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Abstract: A line ℓ is a *transversal* to a family $\mathcal{F}$ of convex objects in $\mathbb{R}^d$ if it intersects every member of $\mathcal{F}$. In this paper we show that for every integer $d \geq 3$ there exists a family of $2d-1$ pairwise disjoint unit balls in $\mathbb{R}^d$ with the property that every subfamily of size $2d-2$ admits a transversal, yet any line misses at least one member of the family. This answers a question of Danzer from 1957.

Key-words: Discrete Geometry, Geometric Transversal, Helly-type Theorem

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Lower Bounds for Pinning Lines by Balls

Résumé : Une droite ℓ est une *transversale* à une famille $\mathcal{F}$ de convexes de $\mathbb{R}^d$ si elle coupe chaque membre de $\mathcal{F}$. Dans cet article, nous montrons que pour tout entier $d \geq 3$, il existe une famille de $2d-1$ boules disjointes de même rayon dans $\mathbb{R}^d$ sans droite transversale et telle que toute sous-famille de taille $2d-2$ admet une droite transversale. Cela répond à une question de Danzer de 1957.

Mots-clés : Géométrie Discrète, Transversales Géométriques, Théorèmes à la Helly

1 Introduction

A straight line that intersects every member of a family $\mathcal{F}$ of compact convex sets in $\mathbb{R}^d$ is called a *line transversal* to $\mathcal{F}$. An important problem in *geometric transversal theory* is to give sufficient conditions on $\mathcal{F}$ that guarantee the existence of a transversal. As an example consider the following result due to Danzer [3].

Theorem (Danzer, 1957)**.** *A family $\mathcal{F}$ of pairwise disjoint congruent disks in the plane has a transversal if and only if every subfamily of $\mathcal{F}$ of size at most 5 has a transversal.*

Simple examples show that the disjointness or congruence can not be dropped, nor can the number 5 be reduced. Danzer's theorem has been very influential on geometric transversal theory. In 1958, Grünbaum [8] showed that the same result holds when *congruent disks* is replaced by *translates of a square*, and conjectured that the result holds also for families of disjoint translates of an arbitrary planar convex body. This long-standing conjecture was finally proven by Tverberg [12] after partial results were obtained by Katchalski [11].

Theorem (Tverberg, 1989)**.** *A family $\mathcal{F}$ of pairwise disjoint translates of a compact convex set in the plane has a transversal if and only if every subfamily of $\mathcal{F}$ of size at most 5 has a transversal.*

In a different direction, Danzer's theorem was recetly generalized by the present authors together with S. Petitjean [2]. This is a higher-dimensional analogue of Danzer's theorem, and it solves a problem which dates back to Danzer's original article.

Theorem (Cheong-Goaoc-Holmsen-Petitjean, 2008)**.** *A family $\mathcal{F}$ of disjoint congruent balls in $\mathbb{R}^d$ has a transversal if and only if every subfamily of size at most $4d-1$ has a transversal.*

It should be noted that there are examples which show that Tverberg's theorem does not extend to dimensions greater than two [10]. The theorem just stated provides an upper bound on the *Helly-number* for line transversals to disjoint congruent balls in $\mathbb{R}^d$. However, a missing piece in this particular line of research has been a matching lower bound, a problem which again dates back to Danzer's original article. The main result of this paper is the following.

Theorem 1. *For every $d \geq 3$, there exists a family of disjoint congruent balls in $\mathbb{R}^d$ which does not have a transversal but where every subfamily of size at most $2d-2$ has a transversal.*

Thus the Helly-number for line transversals to disjoint unit balls in $\mathbb{R}^d$ is determined up to a factor of 2.

The crucial idea for the proof of Theorem 1 is the notion of a *pinning*, which was also used in [2]. Intuitively, a line transversal l to a family $\mathcal{F}$ is *pinned* if every line l' sufficiently close to, but distinct from l fails to be transversal to $\mathcal{F}$. In [2] we showed that if a line is pinned by a family $\mathcal{F}$ of disjoint balls then there is a subfamily $\mathcal{G} \subset \mathcal{F}$ of size at most $2d-1$ such that l is pinned by $\mathcal{G}$. Here we will show that there exists *minimal pinning configuration* of disjoint (congruent) balls in $\mathbb{R}^d$ of size $2d-1$. By this we mean a family of $2d-1$

disjoint balls with a unique transversal l which is pinned but where no proper subfamily pins l. Theorem 1 then follows by slightly shrinking each member of the pinning configuration about its center.

There are many surveys that cover geometric transversal theory, among others [4, 5, 7, 13]. For detailed information on the transversal properties to families of disjoint balls the reader should consult [6].

2 Existence of stable pinnings

Let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^d$. The set $\mathfrak{T}(\mathcal{F})$ of all line transversals to $\mathcal{F}$ forms a subspace of the affine Grassmanian, which is called the *space of transversals* to $\mathcal{F}$. A set $\mathcal{F}$ *pins* (or is a *pinning* of) a line ℓ if ℓ is an isolated point of $\mathfrak{T}(\mathcal{F})$; we also say that ℓ is *pinned* by $\mathcal{F}$, or that the pair $(\mathcal{F}, \ell)$ is a *pinning configuration.* If $\mathcal{F}$ pins ℓ and no proper subset of $\mathcal{F}$ does, then $\mathcal{F}$ is a *minimal* pinning of ℓ. A minimal pinning configuration consisting of pairwise disjoint balls in $\mathbb{R}^d$ has size at most $2d-1$ [1, 2]. Our goal is to show that this constant is best possible in all dimensions.

Theorem 2. *For any $d \geq 2$, there exists a minimal pinning by $2d-1$ disjoint congruent balls in $\mathbb{R}^d$.*

A pinning configuration $(\mathcal{F}, \ell)$ consisting of disjoint balls $B_1, \ldots, B_n$ in $\mathbb{R}^d$ is *stable* if there exists an $\varepsilon > 0$ such that any configuration $\mathcal{F}' = \{B'_1, \ldots, B'_n\}$, where the center of B'_i has distance at most ε from the center of B_i and B'_i is tangent to ℓ, is also a pinning of ℓ.

Pinning patterns. A *halfplane pattern* is a sequence $\mathcal{H} = (H_1, \ldots, H_n)$ of halfplanes in $\mathbb{R}^2$ bounded by lines through the origin. A halfplane pattern is a *pinning pattern* if no two halfplanes are bounded by the same line, and if for every directed line ℓ not meeting the origin and intersecting each halfplane there exist indices $i < j$ such that ℓ exits H_j before entering H_i.

We first observe that pinning patterns are invariant under small perturbations of the halfplanes (that is, if each halfplane is rotated about the origin by a sufficiently small angle). More precisely, two halfplane patterns are equivalent with respect to the pinning pattern property if the cyclic order of the halfplane boundaries and their orientation is the same, or, equivalently, if the cyclic order of the inward and outward normals of the halfplanes is identical.

Let $n_i \in \mathbb{S}^1$ denote the outward normal of H_i (throughout the paper, we let $\mathbb{S}^{d-1}$ denote the set of unit vectors or, equivalently, directions in $\mathbb{R}^d$). We call a halfplane pattern of five halfplanes a *σ_5-pattern* if the outward and inward normals appear in the order (see Figure 1)

$$n_1, -n_3, n_5, n_2, -n_4, -n_1, n_3, -n_5, -n_2, n_4.$$

It is easy (but a bit tedious) to verify manually that any σ_5-pattern is a pinning pattern. We will give a somewhat more elegant argument below, but let us first understand the significance of this fact.

Stable pinnings from pinning patterns. The existence of a pinning pattern in the plane allows us to prove the existence of a stable pinning of a line by five disjoint balls in $\mathbb{R}^3$.

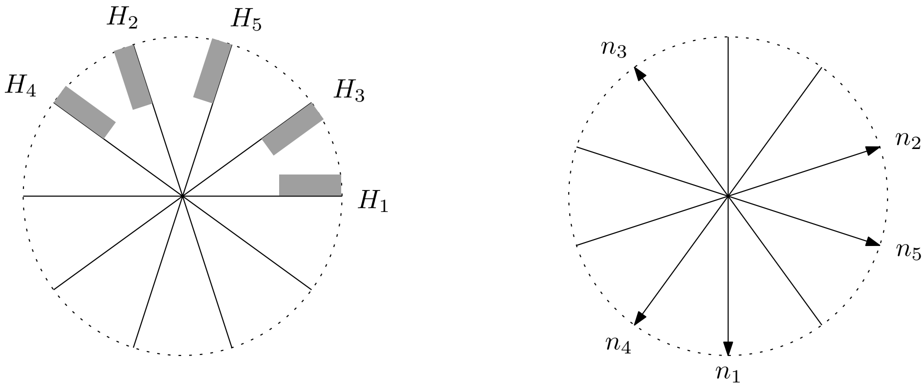

Figure 1: A σ_5-patterns, as an arrangement of halfplanes through the origin (left) and as a cyclic order of outward and inword normals on $\mathbb{S}^1$ (right).

Let $\mathcal{C} = (B_1, \ldots, B_n)$ be a sequence of balls tangent to a directed line ℓ, which touches the balls in the order $B_1, \ldots, B_n$. We choose a coordinate system where ℓ is the positive z-axis. Projecting ball B_i on the xy-plane results in a disk whose boundary contains the origin; we let H_i denote the halfplane (in the xy-plane) containing this disk and bounded by its tangent in the origin. We call the halfplane pattern $\mathcal{H} = (H_1, \ldots, H_n)$ the *projection* of $\mathcal{C}$ along ℓ.

Lemma 1. *Let $\mathcal{C}$ be a sequence of disjoint balls in $\mathbb{R}^3$ touching a line ℓ in the order of the sequence. If the projection of $\mathcal{C}$ along ℓ is a pinning pattern, then $\mathcal{C}$ is a pinning of ℓ.*

Proof. We show that no line other than ℓ intersects the members of $\mathcal{C} = (B_1, \ldots, B_n)$ in the same order, implying that ℓ is pinned by $\mathcal{C}$. Let $\mathcal{H} = (H_1, \ldots, H_n)$ be the projection of $\mathcal{C}$, and assume that such a line g exists. If g is neither parallel nor meets ℓ, its projection g' on the xy-plane does not go through the origin. Since g meets each B_i, g' intersects each halfplane H_i. Since $\mathcal{H}$ is a pinning pattern, there must then be indices $i < j$ such that g' exits H_j before entering H_i. But this implies that g must intersect B_j before B_i, a contradiction.

If g is parallel to ℓ then its projection on the xy-plane is a point lying in $\bigcap_{1 \leq i \leq n} H_i$. Since $\mathcal{H}$ is a pinning pattern, this intersection must have empty interior as otherwise any line pointing into the sector and not meeting the origin does not exit any halfplane; as no two halplanes in $\mathcal{H}$ are bounded by the same line, we get $\bigcap_{1 \leq i \leq n} H_i = \{O\}$, and so g is ℓ, a contradiction.

If g meets ℓ, we argue that there exists a line that intersects the balls in the same order as ℓ and is neither parallel to nor secant with ℓ, which brings us back to the first case above. Specifically, let s_i be a segment joining a point in $B_i \cap \ell$ and a point in $B_i \cap g$ and g_1 a line through $g \cap \ell$ and the interior of one of the s_i. Since g and ℓ intersect the balls in the same order, so does g_1. If no s_i is reduced to a single point, g_1 intersects the open balls and can be perturbed into the desired line. If some s_i is reduced to a single point, that point is $g \cap \ell$ and g_1 meets every other segment in its interior; we can thus translate g_1 to (i) keep intersecting all balls other than B_i, (ii) move closer to the center of B_i and (iii) stop intersecting ℓ; this yields the desired line. $\square$

In fact, we can strengthen the lemma as follows.

Lemma 2. *Let $\mathcal{C}$ be a sequence of disjoint balls in $\mathbb{R}^3$ touching a line ℓ in order of the sequence. If the projection of $\mathcal{C}$ along ℓ is a pinning pattern, then $\mathcal{C}$ is a stable pinning of ℓ.*

Proof. Consider moving the center of a ball B_i in the collection $\mathcal{C}$. In the projection $\mathcal{H}$, the halfplane H_i remains unchanged or rotates about the origin. Since we observed above that pinning patterns are invariant under sufficiently small rotations of each halfplane, the resulting collection is a pinning by Lemma 1. And so $\mathcal{C}$ is a stable pinning of ℓ. $\square$

σ_5-patterns are pinning patterns. The following lemma characterizes pinning patterns.[1] In addition to proving that σ_5-patterns are pinning patterns, we have used a higher-dimensional version of the sufficient condition to experimentally find pinning patterns in $\mathbb{R}^3$.

Lemma 3. *A halfplane pattern $\mathcal{H}$ is a pinning pattern if and only if for any direction $u \in \mathbb{S}^1$ there exist indices $i < j < k$ such that $\{n_i, n_j, n_k\}$ positively span[2] the plane, $\langle u, n_i \rangle < 0$, and $\langle u, n_k \rangle > 0$.*

Proof. A directed line with direction u exits halfplane H_i if and only if $\langle u, n_i \rangle > 0$ and enters H_i if and only if $\langle u, n_i \rangle < 0$. We first prove that the condition implies that $\mathcal{H}$ is a pinning pattern. Let g be a line not meeting the origin and meeting each H_i, let u be its direction, and let $i < j < k$ be a triple satisfying the conditions. Since $\{n_i, n_j, n_k\}$ positively span the plane, we have $H_i \cap H_j \cap H_k = \{0\}$. As g does not contain the origin, $g \cap H_j$ and $g \cap (H_i \cap H_k)$ are disjoint. From $\langle u, n_i \rangle < 0$ and $\langle u, n_k \rangle > 0$, we get that g enters H_i and exits H_k. We are thus in one of three cases: (i) g does not intersect $H_i \cap H_k$, and so exits H_k before entering H_i, (ii) g intersects $H_i \cap H_k$ before H_j, and thus exits H_k before entering H_j, or (iii) g intersects $H_i \cap H_k$ after H_j, and thus exits H_j before entering H_i. In each of these cases, g exits H_u before entering H_v for some $u < v$. Since this holds for any line g not containing the origin, it follows that $\mathcal{H}$ is a pinning pattern.

We now prove the other implication. Assume that $\mathcal{H}$ is a pinning pattern and let u be a direction. We let g_1 and g_2 be two lines with direction u such that the origin lies in between these two lines. Since $\mathcal{H}$ is a pinning pattern, there exist indices $a < b$ and $\alpha < \beta$ such that g_1 exits H_b before entering H_a, and g_2 exits H_β before entering H_α. Assume first that no two elements in $\{a, b, \alpha, \beta\}$ are equal and consider the arrangement of $\{H_a, H_b, H_\alpha, H_\beta\}$; up to exchanging the roles of g_1 and g_2, we are in one of the situations (i)–(iii) depicted in Figure 2. In each case, we give an unordered triple of indices whose halfplanes have outer normals that positively span the plane (or, equivalently, intersect in exactly the origin):

- In situation (i), the triple is (a, α, β) if $a < \alpha$ and (α, β, b) otherwise,
- In situation (ii), the triple is (a, β, b) if $a < \beta$ and (α, β, b) otherwise,

[1]The necessary condition is actually not used in this paper, and only included for completeness

[2]The vectors $\{v_1, \ldots, v_k\}$ positively span the plane if any vector in $\mathbb{R}^2$ can be written as a linear combination of the v_i with non-negative coefficients.

- In situation (iii), the triple is (a, α, β) if $a < \alpha$ and (α, a, b) otherwise.

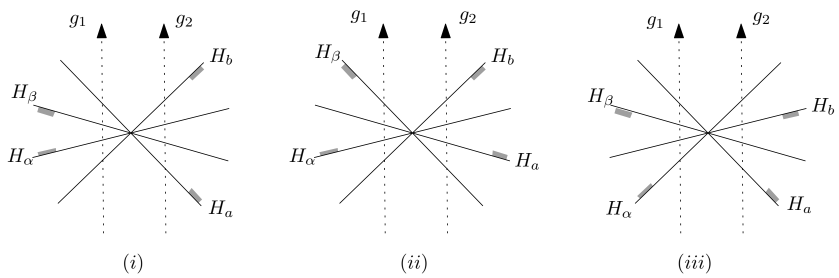

Figure 2: The three possible situations for H_a, H_b, H_α and H_β.

The smallest element must belong to $\{a, \alpha\}$ and the largest to $\{b, \beta\}$. Since g_1 and g_2 enters (resp. exit) H_a and H_α (resp. H_b and H_β), it follows that u makes a negative (resp. positive) dot product with the outer normal of the halfplane with lowest (resp. highest) index; this implies the condition.

Consider now the case where $\{a, \alpha, b, \beta\}$ are not all distinct. Since g_2 exits H_b *after* entering H_a, at least three elements of $\{a, b, \alpha, \beta\}$ are pairwise distinct; for the same reasons as above, this triple of indices satisfies the condition. □

Lemma 4. *Any σ_5-pattern is a pinning pattern.*

Proof. There are four triples of indices of outer normals in a σ_5-pattern that positively span the plane: $\{1, 2, 3\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$ and $\{3, 4, 5\}$. Figure 3 shows, for each triple, the interval of directions that enter the first and exit the last member. The union of these (open) intervals covers $\mathbb{S}^1$. By Lemma 3, such a pattern is a pinning pattern.

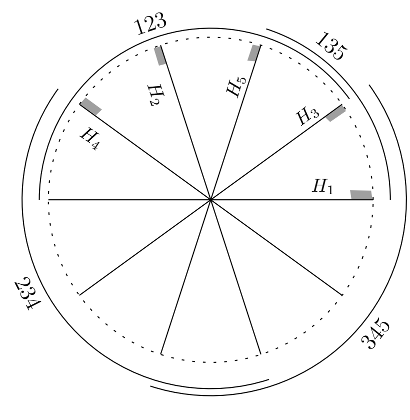

Figure 3: A σ_5-pattern satisfies the condition of Lemma 3.

□

Combining Lemmas 2 and 4 we obtain:

Theorem 3. *There exist sequences of five disjoint congruent balls in $\mathbb{R}^3$ that are stable pinnings.*

Higher dimensions. We now show the existence of stable pinnings by finite families of disjoint balls in arbitrary dimension.

Theorem 4. *For any $d \geq 2$, there exists a stable pinning of a line by finitely many disjoint congruent balls in $\mathbb{R}^d$.*

Proof. Let ℓ be the x_d-axis in $\mathbb{R}^d$ and let Γ be the space of all three-dimensional flats containing ℓ. The natural homeomorphism between Γ and the space of two-dimensional linear subspaces of $\mathbb{R}^{d-1}$ implies that Γ is compact.

For every $T \in \Gamma$ we can construct a quintuple Q_T of disjoint balls in $\mathbb{R}^d$ tangent to ℓ such that their restriction to T projects along ℓ to a σ_5-pattern. By construction, Q_T pins ℓ in T. By continuity, there exists a neighborhood $\mathcal{N}_T$ of T in Γ such that Q_T pins ℓ in any $T' \in \mathcal{N}_T$. The union of all $\mathcal{N}_T$ covers Γ. Since Γ is compact, there exists a finite sub-family $\{T_1, \ldots, T_n\}$ such that the union of the $\mathcal{N}_{T_i}$ cover Γ. Let $\mathcal{C}$ denote the union of the Q_{T_i}.

By construction, $\mathcal{C}$ is a finite collection of balls such that the intersection of $\mathcal{C}$ with any 3-flat $T \in \Gamma$ is a stable pinning of ℓ in T. Let $\varepsilon > 0$ be such that any collection $\mathcal{C}'$ obtained by perturbing $\mathcal{C}$ by at most ε remains a pinning of ℓ in each $T \in \Gamma$. If such a perturbation $\mathcal{C}'$ of $\mathcal{C}$ does not pin ℓ then there is another transversal ℓ' of $\mathcal{C}'$ in $\mathbb{R}^d$ with the same order. There is a three-dimensional affine subspace T containing both ℓ and ℓ'. Since the set of line transversals with a fixed ordering on family of disjoint balls is connected [2], this implies that ℓ is not pinned by $\mathcal{C}' \cap T$ in T, and since $T \in \Gamma$ this is a contradiction. Thus, any such perturbation $\mathcal{C}'$ pins ℓ in $\mathbb{R}^d$, implying that $\mathcal{C}$ is a stable pinning.

Finally, we observe that we can replace a ball $B \in \mathcal{C}$ touching ℓ in point p by moving the center of B on the segment towards p. Since this does not change the halfplane pattern in the projection, $\mathcal{C}$ remains a stable pinning, and so we can choose $\mathcal{C}$ to consist of pairwise disjoint congruent balls. $\square$

By further shrinking the balls, we could even enforce that any two are separated by a hyperplane orthogonal to ℓ.

3 The size of stable pinnings

In this section, we will show that families of $k < 2d - 1$ balls cannot be stable pinnings of a line in $\mathbb{R}^d$. Instead of balls, we will work with simpler objects we call screens (half-hyperplanes orthogonal to the line to be pinned), and the lower bound we obtain will carry over to balls.

Screens and lines. Let ℓ be the positively oriented x_d-axis in $\mathbb{R}^d$. For $\lambda \in \mathbb{R}$ and direction vector $n \in \mathbb{S}^{d-2}$, consider the set

$$\mathcal{S}(\lambda, n) := \{(x, \lambda) \in \mathbb{R}^d \mid x \in \mathbb{R}^{d-1},\ \langle n, x \rangle \leq 0\},$$

where the notation (a, b) denotes a vector whose coordinates are obtained as the concatenation of the coordinates of the vectors a and b.

We call $\mathcal{S}(\lambda, n)$ a *screen.* A screen is a $(d-1)$-dimensional halfspace of a hyperplane orthogonal to ℓ; the screen $\mathcal{S}(\lambda, n)$ is tangent to ℓ in the point $(0, \lambda)$. We identify $\mathfrak{S} = \mathbb{R} \times \mathbb{S}^{d-2}$ with the space of all possible screens.

Consider now the space $\mathfrak{L}$ of lines not orthogonal to ℓ. Any line in $\mathfrak{L}$ must intersect the planes $x_d = 0$ and $x_d = 1$. We identify $\mathfrak{L}$ with $\mathbb{R}^{2d-2}$ by identifying the line meeting the points $(u_0, 0)$ and $(u_1, 1)$ with the point $(u_0, u_1) \in \mathbb{R}^{2d-2}$.

For $(\lambda, n) \in \mathfrak{S}$, let $H(\lambda, n) \subset \mathfrak{L}$ denote the set of those lines $g \in \mathfrak{L}$ that intersect $\mathcal{S}(\lambda, n)$.

Lemma 5. *For $(\lambda, n) \in \mathfrak{S}$, the set $H(\lambda, n)$ is the halfspace of $\mathbb{R}^{2d-2}$ through the origin with outer normal $\Phi(\lambda, n) := ((1-\lambda)n, \lambda n)$.*

Proof. The line $(u_0, u_1) \in \mathfrak{L}$ intersects the hyperplane $x_d = \lambda$ in the point $((1-\lambda)u_0 + \lambda u_1, \lambda)$. This point lies in $\mathcal{S}(\lambda, n)$ if and only if

$$\langle n, (1-\lambda)u_0 + \lambda u_1 \rangle \leq 0,$$

and since

$$\langle n, (1-\lambda)u_0 + \lambda u_1 \rangle = \langle (1-\lambda)n, u_0 \rangle + \langle \lambda n, u_1 \rangle = \langle ((1-\lambda)n, \lambda n), (u_0, u_1) \rangle$$

the lemma follows. □

Let $\mathfrak{N} \subset \mathbb{R}^{2d-2}$ denote the set of vectors $\Phi(\lambda, n)$, for some $(\lambda, n) \in \mathfrak{S}$. The function Φ is a bicontinuous bijection from $\mathfrak{S}$ to $\mathfrak{N}$, and so $\mathfrak{S}$ and $\mathfrak{N}$ are homeomorphic. In particular, $\mathfrak{N}$ is locally homeomorphic to $\mathbb{R}^{d-1}$, and so $\mathfrak{N}$ is a $(d-1)$-dimensional manifold in $\mathbb{R}^{2d-2}$. We need to argue that it is nowhere contained in a hyperplane, that is, that there is no neighborhood of a point in $\mathfrak{N}$ that is contained in a hyperplane.

Lemma 6. *$\mathfrak{N}$ is nowhere locally contained in a hyperplane of $\mathfrak{L}$.*

Proof. We assume, by way of contradiction, that $\mathfrak{N}$ is contained in a hyperplane in a neighborhood of the point $\Phi(\lambda, n) = ((1-\lambda)n, \lambda n)$. Let this hyperplane be $\langle (a, b), (u_0, u_1) \rangle = c$, where $a, b \in \mathbb{R}^{d-1}$ and $c \in \mathbb{R}$. This means that for $\varepsilon \in \mathbb{R}$ sufficiently small and $\eta \in \mathbb{S}^{d-2}$ sufficiently close to n,

$$\langle (a, b), ((1-\lambda-\varepsilon)\eta, (\lambda+\varepsilon)\eta) \rangle = c.$$

Separating out the terms with ε, we obtain

$$\langle (a, b), ((1-\lambda)\eta, \lambda\eta \rangle + \varepsilon \langle b - a, \eta \rangle = c.$$

Since this holds for any ε small enough, we must have $\langle b - a, \eta \rangle = 0$. Since no neighborhood on $\mathbb{S}^{d-2}$ can lie in a hyperplane, it follows that $b = a$. We thus have

$$\langle a, (1-\lambda)\eta + \lambda\eta \rangle = c,$$

which implies $\langle a, \eta \rangle = c$. Again, no neighborhood on $\mathbb{S}^{d-2}$ lies in a hyperplane, a contradiction. □

Strict transversals to screens. Given a family $\mathcal{F} \subset \mathfrak{S}$ of k screens, a line $g \in \mathfrak{L}$ is a *strict transversal* of $\mathcal{F}$ if it meets the relative interior of each screen. Recall that the line ℓ meets every screen of $\mathcal{F}$, but since ℓ only touches their boundary, it is not a strict transversal. If $g \in \mathfrak{L}$ is a strict transversal of $\mathcal{F}$, then any line $g' \in \mathfrak{L}$ sufficiently close to g must also be a strict transversal. Indeed, $\mathcal{F}$ has a strict transversal if and only if the intersection of the halfspaces $H(\lambda, n)$ for $\mathcal{S}(\lambda, n) \in \mathcal{F}$ has non-empty interior.

Lemma 7. *Let $\mathcal{F}$ be a family of $k \leq 2d-2$ screens. If $\mathcal{F}$ has no strict transversal, then the k normals $\Phi(\lambda, n)$, for $(\lambda, n) \in \mathcal{F}$, are linearly dependent.*

Proof. If $\mathcal{F}$ has no strict transversal, then the intersection of the halfspaces $H(\lambda, n)$ for $(\lambda, n) \in \mathcal{F}$ has empty interior. However, the intersection of $k \leq 2d-2$ halfspaces through the origin in $\mathbb{R}^{2d-2}$ can have empty interior only if the outward normals of the halfspaces are linearly dependent. It follows that the normals $\Phi(\lambda, n)$, for $(\lambda, n) \in \mathcal{F}$, are linearly dependent. $\square$

We can represent[3] a family of m screens as a point in $\mathfrak{S}^m$. Let $\mathfrak{X}_m \subset \mathfrak{S}^m$ be the space of those families $\mathcal{F}$ of m screens that have a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of at most $2d-2$ screens with *no strict transversal*.

Lemma 8. *$\mathfrak{X}_m \subset \mathfrak{S}^m$ has empty interior.*

Proof. Let $\mathcal{F}$ be a family in $\mathfrak{X}_m$. We perturb $\mathcal{F}$, element by element, into a family $\mathcal{F}'$ with no subset of at most $2d-2$ screens with linearly dependent vectors. The first element of $\mathcal{F}$ need not be changed. Assume we already perturbed the first i elements of $\mathcal{F}$. Every subset of at most $2d-3$ among these i already fixed normals span a linear subspace of $\mathfrak{L}$. By Lemma 6, $\mathfrak{N}$ lies nowhere locally inside a hyperplane or, since it is a $d-1$-manifold, locally inside a finite union of hyperplanes. Thus we can choose the $(i+1)^{th}$ element outside of each of these subspaces, and by induction obtain the desired perturbation of $\mathcal{F}$. $\square$

A necessary condition for pinning Consider now a collection $\mathcal{C}$ of balls tangent to the line ℓ (still assumed to be the x_d-axis). If (p, λ) is the center of ball $B \in \mathcal{C}$, we consider the screen $\mathcal{S}(B) = \mathcal{S}(\lambda, -p/||p||)$. This screen touches ℓ in the same point that B does, and its boundary is contained in the tangent hyperplane to B at this point.

Lemma 9. *Let $\mathcal{C}$ be a collection of balls tangent to ℓ. If the family of screens $\mathcal{F} := \{\mathcal{S}(B) \mid B \in \mathcal{C}\}$ has a strict transversal, then $\mathcal{C}$ does not pin ℓ.*

Proof. If $\mathcal{F}$ has a strict transversal, then the halfspaces $H(\lambda, n)$ for $(\lambda, n) \in \mathcal{F}$ intersect with non-empty interior. This implies that there exists a segment τ in $\mathfrak{L}$ with one endpoint at the origin and which is, except for that point, contained in the interior of each halfspace $H(\lambda, n)$.

Consider moving a line $g \in \mathfrak{L}$ along τ. The trace of g on the hyperplane $x_d = \lambda$ is a straight segment. Since τ lies in the interior of $H(\lambda, n)$, this trace lies in the relative interior of $\mathcal{S}(\lambda, n)$. But this implies that if we make τ sufficiently short, the trace also lies in the interior of each ball B. It is therefore possible to move a line g, starting with $g = \ell$, while intersecting each ball B. It follows that $\mathcal{C}$ is not a pinning of ℓ. $\square$

We now obtain the desired lower bound on the size of stable pinning configurations of balls:

Theorem 5. *Any pinning of a line by $k \leq 2d-2$ balls in $\mathbb{R}^d$ is instable.*

[3]Note that this is not a bijection as not every point in $\mathfrak{S}^m$ represents a set of *distinct* screens.

Proof. Let $\mathcal{C}$ be a pinning of ℓ. Let $\mathcal{F} := \{\mathcal{S}(B) \mid B \in \mathcal{C}\}$ be the corresponding family of screens. By Lemma 9, $\mathcal{F}$ does not have a strict transversal, and so $\mathcal{F} \in \mathfrak{X}_k$. By Lemma 8, $\mathfrak{X}_k$ has empty interior, and so we can find $\mathcal{F}' \in \mathfrak{S}^k \setminus \mathfrak{X}_k$ arbitrarily close to $\mathcal{F}$. Since $\mathcal{F}'$ can be realized as the set of screens of a perturbation of $\mathcal{C}$, the theorem follows. ☐

4 Consequences

Lower bound for minimal pinnings. Theorem 2 follows immediately from Theorem 4 and the following lemma:

Lemma 10. *If $\mathcal{C}$ is a stable pinning of a line by finitely many balls in $\mathbb{R}^d$, then there exist minimal pinnings of ℓ by $2d-1$ balls arbitrarily close to some subset of $\mathcal{C}$ of size $2d-1$.*

Proof. Let m denote the number of balls in $\mathcal{C}$, and let $\mathcal{F} := \{\mathcal{S}(B) \mid B \in \mathcal{C}\}$ be the corresponding family of screens. Since $\mathfrak{X}_m \subset \mathfrak{S}^m$ has empty interior, we can find a family $\mathcal{F}' \subset \mathfrak{S}^m \setminus \mathfrak{X}_m$ arbitrarily close to $\mathcal{F}$. Let $\mathcal{C}'$ be the correspondingly perturbed family of balls.

By definition of $\mathfrak{X}_m$ and Lemma 9, no subfamily of at most $2d-2$ balls of $\mathcal{C}'$ is a pinning. However, any minimal pinning of a line by disjoint balls in $\mathbb{R}^d$ has size at most $2d-1$ [2], so there must be a subfamily $\mathcal{C}'' \subset \mathcal{C}'$ of $2d-1$ balls that is minimally pinning. ☐

Helly number for transversals to disjoint unit balls. Hadwiger's transversal theorem [9] can be extended to families of disjoint balls in arbitrary dimension [1, 2]: a family $\mathcal{F}$ of disjoint balls in $\mathbb{R}^d$ has a line transversal if and only if there is an ordering on $\mathcal{F}$ such that every h_d members have a line transversal consistent with that ordering. The smallest such constant h_d is at most $2d$ and at least the size of the largest minimal pinning family of disjoint balls in $\mathbb{R}^d$; Theorem 2 implies that this number is $2d-1$ or $2d$. Similarly, we obtain that $2d-1$ is a lower bound for the Helly number of the generalization of Helly's theorem to sets of transversals to disjoint (unit) balls.

Proof of Theorem 1. The proof of Theorem 4 shows that there exists a stable pinning of a line ℓ by finitely many disjoint unit balls in $\mathbb{R}^d$ such that any two balls can be separated by a hyperplane orthogonal to ℓ. Lemma 10 now implies that there exists a minimal pinning $\mathcal{F}$ of a line ℓ by $2d-1$ disjoint unit balls in $\mathbb{R}^d$ such that any line intersecting a subset of the balls does so in an order consistent with the geometric permutation induced by ℓ. Since $\mathcal{F}$ is a pinning of ℓ by disjoint balls, $\mathcal{F}$ has no other transversal consistent with the geometric permutation induced by ℓ. The statement then follows in the case of open balls. Reducing the radii of the balls slightly gives a similar construction for closed balls. ☐

5 Final remarks

Our lower bound construction is hardly effective, given its use of the compactness of the set of 3-spaces through a fixed line; actually constructing minimal pinnings of size $2d-1$ (or any given size) seems challenging. Another natural question is whether any of the results obtained for pinning lines by disjoint balls extend to more general pinnings, for instance pinnings of lines by (disjoint) convex sets. In that direction, little is known, not even whether the size of minimal pinnings is bounded by a function of d.

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