On the stability of walking systems

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Abstract

We reconsider here the stability criteria usually proposed for the analysis of walking systems, exhibiting their limits and their ambiguity. We propose then some new criteria based on a thorough analysis of the dynamics of walking systems and precise definitions concerning their stability. Numerical methods are presented then to deal with these new criteria.

1 Introduction

A mechanical system is walking if it has multiple contacts with the ground which are regularly broken and recovered in order to obtain a displacement of the whole structure. There lies an ability to come across obstacles with great versatility, but there also lies an intrinsic instability. Apart from the need for very high-technology, it is this instability that has been and is still the main bridle to the developpment of walking systems. This has been the object of many diverse analyses [3, 5, 6, 9], but without obtaining so far a complete understanding of the question.

Here, after presenting the usual stability criteria in section 2, sections 3 and 4 precise the movements that walking systems can realize; section 5 proposes new definitions concerning their stability properties, while section 6 presents some numerical methods to deal with them.

2 Usual stability criteria

2.1 The basic laws of mechanics

The basic laws of mechanics state that the dynamic wrench of an object is strictly equal to the total wrench of the exterior forces acting on it. Therefore, for a specific movement to be realized, appropriate exterior forces will have to be applied. As long as generic walking systems are considered (no thrusters, for example), exterior forces are solely gravity and contact forces, and gravity being unalterable, appropriate contact forces will be the only way for the system to effectively realize any specific movement.

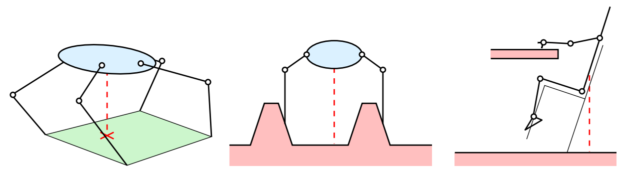

Figure 1: A walking system is generally supposed to be statically balanced if and only if its center of mass projects vertically inside the convex hull of its contact points (left). There are situations where it does but the system is not balanced (center), and situations where it doesn't but the system is balanced (right).

Note that when walking systems are made up of rigid bodies, their dynamic wrench can be written as:

$$\begin{bmatrix} \sum m_k \ddot{x}_k \\ \sum x_k \times m_k \ddot{x}_k + R_k \mathbb{I}_k \, \dot{\omega}_k - R_k \big((\mathbb{I}_k \, \omega_k) \times \omega_k \big) \end{bmatrix} \quad (1)$$

where x_k is the position of the center of mass of the k^{th} solid, R_k the rotation matrix associated to it, ω_k its rotation speed, m_k its mass and $\mathbb{I}_k$ its inertia matrix [11].

2.2 The projection of the center of mass

For such a system to remain static, i.e. with a dynamic wrench equal to 0, the total wrench of gravity and contact forces must therefore be equal to 0. If all the contact points are on the same horizontal plane, the horizontal and vertical components of the forces and momenta can be decoupled. Considering then the vertical translation momentum and the horizontal rotation momentum, we must have:

$$\begin{cases} -\sum m_k g \, \mathbf{n} + \sum f_k \, \mathbf{n} = 0 \\ -\sum x_k \times m_k g \, \mathbf{n} + \sum p_k \times f_k \, \mathbf{n} = 0 \end{cases}$$

with $\mathbf{n}$ a vertical unit vector, p_k the position of the k^{th} contact point and f_k the vertical component of the associated contact force. With $m = \sum m_k$ the total mass of the system and $x_G = \sum m_k\, x_k / m$ its center of mass, a direct rewriting states that we must have:

$$x_G \times \mathbf{n} = \frac{\sum f_k\, p_k \times \mathbf{n}}{\sum f_k}$$

what, considering that contact forces can only point upwards ($f_k \geq 0$), can be interpreted as (figure 1, left):

> A walking system can remain static if and only if its center of mass projects vertically inside the convex hull of the contact points.

But if the contacts are made on tilted surfaces (figure 1, center) or not on the same horizontal plane, horizontal and vertical components cannot be decoupled, disallowing the whole analysis. Consider for example the case of a student swaying back on his chair: he can maintain his center of mass far behind the contacts of his chair with the ground and still keep his balance thanks to friction of his hands on his desk in front of him (figure 1, right).

The “projection of the center of mass” criterion cannot therefore discriminate correctly cases where the system can remain static from cases where it can’t.

2.3 The Zero Moment Point

Now, for the system to realize a specified movement, the total wrench of gravity and contact forces must be equal to the dynamic wrench (1). Redoing the same analysis as in section 2.2, if all the contact points are on the same horizontal plane, we must have:

$$\left\{ \begin{array}{r} -\sum m_k g\, \mathbf{n} + \sum f_k\, \mathbf{n} = (\mathbf{T}.\mathbf{n})\, \mathbf{n} \\ -\sum x_k \times m_k g\, \mathbf{n} + \sum p_k \times f_k\, \mathbf{n} = \mathbf{n} \times \mathbf{R} \times \mathbf{n} \end{array} \right.$$

with $\mathbf{T}$ and $\mathbf{R}$ the translation and rotation parts of (1), $\mathbf{n}\times\mathbf{R}\times\mathbf{n}$ being the horizontal component of $\mathbf{R}$. The same direct rewriting states here that we must have:

$$\frac{mg\, x_G \times \mathbf{n} + \mathbf{n} \times \mathbf{R} \times \mathbf{n}}{mg + \mathbf{T}.\mathbf{n}} = \frac{\sum f_k\, p_k \times \mathbf{n}}{\sum f_k}$$

what can be interpreted similarly:

> A walking system can realize a specified movement if and only if the related point defined by:
>
> $$\frac{mg\, x_G + \mathbf{n} \times \mathbf{R}}{mg + \mathbf{T}.\mathbf{n}}$$
>
> projects vertically inside the convex hull of the contact points.

Let’s consider then the horizontal rotation momentum of gravity and “dynamic” forces around the projection of this point on the ground:

$$-\left[\frac{mg\,\widehat{x_G} + \mathbf{n}\times\mathbf{R}}{mg + \mathbf{T}.\mathbf{n}}\right] \times (-mg\,\mathbf{n} - \mathbf{T}) - x_G \times mg\,\mathbf{n} - \mathbf{n}\times\mathbf{R}\times\mathbf{n}$$

$\widehat{x_G}$ being the vertical projection of x_G, having therefore only horizontal components, we can replace here $\mathbf{T}$ by $(\mathbf{T}.\mathbf{n})\,\mathbf{n}$ thanks to orthogonality properties, so that this momentum can be trivially shown to be equal to zero. The projection on the ground of the point considered here is therefore nothing else but what is usually known as the *Zero Moment Point* [5, 8, 9]. Considering as an equal definition of this point $\sum f_k\, p_k / \sum f_k$, we could have shown just as easily that the horizontal rotation momentum of contact forces is also equal to zero around this projected point, what could lead to call it also the *Center of Pressure* [5, 8].

But for the same reason as in section 2.2, this analysis is disallowed when the contacts are not coplanar: the “ZMP/CoP” criterion cannot discriminate correctly cases where the system is dynamically balanced from cases where it’s not in situations even as simple as climbing upstairs.

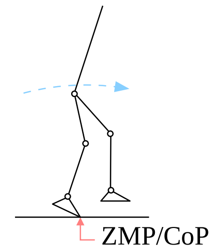

Figure 2: Tipping over an edge of contact points is not an unusual situation in stable walking (here walking forward and tipping around the toes of the foot in the back).

2.4 Stability Margins

Walking systems being particularly suited to locomotion on irregular grounds, restricting their analysis to the only case of a flat level ground can be a severe shortcoming. Still, these two criteria have been widely used to check the static or dynamic balance of walking systems.

And besides checking their balance, it has even been usual to consider that walking systems are dynamically (resp. statically) stable if their ZMP/CoP (resp. the projection of their CoM) lies strictly inside the convex hull of the contact points, unstable if it lies on the edge of the convex hull. This led to measure in many different ways the distance from these points to the edge of this convex hull, deducing a Static Stability Margin, an Energy Stability Margin, a Dynamic Stability Margin, a Tumble Stability Margin and other equivalent propositions (see [3] for a complete description of these and others).

But all of these stability margins agree to conclude that tipping over an edge of contact points is an unstable situation for a walking system, even though this happens to be not an unusual situation in stable walking (figure 2), what raises a problematic mismatch about the use of the word “stable”! We will propose then in section 5 a couple of definitions that will allow to develop a less ambiguous analysis of the stability of walking systems.

3 The dynamics of walking

When walking systems are systems of articulated rigid bodies, their complete dynamics can be written as a classical set of Euler-Lagrange equations [11]:

$$M(q)\,\ddot{q} + N(q,\dot{q})\,\dot{q} + G(q) = T(q)\,u + C(q)^T\lambda \tag{2}$$

where $T(q)\,u$ are the actuation forces and $C(q)^T\lambda$ the contact forces.

But as soon as locomotion systems are considered, the configuration vector q has to account for two different informations [2, 7, 11]: the shape of the system, that can be described by the joint positions q_1, and its position and orientation in the space, that can be described by the position and orientation q_2 of a frame attached to some part of the system. The vector q manifesting a structure:

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

we can split the dynamics (2) to exhibit the same structure:

$$\begin{bmatrix} M_1(q) \\ M_2(q) \end{bmatrix} \ddot{q} + \begin{bmatrix} N_1(q,\dot{q}) \\ N_2(q,\dot{q}) \end{bmatrix} \dot{q} + \begin{bmatrix} G_1(q) \\ G_2(q) \end{bmatrix} = \dots$$

$$\dots \begin{bmatrix} T_1(q) \\ 0 \end{bmatrix} u + \begin{bmatrix} C_1(q)^T \\ C_2(q)^T \end{bmatrix} \lambda$$

where the actuation forces don’t appear in the lower part [2, 7, 11]:

$$M_2(q)\,\ddot{q} + N_2(q,\dot{q})\,\dot{q} + G_2(q) = C_2(q)^T\lambda \tag{3}$$

A walking system can realize a movement $q(t)$ if and only if equation (2) is satisfied with appropriate actuation and contact forces $u(t)$ and $\lambda(t)$. Now, whatever the possibilities of the actuation forces, the lower part (3) has to be satisfied with the only action of contact forces, and the physics of contact is such that these forces have limitations: in the general case (no gluing, for example), contacting solids can push one another but they can’t pull one another (what is referred to as the *unilaterality* of contacts), and friction between them is limited [2, 4, 6, 11].

Seperating the tangential and normal components of the contact forces, f_t and f_n (figure 3), unilaterality appears as a simple bound:

$$f_n \geq 0$$

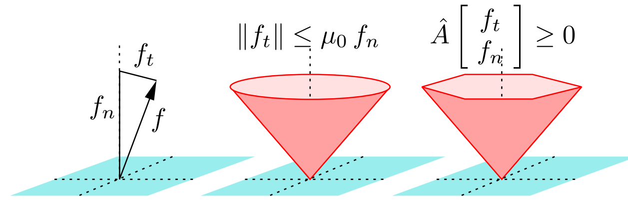

Figure 3: The contact forces can be seperated in tangential and normal components, f_t and f_n (left). Coulomb’s law of friction state then that these forces are limited to a revolution cone (center), of which we will consider a linear (convex polyhedral) approximation (right).

and according to Coulomb’s law, the limit of friction appears as a revolution cone:

$$\|\, f_t \,\| \leq \mu_0\, f_n$$

These limits apply at each contact point, but with forces defined as in (2), this can be expressed as a single vector inequality:

$$\mathcal{A}(\lambda) \geq 0$$

Considering this restriction of contact forces together with the lower part of the dynamics (3), a necessary condition for a walking system to realize a movement $q(t)$ is that there exist contact forces $\lambda(t)$ such that:

$$\begin{cases} M_2(q)\,\ddot{q} + N_2(q,\dot{q})\,\dot{q} + G_2(q) = C_2(q)^T\lambda \\ \mathcal{A}(\lambda) \geq 0 \end{cases} \tag{4}$$

And if the actuation forces are adequate enough to cope with the upper part of the dynamics, this condition is necessary and sufficient.

4 About realizable movements

4.1 A generalization of previous criteria

It can be shown [7, 11] that the lower part of the dynamics (3) is nothing else but the concatenation of the Newton and Euler equations of the whole system: except for a residual pre-multiplication by the jacobian of a rotation matrix, $M_2(q)\,\ddot{q} + N_2(q,\dot{q})\,\dot{q}$ is strictly equal to the dynamic wrench (1), and $C_2(q)^T\lambda - G_2(q)$ is strictly equal to the total wrench of contact and gravity forces.

Condition (4), forged around equation (3) to discriminate realizable movements from unrealizable ones, is therefore of the same nature as the criteria presented in sections 2.2 and 2.3: it simply relates the dynamic and gravity wrenches to the set of wrenches that can be obtained from contact forces. But contrary to these criteria, whole wrenches are considered here without having to rely on any decoupling between orthogonal components.

Making no particular assumption, condition (4) appears as a complete generalization of these criteria.

Note that this analysis could have been developped in the same setting as in section 2 since the same concepts are at work, but the Euler-Lagrange setting is preferred here since it allows a seamless integration of condition (4) at the heart of the design of control laws [2, 4, 10, 11].

4.2 The computation issue

But this completely general criterion may be complex to deal with since it needs to answer the question: does there exist a λ such that (4) is satisfied?

Numerical methods stemming from optimization theory are able to answer to this kind of question, but at a computational expense that can be hindering: it would be of the utmost interest (and it will proove to be most valuable in section 6) to turn this condition into a more straightforward criterion, a point-in-set test, for example (as in sections 2.2 and 2.3), or a set of inequalities.

Condition (4) already presents such a structure, checking whether the wrench $M_2(q)\,\ddot{q} + N_2(q,\dot{q})\,\dot{q} + G_2(q)$ is inside the set of wrenches $\mathcal{W} = \{C_2(q)^T\lambda,$ for λ s.t. $\mathcal{A}(\lambda) \geq 0\}$. What we need then is a more straightforward description of this set, but it is the projection by the linear operator $C_2(q)^T$ of the set of λs such that $\mathcal{A}(\lambda) \geq 0$, and this latter is a cartesian product of revolution cones (see section 3), what's not easy to handle.

4.3 A polyhedral approximation

If we consider a convex polyhedral approximation of the revolution cones (figure 3), their cartesian product is also a convex polyhedral cone, the projection of which through a linear operator is still a convex polyhedral cone. Algorithms dealing with convex polyhedrons, able to compute their projections, are familiar and efficient, so we can easily compute this way a convex polyhedral approximation $\widehat{\mathcal{W}}$ of $\mathcal{W}$, the precision of which directly reflects the precision of the primary approximation of revolution cones.

Now, convex polyhedrons can be simply represented by linear inequalities:

$$w \in \widehat{\mathcal{W}} \iff B_2(q)\,w \geq 0$$

where we have stressed the dependence of the linear inequalities on q since $\mathcal{W}$, and therefore $\widehat{\mathcal{W}}$ results from the linear operator $C_2(q)^T$. We can propose then a set of inequalities approximating condition (4) with a precision that can be controlled easily:

A walking system can realize a movement $q(t)$ if and only if:

$$B_2(q)\Big[M_2(q)\,\ddot{q} + N_2(q,\dot{q})\,\dot{q} + G_2(q)\Big] \geq 0$$

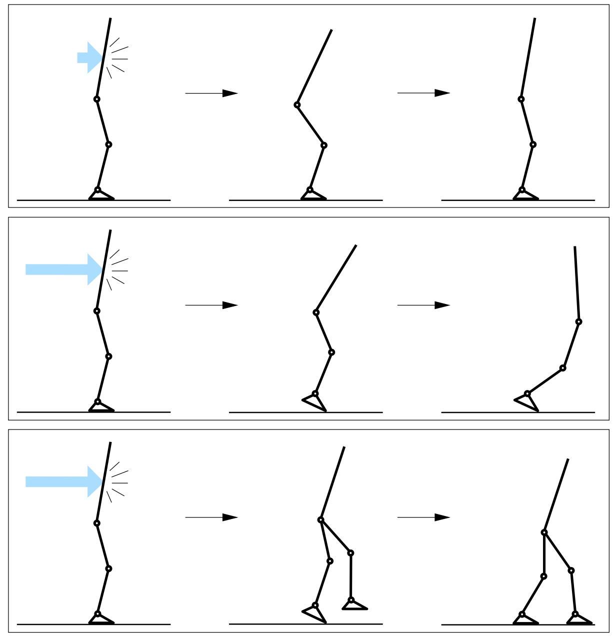

Figure 4: An erected posture with both feet on the ground can be maintained as long as perturbations are not too strong (top) otherwise available contact forces might be too limited to counter them (middle), forcing the system to make a step to avoid to fall (bottom).

5 The stability of walking systems

5.1 Avoiding to fall

Considering the example of a biped system whose objective is to maintain an erected posture with both feet on the ground (figure 4), we can observe that the available contact forces may be too limited to achieve this objective when strong perturbations occur, in which case the only way to avoid to fall may be through a momentary change of objective, making a step for example, what will allow to fulfill the initial objective later.

The point is that walking systems, relying strongly on available contact forces, can only realize movements that comply with condition (4), what may seriously interfere with the execution of any prescribed objective.

Observing that falling induces a significant risk to definitively disrupt any possibility to achieve any objective at all (in case of a major breakage, for example), and that an objective that cannot be achieved at a given moment can usually be postponed without particular contraindications, we can conclude that in the general case [10, 11]:

The major issue for walking systems is to avoid to fall, and more precise objectives can be taken care of only when this point is guaranteed.

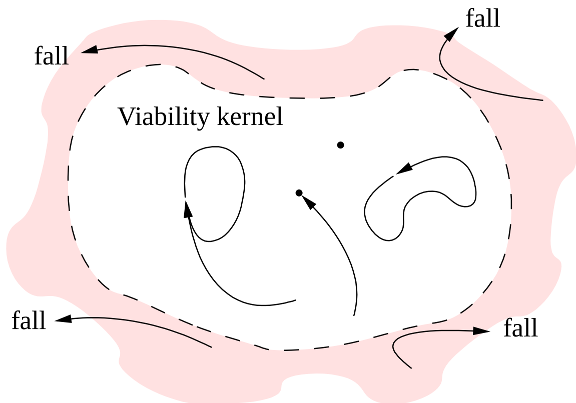

Figure 5: The viability kernel gathers all the states from which it is possible to avoid to fall, for example equilibrium points and cyclic movements. By definition, it is always possible to stay inside it and avoid to fall, while leaving it immediately implies an unavoidable fall.

5.2 Viability considerations

With $\mathcal{F}$ the set of positions where the system is considered as having fallen (where a part of the system other than the feet is in contact with the ground, for example), avoiding to fall means avoiding to be in a position $q \in \mathcal{F}$, what naturally induces the following *viability* condition [1, 10, 11]:

A state $(q, \dot{q})$ is *viable* if and only if the system is able to realize a movement $q(t)$ starting from this state that never gets inside the set $\mathcal{F}$.

This way, the state space can be split in two categories: viable states from which it is possible to avoid to fall, and non-viable states from which a fall is unavoidable. Considering the union of all viable states, what is called the *viability kernel*, the mandatory requirement for walking systems is therefore to stay inside it, what is always possible by definition, since leaving it immediately implies an unavoidable fall (figure 5).

A Viability Margin can be defined then as the distance between the state of the system and the closest non-viable state (the stability margins of section 2 being obviously of little help here).

But such a distance is most probably impossible to compute: numerous viable states can be pointed out, equilibrium points, cyclic movements or trajectories leading to one of these, but the complexity of the dynamics of walking systems is such that in the general case, it may be numerically extremely expensive or even impossible to check whether a given state is viable or not.

Adequately describing the requirements of walking systems, this notion of viability (sometimes called weak invariance, controlled invariance or conditional invariance) appears unfortunately as mostly theoretical and out of reach of numerical computations.

5.3 Invariance considerations

Still, the primary goal of a control law should be to make the viability kernel completely invariant, meaning that as long as the system is able to avoid to fall, it manages to do so (see [10, 11] for more on what this implies on the analysis and design of control laws).

In the more probable case of a control law that doesn't perfectly manage to do so, let's focus on the set of states from which a fall is effectively avoided: this subset of the viability kernel is by definition an invariant set for the control law, the largest invariant set that doesn't intersect $\mathcal{F}$.

An Invariance Margin can be defined then as the distance between the state of the system and the boundary of this largest invariant set. This margin can be more judicious than the Viability Margin defined earlier since it refers to the stability that is effectively obtained. Unfortunately, it may be just as impossible to compute.

5.4 Lyapunov stability considerations

A lower estimate of this Invariance Margin can be obtained though, with a glimpse of Lyapunov stability theory: suppose we have a function $V(q, \dot{q})$ that doesn't increase with time when the control law is active, and consider the sets $\mathcal{V}_\alpha = \{(q, \dot{q}) \text{ s.t. } V(q, \dot{q}) < \alpha\}$. The important point is that if the control law is effective on a whole set $\mathcal{V}_\alpha$, then this set is invariant.

The sets $\mathcal{V}_\alpha$ that don't intersect $\mathcal{F}$ and such that the control law is effective on the whole of them are therefore invariant sets of states for which a fall is avoided. Considering $\mathcal{V}_\Omega$, the largest of such sets, the distance between the state of the system and the boundary of $\mathcal{V}_\Omega$ provides an interesting lower estimate of the Invariance Margin defined earlier, and a formula as simple as $\Omega - V(q, \dot{q})$ presents a particularly convenient way to measure it.

This measure can be referred to as a Stability Margin, in the Lyapunov sense. Contrary to the stability margins of section 2, this one strictly relates to one control law: it can be used to measure the stability of a reference posture or trajectory, but only once a control law has been chosen to track this reference.

6 Computing a Lyapunov Stability Margin

6.1 Time-invariant control laws

Computing the stability margin defined in section 5.4 amounts to checking whether a control law is effective over a whole set $\mathcal{V}_\alpha$: we will suppose here that no particular restriction interferes with the realization of a control law, except for the irremovable condition (4). To cope efficiently with this condition, it will be preferable then to consider any time-invariant feedback $u(q, \dot{q})$ in the form of the resulting acceleration $\ddot{q} = \mathcal{U}(q, \dot{q})$ (what is not unnatural for control laws such as feedback linearizations).

Considering that the non-intersection of the sets $\mathcal{V}_\alpha$ with $\mathcal{F}$ is already taken care of, we will focus here on checking whether the control law $\mathcal{U}(q, \dot{q})$ satisfies condition (4) over a whole set $\mathcal{V}_\alpha$, looking for the largest set for which it is so:

$$\begin{aligned} \mathcal{V}_\Omega &= \max\ \mathcal{V}_\alpha \\ \text{s.t. } (q, \dot{q}) \in \mathcal{V}_\alpha &\implies \exists\,\lambda \text{ s.t.} \ldots \\ \ldots &\begin{cases} M_2(q)\,\mathcal{U}(q,\dot{q}) + N_2(q,\dot{q})\,\dot{q} + G_2(q) = C_2(q)^T \lambda \\ \mathcal{A}(\lambda) \geq 0 \end{cases} \end{aligned}$$

Reintroducing the definition of the sets $\mathcal{V}_\alpha$, and using the polyhedral approximation of condition (4) that has been proposed in section 4.3, this problem can be turned into:

$$\begin{aligned} \Omega &= \max\ \alpha \\ \text{s.t. } V(q,\dot{q}) < \alpha &\implies \ldots \\ \ldots B_2(q)&\Big[M_2(q)\,\mathcal{U}(q,\dot{q}) + N_2(q,\dot{q})\,\dot{q} + G_2(q)\Big] \geq 0 \end{aligned} \tag{5}$$

Let's consider then the alternative optimization problems:

$$\omega = \min_k\ \omega_k \tag{6}$$

with:

$$\begin{aligned} \omega_k = \inf_{q,\,\dot{q}}\ & V(q,\dot{q}) \\ & B_2^k(q)\Big[M_2(q)\mathcal{U}(q,\dot{q}) + N_2(q,\dot{q})\,\dot{q} + G_2(q)\Big] < 0 \end{aligned} \tag{7}$$

$B_2^k(q)$ being the k^{th} row of $B_2(q)$, and let's introduce some notations: $\mathcal{B}$ for the set of states such that $B_2(q)\ldots \geq 0$, $\mathcal{B}^k$ for the set of states such that $B_2^k(q)\ldots \geq 0$, and $\overline{\mathcal{B}^k}$ for the set of states such that $B_2^k(q)\ldots < 0$.

Since ω_k is the infimum of $V(q,\dot{q})$ over $\overline{\mathcal{B}^k}$, $V(q,\dot{q})$ takes values less than ω_k only outside of this set, that is in $\mathcal{B}^k$ (figure 6). In the same way, ω is the infimum of $V(q,\dot{q})$ over the union $\bigcup_k \overline{\mathcal{B}^k}$, so that $V(q,\dot{q})$ takes values less than ω only outside of this union, that is in the intersection $\bigcap_k \mathcal{B}^k = \mathcal{B}$.

We can observe then that "$V(q,\dot{q})$ takes values less than $(\cdot)$ only in $\mathcal{B}$" is an exact retranscription of the constraint of problem (5). Taking the infimum on one side and the maximum on the other side, problems (6)-(7) and problem (5) are therefore strictly equivalent:

$$\boxed{\Omega = \omega}$$

The stability margin defined in section 5.4 can therefore be computed fairly easily with the optimization problems (6)-(7), at least when the function $V(q,\dot{q})$ and the control law $\mathcal{U}(q,\dot{q})$ are continuous and differentiable (in which

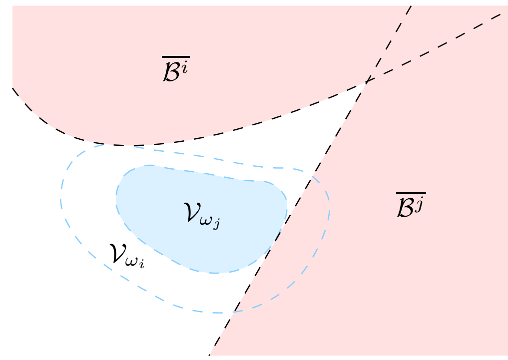

Figure 6: If ω_i is the infimum of $V(q,\dot{q})$ over $\overline{\mathcal{B}^i}$, $V(q,\dot{q})$ takes values less than ω_i only outside of this set. If ω_j is the infimum of $V(q,\dot{q})$ over $\overline{\mathcal{B}^j}$, and $\omega_j \leq \omega_i$, then $V(q,\dot{q})$ takes values less than ω_j only outside $\overline{\mathcal{B}^i}$ and $\overline{\mathcal{B}^j}$.

case the strict inequality in (7) can moreover be relaxed). Note though that great care must be taken when solving these problems since local minima might induce an overestimation of the stability margin, what can be dangerous for the safety of the system!

6.2 Time-varying control laws

For a time-varying feedback $u(t,q,\dot{q})$, such as a trajectory tracking control law, the function $V(t,q,\dot{q})$ is usually also time-varying, and so are the sets $\mathcal{V}_\alpha(t)$: the analysis of section 5.4 has therefore to be slightly adapted. When the function $V(t, q(t), \dot{q}(t))$ doesn't increase, the invariance property becomes:

$$\begin{gathered} (q(t), \dot{q}(t)) \in \mathcal{V}_\alpha(t) \\ \Downarrow \\ (q(s), \dot{q}(s)) \in \mathcal{V}_\alpha(s) \ \text{ for } \ s \geq t \end{gathered}$$

But since by definition we have $\mathcal{V}_\alpha(t) \subset \mathcal{V}_\beta(t)$ for $\beta \geq \alpha$, this property can be directly generalised to non-decreasing functions of time $\alpha(t)$, what can be useful here:

$$\begin{gathered} (q(t), \dot{q}(t)) \in \mathcal{V}_{\alpha(t)}(t) \\ \Downarrow \\ (q(s), \dot{q}(s)) \in \mathcal{V}_{\alpha(s)}(s) \ \text{ for } \ s \geq t \end{gathered} \tag{8}$$

Then, just as in section 5.4, we can consider at each instant t the largest set $\mathcal{V}_\alpha(t)$ that does not intersect $\mathcal{F}$ and such that the control law is effective at the instant t on the whole of it, and call it $\mathcal{V}_{\Omega^i(t)}(t)$. Since the instantaneous measure $\Omega^i(t)$ may not be non-decreasing, the sets $\mathcal{V}_{\Omega^i(t)}(t)$ may not satisfy as such the invariance property (8), so we must extract first its maximal non-decreasing part (figure 7):

$$\Omega(t) = \inf_{s \geq t}\ \Omega^i(s)$$

to obtain sets $\mathcal{V}_{\Omega(t)}(t)$ that duly satisfy this property, and are the largest to do so. We can safely consider then the stability margin $\Omega(t) - V(t,q,\dot{q})$, just as before.

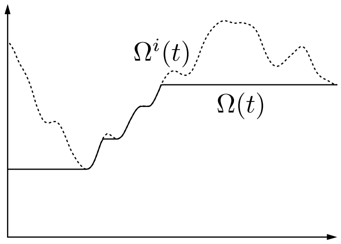

Figure 7: In the case of a time-varying control law, the instantaneous measure $\Omega^i(t)$ may not be non-decreasing, so that its maximal non-decreasing part $\Omega(t)$ must be extracted first to estimate a safe stability margin.

Now, the same computation as in section 6.1 can be carried out to obtain the instantaneous measure $\Omega^i(t)$, from which the non-decreasing part $\Omega(t)$ is immediate to derive:

$$\Omega^i(t) \;=\; \min_k \; \omega_k(t)$$

with:

$$\omega_k(t)=\inf_{q,\,\dot q}\; V(t,q,\dot q)$$
$$B_2^k(q)\Big[M_2(q)\,\mathcal{U}(t,q,\dot q)+N_2(q,\dot q)\,\dot q+G_2(q)\Big]<0$$

7 Conclusion

We have shown in section 2 that the criteria usually used to check whether a movement is realizable by a walking system, the "projection of the center of mass" criterion and the Zero Moment Point criterion, are only valid on a flat level ground. We have shown also that the stability analyses usually deduced from these criteria induce an ambiguous usage of the words "stable" and "unstable".

We have shown then in sections 3 and 4 how to conceive a complete generalization of these criteria in the classical setting of Euler-Lagrange equations, and how to derive a computation-friendly approximation of the resulting criterion.

We have proposed in section 5 to clarify the notion of "stability" of walking systems, what has led us to introduce some viability and invariance considerations. The classical notion of Lyapunov stability has asserted itself as the most convenient tool, but only to focus very specifically on concomitant invariance properties, what has led to introduce a Lyapunov Stability Margin. Numerical methods to measure this stability margin efficiently have been proposed then in section 6.

The analysis presented here clearly opens the way to proove the stability of walking systems, but there is still a long way to go before this may be fulfilled: **all** the control laws proposed so far rely on strong suppositions on the states of the contacts between the feet and the ground [6, 11]. The variation of the state of these contacts is a very difficult issue to deal with (it has been completely ignored here), and the theory needed is only emerging [6]. Still, to proove the stability of walking systems will be impossible as long as this question is not considered more thoroughly.

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