Register allocation by graph coloring under full live-range splitting

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Live-Range Unsplitting for Faster Optimal Coalescing

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Abstract

Register allocation is often a two-phase approach: spilling of registers to memory, followed by coalescing of registers. Extreme live-range splitting (i.e. live-range splitting after each statement) enables optimal solutions based on ILP, for both spilling and coalescing. However, while the solutions are easily found for spilling, for coalescing they are more elusive. This difficulty stems from the huge size of interference graphs resulting from live-range splitting.

This paper focuses on coalescing in the context of extreme live-range splitting. It presents some theoretical properties that give rise to an algorithm for reducing interference graphs. This reduction consists mainly in finding and removing useless splitting points. It is followed by a graph decomposition based on clique separators. The reduction and decomposition are general enough, so that any coalescing algorithm can be applied afterwards.

Our strategy for reducing and decomposing interference graphs preserves the optimality of coalescing. When used together with an optimal coalescing algorithm (e.g. ILP), optimal solutions are much more easily found. The strategy has been tested on a standard benchmark, the optimal coalescing challenge. For this benchmark, the cutting-plane algorithm for optimal coalescing (the only optimal algorithm for coalescing) runs 300 times faster when combined with our strategy. Moreover, we provide all the optimal solutions of the optimal coalescing challenge, including the three instances that were previously unsolved.

Categories and Subject Descriptors D.3.4 [*Programming languages*]: Processors - compilers, optimization; G.2.2 [*Graph theory*]: Graph algorithms

General Terms algorithms, languages

Keywords register allocation, coalescing, graph reduction

1. Introduction

Register allocation determines at compile time where each variable will be stored at execution time: either in a register or in memory. Register allocation is often a two-phase approach: spilling of registers to memory, followed by coalescing of registers [3, 8, 16, 5]. Spilling generates loads and stores for live variables[1] that cannot be stored in registers. Coalescing allocates unspilled variables to registers in a way that leaves as few as possible `move` statements (i.e. register copies). Both spilling and coalescing are known to be NP-complete [24, 6].

Classically, register allocation is modeled as a graph coloring problem, where each register is represented by a color, and each variable is represented by a vertex in an interference graph. Given a number k of available registers, register allocation consists in finding (if any) a k-coloring of the interference graph. When there is no k-coloring, some variables are spilled to memory. When there is a k-coloring, coalescing consists in choosing a k-coloring that removes most of the `move` statements.

ILP-based approaches have been applied to register allocation in order to provide optimal solutions. Register allocation is one of the most important passes of a compiler. A compiler performing an optimal register allocation can thus generate efficient code. Having such a code is often of primary concern for embedded software, even if the price to pay is to use an external ILP-solver. Appel and George formulate spilling as an integer linear program (ILP) and provide optimal and efficient solutions [3]. Their process to find optimal solutions for spilling requires live-range splitting, an optimization that leaves more flexibility to the register allocation (i.e. avoids to spill a variable everywhere). While the solutions are easily found for spilling in this context, for coalescing they are more elusive. Indeed, live-range splitting generates huge interference graphs (due to many `move` statements) that make the coalescing harder to solve.

Splitting the live-range of a variable v consists in renaming v to different variables having shorter live-ranges than v and adding `move` statements connecting the variables originating from v. Recent spilling heuristics benefit from live-range splitting: when a variable is spilled because it has a long live-range, splitting this live-range into smaller pieces may avoid to spill v. If the live-range of v is short, it is easier to store v in a register, as the register needs to hold the value of v only during the live-range of v.

There exists several ways of splitting live-ranges (e.g. region spilling, zero cost range splitting, load/store range analysis) [10, 7, 18, 19, 4, 11, 20]. Splitting live-ranges often reduces the interferences with other live-ranges. Thus, most of the splitting heuristics have been successful in improving the spilling phase. The differences between these heuristics stem from the number of splitting points (i.e. program points where live-ranges are split) as well as the sizes of the split live-ranges. These heuristics are sometimes difficult to implement.

The most aggressive live-range splitting is *extreme live-range splitting*, where live-ranges are split after each statement. Its main advantage is the accuracy of the generated interference graph. Indeed, a variable is spilled only at the program points where there is no available register for that variable. As in a SSA form, each variable is defined only once. Furthermore, contrary to other splitting heuristics, extreme live-range splitting is easy to implement.

[1] A variable that may be potentially read before its next write is called a *live* variable.

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Extreme live-range splitting helps in finding optimal and efficient solutions for spilling [3]. However, it generates programs with huge interference graphs. Each renaming of a variable v to v_1 results in adding a vertex in the interference graph for v_1 and, consequently, some edges incident to that vertex. Thus, interference graphs become so huge that coalescing heuristics often fail. The need for a better algorithm for the optimal coalescing problem gave rise to a benchmark of interference graphs called the optimal coalescing challenge (written OCC in the sequel of this paper) [2].

Recently, Grund and Hack have formulated coalescing as an ILP [16]. They introduce a cutting-plane algorithm that reduces the search space (i.e. the space of potential solutions). Thus, their ILP formulation needs less time to compute optimal solutions. As a result, they provide the first optimal and efficient solutions of the OCC. These solutions are 50% better than the solutions computed by the best coalescing heuristics.

Grund and Hack conclude in [16] that their cutting-plane algorithm fails when applied to the largest graphs of the OCC. Because of extreme live-range splitting (that was required for optimal spilling), optimal solutions for coalescing cannot be efficiently computed on these graphs. Owing to the extreme amount of copies, coalescing is hard to solve optimally. In this paper, we study the impact of extreme live-range splitting on coalescing and provide a strategy for reducing interference graphs before coalescing. ILP is commonly composed with algorithms in order to let the solver, that uses an exponential algorithm, only deal with the core of the problem. The work we present follows this classical methodology.

Our strategy performs mainly two steps. The first step is the main step of the strategy and we call this step live-range unsplitting; it identifies most of the splitting points that are not useful for coalescing, and updates the graph consequently. Then, a second step finds clique separators in the updated graph and thus decomposes it into several subgraphs. Coalescing on each subgraph is then solved separately. As the subgraphs are much smaller than the original graph, ILP solutions are much easier and faster to find. Moreover, both steps do not break the optimality of coalescing.

The remainder of this paper is organized as follows. Section 2 recalls some definitions form graph theory and defines some terms that are computed by our strategy for optimal coalescing (e.g. *split interference graphs*, that are the interference graphs resulting from extreme live-range splitting). Section 3 presents the main contribution of this paper. It details our algorithm for live-range unsplitting, that reduces split interference graphs. Then, section 4 defines our whole strategy for optimal coalescing. This strategy performs mainly live-range unsplitting and then a graph decomposition.

Section 5 presents experimental results on the OCC. A first result is that our live-range unsplitting reduces the size of original graphs (i.e. before extreme live-range splitting) by a factor of up to 10, and thus extreme live-range splitting does not make coalescing harder anymore. A second result is that Grund and Hack's cutting-plane algorithm for optimal coalescing runs 300 times faster when combined with our strategy, thus enabling to solve all the instances of the OCC, including the 3 instances that were previously unsolved. Related work is discussed in Section 6, followed by concluding remarks in Section 7.

2. Foundation

This section defines split interference graphs and SB-cliques, as well as some concepts from graph theory.

Register allocation is performed on an interference graph. There are two kinds of edges in an interference graph: interference (or conflict) edges and affinity (or preference) edges. Two variables *interfere* if there exists a program point where one variable is live at the exit of a statement that defines the other variable (and the statement differs from a `move` statement between both variables) [1]. An *affinity edge* between two variables represents a `move` statement between these variables (that should be stored in a same register or at the same memory location). Weights are associated to affinity edges, taking into account the frequency of execution of the `move` statements.

Given a number k of registers, register allocation consists in satisfying all interference edges as well as maximizing the sum of weights of affinity edges such that the same color is assigned to both endpoints. Satisfying most of the affinity edges is the goal of register coalescing.

Interference graphs are built after a liveness analysis [9]. In an interference graph, a variable is described by a unique live-range. Consequently, spilling a variable means spilling it everywhere in the program, even if it could have been spilled on a shorter live-range. Figure 1 illustrates this problem on a small program consisting of a `switch` statement with 3 branches (see [21] for more details). The program has 3 variables but only 2 variables are updated in each branch of the `switch` statement. Thus, its corresponding interference graph is a 3-clique, that is not 2-colorable, although only 2 registers are needed.

```
switch(...){
case 0:              case 1:              case 2:
l1 : a := ...        l4 : a := ...        l7 : b:= ...
l2 : b := ...        l5 : c := ...        l8 : c:= ...
l3 : ... := a + b    l6 : ... := a + c    l9 : ... := b + c
}
```

Figure 1. Excerpt of a small program such that its interference graph is a 3-clique.

The usual way to overcome the previous problem is to perform live-range splitting. Extreme live-range splitting splits live-ranges after each statement, and thus generates some renamings that are not useful for coalescing. When v is renamed to v_1 and v_2, if after optimal coalescing v_1 and v_2 share a same color, then the renaming of v_2 is useless: v_2 can be replaced by v_1 while preserving the optimality of coalescing.

Moreover, the number of affinity edges blows up during extreme live-range splitting since there is an affinity edge between any two vertices that represent the same variable in two consecutive statements. Figure 2 shows the split interference graph of a small program given in [1]. In the initial interference graph, each vertex represents a variable of the initial program (the array `mem` is stored in memory); the affinity edges correspond to both assignments `d:=c` and `j:=b`.

The bottom of Figure 2 is the split interference graph resulting from extreme live-range splitting on the graph at the top of the Figure. The construction of the split interference graph is inspired from the construction of the OCC by Appel and George. Every variable v_i that is live-in and live-out[2] at program point p is renamed into v_{i+1} at this program point (i.e. the variables are renamed in parallel with the statement of p). For instance, $k0$, $k1$ and $k2$ are copies of k. As k is live initially, it is renamed (at program point p_0) to $k0$ (and so is j). Similarly, g and $j0$ are renamed to respectively $g0$ and $j1$ at program point p_1, but the variable k is not renamed, as it dies (i.e. it is not live-out) at p_1. In order to show how the edges of the first graph (on the top of the figure) are transformed into the edges of the second graph (bottom of the figure), the edges (j, k) and (j, b) of the first graph and their corresponding edges in the second graph are bold edges. The renamings are done in parallel in order to limit the interferences between the newly created vari-

[2] Given a program point p, a variable is *live-in* (resp. *live-out*) if it is live just before (resp. after) p.

```
Live-in : k j
    (p0)   g := mem[j+12]
    (p1)   h := k - 1
    (p2)   f := g * h
    (p3)   e := mem[j+8]
    (p4)   m := mem[j+16]
    (p5)   b := mem[f]
    (p6)   c := e + 8
    (p7)   d := c
    (p8)   k := m + 4
    (p9)   j := b
   (p10)
Live-out : d k
```

```
Live-in : k j
        (p0)   k0 := k || j0 := j || g := mem[j+12]
        (p1)   j1 := j0 || g0 := g || h := k0 - 1
        (p2)   j2 := j1 || f := g0 * h
        (p3)   f0 := f || j3 := j2 || e := mem[j2+8]
        (p4)   e0 := e || f1 := f0 || m:=mem[j3+16]
        (p5)   e1 := e0 || m0 := m || b:=mem[f1]
        (p6)   b0 := b || m1 := m0 || c := e1 + 8
        (p7)   b1 := b0 || m2 := m1 || d := c
        (p8)   b2 := b1 || d0 := d || k1 := m2 + 4
        (p9)   d1 := d0 || k2 := k1 || j4 := b2
       (p10)
Live-out : d1 k2
```

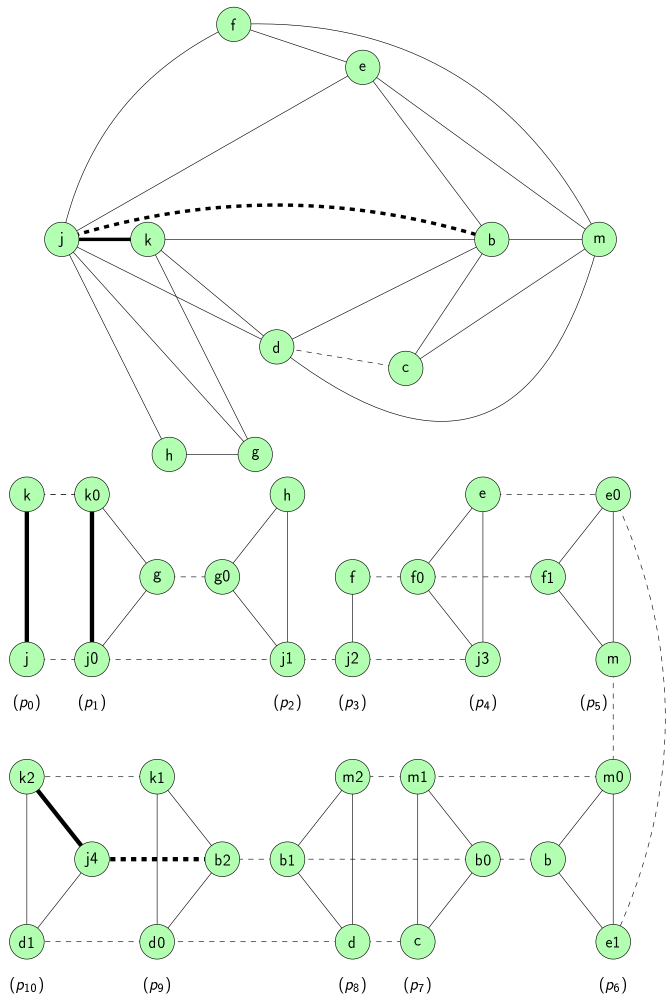


Figure 2. A small program and its interference graph (top). Full edges are interference edges. Affinity edges are dashed edges. The same program after extreme live-range splitting and its split interference graph (bottom). The weights of affinity edges are omitted in the Figure. The program points $(p_0) \dots (p_{10})$ of both programs are shown in the Figure.

ables and the other variables. For instance, in the first statement (at program point p_0), $k0$ interferes with $j0$ but not with j.

Edges corresponding to renamings are added in the split interference graph. Affinity edges between renamed variables are also added, as well as interference edges related to renamed variables. For instance, renaming j to $j0$ generates the affinity edge $(j, j0)$. More precisely, an interference edge (x, y) in the initial graph generates (in the split interference graph) n interference edges between a renaming x' of x (or x itself) and a renaming y' of y (or y itself), where n is the number of program points where x and y interfere in the initial program. For instance, the interference edge (j, k) (of the initial graph) corresponds to the three interference edges (j, k), $(j0, k0)$ and $(j4, k2)$ of the split interference graph, because in the initial program, j and k interfere at program points p_0, p_1 and p_{10}. Similarly, the affinity edge (j, b) of the initial graph corresponds to the affinity edge $(j4, b2)$ of the split interference graph, and this is the only affinity edge originating from (j, b) as there is only one move statement `j:=b` in the initial program.

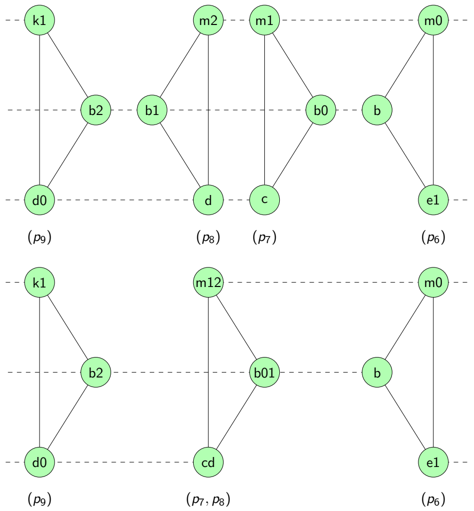
The main drawback of extreme live-range splitting is that it generates huge graphs. There are two kinds of affinity edges in a split interference graph: edges representing coalescing behaviors, and edges added by variable renaming during live-range splitting. A lot of affinity edges and vertices (as well as some associated edges) corresponding to variable renaming are added in the graph. This section gives two properties of these edges and vertices that are useful for reducing the graphs.

Clique. A *clique* of a graph G is a subgraph of G having an edge between every pair of vertices. A clique is said to be *maximal* if there is no other clique containing it.

Interference connected component. Given a graph, an *interference connected component* is a maximal set of vertices such that there exists a path (of interference edges) between any pair of its vertices.

Interference clique. An *interference clique* is a clique containing only interference edges. For instance, the subgraph induced by the vertices $b0, c$ and $m1$ is an interference clique that we call $\mathcal{C}$. The subgraph induced by the vertices $b1, d$ and $m2$ is also an interference clique that we call $\mathcal{C}'$.

Theorem 1. After extreme live-range splitting, a statement corresponds to an interference connected component of the split interference graph. Moreover, such a component is an interference clique, that we call a *statement clique*.



(p_5) e1 := e0 || m0 := m || b:=mem[f1]
(p_6) b0 := b || m1 := m0 || **c := e1 + 8**
(p_7) b1 := b0 || m2 := m1 || **d := c**
(p_8) b2 := b1 || d0 := d || k1 := m2 + 4

(p_5) e1 := e0 || m0 := m || b:=mem[f1]
(p_6, p_7) b1 := b || m2 := m0 || **d := e1 + 8**
(p_8) b2 := b1 || d0 := d || k1 := m2 + 4

Figure 3. Some parallel cliques and the corresponding piece of code (top). The same subgraph (and the corresponding piece of code) after our graph reduction (bottom).

For instance, the statement clique of `d := c` (at program point p_7) is $\mathcal{C}$, and the statement clique of `k := m+4` (at program point p_8) is $\mathcal{C}'$. The split interference graph of Figure 2 shows the correspondence between statement cliques and statements (i.e. program points).

The proof of Theorem 1 relies on the fact that all the variables that are defined in parallel at a program point interfere together, thus they induce an interference clique. Moreover, there is no other interference involving these variables since there is an unique statement where they are live.

Matching. A *matching* is a set of non-adjacent edges. A matching is *maximum* if its cardinality is the maximal cardinality of all matchings. For instance, the set $\{(b0, b), (m0, m1)\}$ is a matching.

Affinity matching. An *affinity matching* is a matching consisting only of affinity edges. For instance, the previous example is also an affinity matching. It is maximum on the graph induced by the vertices of $\mathcal{C}$ and $\mathcal{C}'$.

Parallel clique, dominant and dominated parallel cliques, dominant affinity matching. Let C_1 and and C_2 be two maximal interference cliques. C_1 *dominates* C_2 if there exists an affinity matching M such that :

1. M contains only affinity edges having an endpoint in C_1 and the other in C_2,
2. each vertex of C_2 is reached by M,
3. no affinity edge of M has endpoints precolored [3] with different colors,
4. for any $v \in C_1$ and $v' \in C_2$ which are precolored with the same color, the edge (v, v') belongs to M,
5. for each vertex v of C_2, the weight of the edge of M that reaches v is greater than or equal to the total weight of all others affinity edges reaching v. This condition is not as restrictive as it seems. The weight of an affinity edge is often half the weight of the total weight of incident affinity edges reaching its endpoints. Indeed, the number of copy statements does not change, except when entering in or exiting from a loop.

We also say that C_1 and C_2 are *parallel cliques*, C_1 is a *dominant parallel clique* and C_2 is a *dominated parallel clique*. Moreover, M is called a *dominant affinity matching*.

Figure 3 shows a subgraph of the split interference graph given in Figure 2. This subgraph (on the top of the figure) consists of both cliques $\mathcal{C}$ and $\mathcal{C}'$ and two other cliques. $\mathcal{C}$ dominates $\mathcal{C}'$ (and $\mathcal{C}'$ dominates $\mathcal{C}$). We call $\mathcal{M} = \{(b0, b1), (c, d), (m1, m2)\}$ the dominant affinity matching between $\mathcal{C}$ and $\mathcal{C}'$. Intuitively, the edges of $\mathcal{M}$ represent the variables that will be coalesced. When applied to the subgraph on the top of Figure 3, live-range unsplitting consists in computing $\mathcal{M}$ and then coalescing the variables of affinity edges of $\mathcal{M}$, thus yielding the reduced graph at the bottom of the figure. In this reduced graph, the splitting point p_8 has been removed. In the corresponding piece of code, the copies $b0$ and $m1$ have been removed, and the sequence of statements `c:=e1+8; d:=c` has been simplified into the statement `d:=e1+8`.

Actually, many dominations appear. The following property of dominated parallel cliques enables us to remove the splitting points that have created this domination, without worsening the quality of coalescing (i.e. while preserving the optimality).

Theorem 2. If C_1 and C_2 are two parallel cliques such that C_1 dominates C_2, then there exists an optimal coalescing such that the endpoints of each edge of the dominant affinity matching are colored with a same color.

We prove this result by induction on the size of C_2. If there is only one vertex then C_2 has no coloring constraint. Thus coloring this vertex with the same color than its image in the matching leads to an optimal solution. Assume the property holds for a clique of size p. We decompose a clique of size $p+1$ into a clique of size p and an other vertex. The result holds for the clique and the single vertex. Moreover, there is no constraint between the p vertices and the last one since $p+1$ is lower than or equal to k because spilling has already been done, and there is no other constraints since a statement clique is a connected component. Finally, the optimum is reached since minima of both problems are reached and that the function to optimize is linear.

[3] A *precolored node* is a node that is colored (i.e. preassigned to a register) before register allocation, in order to respect the calling conventions of the processor. Register allocation cannot change the color of such a node, but a precolored node can be coalesced with any ordinary node.

Bipartite graph. A graph G is *bipartite* if there exists a partition (V_1, V_2) of its vertices such that every edge has an endpoint in V_1 and the other in V_2.

SB-clique. The main idea of live-range unsplitting is to reduce the size of the split interference graph by removing most of the splitting points. Any splitting point separates two consecutive sequences of statements. We define a *split-block* as a sequence of statements that is not separated by a splitting point. Hence, initially (i.e. before the reduction) split-blocks are statements. Moreover, removing a splitting point between two split-blocks leads to a new split-block. We also define *SB-cliques* (for split-blocks cliques), as cliques of the interference graph. Thus, SB-cliques are initially statement cliques. The intuition that motivated the following work relies on the correspondence between split-blocks and SB-cliques.

3. Live-range unsplitting

Live-range unsplitting removes the splitting points that could have been useful for spilling but that are useless for coalescing. This reduction relies on dominated parallel cliques representing the splitting points that can be removed from the program. This section details a first algorithm that checks if two cliques are parallel. Then, it details a second algorithm that reduces split interference graphs. This second algorithm merges parallel cliques and removes trivially colorable vertices.

3.1 Detecting dominated parallel cliques

A reduction rule arises from Theorem 2. Furthermore, checking if two cliques are parallel can be done in polynomial time. Algorithm 1 does it in O($k \cdot m_{C_2}$), where m_{C_2} is the number of affinity edges having an endpoint in C_2.

The first part of Algorithm 1 (lines 1 to 7) computes E, the set of affinity edges that may belong to a dominant affinity matching. The edges that cannot respect precoloring constraints (i.e. that have endpoints colored with different colors or an endpoint colored with a color that cannot be a color for the second endpoint) are removed.

Then, the next two loops (lines 8 to 17) remove every affinity edge such that its weight is not high enough to be dominant. More precisely, an affinity edge can be deleted if its weight is not greater than the half of the total weight of its endpoint that belongs to the potential dominated parallel clique. Affinity edges having endpoints precolored with the same color are not deleted since they must belong to any dominant matching. The last part of Algorithm 1 (lines 18 to 23) is a search for a maximum affinity matching included in E. This problem is nothing but the search of a maximum matching in a bipartite graph, which can be solved in polynomial time [23]. Finally, one only needs to check if each vertex of C_2 is an endpoint of an affinity edge of the matching. That can be done by checking the equality between the cardinal of the matching and the number of vertices of C_2 (line 19).

3.2 Merging parallel cliques

Algorithm 2 reduces split interference graphs as long as it is possible. This reduction requires the computation of SB-cliques. First, SB-cliques are initialized to statement cliques. If two SB-cliques are parallel, then they can be merged (resulting in a new SB-clique), since each pair of vertices linked by an edge of the dominant affinity matching (that is computed by Algorithm 1) can be coalesced. Indeed, there exists an optimal solution that assigns the same color to both vertices (see Theorem 2). This merge leads to a graph where new dominations may appear, as well as vertices with no affinity edges. These vertices can be removed from the graph since the interference degree of any vertex is strictly lower than k. We denote such a vertex a *low-degree vertex*. Low-degree vertices can be removed from the graph and put in the coloring stack to be treated later, as many coalescing algorithms do [9, 13].

Algorithm 1 `dominant_affinity_matching` (C_1,C_2)

```
Require: Two maximal interference cliques $C_1$ and $C_2$
Ensure: A dominant affinity matching $M$ if $C_2$ is dominated by
    $C_1$, NULL otherwise
 1: $E$ := {affinity edges having an endpoint in $C_1$ and the other in
    $C_2$}
 2: delete every affinity edge having endpoints precolored with
    different colors from $E$
 3: for all color $c$ do
 4:     if there exist $v_1 \in C_1$ and $v_2 \in C_2$ both colored with $c$
        then
 5:         delete from $E$ every affinity edge reaching $v_1$ or $v_2$
            except $(v_1, v_2)$
 6:     end if
 7: end for
 8: for all $v \in C_2$ do
 9:     $\mathit{Aff\_weight}(v) := \sum_{x \in \mathit{Aff\_Nghb}(v)} \mathit{weight}_{(v,x)}$
10: end for
11: for all $v_2 \in C_2$ do
12:     for all $v_1$ such that $(v_1, v_2) \in E$ do
13:         if $\mathit{weight}_{(v_1,v_2)} < \frac{1}{2} \cdot \mathit{Aff\_weight}(v_2)$ and $v_1$ and
            $v_2$ are not precolored with the same color then
14:             delete $(v_1, v_2)$ from $E$
15:         end if
16:     end for
17: end for
18: $M$ := maximum affinity matching included in $E$
19: if cardinal(M) = number of vertices of $C_2$ then
20:     return $M$
21: else
22:     return NULL
23: end if
```

Merging two SB-cliques is equivalent to removing the splitting point that separates the split-blocks they represent and, hence, removing copies that have been created by the deleted splitting point. In other words, merging two SB-cliques is equivalent to undo a splitting. Moreover, since merging two SB-cliques yields a new SB-clique, the reduction is iterated until the graph is left unchanged, i.e. as long as there remain parallel SB-cliques or low-degree vertices (lines 7 to 22). In order to speed up the process, for each SB-clique i, we first compute the set N_i of SB-cliques j such that there exists an affinity edge having an endpoint in i and the other in j. Then, one only needs to find and merge parallel SB-cliques, delete low-degree vertices, and update the graph. Algorithm 2 details our reduction.

3.3 An example

Figure 4 presents the succession of parallel cliques detected during the application of live-range unsplitting to the second graph of Figure 2, for $k \geq 3$. Each of the steps of Figure 4 is followed by a (traditional) phase of deletion of low-degree vertices that is not detailed in the Figure (except the first one).

At the first step four pairs parallel-cliques are detected (graph (a)), and thus coalesced. In the graph, this coalescing corresponds to the merge of four affinity edges that are in bold in graph (a): $(k, k0), (j, j0), (f, f0), (j2, j3)$. Graph (b) shows the graph resulting from this merge. In Figure 4, when two nodes are merged, the resulting node is identified by one of these two nodes (e.g. $k0$ in graph (b)), not to overload the graphs of Figure 4. Then, low-degree vertices ($h, k0, k1$) are removed, thus yielding graph (c). A new phase of parallel cliques detection is performed on graph (c),

Algorithm 2 `graph_reduction` (G)

Require: A split interference graph G
Ensure: A reduced split interference graph

```
compute statement cliques
for all statement clique $i$ do
    $N_i$ := {statement cliques linked to $i$ with an affinity edge}
end for
$red := 1$
while $red \neq 0$ do
    red := 0
    for all SB-clique $i$ do
        for all $j \in N_i$ such that $|i| \geq |j|$ do
            $M$ := dominant_affinity_matching$(i, j)$
            if $M \neq NULL$ then
                merge each pair of $M$ and compute new weights
                red := red+1
                $N_{ij} := N_i \cup N_j - \{i, j\}$
                for all $k \in N_i \cup N_j$ do
                    $N_k := N_k \cup \{ij\} - \{i, j\}$
                end for
            end if
        end for
    end for
    red := red + card({low-degree vertices})
    remove low-degree vertices
end while
```

thus yielding the graph (d). The process is then iterated, thus yielding graph (e) and an empty graph.

Finally live-range unsplitting yields an empty graph, meaning that this instance can be optimally solved in polynomial time. Indeed, contrary to classical algorithms, it does not always lead to an empty graph. However, when it does, as for this example, it also ensures the optimality of the coalescing. Moreover, the solution requires only three colors while any coalescing technique applied to the the classical interference graph (first graph of Figure 2), and satisfying its two affinities, would require at least four colors. In Figure 4, thanks to live-range splitting, k and $k2$ are not colored with the same color, as well are j and $j5$.

Actually, if j and b are coalesced and if jb is the vertex obtained by coalescing j and b, then $\{e, f, m, jb\}$ form a clique of four vertices. Thus these four vertices must have different colors, and four colors (at least) are needed. It shows, again, that live-range splitting can provide better solutions because variables belonging to split live-ranges may be stored in different registers.

4. A strategy for optimal coalescing

Live-range unsplitting is the first step of our strategy for optimal coalescing. This strategy consists of four steps, and it does not affect the global quality of coalescing (and preserve the optimality of coalescing). It is detailed in Figure 5.

The second step uses clique separators to decompose the graph into several smaller subgraphs. The third step consists in solving coalescing separately on each subgraph using any well-known coalescing algorithm (i.e. a heuristic or an exact algorithm, such as an ILP). The last step is the combination of coalescings of the subgraphs, that is the construction of the global coalescing from the coalescings of all the subgraphs.

4.1 Decomposition by clique separators

Our graph decomposition (the second step) is an enhancement of a decomposition based on clique separators that is commonly used in operations research [22, 17]. Its main idea is to use SB-cliques, that have already been computed by the previous phase as separation sets. A *separation set* is a set of vertices whose removal partitions a connected component into several smaller ones. Thanks to separation cliques, coalescing needs to be solved only on each component resulting from the decomposition (and not on the whole graph), as the following property proves.

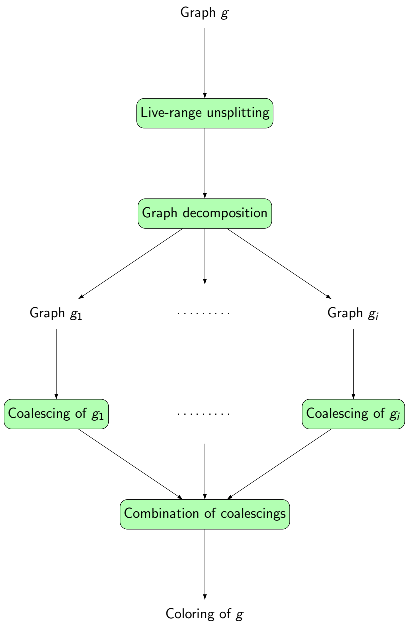


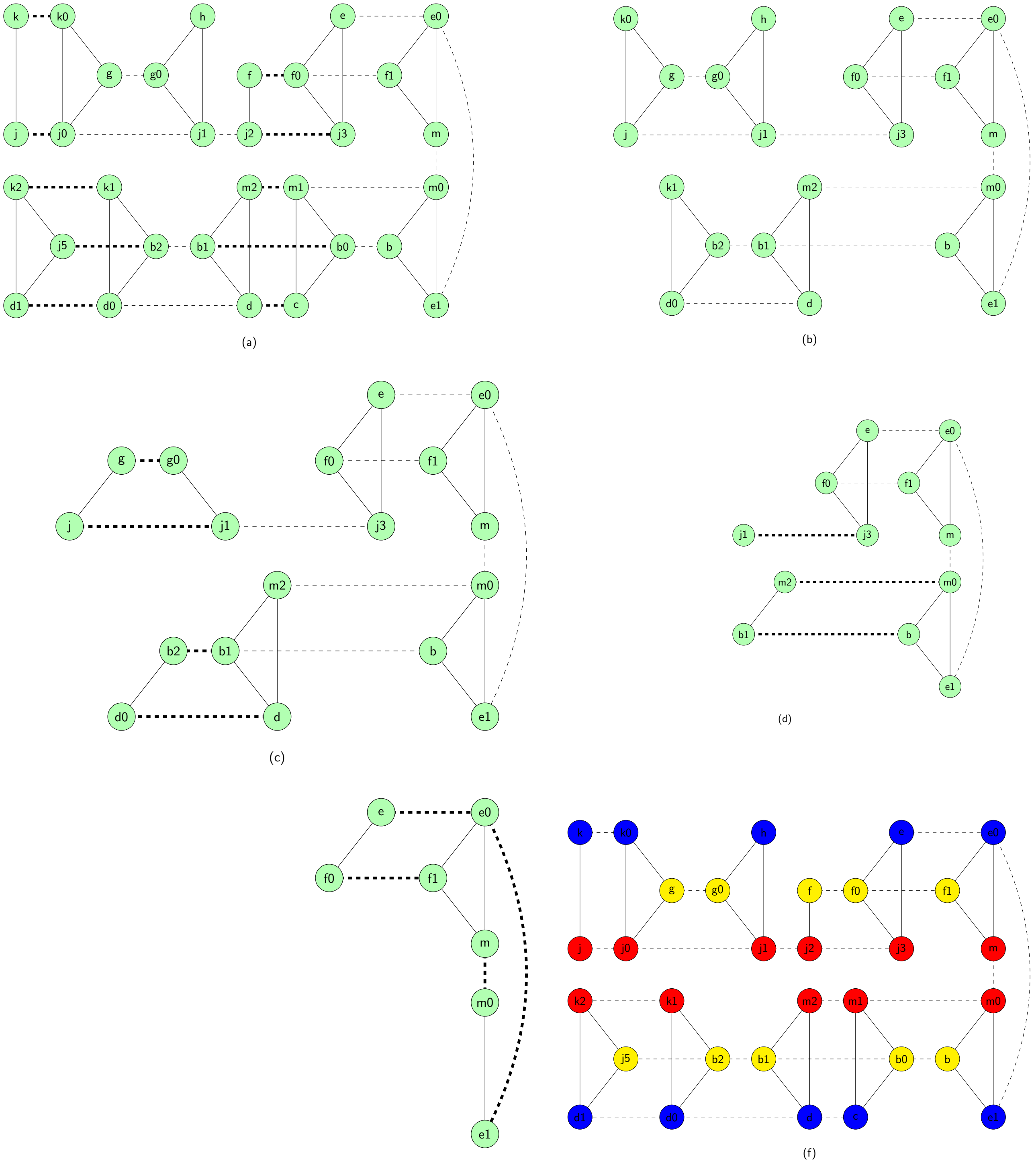
Figure 5. The whole process for coalescing.

Property 1. Let s be an interference clique of g that separates g into components $c_1, \ldots, c_j$ and let $col_1, \ldots, col_j$ be some respective colorings of $c_1 \cup s, \ldots, c_j \cup s$ and col_s a coloring of s. Then, Π_i is defined, for $1 \leq i \leq j$, as a permutation of $\{1, \ldots k\}$ such that for each vertex v of s, $\Pi_i(col_i(v)) = col_s(v)$. Such permutations exist, since all the available colorings of a clique are permutating. The composition of Π_i and col_i is thus a coloring of c_i satisfying exactly the same preferences than col_i. Moreover, the coloring col defined such that $\forall x \in c_i, col(x) = \Pi_i(col_i(x))$ is a coloring of g.

In other words, this property explains that it is always possible to paste the colorings of subgraphs to obtain the coloring of the initial graph. A prerequisite to make the colorings of subgraphs compatible is to take care that the color of any vertex belonging to the separation clique does not change between these colorings. This constraint of compatibility can be either imposed when the coalescing algorithm is called, by forcing the coloring of the vertices of separation cliques, or after the colorings, by permutating colors as Property 1 explains.

In split interference graphs, this decomposition can be done in linear time, rather than in quadratic time. Indeed, the hardest task is to find interference clique separators. This can easily been done in split interference graphs since all interference cliques are disjoint. Hence, to know if a SB-clique is a separator clique, we create a graph where a vertex represents a SB-clique and two vertices are adjacent if there exists an affinity edge between the two SB-cliques that these vertices represent. Then, we compute separator vertices of this graph. A separator vertex of this graph corresponds to a separator clique of the split interference graph.

Furthermore, our live-range unsplitting is based on cliques merging (see Algorithm 1) and thus makes cliques more likely



(a)

(b)

(c)

(d)

(e)

(f)

Figure 4. Application of the live-range coalescing algorithm to the example of Figure 2. Dominant matchings are in bold at each iteration.

$$
(P)\left\{\begin{array}{llcl}
Min & \displaystyle\sum_{(i,j)\in A} w_{ij}\cdot y_{ij} & & \\
\text{under constraints} & & & \\
(C_1)\ \forall i\in\{1,\dots,n(G)\}, & \displaystyle\sum_{c=1}^{k} x_{ic} & = & 1 \\
(C_2)\ \forall (i,j)\in I,\ \forall c\in\{1,\dots,k\}, & x_{ic}+x_{jc} & \leq & 1 \\
(C_3)\ \forall (i,j)\in A,\ \forall c\in\{1,\dots,k\}, & x_{ic}-x_{jc} & \leq & y_{ij} \\
(C_4)\ \forall i\in\{1,\dots,n(G)\},\ \forall c\in\{1,\dots,k\}, & x_{ic}\in\{0,1\} & &
\end{array}\right.
$$

Figure 6. ILP formulation of Grund and Hack for coalescing. Variables are x_{ic} which are set to 1 iff the vertex i is of color c and y_{ij} which are set to 1 iff vertices i and j have the same color. w_{ij} is the weight of the affinity edge (i,j), I is the set of interference edges, A is the set of affinity edges and $n(G)$ is the number of vertices of G.

to be separators. Indeed, if a union of two cliques is a separable set, then the clique obtained by merging these two cliques is a separation clique, even if the two cliques alone are not.

The strength of this decomposition is that it gets rid of solutions that are permutations of previous solutions, while reducing the size of each subproblem we need to solve. For a coloring problem, the huge number of such permutations makes this problem hard to deal with. Thus imposing a same coloring for the clique separators reduces a lot the number of solutions. Since ILP solvers are very sensitive to permutations, deleting some of them may lead to much faster optimal computations.

4.2 Impact on the cutting-plane algorithm for coalescing

Even if any coalescing algorithm can be used after our strategy, this section focuses on the most efficient optimal algorithm, the cutting-plane algorithm of [16]. This algorithm consists on an ILP formulation of coalescing to which are added some efficient inequalities called cuts. Figure 6 recalls the formulation of [16] that we reuse.

At each iteration where a dominated parallel clique of size s is found, the size of the graph decreases of s vertices, s^2 interference edges and at least s affinity edges. On the ILP formulation of [16], it involves a reduction of at least $k \cdot s + s$ variables ($k \cdot s$ for vertices and at least s for affinity edges) and at least $s^2 + k \cdot \frac{s\cdot(s-1)}{2} + s$ constraints (s^2 for constraints C_2, $k \cdot \frac{s\cdot(s-1)}{2}$ for constraints C_3 and s for constraints C_1). Such a reduction is quite significant, especially when applied many times as live-range unsplitting.

Moreover, the number of cut inequalities generated for the cutting-plane algorithm and the number of variables involved in them decrease with the size of the graph. The more cut inequalities are generated, the more the solver takes time to find efficient ones for each iteration of the simplex algorithm (on which solvers rely). Following the same idea, the more variables are involved in a cut inequality, the more it is difficult to find values for these variables. For instance, a path cut [16] is more efficient if it concerns a path of three affinity edges than if it concerns a path of ten affinity edges. For these reasons, the computation of cut inequalities and the solution are speeded up when using our strategy.

5. Experimental results

As mentioned previously, we use the OCC as benchmark. The OCC is a set of 474 large interference graphs that result from a spilling phase. The graphs in OCC are from compiling real programs in SML/NJ. The mix of easy and hard problems in OCC exactly corresponds to (inlined) functions in a set of real programs. Our strategy is applied on the OCC graphs and generates simplified graphs that are given as input to the ILP formulation (and the associated cutting-plane algorithm) defined in [16]. We use the AMPL/CPLEX 9.0 solver (as in [16]) on a PENTIUM 4 2.26Ghz.

Initial nb of vertices	Nb of instances	Vertex nb ratio	Edge nb ratio
0-499	292	18%	33%
500-999	97	14%	27%
1000-2999	63	13%	27%
over 3000	22	13%	8%

Figure 7. Size reductions for OCC graphs. The vertex (resp. edge) number ratio is the ratio between the number of vertices (resp. edges) of the largest subgraph after reduction and the one before reduction.

The first part of this section measures the efficiency of our reduction. Then, the section details respectively optimal and near-optimal solutions.

5.1 Reduction and decomposition

The first measure is the ratio between the sizes of the OCC graph and the largest subgraph on which coalescing has to be solved (i.e. resulting from our decomposition). We focus on this subgraph because its solution requires almost the whole computation time. These results are detailed in Figure 7.

The average reduction is quite significant since the vertex (resp. edge) number is divided by 7 (resp. 4.5) when using our strategy. The reduction rates are also quite significant on the largest graphs of OCC (they are even better on the largest graphs). Let us note that the precolored vertices are always kept (because they model the calling conventions of the processor), thus involving a smaller reduction ratio for small graphs. 90% of the reduction arises from the live-range unsplitting, i.e. from the deletion of a set of splitting points. Hence, this coalescing strategy is both optimality conservative and very efficient. Moreover, our algorithms run very fast since they only take 6 seconds when applied to all the instances of OCC.

5.2 Optimal solutions

We compute optimal solutions for each component of the decomposition using the cutting-plane algorithm of Grund and Hack [16]. For each interference edge, we only compute the path cut corresponding to the shortest path of affinity edges linking its endpoints. Figure 8 shows a fall of computation times between the solution with and without our strategy. Indeed, the cutting-plane algorithm finds only 430 optimal solutions within 5 minutes for each when applied to the OCC graphs. When using our strategy combined with the cutting-plane algorithm, we find the optimal solution for 436 instances withing one second for each. Furthermore, only 6 instances are solved in more than one minute, and only 3 of them are solved in more than 150 seconds.

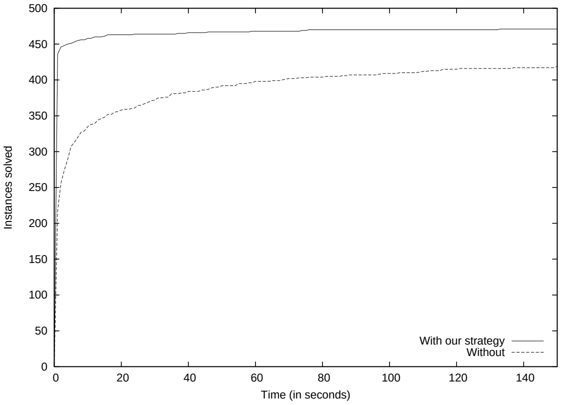


Figure 8. Number of instances of the OCC solved within a short time limit : comparison of the cutting-plane algorithm efficiency when using (or not) our strategy.

Nb of solved instances	Time with our strategy	Time without our strategy	Time ratio without/with
436	1	298	298
446	2	636	318
448	3	732	244
450	4	1198	300
451	5	1228	245
453	6	3153	525
455	7	3321	474
456	8	3574	447
458	10	4022	402
460	12	5630	469
461	15	7214	480
463	16	8709	544
464	24	10397	433
465	37	11757	317
466	40	14941	373
467	59	24916	422
468	74	25490	344
469	76	25663	337
470	133	69956	525
471	11026	71208	7

Figure 9. Time comparisons (in seconds) when using (or not) our strategy for the 471 instances of the OCC solved in both cases.

Figure 9 provides more details about computations times. It contains all the successive numbers of instances solved when using our strategy and their corresponding computation time. We compare these times with the ones required without our strategy to solve the same number of instances. This comparison only considers the 471 instances of the OCC both solved with and without our strategy. One could point out that the cutting-plane algorithm runs about 300 times faster when combined with our strategy.

Moreover, we are the first to solve the whole OCC instances optimally. Indeed, in [16] 3 solutions are far too slow to compute and thus their optimality was not certain. We have found a strictly better solution for one instance and proved that the two other solutions are optimal.

5.3 Near-optimal solutions

Many problems are solved within a few seconds. We adapt our approach to the other problems in order to avoid combinatorial explosion. Thus, we tune the ILP solver for the six instances that take more than one minute to be solved. Numerical results are presented Figure 10.

A first way of tuning the solver is to give it a time limit. Finding the optimal solution (or a near optimal one) often takes less than 10% of the computation time. The ILP formulation can call the solver a lot, even if the solver has a time limit. Thus, the computation can take more than the time limit. However, it never exceeds this limit too much since there is empirically only one call to the solver that reaches the time limit. In addition, this method can fail if no integer solution is found within the time limit.

A better way to tune the solver is to limit the gap between the expected solution and the optimum. Indeed, the solver can give at any time the gap between the current best solution and the best potential one using a bound of the latter. This method is the opposite of choosing a time limit: it sets the quality of the expected solution and evaluates the time spent to find it, instead of setting a time limit and evaluating the quality of the solution.

Results of Figure 10 give a flavor of the quality of coalescing on split interference graphs. The numbers of the first line are graph numbers that correspond to the six hardest graphs. First, a short time limit of 20 seconds provides solutions of good quality when it does not fail. Second, a time limit of 30 seconds leads to near-optimal coalescing. The gap between the corresponding solutions and the optimum is never greater than 20%. The failure that occurs for some instances is quite prohibitive but the time limit gives a good idea of the difficulty for solving an instance.

Last, using a gap limit seems very powerful, especially when it is large enough to avoid combinatorial explosion. Here, a limit of 10% leads to solutions of very good quality (under 5% of gap with the optimum) and within a quite short time (less than two minutes). Giving a too restricted limit (such as 5% or less) leads to good solutions too but these solutions may be quite slower, as for the instance 387 that goes from 115 to 1187 seconds when the gap goes from 10% to 5%.

6. Related work

Goodwin and Wilken were the first using ILP to solve register allocation [14, 15]. Their model solved in a single step both spilling and coalescing, but it failed to provide solutions for large programs. Furthermore, their model was quite difficult to handle since they tackled the problem with a machine level point of view. For instance, they did not model interference graphs. Since then, some improvements were added, in particular by Fu and Wilken [12], Appel and George [3], or Grund and Hack [16]. Appel and George optimally solved spilling by ILP and empirically showed that separating spilling and coalescing does not significantly worsen the quality of register allocation. Because their ILP formulation requires extreme live range splitting, they were not able to solve coalescing optimally. More recently, Grund and Hack proposed a cutting-plane algorithm to solve coalescing and were the first to solve the OCC [16].

Our study reuses this previous work and introduces split interference graphs, as well as properties of these graphs. Split interference graphs model precisely the live-ranges of variables. In particular, split interference graphs are more accurate than the interference graphs obtained from programs in SSA forms (i.e. where each variable is defined only once) or from similar forms (e.g. SSI or SSU).

Concerning coalescing, our strategy divides significantly the size of the interference graphs (by up to ten when measured on the OCC graphs), thus enabling us to find in a faster way more solutions that are optimal. In addition, our strategy can be combined

Instance	144	304	371	387	390	400
Optimum	129332	6109	1087	3450	339	1263
20s	129333	no	no	no	417	1388
30s	129333	no	1285	no	365	1263
10% gap	132040	6448	1094	3550	339	1263
5% gap	129342	6273	1094	3450	339	1263

(a) Values of solutions

Instance	144	304	371	387	390	400
Optimum	11026	1058	132	29543	102	75
20s	20	20	20	20	20	20
30s	30	30	30	30	30	30
10% gap	15	36	62	115	86	21
5% gap	17	64	62	1187	92	21

(b) Computation time

Figure 10. Comparison between different approaches for solving the hardest instances of OCC. *no* means that no solution is computed within the time limit. Times are in seconds.

with any coalescing algorithm. Iterated register coalescing, a commonly used algorithm, becomes of better quality when embedded in our strategy. In our experiments, the cost of the solution was divided by three on average. One could expect that more recent and efficient heuristics lead to better solutions with the help of our strategy and we need to conduct more experiments in that direction.

7. Conclusion

Our main motivation was to improve register coalescing using ILP techniques. Solving an ILP problem is exponential in time and thus reducing the size of the formulation can drastically speed up the solution. Rather than reasoning on the ILP model, we have studied the impact of extreme live-range splitting on register coalescing. The main drawback of extreme live-range splitting is that it yields huge interference graphs. However, no liveness information is lost when performing register allocation on these graphs. Thus, we have defined a strategy for simplifying these graphs in order to reduce significantly the size of the ILP formulation for coalescing. This strategy relies on a graph reduction and a graph decomposition, that exploit results from graph theory. Our strategy is general enough so that any coalescing algorithm can be used together with it.

As said in [16], all the optimizations must go hand in hand to achieve top performance. When our graph reduction and graph decomposition are combined with a cutting-plane algorithm, we solve the whole optimal coalescing challenge optimally and more efficiently than previously.

Moreover, this work on extreme live-range splitting raises many questions. Indeed, it can be interesting to relax some constraints on split-blocks merging in order to design new heuristics, or to wonder if unsplitting could be done before spilling. Finally, since finding optimal solutions for spilling and coalescing separately is not elusive anymore, one could expect to solve both simultaneously and to evaluate the real gap arising from the separation.

This work is part of an on-going project that investigates the formal verification of a realistic C compiler usable for critical embedded software. Future work concern the formal verification of the optimizations described in this paper.

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References

[1] Andrew W. Appel. *Modern Compiler Implementation in ML.* Cambridge University Press, 1998.

[2] Andrew W. Appel and Lal George. Optimal coalescing challenge, 2000. `http://www.cs.princeton.edu/~appel/coalesce`.

[3] Andrew W. Appel and Lal George. Optimal spilling for CISC machines with few registers. In *PLDI'01*, pages 243–253, 2001.

[4] Peter Bergner, Peter Dahl, David Engebretsen, et al. Spill code minimization via interference region spilling. In *PLDI '97*, pages 287–295, 1997.

[5] Florent Bouchez. *A Study of Spilling and Coalescing in Register Allocation as Two Separate Phases*. PhD thesis, ENS Lyon, France, dec 2008.

[6] Florent Bouchez, Alain Darte, and Fabrice Rastello. On the complexity of register coalescing. In *CGO'07*, mar 2007.

[7] Preston Briggs. *Register Allocation via Graph Coloring*. PhD thesis, Rice University, april 1992.

[8] Philip Brisk, F.Dabiri, J.Macbeth, et al. Polynomial time graph coloring register allocation. In *Workshop on Logic and Synthesis*, 2005.

[9] G J Chaitin. Register allocation and spilling via graph coloring. *Symposium on Compiler Construction*, 17(6):98 – 105, 1982.

[10] Fred C. Chow and John L. Hennessy. The priority-based coloring approach to register allocation. *ACM TOPLAS*, 12(4):501–536, 1990.

[11] Keith D. Cooper and L. Taylor Simpson. Live range splitting in a graph coloring register allocator. In *CC '98*, pages 174–187, 1998.

[12] Changqing Fu and Kent Wilken. A faster optimal register allocator. In *MICRO 35*, pages 245–256, 2002.

[13] Lal George and Andrew W. Appel. Iterated register coalescing. *ACM TOPLAS*, 18(3):300–324, 1996.

[14] David Goodwin and Kent Wilken. Optimal and near-optimal global register allocations using 0-1 integer programming. *Softw. Pract. Exper.*, 26(8):929–965, 1996.

[15] David W. Goodwin. *Optimal and near-optimal global register allocation*. PhD thesis, University of California, 1996.

[16] Daniel Grund and Sebastian Hack. A fast cutting-plane algorithm for optimal coalescing. In *CC'07*, LNCS, pages 111–125, 2007.

[17] Rajiv Gupta, Mary Lou Soffa, and Denise Ombres. Efficient register allocation via coloring using clique separators. *ACM TOPLAS.*, 16(3):370–386, 1994.

[18] Priyadarshan Kolte and Mary Jean Harrold. Load/store range analysis for global register allocation. In *PLDI'93*, pages 268–277, 1993.

[19] Guei-Yuan Lueh and Thomas Gross. Fusion-based register allocation. *ACM TOPLAS*, 22:2000, 1997.

[20] Jinpyo Park and Soo-Mook Moon. Optimistic register coalescing. In *PACT '98*, page 196, 1998.

[21] Vivek Sarkar and Rajkishore Barik. Extended linear scan: An alternate foundation for global register allocation. In *CC'07*, LNCS, pages 141–155, 2007.

[22] Robert Endre Tarjan. Decomposition by clique separators. *Discrete Mathematics*, 55(2):221–232, 1985.

[23] Douglas B. West. *Introduction to Graph Theory (2nd Edition)*. Prentice Hall, August 2000.

[24] Mihalis Yannakakis and Fanica Gavril. The maximum k-colorable subgraph problem for chordal graphs. *Inf. Process. Lett.*, 24(2):133–137, 1987.