

# An observation algorithm for nonlinear systems with unknown inputs

Jean-Pierre Barbot, Driss Boutat, Thierry Floquet

► **To cite this version:**

Jean-Pierre Barbot, Driss Boutat, Thierry Floquet. An observation algorithm for nonlinear systems with unknown inputs. *Automatica*, Elsevier, 2009, 45 (8), pp.1970-1974. 10.1016/j.automatica.2009.04.009 . inria-00391819

**HAL Id: inria-00391819**

**<https://hal.inria.fr/inria-00391819>**

Submitted on 4 Jun 2009

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# An observation algorithm for nonlinear systems with unknown inputs

Jean-Pierre Barbot<sup>1,4</sup>, Driss Boutat<sup>2</sup> and T. Floquet<sup>3,4</sup>

<sup>1</sup>Equipe Commande des Systèmes (ECS), ENSEA,  
6 Avenue du Ponceau, 95014 Cergy, France.

<sup>2</sup>LVR/ENSIB, 10 Boulevard de Lahitolle, 18020 Bourges, France.

<sup>3</sup>LAGIS UMR CNRS 8146, Ecole Centrale de Lille,  
BP 48, Cité Scientifique, 59651 Villeneuve-d'Ascq, France

<sup>4</sup>Equipe Projet ALIEN, INRIA Lille - Nord Europe, France.

## Abstract

This paper provides some new developments in the design of unknown input observers for nonlinear systems. An algorithm which states if the state and the unknown input of the system can be recovered in finite time is introduced. This algorithm leads to the transformation of the system into an extended block triangular observable form suitable for the design of finite time observers. The proposed method is useful to relax some restrictive conditions of existing nonlinear unknown input observer design procedures.

## 1 Introduction

It is of importance to design observers for multivariable linear or nonlinear systems partially driven by unknown inputs. Such a problem arises in systems subject to disturbances or with inaccessible inputs and in many applications such as parameter identification, fault detection and isolation or cryptography. The design of observers for nonlinear systems is a challenging problem, even for accurately known systems. It has received a considerable amount of attention in the literature. In many approaches, nonlinear coordinate transformations are used to transform the system into suitable observer canonical forms. Then, observers with linearizable error dynamics (see e.g. [9, 20, 21]), high gain observers [8] or backstepping observers [15] can be designed. Few works deal with the design of unknown input observers for nonlinear systems. Some of them are concerned with applications in the field of fault detection and identification, and in particular the nonlinear Fundamental Problem of Residual Generation [4, 10, 19]. Other ones deal with nonlinear systems subject to exogenous perturbations and are based on sliding mode considerations [22]. The convergence is obtained in finite time under the assumptions that the system can be put

into a set of triangular observable forms, where the unknown inputs act only on the last dynamics of each triangular form. This assumption is known as the observability matching condition. The present work aims at the development of a systematic method leading to the finite time observation of a class of nonlinear systems with unknown inputs even if the observability matching condition is not fulfilled. To this end, the procedure given in [5] is extended to the nonlinear case. It also results in a constructive algorithm that transforms the system into a similar type of block triangular observable forms. This transformation relies on the introduction of suitable fictitious outputs. Sufficient conditions for the existence of such auxiliary variables are given. After transformation of the system, it is shown that it is possible to design observers that provide the finite time estimation of both the state variables and the unknown inputs. An illustrative example highlights the efficiency of the proposed methodology.

## 2 Problem statement and motivations

Consider, on an open set  $U$ , the nonlinear system:

$$\begin{cases} \dot{x} = f(x) + g(x)w = f(x) + \sum_{i=1}^m g_i(x)w_i \\ y = h(x) = [h_1(x), \dots, h_p(x)]^T \end{cases} \quad (1)$$

where  $x \in U \subset \mathbb{R}^n$  is the state vector,  $y \in \mathbb{R}^p$  is the output vector and where  $w = [w_1, \dots, w_m] \in \mathbb{R}^m$  represents the unknown inputs. The vector fields  $f$  and  $g_1, \dots, g_m$ , and the functions  $h_1, \dots, h_p$ , are assumed to be sufficiently smooth on  $U$ . Without loss of generality, it is assumed that  $p \geq m$ , and that for all  $x \in U$ , the distribution  $G = \text{span}\{g_1, \dots, g_m\}$  and the codistribution  $\text{span}\{dh_1, \dots, dh_p\}$  are nonsingular on  $U$ . Like in [17], let us first define the unknown input characteristic indexes  $\{\rho_1, \dots, \rho_p\}$  such that, for  $1 \leq i \leq p$ :

$$\begin{aligned} L_{g_j} L_f^k h_i(x) &= 0, \text{ for } k < \rho_i - 1, \text{ and for all } 1 \leq j \leq m, \\ L_{g_j} L_f^{\rho_i - 1} h_i(x) &\neq 0, \text{ for at least one } 1 \leq j \leq m, \end{aligned}$$

for all  $x \in U$ . The system (1) with  $w = 0$  is supposed to be locally weakly observable on  $U$  [11]. Thus, there exists a change of coordinates

$$\phi = \left( h_1, \dots, L_f^{\nu_1 - 1} h_1, \dots, h_p, \dots, L_f^{\nu_p - 1} h_p \right)^T$$

such that, after a suitable reordering of the state components, the system (1) is locally transformed into

$$\dot{\xi}_i = A_{\delta_i} \xi_i + H_{\delta_i} V_{\delta_i}(x, w) \quad (2)$$

$$\dot{\eta} = a(\xi, \eta) + b(\xi, \eta)w \quad (3)$$

$$y_i = C_{\delta_i} \xi_i$$

with  $\xi = (\xi_1^T, \dots, \xi_p^T)^T$ ,  $\xi_i \in \mathbb{R}^{\delta_i}$ ,  $1 \leq i \leq p$

$$V_{\delta_i}(x, w) = L_f^{\delta_i} h_i(x) + \sum_{j=1}^m L_{g_j} L_f^{\delta_i-1} h_i(x) w_j \in \mathbb{R}$$

$$A_{\delta_i} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{\delta_i \times \delta_i}$$

$$H_{\delta_i} = (0 \ 0 \ \dots \ 1)^T \in \mathbb{R}^{\delta_i}, \quad C_{\delta_i} = (1 \ 0 \ \dots \ 0) \in \mathbb{R}^{1 \times \delta_i}$$

and  $a$ ,  $b$  are smooth vector fields on  $U$ . The integers  $(\nu_1, \nu_2, \dots, \nu_p)$  are the so-called observability indices of system (1) (see [14] for a definition) and thus satisfy  $\nu_1 + \dots + \nu_p = n$ . From the definition of the  $\rho_i$ , one has  $\delta_i = \min(\nu_i, \rho_i)$  and  $g_{\delta_i} = 0$  if  $\nu_i < \rho_i$  (i.e.  $\delta_i = \nu_i$ ).

## 2.1 A triangular observable form

Most existing nonlinear observers for system (1) are designed under the assumption that the system satisfies the so-called observability matching condition. This condition was first formulated for SISO linear systems in [18], Chapter 4. In the general case of MIMO nonlinear systems, a necessary and sufficient condition is given by:

$$\nu_i \leq \rho_i \text{ for all } 1 \leq i \leq p. \quad (4)$$

Then,  $\delta_i = \nu_i$  for all  $1 \leq i \leq p$ , and under the change of coordinates  $\xi = \phi(x)$  the system (1) is transformed into the form:

$$\begin{aligned} \dot{\xi}_i &= A_{\nu_i} \xi_i + H_{\nu_i} V_{\nu_i}(\xi, w) \\ y_i &= C_{\nu_i} \xi_i \end{aligned} \quad (5)$$

with  $\sum_{j=1}^m L_{g_j} L_f^{\nu_i-1} h_i(x) w_j \neq 0$  if and only if  $\nu_i = \rho_i$ . Each subsystem of (5) is in the so-called triangular observable form. Finite time observers for such a form can be found in the literature, based on step-by-step sliding mode techniques [22], higher order sliding modes [6, 16], numerical approaches [3, 13], or algebraic methods [2]. The design of such observers is left to the reader, since they are straightforward applications of existing results. Nevertheless, depending on the choice of the observer, some assumptions have to be introduced. For instance, in all the previously mentioned works, the unknown input  $w$  has, at least, to be bounded. A necessary and sufficient condition for the recovery of the unknown inputs is that

$$\Gamma(x) = \begin{pmatrix} L_{g_1} L_f^{\rho_1-1} h_1(x) & \dots & L_{g_m} L_f^{\rho_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{\rho_p-1} h_p(x) & \dots & L_{g_m} L_f^{\rho_p-1} h_p(x) \end{pmatrix}$$

has rank  $m$ . In this case,  $\rho = \{\rho_1, \dots, \rho_p\}$  is the vector relative degree as defined in [12], p. 220, when  $m = p$ .

## 2.2 An extended triangular observable form

The aim of this paper is to provide an observation algorithm that allows for the finite time estimation of both the state and the unknown inputs of (1) even if  $\nu_j > \rho_j$  for at least a  $j$  in  $\{1, \dots, p\}$ . Consider again the general form (2-3). Applying any of the aforementioned finite time observers: (i) for  $1 \leq i \leq p$ ,  $\xi_i$  can be estimated in finite time; (ii) one can also recover in finite time the last component  $V_{\delta_i}$  of each subsystem of (2). The problem is to recover the remaining state  $\eta$ . Denote:

$$V(x) = \begin{pmatrix} V_{\delta_1}(x, w) \\ V_{\delta_2}(x, w) \\ \vdots \\ V_{\delta_p}(x, w) \end{pmatrix} = \begin{pmatrix} L_f^{\delta_1} h_1(x) \\ L_f^{\delta_2} h_2(x) \\ \vdots \\ L_f^{\delta_p} h_p(x) \end{pmatrix} + \Gamma_\delta(x)w$$

where

$$\Gamma_\delta(x) = \begin{pmatrix} L_{g_1} L_f^{\delta_1-1} h_1(x) & \dots & L_{g_m} L_f^{\delta_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{\delta_p-1} h_p(x) & \dots & L_{g_m} L_f^{\delta_p-1} h_p(x) \end{pmatrix}.$$

Let  $\mathcal{L}$  be the commutative algebra of the measured outputs and their successive Lie derivatives up to order  $\delta_i$ :  $\mathcal{L} = \text{span}\{h_1, \dots, L_f^{\delta_1-1} h_1, \dots, h_p, \dots, L_f^{\delta_p-1} h_p\}$  and let  $d\mathcal{L}$  be the codistribution:

$$d\mathcal{L} = \text{span}\{dh_1, \dots, dL_f^{\delta_1-1} h_1, \dots, dh_p, \dots, dL_f^{\delta_p-1} h_p\}.$$

Assume there exists a  $1 \times p$  row vector  $K(x) = (k_1(x), \dots, k_p(x)) \neq 0$ ,  $k_i \in \mathcal{L}$  for  $1 \leq i \leq p$ , such that:

$$K(x)\Gamma_\delta(x) = 0 \text{ for all } x \in U \quad (6)$$

and set:

$$\bar{y} = \bar{h}(x) = K(x)V(x) = \sum_{i=1}^p k_i(x) L_f^{\delta_i} h_i(x) \quad (7)$$

Note that  $\bar{y}$  is an available information (after a finite time) and is not affected by the unknown inputs. Therefore, if  $d\mathcal{L} + \text{span}\{d\bar{h}\} \subsetneq d\mathcal{L}$ ,  $\bar{y}$  can be considered as an additional fictitious output. Then, let  $\bar{\rho}_i$  and  $\bar{\nu}_i$  be the unknown input characteristic indexes and the observability indices of (1) with respect to the extended output  $[y^T, \bar{y}]^T = [h^T, \bar{h}(x)]^T$ . If  $\bar{\nu}_i \leq \bar{\rho}_i$  for all  $1 \leq i \leq p+1$ , the

system (1) can be transformed into:

$$\begin{aligned}\dot{\xi}_i &= A_{\bar{\nu}_i} \xi_i + H_{\bar{\nu}_i} V_{\bar{\nu}_i}(\tilde{\xi}, w) \text{ for } 1 \leq i \leq p \\ y_i &= C_{\bar{\nu}_i} \xi_i \\ \dot{\bar{\xi}} &= A_{\bar{\nu}_{p+1}} \bar{\xi} + H_{\bar{\nu}_{p+1}} V_{\bar{\nu}_{p+1}}(\tilde{\xi}, w) \\ \bar{y} &= C_{\bar{\nu}_{p+1}} \bar{\xi}\end{aligned}$$

where  $\tilde{\xi} = (\xi_1^T, \dots, \xi_p^T, \bar{\xi}^T)^T \in \mathbb{R}^n$ . Then, it is possible to recover both the state and the unknown inputs in finite time. Let us give sufficient conditions for the existence of a suitable fictitious output  $\bar{y}$ . For this, the following notations are introduced:

- i)  $G^\perp = \text{span}\{\varpi_1, \dots, \varpi_{n-m}\}$ , the annihilator of  $G$  ( $\varpi_i$  are 1-forms such that  $\iota_{g_k} \varpi_i = 0$ , where  $\iota_g \varpi$  is the inner product of the vector field  $g$  and the 1-form  $\varpi$ ).
- ii)  $\Omega_{\mathcal{L}}$ , the module spanned by  $d\mathcal{L}$  over  $\mathcal{L}$ .

**Proposition 1** *The following conditions are equivalent:*

- i) Equation (6) has a solution  $K$  and  $KV \notin \mathcal{L}$ .
- ii)  $\Xi = \text{span}\{\varpi \in G^\perp \cap \Omega_{\mathcal{L}} \text{ such that } \iota_f \varpi \notin \mathcal{L}\} \neq \{0\}$ .

**Proof:** Set  $\varpi = \sum_{i=1}^p k_i dL_f^{\delta_i-1} h_i$  with  $k_i \in \mathcal{L}$ . Clearly,  $\varpi \in \Omega_{\mathcal{L}}$  and

$$\iota_f \varpi = \iota_f \sum_{i=1}^p k_i dL_f^{\delta_i-1} h_i = \sum_{i=1}^p k_i L_f^{\delta_i} h_i = KV = \bar{y}.$$

$$K\Gamma_\delta = K \begin{bmatrix} dL_f^{\delta_1-1} h_1 \\ \vdots \\ dL_f^{\delta_p-1} h_p \end{bmatrix} [g_1 \quad \cdots \quad g_m] = \varpi \cdot [g_1 \quad \cdots \quad g_m]$$

Thus:  $\{K \text{ is a solution of (6) such that } KV \notin \mathcal{L}\} \iff \{\varpi \in \Omega_{\mathcal{L}}, \iota_\tau \varpi = 0 \text{ for any } \tau \in G \text{ and } \iota_f \varpi \notin \mathcal{L}\} \iff \{\varpi \in G^\perp \cap \Omega_{\mathcal{L}} \text{ and } \iota_f \varpi \notin \mathcal{L}\} \iff \Xi \neq \{0\}$ .  $\square$

The discussion above can be recursively generalized as follows. Assume that the condition (4) is not still satisfied with the extended output obtained with the solutions of (6). On the basis of this new output, the corresponding matrix  $\Gamma_{\bar{\delta}}$  can be computed and another set of fictitious outputs can eventually be found. One can iterate this procedure until the condition (4) is fulfilled for a new extended output. Then, the original system can be put into an extended

block triangular observable form<sup>1</sup>:

$$\begin{aligned}
\dot{\xi}_i^1 &= A_{\nu_i^1} \xi_i^1 + H_{\nu_i^1} V_{\nu_i^1}(\xi, w) \\
y_i^1 &= y_i = C_{\nu_i^1} \xi_i^1, \quad 1 \leq i \leq p^1 \\
\dot{\xi}_i^2 &= A_{\nu_i^2} \xi_i^2 + H_{\nu_i^2} V_{\nu_i^2}(\xi, w) \\
y_i^2 &= C_{\nu_i^2} \xi_i^2, \quad 1 \leq i \leq p^2 \\
&\vdots \\
\dot{\xi}_i^{k^*} &= A_{\nu_i^{k^*}} \xi_i^{k^*} + H_{\nu_i^{k^*}} V_{\nu_i^{k^*}}(\xi, w) \\
y_i^{k^*} &= C_{\nu_i^{k^*}} \xi_i^{k^*}, \quad 1 \leq i \leq p^{k^*}
\end{aligned} \tag{8}$$

where the integers  $\nu_i^j$  are the observability indices of the system (1) with the new outputs  $y_i^j$ . The first subsystem is fed by the original outputs of the system. A finite time observer is designed to estimate the state of this subsystem and to provide in finite time the knowledge of the fictitious outputs  $y_i^2$ ,  $1 \leq i \leq p^2$ . Then, the state of the second triangular observable form can be estimated as well as the fictitious outputs  $y_i^3$ . Thus, one can recursively obtain the whole state of the system in finite time.

**Remark 1** *The finite time property is required to ensure that one obtains a fast and accurate estimation of the fictitious outputs (for instance, via the equivalent output injection in the case of sliding mode observers, see [6] and the references therein). Furthermore, this property is often desirable in the framework of observation and particularly for the purpose of observer-based controller design for nonlinear systems. Then, for a large class of nonlinear systems, the observer can be designed separately from the controller and the separation principle does not need to be proved. It can also be of paramount importance in applications that require fast estimations of some unknown inputs like fault detection and identification or on-line parameter identification.*

### 3 Nonlinear unknown input observer algorithm

An algorithm that states if the system can be transformed into (8) and that provides the integers  $p^j$ ,  $\nu_i^j$  and the auxiliary outputs  $y_i^j$  ( $j = 1, \dots, k^*$ ) is now given.

**Step 0:** Compute  $G$  and its annihilator  $G^\perp$ . Set  $p^1 = p$ ,  $[h_1^1, \dots, h_{p^1}^1]^T = [h_1, \dots, h_p]^T$ ,  $\mu^0 = 0$ ,  $z_1^0 = \dots = z_{\mu^0}^0 = 0$ .

**Step  $\alpha$ :** [a] Consider  $y^\alpha = [h_1^\alpha, \dots, h_{p^\alpha}^\alpha]^T \in \mathbb{R}^{p^\alpha}$  and reorder its components as follows:

$$y^\alpha = [h_1^\alpha, \dots, h_{l_\alpha}^\alpha, h_{l_\alpha+1}^\alpha, \dots, h_{p^\alpha}^\alpha]^T$$

---

<sup>1</sup>Systems that admit such a form belong to the class of left invertible systems with trivial zero dynamics (see [2]).

such that for  $1 \leq j \leq l^\alpha$ :

$$\forall i \in [1, \dots, m], \forall k \in \mathbb{N} \quad L_{g_i} L_f^k h_j^\alpha = 0$$

and for  $1 \leq j \leq p^\alpha - l^\alpha$ , there exists an integer  $\rho_j^\alpha$  such that:

$$\begin{aligned} \forall i \in [1, \dots, m] \quad L_{g_i} L_f^k h_{l^\alpha+j}^\alpha &= 0 \quad \forall k < \rho_j^\alpha - 1 \\ \exists i \in [1, \dots, m] \quad L_{g_i} L_f^{\rho_j^\alpha-1} h_{l^\alpha+j}^\alpha &\neq 0. \end{aligned}$$

[b] Define  $\Phi^\alpha = \{dh_1^\alpha, \dots, dL_f^{n-1} h_1^\alpha, \dots, dh_{l^\alpha}^\alpha, \dots, dL_f^{n-1} h_{l^\alpha}^\alpha\}$ . Compute

$$I^\alpha = \text{span} \left\{ \left( dz_1^{\alpha-1}, \dots, dz_{\mu^{\alpha-1}}^{\alpha-1} \right) \cup \Phi^\alpha \right\}.$$

Let  $\dim I^\alpha = \mu^{\alpha-1} + \varphi^\alpha$ .  $I^\alpha$  can be written as follows:

$$I^\alpha = \text{span} \{ dz_1^{\alpha-1}, \dots, dz_{\mu^{\alpha-1}}^{\alpha-1}, dh_1^\alpha, \dots, dL_f^{\varphi_1^\alpha-1} h_1^\alpha, \dots, dh_{l^\alpha}^\alpha, \dots, dL_f^{\varphi_{l^\alpha}^\alpha-1} h_{l^\alpha}^\alpha \}$$

with  $\sum_{i=1}^{l^\alpha} \varphi_i^\alpha = \varphi^\alpha$ . If  $\mu^{\alpha-1} + \varphi^\alpha = n$ , set

$$\begin{aligned} \mu^{k^*} &= \mu^{\alpha-1} + \varphi^\alpha \\ \{ dz_1^{k^*}, \dots, dz_{\mu^{k^*}}^{k^*} \} &= \{ dz_1^{\alpha-1}, \dots, dz_{\mu^{\alpha-1}}^{\alpha-1}, dh_1^\alpha, \dots, dL_f^{\varphi_1^\alpha-1} h_1^\alpha, \dots, dh_{l^\alpha}^\alpha, \dots, dL_f^{\varphi_{l^\alpha}^\alpha-1} h_{l^\alpha}^\alpha \} \end{aligned}$$

and stop the algorithm.

[c] If  $\mu^{\alpha-1} + \varphi^\alpha < n$ , consider the outputs affected by the unknown inputs and define:

$$\Upsilon^\alpha = \{ dh_{l^\alpha+1}^\alpha, \dots, dL_f^{\rho_1^\alpha-1} h_{l^\alpha+1}^\alpha, \dots, dh_{p^\alpha}^\alpha, \dots, dL_f^{\rho_{p^\alpha-l^\alpha}^\alpha-1} h_{p^\alpha}^\alpha \}$$

Compute the codistribution  $\Omega^\alpha = \text{span} \{ I^\alpha \cup \Upsilon^\alpha \}$ . Let  $\dim \Omega^\alpha = \mu^{\alpha-1} + \varphi^\alpha + \kappa^\alpha = \mu^\alpha$  and write  $\Omega^\alpha = \text{span} \{ dz_1^\alpha, \dots, dz_{\mu^\alpha}^\alpha \}$  with

$$\begin{aligned} \{ dz_1^\alpha, \dots, dz_{\mu^\alpha}^\alpha \} &= \{ dz_1^{\alpha-1}, \dots, dz_{\mu^{\alpha-1}}^{\alpha-1}, dh_1^\alpha, \dots, dL_f^{\varphi_1^\alpha-1} h_1^\alpha, \\ &\dots, dh_{l^\alpha}^\alpha, \dots, dL_f^{\varphi_{l^\alpha}^\alpha-1} h_{l^\alpha}^\alpha, dh_{l^\alpha+1}^\alpha, \dots, dL_f^{\kappa_1^\alpha-1} h_{l^\alpha+1}^\alpha, \\ &\dots, dh_{p^\alpha}^\alpha, \dots, dL_f^{\kappa_{p^\alpha-l^\alpha}^\alpha-1} h_{p^\alpha}^\alpha \}. \end{aligned}$$

and  $\sum_{i=1}^{p^\alpha-l^\alpha} \kappa_i^\alpha = \kappa^\alpha$ . If  $\mu^\alpha = n$ , the algorithm stops.

[d] Otherwise,  $\mu^\alpha < n$ . Define

$$\begin{aligned} \mathcal{L}^\alpha &= \text{span} \{ z_1^\alpha, \dots, z_{\mu^\alpha}^\alpha \} \\ \Omega_{\mathcal{L}}^\alpha &= \text{span} \left\{ \sum_{i=1}^{\mu^\alpha} \phi_i dz_i^\alpha, \phi_i \in \mathcal{L}^\alpha \right\} \\ \Xi^\alpha &= \text{span} \{ \varpi \in G^\perp \cap \Omega_{\mathcal{L}}^\alpha \text{ such that } \iota_f \varpi \notin \mathcal{L}^\alpha \}. \end{aligned}$$



Let  $p^{\alpha+1} = \dim \Xi^\alpha$ . If  $p^{\alpha+1} = 0$ , the state of the system (1) can not be recovered with the method described in this paper and the algorithm stops. Otherwise, there exist  $p^{\alpha+1}$  one-forms  $\varpi_i$  such that  $\Xi^\alpha = \text{span}\{\varpi_1, \dots, \varpi_{p^{\alpha+1}}\}$  and one can define the following vector of fictitious outputs, suitable to the problem (see Proposition 1):  $y^{\alpha+1} = [\iota_f \varpi_1, \dots, \iota_f \varpi_{p^{\alpha+1}}]^T$ . Set  $[h_1^{\alpha+1}, \dots, h_{p^{\alpha+1}}^{\alpha+1}]^T = [\iota_f \varpi_1, \dots, \iota_f \varpi_{p^{\alpha+1}}]^T$ . Go to [a].

If the algorithm stops for some  $\mu^{k^*} = n$ , the change of coordinates

$$\phi = \left( z_1^{k^*}, \dots, z_{\mu^{k^*}}^{k^*} \right)^T$$

is well defined and transforms the system into a set of block triangular observable forms similar to (8). The functions  $h_j^i$ , and the integers  $p^i, l^i, \varphi_j^i, \kappa_j^i$  are obtained in each  $i$ -th iteration of the algorithm.

**Remark 2** *If the condition (4) is satisfied for the measured outputs of the system (1),  $\mu^1 = n$  after the first iteration of the algorithm and the system is exactly transformed into the form (5) that is usually considered for the design of asymptotic (see e.g. [8, 15]) or robust finite time ([6, 22]) nonlinear observers.*

## 4 Estimation of the unknown inputs

If the algorithm ends in a positive way, the unknown inputs can also be obtained in a finite time. Indeed, the use of a finite time observer provides an estimation of the state, say  $\tilde{x}$ , and the knowledge of the following quantities (from the last line of each block of the triangular form):

$$\tilde{\theta}_j^i = L_f^{\kappa_j^i} h_{l^i+j}^i(\tilde{x}) + \sum_{s=1}^m L_{g_s} L_f^{\kappa_j^i-1} h_{l^i+j}^i(\tilde{x}) w_s, \quad (9)$$

for  $1 \leq j \leq p^i - l^i$ , and  $1 \leq i \leq k^*$ . The relations (9) can be rewritten as:  $\Lambda(\tilde{x})w = \Theta(\tilde{x})$  with

$$\Lambda(\tilde{x}) = \begin{pmatrix} L_{g_1} L_f^{\kappa_1^1-1} h_{l^1+1}^1(\tilde{x}) & \dots & L_{g_m} L_f^{\kappa_1^1-1} h_{l^1+1}^1(\tilde{x}) \\ \vdots & & \vdots \\ L_{g_1} L_f^{\kappa_{p^{k^*}}^{k^*}-l^{k^*}-1} h_{p^{k^*}}^{k^*}(\tilde{x}) & \dots & L_{g_m} L_f^{\kappa_{p^{k^*}}^{k^*}-l^{k^*}-1} h_{p^{k^*}}^{k^*}(\tilde{x}) \end{pmatrix}$$

$$\Theta(\tilde{x}) = \begin{pmatrix} \tilde{\theta}_1^1 - L_f^{\kappa_1^1} h_{l^1+1}^1(\tilde{x}) \\ \vdots \\ \tilde{\theta}_{p^{k^*}-l^{k^*}}^{k^*} - L_f^{\kappa_{p^{k^*}}^{k^*}-l^{k^*}} h_{p^{k^*}}^{k^*}(\tilde{x}) \end{pmatrix}$$

Since the distribution  $\text{span}\{g_1, \dots, g_m\}$  is assumed to be nonsingular, the matrix

$$\Lambda(\tilde{x}) = \begin{bmatrix} dL_f^{\kappa_1^1-1} h_{l_1+1}^1(\tilde{x}) \\ \vdots \\ dL_f^{\kappa_{p^k}^k-l^*-1} h_{p^k}^{k^*}(\tilde{x}) \end{bmatrix} [g_1 \quad \dots \quad g_m]$$

has rank  $m$  on every subset of  $U$  where at least  $m$  one-forms  $dL_f^{\kappa_j^i-1} h_{l_i+j}^i$  does not belong to  $G^\perp$ . Thus, an estimation of the unknown input is given by:  $\tilde{w} = \Lambda^+(\tilde{x})\Theta(\tilde{x})$  where  $\Lambda^+$  is a well defined pseudo-inverse of  $\Lambda$ .

## 5 Example

As a way of illustration<sup>2</sup>, consider the following nonlinear system subject to the unknown input  $w = [w_1, w_2]^T$

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^3 \\ \dot{x}_2 &= x_3 + x_2^2 - x_2^3 + a(x_3, x_4)w_1 \\ \dot{x}_3 &= x_5 \\ \dot{x}_4 &= -x_4 + x_2^2 + b(x_2, x_3)w_1 \\ \dot{x}_5 &= -x_3 + x_2w_2 \\ \dot{x}_6 &= -x_6 + w_2 \end{aligned} \tag{10}$$

with outputs  $y_1 = x_1, y_2 = x_4$  and  $y_3 = x_6$ . The scalar functions  $a$  and  $b$  are such that  $a(x_3, x_4) = a_1(x_3)a_2(x_4)$ ,  $b(x_2, x_3) = a_1(x_3)b_2(x_2)$  and where  $a(0, 0) \neq 0$  and  $b(0, 0) \neq 0$ , and  $b_2(x_2) \neq 0, \forall x_2 \in \mathbb{R}$ . For this system, one has  $\rho_1 = 2, \rho_2 = \rho_3 = 1$  and  $\nu_1 = 4, \nu_2 = \nu_3 = 1$  (as a consequence  $\delta_1 = 2, \delta_2 = \delta_3 = 1$ ). Thus, the necessary and sufficient condition (4) for a system to be transformed into a form similar to (5) is not fulfilled. Furthermore, the distribution  $G$  is not involutive. As a consequence, the sliding mode observer proposed in [22], where the distribution spanned by the unknown input channels has to be involutive, can not be designed here. However, the procedure proposed in this paper is applicable. The annihilator of  $G$  is given by:

$$G^\perp = \text{span}\{dx_1, dx_3, b_2 dx_2 - a_2 dx_4, dx_5 - x_2 dx_6\}$$

In the first step of the algorithm,  $I^1$  is empty and

$$\Omega^1 = \text{span}\{dh_1, dL_f h_1, dh_2, dh_3\} = \text{span}\{dx_1, -3x_1^2 dx_1 + dx_2, dx_4, dx_6\}$$

with  $\dim \Omega^1 = 4 < n$ . One has

$$\begin{aligned} \mathcal{L}^1 &= \text{span}\{h_1, L_f h_1, h_2, h_3\} = \text{span}\{x_1, x_2 - x_1^3, x_4, x_6\} \\ &= \text{span}\{x_1, x_2, x_4, x_6\} \end{aligned}$$

---

<sup>2</sup>A physical application of the proposed algorithm in the field of chaotic synchronization for secure communication can be found in [2].

and  $b_2 dx_2 - a_2 dx_4 \in \Omega_{\mathcal{L}}^1$ . Then

$$\begin{aligned} G^\perp \cap \Omega_{\mathcal{L}}^1 &= \text{span}\{dx_1, b_2 dx_2 - a_2 dx_4\} \\ \Xi^1 &= \text{span}\{b_2 dx_2 - a_2 dx_4\}. \end{aligned}$$

$\Xi^1 \neq 0$  and one can define the following fictitious output:

$$\begin{aligned} y^2 &= \iota_f(b_2 dx_2 - a_2 dx_4) \\ &= b_2(x_3 + x_2^2 - x_2^3) - a_2(-x_4 + x_2^2) = b_2(x_2)x_3 \text{ mod } [\mathcal{L}] \end{aligned}$$

Since  $b_2(x_2) \neq 0$  for all  $x_2$ , the knowledge of  $y^2$  is equivalent to the knowledge of  $x_3$ . The second step starts by defining the output vector:  $y^2 = x_3$ .  $I^2$  is empty and  $\Omega^2 = \text{span}\{dx_1, dx_2, dx_4, dx_6, dx_3, dx_5\}$  with  $\dim \Omega^2 = 6$ . Then, one can define the change of coordinates:  $z = \psi(x) = [x_1, x_2, x_4, x_6, x_3, x_5]^T$ . In the new coordinates, the system is rewritten as:

$$\begin{aligned} \dot{z}_1 &= z_2 - z_1^3 \\ \dot{z}_2 &= z_5 + z_2^2 - z_2^3 + a_1(z_5)a_2(z_3)w_1 \\ \dot{z}_3 &= -z_3 + z_2^2 + a_1(z_5)b_2(z_2)w_1 \\ \dot{z}_4 &= -z_4 + w_2 \\ \dot{z}_5 &= z_6 \\ \dot{z}_6 &= -z_5 + z_2 w_2 \end{aligned} \tag{11}$$

By the means of a finite time observer fed by the original outputs of system (10), i.e.  $y_1 = z_1$ ,  $y_2 = z_3$  and  $y_3 = z_4$ , one can recover the state  $z_2$  and

$$\begin{aligned} V_1 &= z_5 + z_2^2 - z_2^3 + a_1(z_5)a_2(z_3)w_1 \\ V_2 &= -z_3 + z_2^2 + a_1(z_5)b_2(z_2)w_1 \\ V_3 &= -z_4 + w_2 \end{aligned}$$

For the sake of place, the design of the observer is not given here. However, the reader can for instance refer to [7] where an example of higher order sliding mode observer, designed for a similar problem in the linear case, can be extended to the problem of nonlinear systems. Then,

$$\begin{aligned} y^2 &= \iota_f(b_2(z_2)dz_2 - a_2(z_3)dz_3) \\ &= b_2(z_2)(z_5 + z_2^2 - z_2^3 + a_1(z_5)a_2(z_3)w_1) \\ &\quad - a_2(z_3)(z_2^2 + a_1(z_5)b_2(z_2)w_1) = b_2(z_2)V_1 - a_2(z_3)V_2 \\ &= b_2(z_2)(z_5 + z_2^2 - z_2^3) - a_2(z_3)(-z_3 + z_2^2) \end{aligned}$$

is known after a finite time and  $z_5 = \frac{y^2 + a_2(z_3)(-z_3 + z_2^2)}{b_2(z_2)} + z_2^3 - z_2^2$  is an available information. Again, a finite time observer leads to the recovery of  $z_6$ . Then, the unknown input can also be obtained since:  $w_1 = \frac{V_1 - z_5 - z_2^2 + z_2^3}{a_1(z_5)a_2(z_3)}$  and  $w_2 = V_3 + z_4$ .

## 6 Conclusion

In this paper, the problem of the observation of nonlinear systems subject to unknown inputs was considered. An observation algorithm that determines whether it is possible to recover the state and unknown input in finite time was introduced. When the answer is yes, the algorithm provides a change of coordinates that transforms the system in a new type of block triangular observable form well suited to the design of finite time observers. The observability matching condition usually required for the design of nonlinear unknown input observers is relaxed with the proposed method.

## References

- [1] J.-P. Barbot, I. Belmouhoub and L. Boutat-Baddas, “Observability Normal Forms”, in: *New trends in Nonlinear dynamics and control*, LNCIS 295, W. Kang et al., Eds, Springer Verlag, 2003, pp. 1–24.
- [2] J.-P. Barbot, M. Fliess, T. Floquet, An algebraic framework for the design of nonlinear observers with unknown inputs, *IEEE Conf. on Decision and Control*, New-Orleans, USA, 2007.
- [3] S. Diop, J. W. Grizzle and F. Chaplais, On numerical differentiation algorithms for nonlinear estimation, in *IEEE Conference on Decision and Control*, pp. 1133–1138, 2000.
- [4] T. Floquet, J.-P. Barbot, W. Perruquetti and M. Djemai, On the robust fault detection via a sliding mode disturbance observer, *International Journal of control*, vol. 77, 2004, pp. 622–629.
- [5] T. Floquet and J.-P. Barbot, An observability form for linear systems with unknown inputs, *International Journal of control*, vol. 79, 2006, pp. 132–139.
- [6] T. Floquet, J.P. Barbot, Super twisting algorithm based step-by-step sliding mode observers for nonlinear systems with unknown inputs, *International Journal of Systems Science*, vol. 38, 2007, pp. 803–815.
- [7] T. Floquet and J.P. Barbot, “A canonical form for the design of unknown input sliding mode observers”, in *Advances in Variable Structure and Sliding Mode Control*, Lecture Notes in Control and Information Sciences, Vol. 334, C. Edwards, E. Fossas Colet, L. Fridman, (Eds.), Springer Edition, 2006.
- [8] J. P. Gauthier, H. Hammouri and S. Othman, A simple observer for nonlinear systems with applications to bioreactors, *IEEE Transactions on Automatic Control*, Vol. 37, 1992, pp. 875–880.

- [9] A. Glumineau, C. H. Moog, and F. Plestan, New Algebro-Geometric Conditions for the Linearization by Input-Output Injection, *IEEE Transactions on Automatic Control*, vol. 41, 1996, pp. 598–603.
- [10] H. Hammouri, M. Kinnaert, and E. H. El Yaagoubi, Observer-Based Approach to Fault Detection and Isolation for Nonlinear Systems, *IEEE Transactions on Automatic Control*, vol. 44, 1999, pp. 1879–1884.
- [11] R. Hermann and A. J. Krener, “Nonlinear controllability and observability”, *IEEE Transactions on Automatic Control*, Vol. 22, pp. 728–740, 1977.
- [12] A. Isidori, *Nonlinear Control Systems, Communication and Control Engineering Series*, Third edition, Springer-Verlag, 1995.
- [13] W. Kang, Moving Horizon Numerical Observers of Nonlinear Control Systems, *IEEE Transactions on Automatic Control*, Vol. 51, No. 2, pp. 344–350, 2006.
- [14] A. J. Krener and W. Respondek, Nonlinear observers with linearizable error dynamics, *SIAM J. Control Optim.*, Vol. 23, 1985, pp. 197–216.
- [15] A. J. Krener and W. Kang, Locally convergent nonlinear observers, *SIAM J. Control Optim.*, Vol. 42, 155–177, 2003.
- [16] A. Levant, “Robust Exact Differentiation via sliding mode technique”, *Automatica*, Vol. 34, No. 3, pp. 379–384, 1998.
- [17] R. Marino, W. Respondek, A. J. Van der Schaft, Almost disturbance decoupling for single-input single-output nonlinear systems, *IEEE Transactions on Automatic Control*, Vol. 34, pp. 1013–1017, 1989.
- [18] W. Perruquetti and J.-P. Barbot, *Sliding Mode Control in Engineering*, Ed. Marcel Dekker, 2002.
- [19] C. de Persis and A. Isidori, A geometric approach to nonlinear fault detection and isolation, *IEEE Transactions on Automatic Control*, vol. 46, 2001, pp. 853–865.
- [20] W. Respondek, A. Pogromsky, H. Nijmeijer, “Time scaling for observer design with linearizable error dynamics”, *Automatica*, Vol. 40, No. 2, pp. 277–285, 2004.
- [21] X. H. Xia and W. B. Gao, Nonlinear observer design by observer error linearization, *SIAM J. Control Optim.*, vol. 27, 1989, pp. 199–216.
- [22] Y. Xiong and M. Saif, Sliding Mode Observer for Nonlinear Uncertain Systems, *IEEE Transactions on Automatic Control*, vol. 46, 2001, pp. 2012–2017.