *is said* quasi-periodic *if there exists* $m \in \mathbb{N}$ *and a periodic function* $F \in C^0(\mathbb{R}^{dm})$ *such that*

$$f(x) = F(x, x, \ldots, x).$$

It is well-known that the set of quasi-periodic functions is an algebra (any (finite) linear combination and product of quasi-periodic functions is a quasi-periodic function), and that any quasi-periodic function has a mean value defined by

$$\langle f \rangle = \lim_{R \to +\infty} \frac{1}{|B_R|} \int_{B_R(x)} f(y) dy,$$

where $B_R(x)$ is the ball of radius R of center x. Note that this convergence is uniform with respect to x, hence we also have the convergence of the rescaled functions

$$f\left(\frac{y}{\varepsilon}\right) \xrightarrow[\varepsilon \to 0]{*} \langle f \rangle,$$

in L^∞.

A few well-known examples of quasi-periodic functions are:

- *trigonometric polynomials:* any function f of the form

$$f(x) = \mathrm{Re} \left(\sum_{k=1}^{K} c_k e^{i\omega_k \cdot x} \right),$$

  where $\omega_k \in \mathbb{R}^d$ and $c_k \in \mathbb{C}$, is called a trigonometric polynomial and is quasi-periodic.

- *sums and products of periodic functions:* if f and g are periodic functions, then $f + g$ and fg are quasi-periodic functions.

- *composition of periodic functions:* if g is periodic, and if ϕ is a diffeomorphism such that $\nabla \phi$ is periodic, then $g \circ \phi$ is quasi-periodic.

Note that the space of quasi-periodic functions is not closed for the norm $\| \cdot \|_{C^0(\mathbb{R}^d)}$. It is therefore natural to introduce the notion of almost periodic functions:

Definition 4.2 *The space of almost periodic functions in the sense of Bohr is the closure of the space of quasi-periodic functions for the norm* $\| \cdot \|_{C^0(\mathbb{R}^d)}$.

Other definitions of almost periodic functions exist in the literature. They depend on the specific norm chosen. The most commonly known variants of the above definition are the *Stepanov almost periodic functions* [24], the *Weyl almost periodic functions* [26], and the *Besikovitch almost periodic functions* [4, 29].

In the sequel, we assume that $A = A_{\mathrm{q-per}}$ is quasi-periodic in the sense of Definition 4.1 and will manipulate almost periodic functions in the sense of Definition 4.2.

4.1 One-dimensional case

As in the periodic case, the corrector problem can be explicitly solved in dimension one. It is immediate to see that its solution reads

$$w_p'(y) = \frac{p}{\left(\int_Q A_{\mathrm{q-per}}^{-1}\right) A_{\mathrm{q-per}}(y)} - p = \frac{p}{\langle A_{\mathrm{q-per}}^{-1}\rangle A_{\mathrm{q-per}}(y)} - p,$$

and thus that A^* admits an explicit expression (3.1):

$$A^* = \langle A_{\mathrm{q-per}}^{-1}\rangle, \tag{4.1}$$

similar to (3.1). Likewise, the filtered problem may be solved explicitly:

$$w_{p,R}'(y) = \frac{p}{\left(\int_{Q_R} \varphi_R\,(y)\, A_{\mathrm{q-per}}^{-1}(y)dy\right) A_{\mathrm{q-per}}(y)} - p,$$

and

$$A_R^* = \left(\int_{Q_R} A_{\mathrm{q-per}}^{-1}(y)\,\varphi_R\,(y)\,dy\right)^{-1}. \tag{4.2}$$

We then have

Proposition 4.3 *Assume that $d=1$, and that φ satisfies* (1.9) *with $k \geq 1$. Let $A_\varepsilon = A_{\mathrm{q-per}}(x/\varepsilon)$ satisfy* (1.2) *and* (1.3), *with $A_{\mathrm{q-per}}(x) = F(x,\ldots,x)$, where F is periodic in $\mathbb{R}^m$ and $F \in H^s_{\mathrm{loc}}(\mathbb{R}^m)$ for some $s > k + (m+1)/2$. Then, A^* and A_R^* are given by* (4.1) *and* (4.2) *respectively, and satisfy* (1.15), *namely:*

$$\exists C > 0, \quad |A_R^* - A^*| \leq \frac{C}{R^k}.$$

Proof: Note that, according to the results of [9], F may be chosen such that $F \geq a > 0$ for some constant a. Hence, $B = 1/F$ is in $H^s_{\mathrm{loc}}(\mathbb{R}^m)$, and

$$(A_{\mathrm{q-per}}(x))^{-1} = B(x,\ldots,x).$$

Without loss of generality, it is possible to assume that B is periodic of period T_i in its i^{th} variable, with $(T_1,\ldots,T_m)$ linearly independent on $\mathbb{Z}$ (see [9, 16]). Setting

$$\omega = \left(\frac{1}{T_1},\ldots,\frac{1}{T_m}\right),$$

we have

$$(A_{\mathrm{q-per}}(x))^{-1} = \sum_{j\in\mathbb{Z}^m} c_j e^{2i\pi(\omega\cdot j)x},$$

where the c_j are the Fourier coefficients of B. Note that $c_0 = \langle A_{\mathrm{q-per}}^{-1}\rangle$. Moreover, since the periods T_i are linearly independent on $\mathbb{Z}$, we have

$$\exists C > 0, \quad \forall \eta > 0, \quad (|j\cdot\omega| \leq \eta,\ j \neq 0) \Rightarrow |j| \geq \frac{C}{\eta}.$$

Thus, we have

$$\begin{aligned}
\frac{1}{A_R^*} - \frac{1}{A^*} &= \sum_{j \neq 0} c_j \int_{\mathbb{R}} e^{2i\pi(\omega \cdot j)x} \frac{1}{R} \varphi\left(\frac{x}{R}\right) dx \\
&= \sum_{j \neq 0} c_j \widehat{\varphi}(Rj \cdot \omega).
\end{aligned}$$

Here again, we use the fact that φ is C^k, hence satisfies (3.3). Since we also have $|\widehat{\varphi}(\xi)| \leq \|\varphi\|_{C^0} \leq \|\varphi\|_{C^k}$, we infer

$$|\widehat{\varphi}(\xi)| \leq \|\varphi\|_{C^k} \frac{2}{1 + (2\pi|\xi|)^k},$$

whence

$$\left| \frac{1}{A_R^*} - \frac{1}{A^*} \right| \leq 2\|\varphi\|_{C^k} \sum_{j \neq 0} \frac{|c_j|}{1 + (2\pi R)^k |j \cdot \omega|^k}.$$

We estimate the sum over j as follows:

$$\begin{aligned}
\sum_{j \neq 0} \frac{|c_j|}{1 + (2\pi R)^k |j \cdot \omega|^k} &= \sum_{l \geq 0} \left(\sum_{\frac{1}{l+1} \leq |j \cdot \omega| \leq \frac{1}{l}} \frac{|c_j|}{1 + (2\pi R)^k |j \cdot \omega|^k} \right) \\
&\leq \left(\sum_{l \geq 0} \sum_{\frac{1}{l+1} \leq |j \cdot \omega| \leq \frac{1}{l}} |j|^{2s} |c_j|^2 \right)^{1/2} \\
&\quad \times \left(\sum_{l \geq 0} \sum_{\frac{1}{l+1} \leq |j \cdot \omega| \leq \frac{1}{l}} \frac{1}{|j|^{2s} (1 + (2\pi R)^k |j \cdot \omega|^k)^2} \right)^{1/2} \\
&\leq \frac{\|B\|_{H^s}}{(2\pi R)^k} \left(\sum_{l \geq 0} (l+1)^{2k} \sum_{|j| \geq \frac{C}{l}} \frac{1}{|j|^{2s}} \right)^{1/2} \\
&\leq \frac{C}{R^k} \left(\sum_{l \geq 0} (l+1)^{2k} \sum_{|j| \geq \frac{C}{l}} \frac{1}{|j|^{2s}} \right)^{1/2} \\
&\leq \frac{C}{R^k} \left(\sum_{l \geq 0} \frac{(l+1)^{2k}}{l^{2s-m}} \right)^{1/2}.
\end{aligned}$$

Since $2k - 2s + m < -1$, the right-hand side is finite. $\square$

Remark 4.4 *Proposition 4.3 is not* in general *valid for almost-periodic coefficients $A_\varepsilon(x) = A_{\text{q-per}}(x/\varepsilon)$. Indeed, consider for instance*

$$\frac{1}{A_{\text{q-per}}(x)} = B_0 + \sum_{j \neq 0} \frac{1}{j^p} e^{i\frac{x}{j}} = A_0^{-1} + 2 \sum_{j \geq 1} \frac{1}{j^p} \cos\left(\frac{x}{j}\right),$$

with $p > 1$, and $B_0 > 0$ sufficiently large to have (1.3)*. Then, it is clear that* (1.2) *is satisfied, but using the same computation as in the proof of Proposition 4.3, we find*

$$\frac{1}{A_R^*} - \frac{1}{A^*} = \sum_{j \neq 0} \frac{1}{j^p} \widehat{\varphi}(\frac{R}{j}) = \frac{1}{R^p} \sum_{j \neq 0} \frac{R^p}{j^p} \widehat{\varphi}\left(\frac{R}{j}\right).$$

The latter sum, a Riemann sum, converges to

$$\int_{\mathbb{R}} \frac{1}{t^p} \widehat{\varphi}\left(\frac{1}{t}\right) dt = \int_{\mathbb{R}} s^{p-2} \widehat{\varphi}(s) ds,$$

provided $1 < p < k+1$*. Hence,*

$$\frac{1}{A_R^*} - \frac{1}{A^*} = \frac{1}{R^p} \int_{\mathbb{R}} s^{p-2} \widehat{\varphi}(s) ds + o\left(\frac{1}{R^p}\right).$$

This rate of convergence does not depend on k, but only on p, which may be chosen arbitrarily close to 1*.*

Let us conclude this section with some remarks on the higher dimensional cases. We expect the formal computation of the periodic case (Subsection 3.2) to hold in the quasi-periodic case, provided appropriate regularity is assumed (similarly to the setting of Proposition 4.3). The reason for this has just been pointed out. It is easily seen that, in addition to the Ansatz, the key ingredients are

1. the filtering function improves the convergence of averages. This is the purpose of Proposition 4.3;
2. the averages of the gradients of the functions considered vanish;
3. elliptic problems similar in type to the corrector problem with (quasi-) periodic conditions are well-posed.

Since all these three ingredients are valid in the quasi-periodic case, we believe, although we did not check the calculations in detail, that the formal argument of Subsection 3.2 applies.

4.2 Numerical experiments

As in the periodic case, we now give some numerical results. Define

$$A_{\mathrm{q-per}}(y_1, y_2) = \begin{pmatrix} \begin{array}{c} 4 + \cos(2\pi(y_1+y_2)) \\ + \cos(2\pi\sqrt{2}(y_1+y_2)) \end{array} & 0 \\ & \\ 0 & \begin{array}{c} 6 + \sin(2\pi y_1)^2 \\ + \sin(2\pi\sqrt{2} y_1)^2 \end{array} \end{pmatrix}.$$

We compute the homogenized coefficient $A^*_{\mathrm{q-per}}$ after solving the corrector problem on a truncated domain of size R. We use two different approaches:

- [i] direct, unfiltered approach: we simply solve the corrector problem on a box of size R, with Neumann boundary conditions (other boundary conditions yield qualitatively the same conclusions),
- [ii] filtered approach: we use a second-order filter, as defined in (3.25).

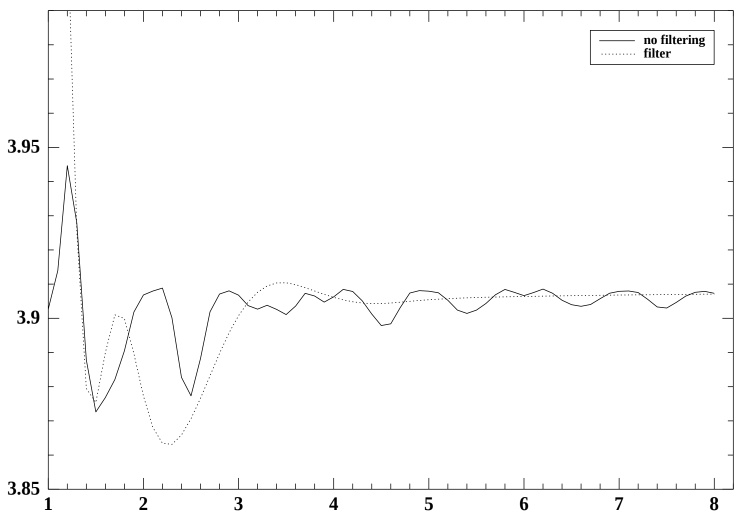

Figure 3: Value of the approximate homogenized coefficient as a function of the box size: unfiltered approach (solid line) and filtered approach (dotted line)

The results obtained are plotted on Figure 3. We display the value of the approximate homogenized coefficient as a function of the box size. The filtered approach converges faster. Recall that in the quasi periodic setting, it is not possible to calculate analytically (since it requires in principle to solve the problem on the entire space) the exact value of the homogenized coefficient. From these results, we therefore deduce an "approximate-exact" homogenized coefficient, using the average of the values given by the filtered problem over a range of sizes between 5 and 8. This is the reference value used in Figure 4 to plot the corresponding relative errors as the box size grows.

5 A nonlinear non-convex example

We give in this section an example of application of our filtering approach to a nonlinear quasiconvex (but non-convex) case. This example is borrowed from [19], and is a two-dimensional hyperelastic problem. We set

$$\begin{array}{rccl} W: & \mathbb{R}^2 \times \mathbb{R}^{2\times 2} & \longrightarrow & \mathbb{R} \\ & (y, A) & \longmapsto & W(y, A), \end{array}$$

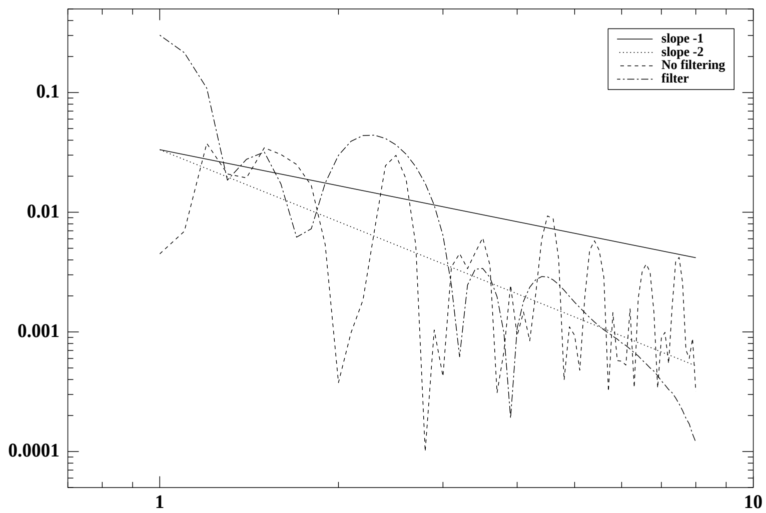

Figure 4: Numerical error (in log-log scale) on the homogenized coefficient as a function of the box size, depending on the numerical approach: comparison of the unfiltered problem (dashed curve) and the filtered problem (dot-dashed curve). See the body of text for more details.

with

$$W(y,A) = J(y)\left[\mathrm{trace}(A^TA)^2 + h(\det(A))\right], \tag{5.1}$$

where

$$J(y) = \mathbf{1}_{(0,1/2)\times(0,1)}(y) + \alpha\mathbf{1}_{(1/2,1)\times(0,1)}(y), \quad \text{and } J \text{ is } \mathbb{Z}^2\text{-periodic}, \tag{5.2}$$

and $\alpha > 0$. The function h is defined by

$$h(\delta) = \begin{cases} \dfrac{8(1+a)^2}{\delta + a} - 8(1+a) - 4 & \text{if } \delta > 0, \\ \dfrac{8(1+a)^2}{a} - 8(1+a) - 4 - \dfrac{8(1+a)^2}{a^2}\delta & \text{if } \delta \leq 0. \end{cases}$$

Since the function h is convex and $J > 0$, it is clear that W is polyconvex, hence quasiconvex. The following lemma is proved in [19]:

Lemma 5.1 (S. Müller, [19]) *Let W be defined by* (5.1)*, and consider*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix},$$

with $\pi/4 < d < 1$. Then, if α in (5.2) *is small enough,*

$$W^{\mathrm{homog}}(A) < W^{\mathrm{homog,per}}(A),$$

where W^{homog} is defined by (1.17)*, and $W^{\mathrm{homog,per}}$ is defined by* (1.19)*.*

In order to compute $W^{\mathrm{homog}}(A)$, one needs in principle to take the limit $N \to \infty$ in (1.17). Numerically, this implies using large values of N.

We thus introduce the corresponding filtered problem, that is,

$$W_N^{\mathrm{homog}}(A) = \frac{1}{N^2}\inf\left\{\int_{(0,N)^2} W(y, \nabla v(y) + A)\varphi_N(y)dy, \right.$$
$$\left. v \in W^{1,p}((0,R)^2;\mathbb{R}^d), \quad \int_{(0,N)^2} \nabla v(y)\,\varphi_N(y) = 0\right\}, \tag{5.3}$$

where φ_R is a filtering function satisfying (1.9). Since W is $\mathbb{Z}^2$-periodic in its first variable, it is natural to use integer values of values of R.

We compute numerically the solutions of (5.3) using a Newton algorithm. The parameters used are $a = 0.25$, $d = 0.8$, $\alpha = 10^{-2}$. The filter is defined by (1.9), with $\varphi = 1$ (no filter) or (5.4) (filter 1) or (3.25) (filter 2), where

$$\varphi(x_1, x_2) = (4 - 2|x_1|)_+ (4 - 2|x_2|)_+ . \tag{5.4}$$

We have used several initial guesses, which are the same for the three values of the filter. From Figure 5, we draw the following conclusions:

(a) in each case, the result depends on the initial guess, and

(b) the result is in general better for the filter of order 1 than for the "no filter" case, and better for the filter of order 2 than for the filter of order 1.

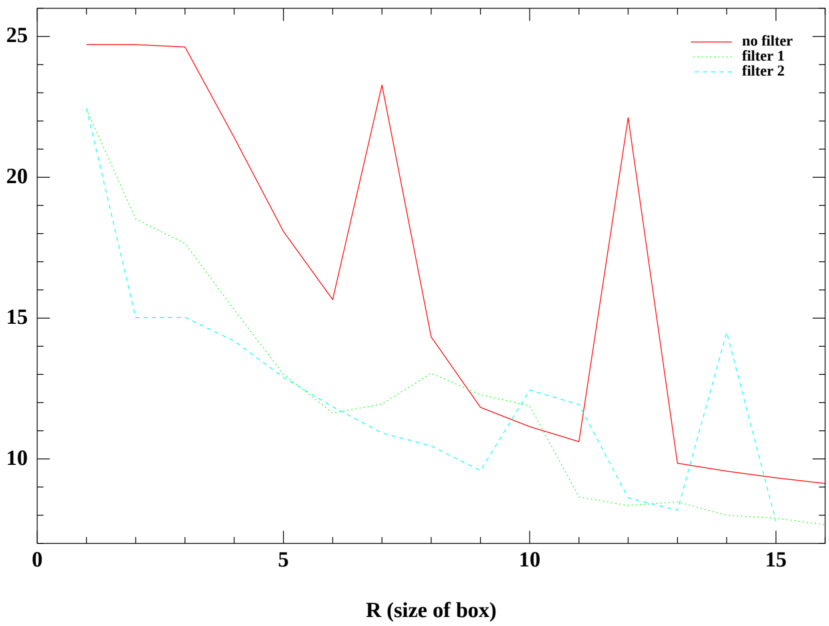

Figure 5: Values of the energy minimum for problem (5.3). The result has been divided by α in order to get significant numbers.

6 Remark on the stochastic case

We briefly investigate here how the above filtering approach performs in the stochastic setting. We shall now see that, in its present state, the method is not efficient there. Consider the one dimensional case, and define

$$A(x, \omega) = a(\tau_x \omega),$$

where $a \in L^1(\Omega)$, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and τ is an ergodic stationary action on Ω. In the sequel, we denote by $\mathbb{E}$ the expectation value on Ω. We refer to [29] for more details on the basic ingredients of stochastic homogenization. The corrector problem

$$\begin{cases} -\frac{d}{dy}\left(A(y,\omega)\left(p + w_p'(y,\omega)\right)\right) = 0, \\ \\ w_p' \text{ is stationnary, } \mathbb{E}(w_p') = 0, \end{cases}$$

is truncated on the interval $[-R, R]$ using the constraint $\int_{-R}^{R} w_p' = 0$, and then solved explicitly. This yields the following approximate value for the homogenized coefficient:

$$A_R^* = \left(\frac{1}{2R}\int_{-R}^{R} A(y,\omega)^{-1} dy\right)^{-1}.$$

The ergodic theorem (see for instance [23]) implies $\lim_{R\to\infty} A_R^* = \left[\mathbb{E}\left(a^{-1}\right)\right]^{-1} := A^*$. The rate of convergence is made precise by the central limit theorem [23]:

$$\sqrt{R}\left(\frac{1}{A_R^*} - \frac{1}{A^*}\right) \underset{R\to+\infty}{\longrightarrow} \mathcal{N}(0,\sigma^2),$$

where $\mathcal{N}(0,\sigma^2)$ is the centered normal distribution with variance

$$\sigma^2 = \mathbb{E}\left(\int_0^\infty \left(A(\omega,0)^{-1} - \mathbb{E}(A^{-1})\right)\left(A(\omega,s)^{-1} - \mathbb{E}(A^{-1})\right) ds\right).$$

Using an obviously defined filtered version of the corrector problem, we can explicitly compute another approximation of A^*, which reads $A_{R,\varphi}^* = \left(\int_{\mathbb{R}} A(Ry,\omega)^{-1}\,\varphi(y)dy\right)^{-1}$, hence

$$\frac{1}{A_{R,\varphi}^*} - \frac{1}{A^*} = \int_{\mathbb{R}} \left(A(Ry,\omega)^{-1} - \mathbb{E}(A^{-1})\right)\varphi(y)dy.$$

Using [6, Lemma 3.1], this quantity is readily seen to satisfy

$$\sqrt{R}\int_{\mathbb{R}} \left(A(Ry,\omega)^{-1} - \mathbb{E}(A^{-1})\right)\varphi(y)dy \underset{R\to+\infty}{\longrightarrow} \mathcal{N}\left(0,\sigma^2\int\varphi^2\right).$$

This proves that the filtering technique does not improve the rate of convergence of the homogenized coefficient. It does not even improve the prefactor (here, the variance) of the convergence either. Indeed, the Cauchy-Schwarz inequality implies $\int\varphi^2 \geq \left(\int\varphi\right)^2 = 1$, showing that the variance is *increased* by the filtering technique. Alternative techniques therefore need to be employed for the stochastic setting. Current efforts are directed toward this goal.

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Appendix

We give now the
Proof of Lemma 2.4: We first point out that it is sufficient to prove this Lemma for $R = 1$. The general case is then easily deduced by a scaling argument. Moreover, according to Lemma 2.3, we may assume that $u \in C^\infty$.

First, we are going to prove

$$\forall u \in \mathcal{H}^1_R, \quad \int_{Q\setminus Q_{1-\delta}} \left[u(x) - u\left(\frac{x}{2}\right)\right]^2 \varphi(x)dx \leq C \int_Q |\nabla u|^2 \varphi, \tag{6.1}$$

where C depends only on φ, and δ is the constant appearing in (2.2). We write:

$$\forall x \in Q, \quad u(x) - u\left(\frac{x}{2}\right) = \int_{1/2}^1 \nabla u(tx) \cdot x dt.$$

This implies

$$\begin{aligned}\int_{Q\setminus Q_{1-\delta}} \left[u(x) - u\left(\frac{x}{2}\right)\right]^2 \varphi(x)dx &\leq \int_{Q\setminus Q_{1-\delta}} \varphi(x) \int_{1/2}^1 |\nabla u(tx)|^2 dtdx \\ &\leq \frac{\inf_{Q_{1-\delta}} \varphi}{\sup_{Q\setminus Q_{1-\delta}} \varphi} \int_{Q\setminus Q_{1-\delta}} \int_{1/2}^1 |\nabla u(tx)|^2 \varphi(tx) dtdx,\end{aligned}$$

where we have used (2.2) (we have assumed, without loss of generality, that $\delta < 1/2$). Thus,

$$\int_{Q\setminus Q_{1-\delta}} \left[u(x) - u\left(\frac{x}{2}\right)\right]^2 \varphi(x)dx \leq \int_{1/2}^1 t^{1-d} \int_{Q_t\setminus Q_{t(1-\delta)}} |\nabla u(x)|^2 \varphi(x) dxdt,$$

which implies (6.1).

We now argue by contradiction: if (2.4) is not true, then one can find a sequence u_n in $\mathcal{H}^1$ such that

$$\int_Q u_n \varphi = 0, \qquad \int_Q u_n^2 \varphi = 1, \qquad \int_Q |\nabla u_n|^2 \varphi \underset{n\to\infty}{\longrightarrow} 0. \tag{6.2}$$

This implies that, up to the extraction of a subsequence,

$$\frac{1}{|Q_{1-\delta}|} \int_{Q_{1-\delta}} u_n \underset{n\to\infty}{\longrightarrow} a \in \mathbb{R},$$

$$\int_{Q_{1-\delta}} |\nabla u_n|^2 \underset{n\to\infty}{\longrightarrow} 0,$$

hence, applying Poincaré-Wirtinger inequality in $Q_{1-\delta}$, u_n converges to a in $L^2(Q_{1-\delta})$. Then, (6.1) implies that $u_n \longrightarrow a$ in $L^2(Q\setminus Q_{1-\delta}, \varphi(x)dx)$. Thus, we have

$$u_n \underset{n\to\infty}{\longrightarrow} a \text{ in } \mathcal{H}.$$

The first equation of (6.2) implies that $a = 0$, which contradicts the second equation of (6.2). □

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