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Abstract: We study the behaviour of the steady-state voltage potential in a material composed of a two-dimensional object surrounded by a very rough thin layer and embedded in an ambient medium. The roughness of the layer is described by a quasi ε–periodic function, ε being a small parameter, while the mean thickness of the layer is of magnitude ε^β, where $\beta \in (0, 1)$. Using the two-scale analysis, we replace the very rough thin layer by appropriate transmission conditions on the boundary of the object, which lead to an explicit characterization of the polarization tensor of Vogelius and Capdeboscq (ESAIM:M2AN. 2003; 37:159-173). This paper extends the previous works Poignard (Math. Meth. App. Sci. 2009; 32:435-453) and Ciuperca *et al.* (Research report INRIA RR-6812), in which $\beta \geq 1$.

Key-words: Asymptotic analysis, Finite Element Method, Laplace equations

* Université de Lyon, Université Lyon 1, CNRS, UMR 5208, Institut Camille Jordan, Bat. Braconnier, 43 boulevard du 11 novembre 1918, F - 69622 Villeurbanne Cedex, France

† Laboratoire Ampère UMR CNRS 5005, Universit de Lyon, École Centrale de Lyon, F-69134 Écully, France

‡ INRIA Bordeaux-Sud-Ouest, Institut de Mathématiques de Bordeaux, CNRS UMR 5251 & Université de Bordeaux1, 351 cours de la Libération, 33405 Talence Cedex, France

Analyse double échelle pour des couches minces très rugueuses. Une caractérisation explicite du tenseur de polarisation

Résumé :

Mots-clés : Analyse Asymptotique, Méthode des Eléments Finis, Equations de Laplace

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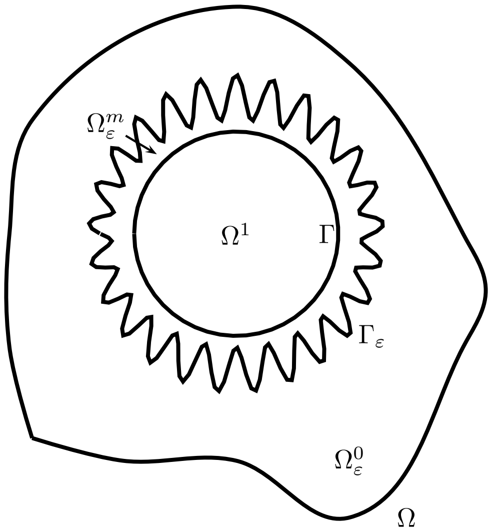


Figure 1: Geometry of the problem.

1 Introduction

Consider a material composed of a two-dimensional object surrounded by a very rough thin layer. We study the asymptotic behaviour of the steady-state voltage potential when the thickness of the layer tends to zero. We present approximate transmission conditions to take into account the effects due to the layer without fully modeling it. This paper ends a series of 3 papers dealing with the steady-state voltage potential in domains with thin layer with a non constant thickness. Unlike [16, 17] in which the layer is weakly oscillating, and unlike [11], which deals with the periodic roughness case, we consider here the case of a very rough thin layer. This means that the period of the oscillations is much smaller than the mean thickness of the layer. More precisely, we consider a period equal to ε, while the mean thickness of the layer is of magnitude ε^β, where β is a positive constant strictly smaller than 1. As for [11], the motivation comes from a collaborative research on the modeling of silty soil, however we are confident that our result is useful for more different applications, particularly in the electromagnetic research area.

1.1 Description of the geometry

For sake of simplicity, we deal with the two-dimensional case, however the three-dimensional case can be studied in the same way up to few appropriate modifications.

Let Ω be a bounded smooth domain of $\mathbb{R}^2$ with connected boundary $\partial\Omega$. For $\varepsilon > 0$, we split Ω into three subdomains: Ω^1, Ω_ε^m and Ω_ε^0. Ω^1 is a smooth domain strictly embedded in Ω. We denote by Γ its connected boundary. The domain Ω_ε^m is the thin oscillating layer surrounding Ω^1 (see Fig. 1). We denote by Γ_ε the oscillating boundary of Ω_ε^m:

$$\Gamma_\varepsilon = \partial\Omega_\varepsilon^m \setminus \Gamma.$$

The domain Ω_ε^0 is defined by

$$\Omega_\varepsilon^0 = \Omega \setminus \overline{(\Omega^1 \cup \Omega_\varepsilon^m)}.$$

We also write

$$\Omega^0 = \Omega \setminus \overline{\Omega^1}.$$

We suppose that the curve Γ is a smooth closed curve of $\mathbb{R}^2$ of length 1, which is parametrized by its curvilinear coordinate:

$$\Gamma = \Big\{\gamma(t),\, t \in \mathbb{T}\Big\},$$

where $\mathbb{T}$ is the torus $\mathbb{R}/\mathbb{Z}$. Denote by ν the normal to Γ outwardly directed to Ω^1. The rough boundary Γ_ε is defined by

$$\Gamma_\varepsilon = \{\gamma_\varepsilon(t), t \in \mathbb{T}\},$$

where

$$\gamma_\varepsilon(t) = \gamma(t) + \varepsilon^\beta f\left(t, \frac{t}{\varepsilon}\right)\nu(t),$$

where $0 < \beta < 1$ and f is a smooth, (1, 1)–periodic and positive function such that $\frac{1}{2} \le f \le \frac{3}{2}$. Observe that the membrane has a fast oscillation compared with the size ε^β of the perturbation.

1.2 Statement of the problem

Define the piecewise regular function σ_ε by

$$\forall x \in \Omega, \quad \sigma_\varepsilon(x) = \begin{cases} \sigma_1, \text{ if } x \in \Omega^1, \\ \sigma_m, \text{ if } x \in \Omega_\varepsilon^m, \\ \sigma_0, \text{ if } x \in \Omega_\varepsilon^0, \end{cases}$$

where σ_1, σ_m and σ_0 are given positive[1] constants and let $\sigma : \Omega \to \mathbb{R}$ be defined by[2]

$$\sigma(x) = \begin{cases} \sigma_1, \text{ if } x \in \Omega^1, \\ \sigma_0, \text{ if } x \in \Omega^0. \end{cases}$$

Let g belong to $H^s(\Omega)$, for $s \ge 1$. We consider the unique solution u_ε to

$$\nabla.\,(\sigma_\varepsilon \nabla u_\varepsilon) = 0, \text{ in } \Omega, \tag{1a}$$

$$u_\varepsilon|_{\partial\Omega} = g|_{\partial\Omega}. \tag{1b}$$

Let u be the unique solution to the limit problem

$$\nabla.\,(\sigma \nabla u) = 0, \text{ in } \Omega, \tag{2a}$$

$$u|_{\partial\Omega} = g|_{\partial\Omega}. \tag{2b}$$

[1] The same following results are obtained if σ_1, σ_m and σ_0 are given complex and regular functions with imaginary parts (and respectively real parts) with the same sign.

[2] σ represents the piecewise-constant conductivity of the whole domain Ω.

Since the domains Ω, Ω^1 and Ω^0 are smooth, the above function u belongs to $H^s(\Omega^1)$ and $H^s(\Omega^0)$. In the following we suppose that $s > 3$ hence by Sobolev embeddings there exists $s_0 > 0$ such that $u \in C^{1,s_0}(\overline{\Omega^1})$ and $u \in C^{1,s_0}(\overline{\Omega^0})$. We aim to give the first two terms of the asymptotic expansion of u_ε for ε tending to zero.

Several papers are devoted to the modeling of thin layers: see for instance [8, 7, 16] for smooth thin layers and [1, 2, 4, 14, 11] for rough layers. However, as far as we know, the case of very rough thin layer has not been treated yet. In [10] Vogelius and Capdeboscq derive a general representation formula of the steady-state potential in the very general framework of inhomogeneities of low volume fraction, including the case of very rough thin layers. However their result involves the polarization tensor, which is not precisely given. This paper can be seen as an explicit characterization of the polarization tensor for very rough thin layers.

Our main result (see Theorem 2.3) is weaker than the results of [16, 11], since we do not prove error estimates. Actually, using variational techniques we prove that the sequence $(u_\varepsilon - u)/\varepsilon^\beta$ weakly converges in $L^p(\Omega)$, for all $p \in (1, 2)$ to a function z. This function z is uniquely determined by the elliptic problem (11), and the convergence does hold in L^s, for $s \geq 1$ far from the layer (see Theorem 2.7).

In the present paper it seems difficult to obtain the H^1 strong convergence in Ω as in [11]. The main reason comes from the fact that according to Bonder *et al.*, the best Sobolev trace constant blows up for ε tending to zero in the case of a very rough layer. Therefore, the analysis performed previously can not be applied. To obtain our present result, we use a variational technique based on the two-scale analysis. We emphasize that this technique can be applied to obtain the limit problems presented in [16, 11], even if the error estimates are more complex to be achieved in such a way. We conclude by observing that the two-scale convergence enables us to draw the target to be reached: another asymptotic analysis as to be performed to obtain error estimates, however the result is sketched.

The outline of the paper is the following. In the next section we present precisely our main results using a variational formulation. Section 3 is devoted to preliminary results. In particular, we show the first two limits easy to be reached. In Section 4, we end the proof of the main theorems by computing the limit of E''_ε defined by (19). We then conclude the paper with numerical simulations, which illustrate the theoretical results. We shall first present our main results.

2 Main results

2.1 Variational formulations

Denote by z_ε the element of $H^1_0(\Omega)$ defined by

$$z_\varepsilon = \frac{u_\varepsilon - u}{\varepsilon^\beta}.$$

We shall obtain the limit of z_ε with the help of variational techniques. Since g belongs to $H^s(\Omega)$, for $s > 3$, we define by $g + H^1_0(\Omega)$ the affine space

$$g + H^1_0(\Omega) = \Big\{ v \in H^1(\Omega) : v|_{\partial\Omega} = g|_{\partial\Omega} \Big\}.$$

The variational formulation of Problem (1) is

$$\text{Find } u_\varepsilon \in g + H^1_0(\Omega) \text{ such that: } \int_\Omega \sigma_\varepsilon \nabla u_\varepsilon \cdot \nabla \varphi = 0, \qquad \forall \varphi \in H^1_0(\Omega),$$

and respectively for Problem (2)

$$\text{Find } u \in g + H^1_0(\Omega) \text{ such that: } \int_\Omega \sigma \nabla u \cdot \nabla \varphi = 0, \qquad \forall \varphi \in H^1_0(\Omega).$$

Taking the difference between the above equalities, z_ε belongs to $H^1_0(\Omega)$ and satisfies

$$\int_\Omega \sigma_\varepsilon \nabla z_\varepsilon \cdot \nabla \varphi = -\frac{1}{\varepsilon^\beta} \int_\Omega (\sigma_\varepsilon - \sigma) \nabla u \cdot \nabla \varphi, \quad \forall \varphi \in H^1_0(\Omega), \tag{3}$$

or equivalently

$$\int_\Omega \sigma \nabla z_\varepsilon \cdot \nabla \varphi = -\int_\Omega (\sigma_\varepsilon - \sigma) \nabla z_\varepsilon \cdot \nabla \varphi - \frac{1}{\varepsilon^\beta} \int_\Omega (\sigma_\varepsilon - \sigma) \nabla u \cdot \nabla \varphi, \quad \forall \varphi \in H^1_0(\Omega). \tag{4}$$

Notation 2.1 (Normal and tangential derivatives)**.** *Denote by $\theta(t)$ the tangent vector to Γ in any point $\gamma(t)$:*

$$\forall t \in \mathbb{T}, \quad \theta(t) = (\gamma_1'(t), \gamma_2'(t))^T.$$

The normal vector ν outwardly directed to Ω^1 is then given by

$$\forall t \in \mathbb{T}, \quad \nu(t) = (\nu_1(t), \nu_2(t))^T = (\gamma_2'(t), -\gamma_1'(t))^T.$$

In the following, for any $x \in \Gamma$ and for any function φ smooth enough, we denote the normal and tangential derivatives of φ respectively by

$$\frac{\partial \varphi^+}{\partial \nu}(x) = \lim_{y \to x, y \in \Omega^0} \nabla \varphi(y) \cdot \nu, \qquad \frac{\partial \varphi^-}{\partial \nu}(x) = \lim_{y \to x, y \in \Omega^1} \nabla \varphi(y) \cdot \nu,$$

$$\frac{\partial \varphi}{\partial \theta}(x) = \nabla \varphi(x) \cdot \theta.$$

We also write

$$\varphi^+(x) = \lim_{y \to x, y \in \Omega^0} \varphi(y), \qquad \varphi^-(x) = \lim_{y \to x, y \in \Omega^1} \varphi(y).$$

Notation 2.2 (Green operator)**.** *We introduce the Green operator $\mathcal{G} : H^{-1}(\Omega) \to H^1_0(\Omega)$ given by $\mathcal{G}(\psi) = \varphi$ iff φ is the unique solution of the problem*

$$-\nabla.(\sigma \nabla \varphi) = \psi \quad \text{in } \Omega, \tag{5a}$$

$$\varphi|_{\partial\Omega} = 0. \tag{5b}$$

It is well known that if $\psi \in L^{p'}(\Omega)$ with $p' > 2$ then $\varphi \in W^{2,p'}(\Omega^k)$, $k = 0, 1$, then by Sobolev embeddings there exists $s_0 > 0$ such that $\varphi \in C^{1,s_0}(\overline{\Omega^1})$ and $\varphi \in C^{1,s_0}(\overline{\Omega^0})$.

2.2 Approximate transmission conditions

Let f_{min} and f_{max} be

$$f_{min} = \min_{t,\tau\in\mathbb{T}} f(t,\tau) \qquad \text{and} \qquad f_{max} = \max_{t,\tau\in\mathbb{T}} f(t,\tau).$$

For sake of simplicity, we suppose that

$$\frac{1}{2} \leq f_{min} \leq f_{max} \leq \frac{3}{2}.$$

For any fixed $t \in \mathbb{T}$ and $s \in \mathbb{R}$ we denote by $Q(s,t)$ the one-dimensional set

$$\forall (s,t) \in \mathbb{R}\times\mathbb{T}, \quad Q(s,t) = \{\tau \in \mathbb{T},\ s \leq f(t,\tau)\},$$

and let $q(s,t)$ be the Lebesgue-measure of $Q(s,t)$:

$$\forall (s,t) \in \mathbb{R}\times\mathbb{T}, \quad q(s,t) = \int_{\mathbb{T}} \chi_{Q(s,t)}(\tau)\,\mathrm{d}\tau, \tag{6}$$

where χ_A is the characteristic function of the set A. Observe that q satisfies $0 \leq q(s,t) \leq 1$, $q(s,t) = 1$ for $s < f_{min}$ and $q(s,t) = 0$ for $s > f_{max}$. Moreover since q is a measurable function it belongs to L^∞. We also write

$$\tilde{f}(t) = \int_0^1 f(t,\tau)\,\mathrm{d}\tau. \tag{7}$$

Our approximate transmission conditions need the two following functions

$$\forall t \in \mathbb{T}, \quad r_1(t) = \int_0^{f_{max}} \frac{q^2(s,t)}{\sigma_m(\gamma(t))q(s,t) + \sigma_0(\gamma(t))[1-q(s,t)]}\,\mathrm{d}s, \tag{8}$$

$$\forall t \in \mathbb{T}, \quad r_2(t) = \int_{f_{min}}^{f_{max}} \frac{q(s,t)[1-q(s,t)]}{\sigma_0(\gamma(t))q(s,t) + \sigma_m(\gamma(t))[1-q(s,t)]}\,\mathrm{d}s. \tag{9}$$

To simplify notations, we still denote by r_k the function of Γ equal to $r_k \circ \gamma^{-1}$, for $k = 1,2$. The aim of the paper is to prove the following theorem.

Theorem 2.3 (Main result)**.** *There exists $z \in \cap_{1<p<2} L^p(\Omega)$ such that z_ε weakly converges to z in $L^p(\Omega)$ for all $p \in (1,2)$. The limit z is the unique solution to*

$$\begin{aligned} &\forall \psi \in \cup_{p'>2} L^{p'}(\Omega), \\ &\int_\Omega z\psi\,dx = \int_\Gamma \left[(\sigma_0-\sigma_m)\left(\tilde{f} + (\sigma_0-\sigma_m)r_1\right)\frac{\partial u^+}{\partial\nu}\frac{\partial\varphi^+}{\partial\nu}\right]\mathrm{d}\Gamma \\ &\qquad\qquad + \int_\Gamma \left[(\sigma_0-\sigma_m)\left(\tilde{f} + (\sigma_0-\sigma_m)r_2\right)\frac{\partial u}{\partial\theta}\frac{\partial\varphi}{\partial\theta}\right]\mathrm{d}\Gamma, \end{aligned} \tag{10}$$

where $\varphi = \mathcal{G}(\psi)$.

Remark 2.4. *The existence and the uniqueness of $z \in \cap_{1<p<2} L^p(\Omega)$ solution of (10) comes from the fact that for any $p' > 2$ the dual of $L^{p'}(\Omega)$ is $L^p(\Omega)$ with $1/p + 1/p' = 1$ and that the expression of the right-hand side of (10) is a continuous linear application from $L^{p'}(\Omega)$ to $\mathbb{R}$ with argument ψ.*

Remark 2.5. *From the uniqueness of z we deduce that the whole sequence z_ε converges to z.*

Remark 2.6 (Strong formulation)**.** *We can write a strong formulation of* (10). *Supposing that z is regular enough on Ω^0 and on Ω^1, and taking in* (10) *appropriate test functions, we infer that z satisfies the following problem*

$$\nabla.(\sigma_k \nabla z) = 0 \quad \text{in } \ \Omega^k, \quad k = 0, 1, \tag{11a}$$

$$z^+ - z^- = \left(1 - \frac{\sigma_m}{\sigma_0}\right)\left[\tilde{f} + (\sigma_0 - \sigma_m) r_1\right] \frac{\partial u^+}{\partial \nu} \qquad \text{on } \Gamma, \tag{11b}$$

$$\sigma_0 \frac{\partial z^+}{\partial \nu} - \sigma_1 \frac{\partial z^-}{\partial \nu} = \frac{\partial}{\partial \theta}\left[(\sigma_0 - \sigma_m)\left(\tilde{f} + (\sigma_0 - \sigma_m) r_2\right)\frac{\partial u}{\partial \theta}\right] \qquad \text{on } \Gamma, \tag{11c}$$

$$z|_{\partial\Omega} = 0. \tag{11d}$$

Moreover, using the regularity of u in $H^s(\Omega^0)$, with $s > 3$, we infer easily the existence and the uniqueness of z in $H^{s-1}(\Omega^0)$ and $H^{s-1}(\Omega^1)$.

Theorem 2.7 (Strong convergence far from the layer)**.** *Let D be an open set such that $\Gamma \subset D$ and $\overline{D} \subset \Omega$. Then the sequence z_ε converges strongly to z in $L^p(\Omega \setminus D)$, for all $p \geq 1$.*

Remark 2.8 (The case of a thin layer with constant thickness)**.** *In the particular case where f is independent on τ, we have $\tilde{f} = f(t)$ and*

$$q(s,t) = \begin{cases} 1 & \text{for} \quad s \leq f(t), \\ 0 & \text{for} \quad s \geq f(t), \end{cases} \tag{12}$$

and

$$r_1(t) = \frac{f(t)}{\sigma_m(\gamma(t))} \qquad \text{and} \qquad r_2(t) = 0.$$

Then (11) *becomes*

$$\nabla.(\sigma_k \nabla z) = 0 \quad \text{in } \ \Omega^k, \quad k = 0, 1, \tag{13a}$$

$$z^+ - z^- = \left(\frac{\sigma_0}{\sigma_m} - 1\right) f \frac{\partial u^+}{\partial \nu} \qquad \text{on } \Gamma, \tag{13b}$$

$$\sigma_0 \frac{\partial z^+}{\partial \nu} - \sigma_1 \frac{\partial z^-}{\partial \nu} = \frac{\partial}{\partial \theta}\left(f(\sigma_0 - \sigma_m)\frac{\partial u}{\partial \theta}\right) \qquad \text{on } \Gamma, \tag{13c}$$

$$z|_{\partial\Omega} = 0. \tag{13d}$$

which is the result obtained in [16, 17].

3 Some preliminary results

3.1 Preliminary estimates

Lemma 3.1. *The following estimates hold.*
i) There exists $C > 0$ such that

$$\|z_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{-\beta/2}.$$

ii) For any $p \in]1, 2[$ there exists $C_p > 0$ such that

$$\|z_\varepsilon\|_{L^p(\Omega)} \leq C_p.$$

Proof. $i)$: Take $\varphi = z_\varepsilon$ in (3) and use the regularity of u.
$ii)$: For any $p \in]1,2[$ we introduce the function $z_{\varepsilon p}$ defined on Ω by $z_{\varepsilon p}(x) = z_\varepsilon(x)|z_\varepsilon(x)|^{p-2}\chi_{\{z_\varepsilon(x)\neq 0\}}$. We have $z_{\varepsilon p} z_\varepsilon = |z_\varepsilon|^p$.
Then we take $\varphi = \mathcal{G}(z_{\varepsilon p})$ as a test function in (4); in the left-hand side we obtain $\|z_\varepsilon\|^p_{L^p(\Omega)}$. Let $p_1 = \frac{p}{p-1} > 2$, then

$$\|\nabla\varphi\|_{L^\infty(\Omega)} \leq C_p \|z_{\varepsilon p}\|_{L^{p_1}(\Omega)} = \|z_\varepsilon\|^{p-1}_{L^p(\Omega)},$$

and using $i)$ we easily see that the right-hand side of (4) can be bounded by a term like $C\|z_\varepsilon\|^{p-1}_{L^p(\Omega)}$. This gives the result. □

3.2 Change of variables

We shall use the change of variables:

$$x = \alpha_\varepsilon(s,t), \tag{14}$$

where $\alpha_\varepsilon : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^2$ is an application given by

$$\alpha_\varepsilon(s,t) = \gamma(t) + \varepsilon^\beta s \nu(t).$$

Denote by κ the curvature[3] of Γ. For $\varepsilon > 0$, we denote by $\boldsymbol{C}_\varepsilon$ the rough cylinder

$$\boldsymbol{C}_\varepsilon = \{(s,t),\ t \in \mathbb{T},\ 0 \leq s \leq f(t, t/\varepsilon)\}.$$

Let d_0 be such that

$$0 < d_0 < \frac{1}{\|\kappa\|_\infty}. \tag{15}$$

For all $\varepsilon \in (0, d_0^{1/\beta})$, α_ε is a diffeomorphism between the rough cylinder $\boldsymbol{C}_\varepsilon$ and Ω^m_ε. The Jacobian matrix A_ε of α_ε equals

$$\forall (s,t) \in (-1,1) \times \mathbb{T}, \quad A_\varepsilon(s,t) = J_0(t) \begin{pmatrix} \varepsilon^\beta & 0 \\ 0 & 1 + \varepsilon^\beta s \kappa(t) \end{pmatrix},$$

where

$$\forall t \in \mathbb{T}, \quad J_0(t) = \begin{pmatrix} \nu_1(t) & -\nu_2(t) \\ \nu_2(t) & \nu_1(t) \end{pmatrix}.$$

According to (15), A_ε is invertible. Denote by B_ε its inverse matrix

$$\forall (s,t) \in (-1,1) \times \mathbb{T}, \quad B_\varepsilon(s,t) = \begin{pmatrix} \varepsilon^{-\beta} & 0 \\ 0 & 1/(1 + \varepsilon^\beta s \kappa(t)) \end{pmatrix} J_0^T(t).$$

For any functions v and w belonging to $H^1(\mathbb{R}^2)$, define the functions v and w by

$$\forall (s,t) \in (-1,1) \times \mathbb{T}, \quad \mathrm{v}(s,t) = v \circ \alpha_\varepsilon(s,t), \quad \mathrm{w}(s,t) = w \circ \alpha_\varepsilon(s,t).$$

[3] κ is the function defined by

$$\forall t \in \mathbb{T},\ \nu'(t) = \kappa(t)\gamma'(t).$$

Let $\nabla_{s,t}$ be the gradient operator $(\partial_s, \partial_t)^T$. Using the change of variables, and since $J_0^T = J_0^{-1}$ we obviously have on $(0,2)\times\mathbb{T}$

$$(\nabla_x v \cdot \nabla_x w)\circ\alpha_\varepsilon = (\nabla_{s,t}\mathrm{v})^T B_\varepsilon (B_\varepsilon)^T \nabla_{s,t}\mathrm{w},$$
$$= \frac{1}{\varepsilon^{-2\beta}}\partial_s\mathrm{v}\partial_s\mathrm{w} + \frac{1}{(1+\varepsilon^\beta s\kappa)^2}\partial_t\mathrm{v}\partial_t\mathrm{w}. \tag{16}$$

Hence $\nabla_x v\circ\alpha_\varepsilon\cdot\nabla_x w\circ\alpha_\varepsilon$ is "close" to $\frac{\partial v}{\partial t}\frac{\partial w}{\partial t} + \varepsilon^{-2\beta}\frac{\partial v}{\partial s}\frac{\partial w}{\partial s}$ on $(0,2)\times\mathbb{T}$.

3.3 First convergence results

For any fixed $\psi \in \cup_{p'>2}L^{p'}(\Omega)$ we take $\varphi = \mathcal{G}(\psi)$ as a test function in (4). We obtain

$$\int_\Omega z_\varepsilon\psi\,dx = (\sigma_0 - \sigma_m)\left(E'_\varepsilon + E''_\varepsilon\right), \tag{17}$$

where

$$E'_\varepsilon = \frac{1}{\varepsilon^\beta}\int_{\Omega_\varepsilon^m}\nabla u\cdot\nabla\varphi\,dx, \tag{18}$$

$$E''_\varepsilon = \int_{\Omega_\varepsilon^m}\nabla z_\varepsilon\cdot\nabla\varphi. \tag{19}$$

We pass to the limit in the left-hand side of (17) thanks to Lemma 3.1. Up to an appropriate subsequence we infer

$$\lim_{\varepsilon\to0}\int_\Omega z_\varepsilon\psi\,dx = \int_\Omega z\psi\,dx. \tag{20}$$

The aim of the paper is to obtain the limits of E'_ε and E''_ε.
It is easy to compute the limit of E'_ε. Actually, using the change of variables (s,t) in the expression of E'_ε we infer, for ϵ small enough[4],

$$E'_\varepsilon = \int_\mathbb{T}\int_0^{f(t,t/\varepsilon)}(1+\varepsilon^\beta s\kappa(t))\nabla u\circ\alpha_\varepsilon(s,t)\cdot\nabla\varphi\circ\alpha_\varepsilon(s,t)\mathrm{d}s\,\mathrm{d}t. \tag{21}$$

The regularity of u and φ implies that

$$\sup_{s\in(0,f_{max})}\left\|\nabla u\circ\alpha_\varepsilon(s,.)\cdot\nabla\varphi\circ\alpha_\varepsilon(s,.) - \left(\left.\frac{\partial u}{\partial\nu}\frac{\partial\varphi}{\partial\nu}\right|_{\gamma^+} + \left.\frac{\partial u}{\partial\theta}\frac{\partial\varphi}{\partial\theta}\right|_{\gamma+}\right)\right\|_{L^2(\mathbb{T})} = O(\varepsilon^\beta).$$

We then deduce from the weak convergence of $f(t,\frac{t}{\varepsilon})$ to $\tilde f$ the limit of E'_ε:

$$\lim_{\varepsilon\to0}E'_\varepsilon = \int_\Gamma\left(\frac{\partial u^+}{\partial\nu}\frac{\partial\varphi^+}{\partial\nu} + \frac{\partial u}{\partial\theta}\frac{\partial\varphi}{\partial\theta}\right)\tilde f\,\mathrm{d}\sigma_\Gamma. \tag{22}$$

Therefore we have proved that up to a subsequence

$$(\sigma_0-\sigma_m)\lim_{\varepsilon\to0}E''_\varepsilon = \int_\Omega z\psi - (\sigma_0-\sigma_m)\int_\Gamma\left(\frac{\partial u^+}{\partial\nu}\frac{\partial\varphi^+}{\partial\nu} + \frac{\partial u}{\partial\theta}\frac{\partial\varphi}{\partial\theta}\right)\tilde f\,\mathrm{d}\sigma_\Gamma. \tag{23}$$

To end the proof of Theorem 2.3, it remains to determine the limit of E''_ε.

[4] *i.e.* such that $\epsilon^\beta < (d_0/f_{max})$.

4 Computation of the limit of E''_ε

The limit of E''_ε is more complex to be achieved. Now for simplicity we still denote by z_ε the composition $z_\varepsilon \circ \alpha_\varepsilon$. Using the change of variables (s,t) we infer:

$$E''_\varepsilon = \varepsilon^\beta \int_{\mathbb{T}} \int_0^{f(t,t/\varepsilon)} (1+\varepsilon^\beta s\kappa) \left(\frac{1}{\varepsilon^{2\beta}} \partial_s z_\varepsilon \partial_s \varphi + \frac{1}{(1+\varepsilon^\beta s\kappa)^2} \partial_t z_\varepsilon \partial_t \varphi \right) \mathrm{d}s\, \mathrm{d}t.$$

Unlike for E'_ε, the derivatives of z_ε inside the brackets do not converge strongly. In the following, we show that for all $M > f_{max}$ these derivatives two-scale converge in the cylinder $P_M = (-M, M) \times \mathbb{T}$, for ε tending to zero such that $\varepsilon^\beta \leq d_0/M$.

Denote by Ω^ε_M the tubular neighbourhood of Γ composed by the points at the distance smaller than $\varepsilon^\beta M$ of Γ. By definition, α_ε is a diffeomorphism from P_M onto Ω^ε_M and $\alpha_\varepsilon(P_M)$ contains Ω^m_ε.

According to Lemma 4.1, in order to obtain the limit of E''_ε we just have to prove the two-scale convergence of the derivatives of z_ε in P_M. Actually we have the following general result on the two-scale convergence.

Lemma 4.1. *Let $M > f_{max}$. Let v_ε be a bounded sequence in $L^2(P_M)$ and let $v \in L^2(P_M \times \mathbb{T}^2)$ be a two-scale limit of v_ε for ε tending to zero such that $\varepsilon^\beta < d_0/M$. Let also ϕ be a regular enough function, defined on $P_M \times \mathbb{T}$. Then we have*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{T}} \int_0^{f(t,t/\varepsilon)} v_\varepsilon \phi\left(s,t,\frac{t}{\varepsilon}\right) \mathrm{d}s\, \mathrm{d}t = \int_{\mathbb{T}} \int_0^{f(t,\tau)} \int_{\mathbb{T}^2} v\, \phi(s,t,\tau)\, \mathrm{d}\tau\, \mathrm{d}y\, \mathrm{d}s\, \mathrm{d}t.$$

Proof. Denote by $b(s,t,\tau) = \phi(s,t,\tau)\chi_{\{0<s<f(t,\tau)\}}$ defined on the set $P_M \times \mathbb{T}$, which is independent on ε. The difficulty comes from the fact that the function b is not regular in τ, so we can not take it directly as a test function in the two-scale convergence. Using the change of variables $s = rf(t, \frac{t}{\varepsilon})$ with $r \in [0,1]$, we infer

$$\int_{\mathbb{T}} \int_0^{f(t,t/\varepsilon)} \left| \phi\left(s,t,\frac{t}{\varepsilon}\right) \right|^2 \mathrm{d}s\mathrm{d}t = \int_{\mathbb{T}} \int_0^1 \left| \phi\left(rf\left(t,\frac{t}{\varepsilon}\right), t, \frac{t}{\varepsilon}\right) \right|^2 f\left(t,\frac{t}{\varepsilon}\right) \mathrm{d}r\mathrm{d}t.$$

By regularity, this last integral converges, when ε tends to 0 to

$$\int_{\mathbb{T}} \int_0^1 \int_{\mathbb{T}} |\phi(rf(t,\tau),t,\tau)|^2 [f(t,\tau)]\, \mathrm{d}\tau\mathrm{d}r\mathrm{d}t = \int_{\mathbb{T}} \int_0^{f(t,\tau)} \int_{\mathbb{T}} |\phi(s,t,\tau)|^2 \mathrm{d}\tau\mathrm{d}s\mathrm{d}t.$$

We thus proved the following result:

$$\int_{P_M} \left| b\left(s,t,\frac{t}{\varepsilon}\right) \right|^2 \mathrm{d}t\mathrm{d}s \to \int_{P_M} \int_{\mathbb{T}} |b(s,t,\tau)|^2 \mathrm{d}\tau\mathrm{d}s\mathrm{d}t \quad \text{for} \quad \varepsilon \to 0. \tag{24}$$

We similarly prove that for any ϕ_1 belonging to $L^2(P_M, C(\mathbb{T}))$ we have[5]

$$\int_{P_M} b\left(s,t,\frac{t}{\varepsilon}\right) \phi_1\left(s,t,\frac{t}{\varepsilon}\right) \mathrm{d}t\mathrm{d}s \to \int_{P_M\times\mathbb{T}} b(s,t,\tau)\phi_1(s,t,\tau) \mathrm{d}\tau\mathrm{d}s\mathrm{d}t \quad \text{for } \varepsilon \to 0. \tag{25}$$

[5] We can interpret (25) as a result of "partial " two-scale convergence of $b(s,t,\frac{t}{\varepsilon})$ to $b(s,t,\tau)$. Moreover (24) says that this two-scale convergence is "strong".

By simply adapting the proof of Theorem 11 of Lukassen *et al.* [15] (see also Allaire [3], Theorem 1.8) we prove that the convergences (24) and (25) imply

$$\lim_{\varepsilon\to 0}\int_{P_M} v_\varepsilon b\left(s,t,\frac{t}{\varepsilon}\right)\mathrm{d}s\,\mathrm{d}t=\int_{P_M}\int_{\mathbb{T}^2} v\, b(s,t,\tau)\,\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t,$$

which is the desired result. □

4.1 Two-scale convergence of $\varepsilon^{-\beta}\partial_s z_\varepsilon$ and $\partial_t z_\varepsilon$

Prove now the two-scale convergence of the derivatives of z_ε.

Lemma 4.2. *Let $p\in(1,2)$. There exist two constants C and C_p such that for any $M>2$, for any $0<\varepsilon^\beta<d_0/M$, we have*

$$i)\quad \left\|\frac{\partial z_\varepsilon}{\partial t}\right\|_{L^2(P_M)}+\left\|\varepsilon^{-\beta}\frac{\partial z_\varepsilon}{\partial s}\right\|_{L^2(P_M)}\le C\varepsilon^{-\beta}.$$

$$ii)\quad \|z_\varepsilon\|_{L^p(P_M)}\le C_p\varepsilon^{-\beta/p}.$$

Proof. According to Lemma 3.1 and with the help of the change of variables (14) we straightforwardly obtain (*ii*). For (*i*) we use the formula (16) with $v=w=z_\varepsilon$. □

By two-scale convergence there exist a subsequence of ε still denoted by ε and $\xi_k^M(s,t,\tau,y)\in L^2(P_M\times]0,1[^2)$, $k=1,2$, such that

$$\frac{\partial z_\varepsilon}{\partial s}\rightarrow\!\!\rightarrow \xi_1^M\qquad\text{in}\quad P_M,$$

and

$$\varepsilon^\beta\frac{\partial z_\varepsilon}{\partial t}\rightarrow\!\!\rightarrow \xi_2^M\qquad\text{in}\quad P_M,$$

where $\rightarrow\!\!\rightarrow$ denotes the two-scale convergence.
For $k=1,2$ let $\hat\xi_k^M(s,t,\tau)=\int_0^1\xi_k^M(s,t,\tau,y)\,\mathrm{d}y$, which are functions defined on the domain $P_M\times\mathbb{T}$. The following estimate is obvious:

$$\exists C>0,\ \forall M>2,\qquad \left\|\hat\xi_k^M\right\|_{L^2(P_M\times]0,1[)}\le C,\quad k=1,2. \tag{26}$$

Moreover if $M_1<M_2$ then the restriction of $\hat\xi_k^{M_2}$ to the set $\{|s|\le M_1\}$ is exactly $\hat\xi_k^{M_1}$ for $k=1,2$.

Lemma 4.3. *For any $M>f_{max}$ the following results hold.*
i) $\hat\xi_1^M$ *is independent on* τ.
ii) $\int_0^1\hat\xi_2^M\,\mathrm{d}\tau=0$ *a.e.* (s,t).

Proof. *i)* Consider $\theta_1(s,t,\tau)$ and $\theta_2(s,t,\tau)$ in $\mathcal{D}\left(P_M\times\mathbb{T}\right)$ arbitrary, such that

$$\frac{\partial\theta_1}{\partial s}+\frac{\partial\theta_2}{\partial\tau}=0. \tag{27}$$

Using the two-scale convergence and also the fact that $\beta<1$, we infer

$$\int_{P_M}\left[\frac{\partial z_\varepsilon}{\partial s}\theta_1\left(s,t,\frac{t}{\varepsilon}\right)+\varepsilon\frac{\partial z_\varepsilon}{\partial t}\theta_2\left(s,t,\frac{t}{\varepsilon}\right)\right]\to\int_{P_M}\int_0^1\hat\xi_1^M\theta_1,\quad\text{for }\ \varepsilon\to 0.$$

On the other hand, by Green formula and according to (27) and to Lemma 4.2(ii):

$$\int_{P_M}\left[\frac{\partial z_\varepsilon}{\partial s}\theta_1\left(s,t,\frac{t}{\varepsilon}\right)+\varepsilon\frac{\partial z_\varepsilon}{\partial t}\theta_2\left(s,t,\frac{t}{\varepsilon}\right)\right]=-\varepsilon\int_{P_M}z_\varepsilon\frac{\partial\theta_2}{\partial t}\left(s,t,\frac{t}{\varepsilon}\right)\to 0,\quad \text{for }\ \varepsilon\to 0.$$

We then infer

$$\int_{P_M}\int_{\mathbb{T}}\hat{\xi}_1^M\theta_1=0,\qquad \text{for any } (\theta_1,\theta_2)\,\text{satisfying (27)}.$$

Using now the De Rham theorem, we deduce that the vector $(\hat{\xi}_1^M,0)$ is a gradient in the variables (s,τ). Hence there exists a function H such that

$$\frac{\partial H}{\partial s}=\hat{\xi}_1^M\qquad\text{and}\qquad \frac{\partial H}{\partial\tau}=0,$$

which proves i).

ii) From Lemma 4.2 (ii), for any $p\in]1,2[$ and $M>0$ fixed we have

$$\varepsilon^\beta z_\varepsilon\to 0\quad \text{in }\ L^p(P_M)-\ \text{strongly}\qquad\text{for }\varepsilon\to 0, \tag{28}$$

which implies

$$\varepsilon^\beta\frac{\partial z_\varepsilon}{\partial t}\to 0\quad \text{in }\ \mathcal{D}'(P_M).$$

On the other hand, from Lemma 4.2 (i) there exists $\tilde{\xi}\in L^2(P_M)$ such that, up to a subsequence of ε, we have

$$\varepsilon^\beta\frac{\partial z_\varepsilon}{\partial t}\rightharpoonup\tilde{\xi}\quad \text{in }\ \mathcal{D}'(P_M).$$

By identification we obtain

$$\tilde{\xi}=0.$$

Since by the two-scale theory

$$\tilde{\xi}=\int_0^1\int_0^1\xi_2^M\,\mathrm{d}\tau\,\mathrm{d}y,$$

we infer the result. □

Define now the space $H^1_{per,0}(P_M)$ by

$$H^1_{per,0}(P_M)=\{\varphi\in H^1(P_M),\ \varphi|_{|s|=M}=0\},$$

and let

$$D_0=[0,2]\times\mathbb{T}\times\mathbb{T}\qquad\text{and}\qquad D=\{(s,t,\tau)\in D_0,\ 0\le s\le f(t,\tau)\}.$$

The next lemma shows that $\hat{\xi}_1^M$ is independent on M, for $0\le s\le 2$.

Lemma 4.4. *For any $M>f_{max}$,*

$$\hat{\xi}_1^M=\frac{(\sigma_0-\sigma_m)q}{\sigma_m q+\sigma_0(1-q)}\frac{\partial u^+}{\partial\nu},\qquad \textit{for}\qquad 0\le s\le 2,$$

where σ_0, σ_m and $\frac{\partial u^+}{\partial\nu}$ are evaluated in $x=\gamma(t)$ and q is defined by (6).

Proof. We take as test function in (3) an element $\varphi \in H^1_0(\Omega)$ with support in $\alpha_\varepsilon(P_M)$. Using the local coordinates (s,t) and (16) we infer

$$\varepsilon^\beta \int_0^1 \int_{-M}^{M} (1+\varepsilon^\beta s\kappa)\sigma_\varepsilon(\alpha_\varepsilon) \left(\frac{1}{\varepsilon^{2\beta}} \partial_s z_\varepsilon \partial_s \varphi + \frac{1}{(1+\varepsilon^\beta s\kappa)^2} \partial_t z_\varepsilon \partial_t \varphi \right) \mathrm{d}s\,\mathrm{d}t = \\ (\sigma_0 - \sigma_m) \int_0^1 \int_0^{f(t,t/\varepsilon)} (1+\varepsilon^\beta s\kappa) \left(\nabla_{s,t}\varphi\right)^T B_\varepsilon \nabla_x u(\alpha_\varepsilon)\,\mathrm{d}s\,\mathrm{d}t. \tag{29}$$

Take in the above equality a test function $\varphi(s,t)$ which is an element of $H^1_{per,0}(P_M)$ and multiply by ε^β. Observe that $J_0^T \nabla_x u(\gamma) = (\frac{\partial u}{\partial \nu}(\gamma), \frac{\partial u}{\partial \theta}(\gamma))^T$ hence

$$\lim_{\varepsilon\to 0} \left[\int_0^1 \int_{-M}^0 \sigma_1 \frac{\partial z_\varepsilon}{\partial s} \frac{\partial \varphi}{\partial s} + \int_0^1 \int_0^{f(t,t/\varepsilon)} \sigma_m \frac{\partial z_\varepsilon}{\partial s} \frac{\partial \varphi}{\partial s} \right. \\ \left. + \int_0^1 \int_{f(t,t/\varepsilon)}^M \sigma_0 \frac{\partial z_\varepsilon}{\partial s} \frac{\partial \varphi}{\partial s} \right] = (\sigma_0 - \sigma_m) \lim_{\varepsilon\to 0} \int_0^1 \int_0^{f(t,t/\varepsilon)} \frac{\partial \varphi}{\partial s} \frac{\partial u^+}{\partial \nu}|_{\gamma(t)^+}.$$

According to Lemma 4.1 with $v_\varepsilon = \dfrac{\partial z_\varepsilon}{\partial s}$ and Φ in appropriate manner (for example for the second integral we take $\Phi(s,t,\tau) = \sigma_m \frac{\partial \varphi}{\partial s}(s,t)$), we infer

$$\int_0^1 \int_0^1 \int_{-M}^0 \sigma_1 \hat{\xi}_1^M \frac{\partial \varphi}{\partial s} + \int_D \sigma_m \hat{\xi}_1^M \frac{\partial \varphi}{\partial s} + \int_0^1 \int_0^1 \int_{f(t,\tau)}^M \sigma_0 \hat{\xi}_1^M \frac{\partial \varphi}{\partial s} = \\ (\sigma_0 - \sigma_m) \int_D \frac{\partial \varphi}{\partial s} \frac{\partial u^+}{\partial \nu}. \tag{30}$$

Let φ be arbitrary such that $\varphi = 0$ for $s \le f_{max}$. We deduce that $\hat{\xi}_1^M$ is independent on s for $s \ge f_{max}$. On the other hand, according to (26), the L^2–norm of $\hat{\xi}_1^M$ is uniformly bounded in M hence

$$\hat{\xi}_1^M = 0, \quad \text{for} \quad s \ge f_{max}. \tag{31}$$

Now choose $\varphi \in H^1_{per,0}(P_M)$ arbitrary such that $\varphi = 0$ for $s \le 0$ or $s \ge 2$. Integrating (30) first in τ and using the independence of $\hat{\xi}_1^M$ on τ, we obtain

$$\int_{\mathbb{T}} \int_0^2 [\sigma_m\, q + \sigma_0\,(1-q)] \hat{\xi}_1^M \frac{\partial \varphi}{\partial s}\,\mathrm{d}s\,\mathrm{d}t = \int_{\mathbb{T}} \int_0^2 (\sigma_0 - \sigma_m) \frac{\partial u^+}{\partial \nu}\, q\, \frac{\partial \varphi}{\partial s}\,\mathrm{d}s\,\mathrm{d}t,$$

which gives

$$\frac{\partial}{\partial s} \left[(\sigma_m q + \sigma_0(1-q))\, \hat{\xi}_1^M \right] = \frac{\partial}{\partial s} \left[(\sigma_0 - \sigma_m) \frac{\partial u^+}{\partial \nu}\, q \right], \quad \text{for} \quad 0 \le s \le 2.$$

Taking into account (31) we obtain the result. ☐

The next lemma gives an useful information about $\hat{\xi}_2^M$.

Lemma 4.5. *For any $M > f_{max}$ and any function $d \in C(\mathbb{T})$ we have*

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \int_0^{f(t,\tau)} d(t) \hat{\xi}_2^M \,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}t = (\sigma_0 - \sigma_m) \int_{\mathbb{T}} \frac{\partial u}{\partial \theta} d(t) r_2(t)\,\mathrm{d}t,$$

where r_2 is defined by (9).

Proof. In (29) we take a test function φ in the form $\varphi(s,t) = \Phi(s,t,\frac{t}{\varepsilon})$ where Φ is an enough regular function defined on $]-M,M[\times\mathbb{T}^2$. Multiplying (29) by ε we obtain

$$\lim_{\varepsilon\to 0}\Bigg[\int_0^1\int_{-M}^0 \sigma_1\varepsilon^\beta \frac{\partial z_\varepsilon}{\partial t}\frac{\partial \Phi}{\partial \tau}\left(s,t,\frac{t}{\varepsilon}\right) + \int_0^1\int_0^{f(t,t/\varepsilon)} \sigma_m\varepsilon^\beta \frac{\partial z_\varepsilon}{\partial t}\frac{\partial \Phi}{\partial \tau}\left(s,t,\frac{t}{\varepsilon}\right) + \int_0^1\int_{f(t,t/\varepsilon)}^M \sigma_0\varepsilon^\beta \frac{\partial z_\varepsilon}{\partial t}\frac{\partial \Phi}{\partial \tau}\left(s,t,\frac{t}{\varepsilon}\right)\Bigg] = (\sigma_0-\sigma_m)\lim_{\varepsilon\to 0}\int_0^1\int_0^{f(t,t/\varepsilon)}\frac{\partial u}{\partial\theta}(\gamma)\frac{\partial \Phi}{\partial \tau}\left(s,t,\frac{t}{\varepsilon}\right).$$

Passing to the limit and using again Lemma 4.1 we obtain

$$\int_0^1\int_0^1\int_{-M}^0 \sigma_1\hat{\xi}_2^M\frac{\partial \Phi}{\partial \tau} + \int_D \sigma_m\hat{\xi}_2^M\frac{\partial \Phi}{\partial \tau} + \int_0^1\int_0^1\int_{f(t,\tau)}^M \sigma_0\hat{\xi}_2^M\frac{\partial \Phi}{\partial \tau} = \int_D (\sigma_0-\sigma_m)\frac{\partial u}{\partial\theta}\frac{\partial \Phi}{\partial \tau}. \tag{32}$$

By density argument, this equation is also valid for Φ not regular in (s,t) but with the H^1-regularity in τ.
Taking first Φ arbitrary such that $\Phi = 0$ for $s \geq 0$, we deduce that $\hat{\xi}_2^M$ is independent on τ. With the help of Lemma 4.3(ii) we obtain

$$\hat{\xi}_2^M = 0, \quad \text{for} \quad s \leq 0. \tag{33}$$

We similarly obtain

$$\hat{\xi}_2^M = 0, \quad \text{for} \quad s \geq f_{max}. \tag{34}$$

Let Φ be a test function such that

$$\sigma_m\frac{\partial \Phi}{\partial \tau} = d(t) + c(s,t) \text{ on } D,$$
$$\sigma_0\frac{\partial \Phi}{\partial \tau} = c(s,t), \text{ on } D_0\setminus D,$$

where $c(s,t)$ must be chosen such that $\int_0^1 \frac{\partial\Phi}{\partial\tau}\,\mathrm{d}\tau = 0$ in order to have the periodicity in τ. Obviously, the function Φ given on D_0 by $\Phi(s,t,\tau) = \int_0^\tau \varphi_1(s,t,\tau')\,\mathrm{d}\tau'$ where

$$\varphi_1 = \begin{cases} \dfrac{d(t)}{\sigma_m} + \dfrac{c(s,t)}{\sigma_m} & \text{on} \quad D, \\ \dfrac{c(s,t)}{\sigma_0} & \text{on} \quad D_0\setminus D, \end{cases}$$

with

$$c(s,t) = -\frac{d\,\sigma_0 q}{\sigma_0 q + \sigma_m(1-q)}, \tag{35}$$

satisfies the required conditions. We then extend Φ on $s<0$ or $s>2$ such that $\Phi = 0$ on $s = \pm M$.
Taking this Φ as a test function in (32) and according to (33)–(34) we infer:

$$\int_D d(t)\hat{\xi}_2^M + \int_{D_0} c(s,t)\hat{\xi}_2^M = \int_D (\sigma_0-\sigma_m)\frac{\partial u}{\partial\theta}\frac{d+c}{\sigma_m}. \tag{36}$$

From Lemma 4.3 (ii) the second integral of this equality is equal to 0, which gives the result, according to (35). □

4.2 Proofs of Theorem 2.3 and Theorem 2.7

We now end the proof of our main results.

4.2.a Proof of Theorem 2.3

To prove Theorem 2.3 it remains to compute the limit of E''_ε. Using local coordinates (s,t), E''_ε equals

$$E''_\varepsilon = \int_{\mathbb{T}} \int_0^{f(t,t/\varepsilon)} (\nabla_x \varphi)^T (\alpha_\varepsilon)(B_\varepsilon)^T \nabla_{s,t} z_\varepsilon \ \det(A_\varepsilon)\,\mathrm{d}s\,\mathrm{d}t.$$

Using the regularity of σ_0, σ_m and φ we infer

$$\lim_{\varepsilon\to 0} E''_\varepsilon = \lim_{\varepsilon\to 0} \int_{\mathbb{T}} \int_0^{f(t,t/\varepsilon)} (\nabla_x \varphi^+)^T(\gamma) J_0 \begin{pmatrix} \dfrac{\partial z_\varepsilon}{\partial s} \\ \varepsilon^\beta \dfrac{\partial z_\varepsilon}{\partial t} \end{pmatrix} \mathrm{d}s\mathrm{d}t.$$

Using now Lemma 4.1 we obtain

$$\lim_{\varepsilon\to 0} E''_\varepsilon = \int_D \frac{\partial \varphi}{\partial \theta}(\gamma)\hat{\xi}_2^M + \int_D \frac{\partial \varphi^+}{\partial \nu}(\gamma)\hat{\xi}_1^M.$$

From Lemma 4.5 with $d(t) = \frac{\partial \varphi}{\partial \theta}(\gamma(t))$, we deduce

$$\int_D \frac{\partial \varphi}{\partial \theta}(\gamma)\hat{\xi}_2^M = (\sigma_0 - \sigma_m) \int_{\mathbb{T}} \frac{\partial u}{\partial \theta}(\gamma) \frac{\partial \varphi}{\partial \theta}(\gamma) r_2(t)\,\mathrm{d}t.$$

The expression of $\hat{\xi}_1^M$ of Lemma 4.4 leads to

$$\int_D \frac{\partial \varphi^+}{\partial \nu}(\gamma)\hat{\xi}_1^M = (\sigma_0 - \sigma_m) \int_{\mathbb{T}} \frac{\partial u^+}{\partial \nu}(\gamma) \frac{\partial \varphi^+}{\partial \nu}(\gamma) r_1(t)\,\mathrm{d}t$$

and this last three equalities give

$$\lim_{\varepsilon\to 0} E''_\varepsilon = (\sigma_0 - \sigma_m) \int_\Gamma \left(\frac{\partial u^+}{\partial \nu}\frac{\partial \varphi^+}{\partial \nu} r_1(t) + \frac{\partial u}{\partial \theta}\frac{\partial \varphi}{\partial \theta} r_2(t) \right) \mathrm{d}\Gamma. \tag{37}$$

Inserting (37) into (23) leads to equality (10) of Theorem 2.3.

4.2.b Proof of Theorem 2.7

Let us show that far away from the thin layer, the sequence z_ε is bounded in H^1. Then using a compacity argument we infer that z is the strong limit of z_ε in L^s, for all $s \geq 1$, which is exactly Theorem 2.7.

Lemma 4.6. *Let D be an open set such that $\Gamma \subset D$ and $\overline{D} \subset \Omega$. Then there exist two positive constants ε_0 and c depending on D such that, for any $\varepsilon \in]0, \varepsilon_0[$ we have*

$$\|z_\varepsilon\|_{H_1(\Omega\setminus D)} \leq c.$$

Proof. We proceed as in [9]. We introduce the linear operator $\mathcal{R} : H^1(\Omega \setminus D) \to H^1(D)$ given by $\mathcal{R}(\psi) = \varphi$ iff φ is the unique solution of the problem

$$\left\{ \begin{array}{ccc} -\nabla \cdot (\sigma \nabla \varphi) = 0 & \text{in} & D \\ \varphi = \psi & \text{on} & \partial D. \end{array} \right. \tag{38}$$

It is clear, by interior regularity, that for any open set D_1 with $\overline{D}_1 \subset D$ there exists a positive constant c_1 depending on D_1 such that

$$\|\mathcal{R}(\psi)\|_{W^{1,\infty}(D_1)} \leq c_1 \|\psi\|_{H^1(\Omega\setminus D)}, \quad \forall\, \psi \in H^1(\Omega \setminus D). \tag{39}$$

We now introduce the function φ_ε defined in Ω by

$$\varphi_\varepsilon = \begin{cases} z_\varepsilon & \text{in } \Omega \setminus D \\ \mathcal{R}(z_\varepsilon) & \text{in } D. \end{cases} \tag{40}$$

It is clear that $\varphi_\varepsilon \in H^1_0(\Omega)$ so we can take it as a test function in the variational formulation (4). We obtain

$$\int_\Omega \sigma \nabla z_\varepsilon \cdot \nabla \varphi_\varepsilon = -\int_{\Omega^m_\varepsilon} (\sigma^\varepsilon - \sigma) \nabla z_\varepsilon \cdot \nabla \mathcal{R}(z_\varepsilon) - \frac{1}{\varepsilon^\beta} \int_{\Omega^m_\varepsilon} (\sigma^\varepsilon - \sigma) \nabla u \cdot \nabla \mathcal{R}(z_\varepsilon). \tag{41}$$

On the other hand, taking $\mathcal{R}(z_\varepsilon) - z_\varepsilon \in H^1_0(D)$ as a test function in (38) with $\psi = z_\varepsilon$, we obtain

$$\int_D \sigma |\nabla \mathcal{R}(z_\varepsilon)|^2\, dx = \int_D \sigma \nabla z_\varepsilon \cdot \nabla \mathcal{R}(z_\varepsilon)$$

so, the left-hand side of (41) becomes

$$\int_{\Omega - D} \sigma |\nabla z_\varepsilon|^2\, dx + \int_D \sigma |\nabla \mathcal{R}(z_\varepsilon)|^2\, dx$$

Now using *i*) of Lemma 3.1 and the inequality (39) we easily control the terms of the right of (41) and with the help of the Poincaré inequality on $\Omega \setminus D$ we obtain the desired result. □

5 Conclusion

In this paper, we have derived appropriate transmission conditions to tackle the numerical difficulties inherent in the geometry of a very rough thin layer. These transmission conditions lead to an explicit characterization of the polarization tensor of Vogelius and Capdeboscq [10]. More precisely, suppose that $\sigma_0 = \sigma_1$ and denote by $G(x, y)$ the Dirichlet solution for the Laplace operator defined in [5] pp33 by

$$\begin{cases} \nabla_x \Big(\sigma_0(x) \nabla_x G(x, y) \Big) = -\delta_y, \text{in } \Omega \\ G(x, y) = 0, \quad \forall x \in \partial\Omega. \end{cases}$$

According to Theorem 2.7, the following equality holds almost everywhere in $\partial\Omega$

$$(u_\varepsilon - u)(y) = \varepsilon^\beta \int_\Omega \Delta_x G(x, y) z(x) \mathrm{d}x + o(\varepsilon^\beta), \quad y \in \partial\Omega.$$

According to (11), simple calculations lead for almost every $y \in \partial\Omega$ to

$$(u_\varepsilon - u)(y) = \varepsilon^\beta \int_\Gamma (\sigma_m - \sigma_0) M(s) \begin{pmatrix} \partial_n u \\ \nabla_\Gamma u \end{pmatrix} \cdot \begin{pmatrix} \partial_n G \\ \nabla_\Gamma G \end{pmatrix} (s, y) \mathrm{d}_\Gamma(s) + o(\varepsilon^\beta),$$

where M is the polarization tensor defined by

$$\forall s \in \Gamma, M(s) = \begin{pmatrix} \tilde{f} + (\sigma_0 - \sigma_m) r_1 & 0 \\ 0 & \tilde{f} + (\sigma_0 - \sigma_m) r_2 \end{pmatrix}.$$

Observe that if f is constant, then $M(s) = \begin{pmatrix} \sigma_0/\sigma_m & 0 \\ 0 & 1 \end{pmatrix}$, which is the polarization tensor given by Beretta *et al.* [6, 7].

One of the main feature of our result is the following. Unlike the case of the weakly oscillating thin membrane (see [16]), if the quasi ε-period of the oscillations of the rough layer is fast compared to its thickness, then the layer influence on the steady-state potential may not be approximated by only considering the mean effect of the rough layer.

Actually, if we were to consider the mean effect of the roughness, the approximate transmission conditions would be these presented in (13), by replacing f by its average $\tilde{f}$ defined in (7). Observe that our transmission conditions (11) are different since they involve parameters r_1 and r_2 quantifying the roughness of Ω_ε^m. More precisely, denote by $\tilde{z}$ the correction, which only takes into account the mean effect of the layer. Then according to (13), $\tilde{z}$ will satisfy (for simplicity, we consider the ε-periodic case):

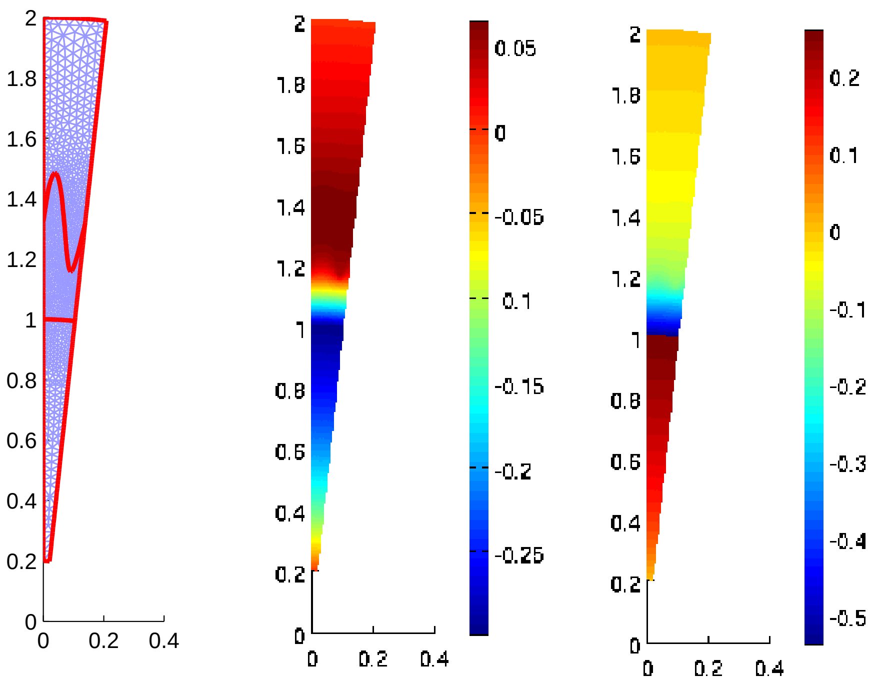
$$\begin{aligned}
&\nabla . (\sigma_k \nabla \tilde{z}) = 0 \quad \text{in} \quad \Omega^k, \quad k = 0, 1, \\
&z^+ - z^- = \left(\frac{\sigma_0}{\sigma_m} - 1 \right) \tilde{f} \frac{\partial u^+}{\partial \nu} \quad \text{on } \Gamma, \\
&\sigma_0 \frac{\partial z^+}{\partial \nu} - \sigma_1 \frac{\partial z^-}{\partial \nu} = \left(\tilde{f} (\sigma_0 - \sigma_m) \frac{\partial^2 u}{\partial \theta^2} \right) \quad \text{on } \Gamma, \\
&z|_{\partial\Omega} = 0.
\end{aligned}$$

To illustrate this assertion, we conclude the paper by numerical simulations obtained using the mesh generator *Gmsh* [13] and the finite element library *Getfem++* [18].

The computational domain Ω is delimited by the circles of radius 2 and of radius 0.2 centered in 0, while Ω^1 is the intersection of Ω with the concentric disk of radius 1. The rough layer is then described by $f(y) = 1 + \frac{1}{2}\sin(y)$ and we choose $\beta = 1/2$. One period of the domain is shown Fig. 2(a). The Dirichlet boundary data is identically 1 on the outer circle and 0 on the inner circle. The conductivities σ_0, σ_1 and σ_m are respectively equal to 1, 1 and 0.1. The computed coefficients for quantifying the roughness are $r_1 = 5.87$ and $r_2 = 0.413$ (three significant digits are kept).

The numerical convergence rates for both the H^1- and the L^2-norms in Ω^1 of the three following errors $u_\varepsilon - u$, $u_\varepsilon - u - \varepsilon^\beta z$ and $u_\varepsilon - u - \varepsilon^\beta \tilde{z}$ as ε goes to zero are given Fig. 3 for[6] $\beta = 1/2$. The numerical convergence rates with the thickness of the layer are comparable between the H^1- and the L^2-norms.

[6]The same numerical simulations have been performed for several values of $\beta < 1$. All the results are very similar, hence we just show here the case $\beta = 1/2$.



(a) One period. (b) Error order 0. (c) Error order 1.

Figure 2: Representation of one period of the domain and the corresponding errors with approximate solutions u and $u + \varepsilon^{\beta} z$. $\varepsilon = 2\pi/60$. Do not consider the error inside the rough layer because a proper reconstruction of the solution in it is not currently implemented.

Observe that they are also similar to the rates shown in [17, 16] and in [11], respectively for the case of constant thickness and for the periodic roughness case. More precisely they are close to 1 for $u_\varepsilon - u$ and for $u_\varepsilon - (u + \varepsilon^\beta \tilde{z})$, whereas the convergence rate is close to 2 for $u_\varepsilon - (u + \varepsilon^\beta z)$. Therefore according to these numerical simulations, the convergence of z_ε to z seems to hold strongly in H^1 far from the layer, even if our method does not lead to such result: another analysis should be performed.

To conclude, Fig. 4 demonstrates that the convergence rate decreases dramatically for $\beta = 1$. This is in accordance with the theory, since the approximate transmission conditions for $\beta = 1$ given in [11, 12] are very different from the conditions proved in the present paper.

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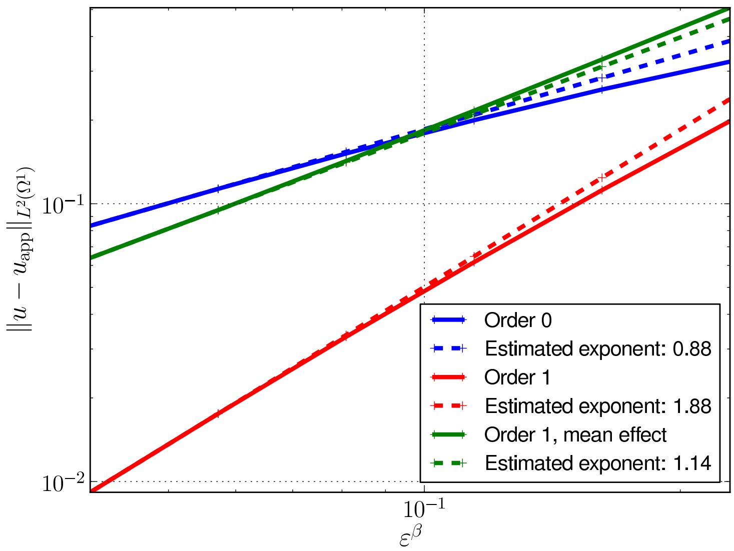
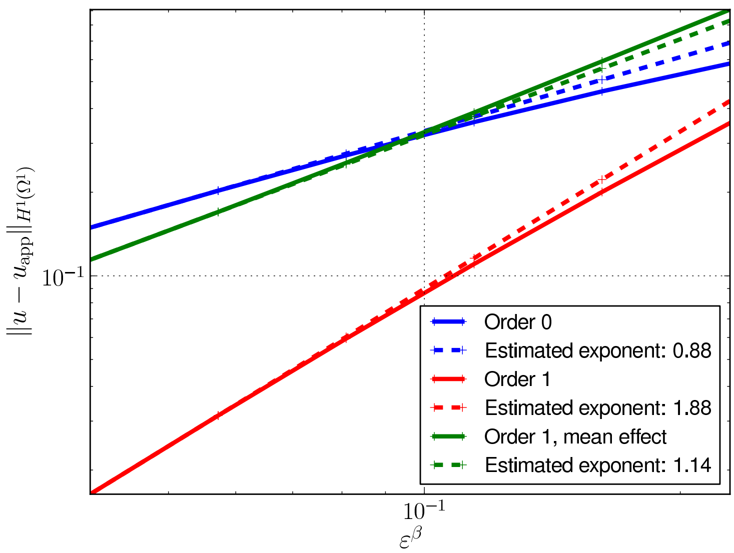


(a) L^2 error



(b) H^1 error

Figure 3: Error in the cytoplasm vs ε^β for three approximate solutions. We choose $\beta = 1/2$.

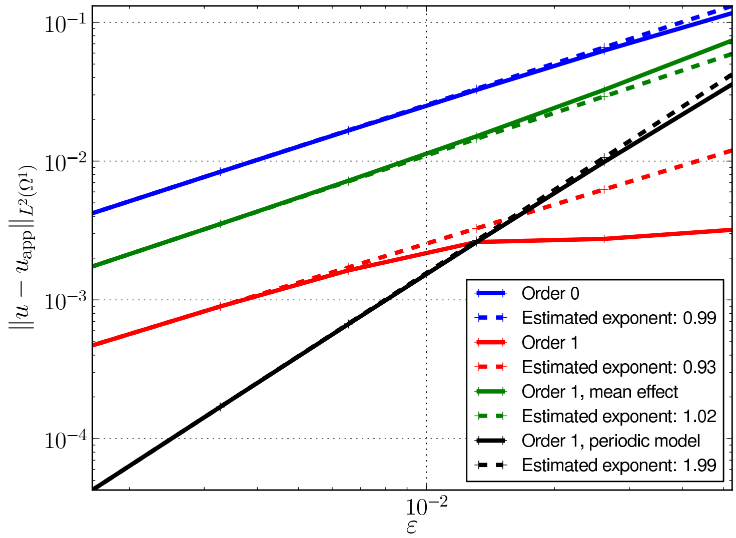


Figure 4: L^2-error in the cytoplasm vs ε for four approximate solutions.

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