Controlled differential equations as Young integrals: a simple approach

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Abstract

The theory of rough paths allows one to define controlled differential equations driven by a path which is irregular. The most simple case is the one where the driving path has finite p-variations with $1 \leqslant p < 2$, in which case the integrals are interpreted as Young integrals. The prototypal example is given by Stochastic Differential Equations driven by fractional Brownian motion with Hurst index greater than 1/2. Using simple computations, we give the main results regarding this theory — existence, uniqueness, convergence of the Euler scheme, flow property, ... — which are spread out among several articles.

Keywords: controlled differential equations, Young integral, fractional Brownian motion, rough paths, flow property, Euler scheme.

1 Introduction

The goal of this article is to solve and study the properties of the rough differential equation

$$y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}x_s \tag{1}$$

where x is a continuous path of finite p-variation with values in a Banach space U and f is a γ-Hölder continuous function with values from V to the

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spaces of linear maps from U to V, under the constraint

$$\gamma + 1 > p. \tag{2}$$

Note that this condition implies that necessarily $p < 2$ so that this approach cannot be used for example for the paths of a Brownian motion.

Controlled differential equations of type (1) have already been subject to several studies. One approach was developed first by T. Lyons in [20], which is one of the seminal paper of the theory of *rough paths*, which allows one to consider paths for which $p > 2$. In this subject, see [11, 16, 17, 21–23]. A special case of (1) which has also been studied intensively from a probabilistic point of view is the one of stochastic differential equation driven by the fractional Brownian motion with Hurst index greater than 1/2. For this, several approaches have been proposed (see [5, 24] for a survey), and one of them relies on the rough paths theory.

In the case $p < 2$, the integral in (1) may be interpreted as a *Young integral*, which was introduced by L.C. Young in [37] in order to define $\int_0^t y_s \, \mathrm{d}x_s$ when x (resp. y) is of finite p-variation (resp. q-variation) with $p^{-1} + q^{-1} > 1$. In order to use the Young integral with $y = f(x)$, the condition (2) is necessary. Other authors have used the definition of Young integrals in order to define solution to differential equations controlled by Hölder continuous paths and the technique used in [35] is rather close to ours. The article [14] also deals with such equations with similar computations, but there, fractional integrals are used. See also [4] for an approach in Besov spaces.

Existence and uniqueness under the assumption that f is continuously differentiable with a γ-Hölder continuous derivative was provided in [20] using a Picard iteration theorem. In [36], D.R.E. Williams have extended this result to the case where x is a Lévy processes. In [19], X.D. Li and T. Lyons have studied the differentiability of the map, called the *Itô map*, which send x to the solution to (1). On the other hand, A.M. Davie have defined in [7] another notion of solution to (1) using the Euler scheme and provided us with several counter-examples to the uniqueness in the case where f is not differentiable with a γ-Hölder continuous derivative, $1 + \gamma > p$.

In this article, we show that the very definition of the Young integral and its properties allows one to recover easily the main results regarding (1) (existence, uniqueness, continuity, flow property, rate of convergence of the Euler scheme). In addition, we are able to deal with the case where f is not bounded, unlike the articles relying on the Picard iteration (however, a global existence result under similar conditions is stated in [7]). Our strategy is the one used in the more complex case for $2 < p \leqslant 3$ for providing bounds and estimation on distances between solutions of rough differential equations [18].

All the results are easily adapted to deal with the controlled differential equation of type

$$y_t = y_0 + \sum_{i=1}^{n} \int_0^t f_i(s, y_s)\,\mathrm{d}x^i_s$$

where the paths x^i are of finite p_i-variations and the vector fields f_i are γ_i-Hölder with $1+\gamma_i > p_i$ in space and uniformly of finite q_i-variation in time with $1/p_i + 1/q_i > 1$. In particular, if B^H is a fractional Brownian motion with Hurst index $H > 1/2$, then one may consider with these methods the stochastic differential equation

$$y_t = y_0 + \int_0^t \sigma(y_s)\,\mathrm{d}B^H_s + \int_0^t b(y_s)\,\mathrm{d}s.$$

Several articles have been written to deal specifically with stochastic differential equations driven by the fractional Brownian motion with Hurst index $H > 1/2$ or with α-Hölder path with $\alpha > 1/2$. We may then recover some of these results: existence of solutions of such equations have been provided in [31]. The properties of such equations are studied for example in [1, 5, 6, 8, 14, 15, 25–30, 32, 35]... Many other articles, including ones using rough paths theory, cover the case $H < 1/2$. On the subject, see among others the review article [5] and the book [25].

Finally, note that in most of the result, we are not bound to work in a finite-dimensional setting. However, to tackle some Stochastic Partial Differential Equations driven by some fractional noise, one needs to use a proper new notion of solution, such as mild solution [13] or variational solution [33, 34].

2 Young integral

2.1 The p-variation (semi-)norm

Let ω be a function from $\Delta^2 := \{0 \leqslant s \leqslant t \leqslant T\}$ to $\mathbb{R}_+$ which is increasing, continuous close to its diagonal and such that

$$\omega(s,r) + \omega(r,t) \leqslant \omega(s,t),\ \forall s \leqslant r \leqslant t \leqslant T. \tag{3}$$

Such a function is called a *control* and condition (3) means that ω is super-additive.

For $1 \leqslant p < 2$, Let us denote by $\Omega_{p,\omega}([0,T], U)$ the set of continuous paths x from $[0,T]$ to a Banach space U such that for some constant C,

$$|x_{s,t}| \leqslant C\omega(s,t)^{1/p},\ x_{s,t} := x_t - x_s,\ (s,t) \in \Delta^2. \tag{4}$$

The smallest constant C such that (4) is denoted by $N_p(x)$ and its called the *p-variation norm of* x. The map $x \mapsto N_p(x)$ is a semi-norm on $\Omega_{p,\omega}([0,T],U)$. It is a norm when restricted to the sub-space of paths x in $\Omega_{p,\omega}([0,T],U)$ with $x_0 = a$ for any a in U.

Remark 1. If x is a α-Hölder continuous path, then one may take $\omega(s,t) = t - s$ for $(s,t) \in \Delta^2$ with $p = 1/\alpha$ and $N_p(x) = H_\alpha(x)$, where $H_\alpha(x)$ is the Hölder constant of x.

Let us recall that a path of finite p-variation is a path $x : [0,T] \to U$ such that

$$V_p(x) := \sup_{\substack{\text{finite partition } \{t_i\}_{i=0,1,\dots,n} \text{ of } [0,T] \\ \text{for any integer } n \geqslant 2}} \left(\sum_{i=0}^{n-1} |x_{t^k_{i+1}} - x_{t^k_i}|^p \right)^{1/p}$$

is finite. With (3), any path in $\Omega_p([0,T],U)$ is of finite p-variation. Conversely, if x is a path of finite p-variation, then for $\omega(s,t) := V_p(x_{|[s,t]})^p$, the path x belongs to $\Omega_{p,\omega}([0,T],U)$ and $N_p(x) \leqslant 1$.

Lemma 1. *Let x be a path of finite p-variation. Then for the control ω defined by $\omega(s,t) := V_p(x_{|[s,t]})^{1/p}$, there exists a sequence of piecewise smooths paths $\{x^n\}_{n\in\mathbb{N}}$ converging to x uniformly and in $\Omega_{r,\omega}([0,T],U)$ for any $r > p$.*

Proof. There exists a non-decreasing, non-negative function φ from $[0,T]$ to $[0,S]$ and a $1/p$-Hölder continuous map from $[0,S]$ to U such that $x = y \circ \varphi$ and $H_{1/p}(y) \leqslant 1$ (See Lemma 3.3 in [3] for example). For this, set $\varphi(t) := V_p(x_{|[0,t]})^p$ and then $\varphi(t) - \varphi(s) \leqslant \omega(0,t) - \omega(0,s) \leqslant \omega(s,t)$. Let y^n be a family of smooth piecewise linear approximations of x along a family of partition whose mesh decreases to 0. From standard computations, $H_{1/p}(y^n) \leqslant 3^{1-1/p}$ and thus $N_p(x^n) \leqslant 3^{1-1/p}$ for $x^n = y^n \circ \varphi$. In addition, x^n converges to x in $\Omega_{r,\omega}([0,T],U)$ for any $r > p$. This last statement follows from an application of the inequality

$$N_r(x^n) \leqslant \|x^n - x\|_\infty^{1-p/r} (N_p(x^n)^{p/r} + N_p(x))^{p/r}), \ r > p$$

and the lemma is then proved. □

If $\{T_i\}_{i=0,\dots,n}$ is an increasing family of times, it follows from standard computations that

$$N_p(x_{|[T_0,T_n]}) \leqslant n^p \max_{i=1,\dots,n} N_p(x_{|[T_{i-1},T_i]}). \tag{5}$$

Thus, the p-variation norm of x on an interval $[0,T]$ may be computed from the p-variation on smaller intervals. This will be useful since many estimates on the solutions of the controlled differential equations will be given first when T is small enough.

Many more informations on paths of finite p-variation may be found in the book [9].

2.2 Young integral

For x in $\Omega_{p,\omega}([0,T],U)$ and y in $\Omega_{q,\omega}([0,T],L(U,V))$ with

$$\frac{1}{p}+\frac{1}{q}>1, \tag{6}$$

the Young integral $t\mapsto\int_0^t y_r\,\mathrm{d}x_r$ is a path in $\Omega_p([0,T],V)$ and extend the notion of usual integral for paths x and y which are smooth.

Theorem 1 ([37]). *Under condition* (6), *for any* $(s,t)\in\Delta^2$, *there exists a family of partitions* $\{t_i^n\}_{i=0,\dots,m_n}$ *of* (s,t) *whose mesh decreases to* 0 *and such that the Riemann sums*

$$\sum_{i=0}^{m_n-1} y_{t_i^n}x_{t_i^n,t_{i+1}^n}$$

converges to an element in V *which we denote by* $\int_s^t y_r\,\mathrm{d}x_r$. *In addition, for any* $0\leqslant s\leqslant r\leqslant t\leqslant T$,

$$\int_s^t y_u\,\mathrm{d}x_u+\int_r^t y_u\,\mathrm{d}x_u=\int_s^t y_u\,\mathrm{d}x_u.$$

The map

$$(x,y)\mapsto\left(t\mapsto\int_0^t y_s\,\mathrm{d}x_s\right)$$

is bilinear and continuous $\Omega_{p,\omega}([0,T],U)\times\Omega_{q,\omega}([0,T],L(U,V))$ *to* $\Omega_{p,\omega}([0,T],V)$. *Finally,*

$$\left|\int_s^t y_r\,\mathrm{d}x_r-y_s(x_t-x_s)\right|\leqslant K(p^{-1}+q^{-1})H_\gamma(f)N_p(x)N_q(y)\omega(s,t)^{1/p+1/q} \tag{7}$$

with $K(\theta):=2^\theta\sum_{j\geqslant 1}1/j^\theta$.

Remark 2. A similar definition may be given for fonctions in Besov spaces [4].

We do not present a complete proof, but the general idea which leads directly to (7) which will be the main inequality. Here, we use the idea given in [21], but alternative points of view are developed in [10] and in [12].

Let $\Pi = \{t_i\}_{i=0,\dots,n}$ be a partition of $[s,t]$ and x in $\Omega_p([0,T],U)$. We set

$$N_p(x|\Pi) := \sup_{u,v\in\Pi} \frac{|x_{u,v}|}{\omega(u,v)^{1/p}} \leqslant N_p(x).$$

Let us consider a partition $\Pi = \{t_i\}_{i=0,\dots,n}$ of $[s,t]$ with $n+1$ points Then there exists a family of partitions $\{\Pi^j\}_{j=1,\dots,n}$ such that $\Pi^1 = \{s,t\}$, $\Pi^{j+1} = \Pi^j \subset \{t_{k(j)}\}$ for some $k(j)$ in $\{0,\dots,n\}$ (which means that Π^j has $j+1$ points) and if u^j and v^j are the two closest points in Π^j of $t_{k(j)}$ with $u^j < t_{k(j)} < v^j$, then

$$\omega(u^j,v^j) \leqslant \frac{2\omega(s,t)}{j-1} \text{ for } j>1.$$

(See Lemma 2.2.1 in [21]).

Let us consider now

$$J_j(s,t) := \sum_{i=0}^{j-1} y_{t_i^j}(x_{t_{i+1}^j} - x_{t_{i+1}^j}) \text{ when } \Pi^j = \{t_i^j\}_{i=0,\dots,j}.$$

Then

$$J_{j+1}(s,t) - J_j(s,t) = (y_{t_{k(j)}} - y_{u_j})(x_{v_j} - x_{t_{t_k(j)}}).$$

and thus

$$|J_{j+1}(s,t) - J_j(s,t)| \leqslant \frac{2^\theta}{j^\theta} N_q(y|\Pi^{j+1}) N_p(x|\Pi^{j+1})\omega(s,t)^\theta,\ \theta := \frac{1}{p} + \frac{1}{q} > 1.$$

Since $\theta > 1$, $K(\theta) := 2^\theta \sum_{j\geqslant 1} 1/j^\theta$ is convergent, and it follows that

$$|J_n(s,t) - J_1(s,t)| \leqslant K(\theta) N_q(y|\Pi) N_p(x|\Pi)\omega(s,t)^\theta. \tag{8}$$

Remark 3. The inequality (8) is called the *Love-Young* estimate. In the orignal article [37], the constant is $K(\theta) := 1 + \sum_{j\geqslant 1} 1/n^\theta$ and the approach is slightly different.

The integral is defined as the limit of the $\{J_n(s,t)\}_{n\geqslant 0}$, which may be proved to be unique.

3 Existence of a solution of the controlled differential equation

Let us consider now a function f from U to $L(U,V)$ which is γ-Hölder continuous, $\gamma \in (0,1]$. We call such a function a Lip(γ)-*vector field* (from U to V).

For y in $\Omega_{p,\omega}([0,T],V)$, $f(y)$ belongs to $\Omega_{p/\gamma}([0,T],L(U,V))$ and

$$N_{p/\gamma}(f(y)) \leqslant H_\gamma(f) N_p(y)^\gamma.$$

Thus, for x in $\Omega_{p,\omega}([0,T],U)$ such that (2) is satisfied, $\int f(y_s)\,\mathrm{d}x_s$ is properly defined as a Young integral.

Definition 1. For a path p in $\Omega_p([0,T],U)$ with $1 \leqslant p < 2$ and a $\mathrm{Lip}(\gamma)$-vector field f with $0 < \gamma \leqslant 1$ and $\gamma + 1 > p$, a solution to (1) is a path y in $\Omega_p([0,T],V)$ such that (1) holds for any $t \in [0,T]$, where the integral $\int_0^t f(y_s)\,\mathrm{d}x_s$ is a Young integral.

3.1 No explosion may occur

Let us start with a first result which assert that any solution is bounded in a finite time.

For practical purpose, let us extend ω to $0 \leqslant s \leqslant t < +\infty$ by setting $\omega(s,t) = \omega(s,T)+t-T$ if $s \leqslant T \leqslant t$. This extension have the same properties as ω.

Let $\varepsilon > 0$ be small enough such that

$$(1 + K(\theta)) N_p(x) H_\gamma(f) \omega(s, s+\tau(s))^{\gamma/p} = \varepsilon \tag{9}$$

has a solution τ for any $s \in [0,T]$ which satisfies $\omega(s, s+\tau(s)) \leqslant 1$. It is possible to find such an ε since ω is increasing and continuous close to its diagonal.

Proposition 1. *If a solution to* (1) *exists with $y_0 = a$, then its p-variation norm and its uniform norm are bounded by constants that depend only on T, p, γ, $N_p(x)$, $H_\gamma(f)$ and a.*

Regarding the uniform norm, we have

$$\sup_{t\in[0,T]} \leqslant C_1 \exp(C_2 \omega(0,T) N_p(x)^{\gamma/p} H_\gamma(f)^{\gamma/p}) \left(|a| + C_2 + C_3 |f(0)|/H_\gamma(f)\right) \tag{10}$$

where C_1, C_2 and C_3 depend only on ε, p and γ.

If f is bounded, we have

$$\sup_{t\in[0,T]} |y_t - a| \leqslant C_4 \omega(0,T), \tag{11}$$

where C_4 depends only on $H_\gamma(f)$, $\|f\|_\infty$, $N_p(x)$, p, γ and ε.

Remark 4. From this, we deduce immediately that any solution remains bounded in a finite time and then no explosion may occurs in a finite time. While A.M. Davie shown local existence in Theorem 2.1 in [7], global existence follows from application of Theorem 6.1 in [7], as well as from the results in [18]. Let us note that here, it is not assume that f is bounded. The bounds given for $\sup_{t\in[0,T]}|y_t - y_a|$ have to be compared to the results in [14], which also deals with unbounded coefficients in a slightly different framework.

Proof. We consider that $N_p(x) > 0$ and $H_\gamma(f) > 0$ (otherwise, (1) is trivially solved). Let us consider a solution y to (1) with $y_0 = a$. From (7),

$$\left|\int_s^t f(y_r)\,\mathrm{d}x_r\right| \leqslant K(\theta)H_\gamma(f)N_p(y)^\gamma N_p(x)\omega(s,t)^\theta + |f(y_s)|N_p(x)\omega(s,t)^{1/p}, \tag{12}$$

with $\theta := (1+\gamma)/p$. Using again the Hölder continuity of f, one gets that

$$\begin{aligned}N_p(y) &= N_p\left(\int_0^\cdot f(y_r)\,\mathrm{d}x_r\right)\\ &\leqslant (1+K(\theta))H_\gamma(f)N_p(y)^\gamma N_p(x)\omega(0,T)^{\gamma/p} + |f(a)|N_p(x).\end{aligned} \tag{13}$$

On the other hand, since $|a|^\gamma \leqslant 1 + |a|$,

$$|f(a)| \leqslant |a|^\gamma H_\gamma(f) + |f(0)| \leqslant |a|H_\gamma(f) + |f(0)| + H_\gamma(f). \tag{14}$$

Short time estimate, f not bounded. Let us set $T = \tau(0)$, where $\tau(0)$ is defined by (9). With (13) and (14),

$$N_p(y) \leqslant \varepsilon N_p(y)^\gamma + |f(0)|N_p(x) + H_\gamma(f)N_p(x) + |a|H_\gamma(f)N_p(x). \tag{15}$$

If $N_p(y) \leqslant 1$, then $N_p(y)^\gamma \leqslant N_p(y)$ and then, if $\varepsilon \leqslant 1/2$,

$$N_p(y) \leqslant 1 + 2|f(0)|N_p(x) + 2H_\gamma(f)N_p(x) + 2|a|H_\gamma(f)N_p(x).$$

Let us note that for $\gamma < 1$, it is not possible it estimate $N_p(y)$ from (15) for values smaller than 1. However, if $\gamma = 1$, we get the better estimate

$$N_p(y) \leqslant 2|f(0)|N_p(x) + 2H_\gamma(f)N_p(x) + 2|a|H_\gamma(f)N_p(x).$$

On the other hand,

$$\sup_{t\in[0,T}|y_t| \leqslant |a| + N_p(y)\omega(0,T)^{1/p}$$

and since $\omega(0, \tau(0)) \leqslant 1$, $\omega(0,T)^{1/p} \leqslant \omega(0,T)^{\gamma/p}$. Thus

$$\sup_{t\in[0,T]} |y_t| \leqslant |a| + 1 + C_5\varepsilon + C_5 \frac{|f(0)|}{H_\gamma(f)}\varepsilon + C_5\varepsilon|a|$$

with $C_5 := 2/(1+K(\theta))$.

Arbitrary time estimate, f not bounded. Now, let us construct a sequence of times T_i by setting $T_{i+1} = \tau(T_i)$. This way,

$$\omega(T_i, T_{i+1}) = \frac{\varepsilon^{p/\gamma}}{(1+K(\theta))^{p/\gamma} N_p(x)^{p/\gamma} H_\gamma(f)^{p/\gamma}}.$$

By the super-additivity of ω,

$$(N+1)\frac{\varepsilon^{p/\gamma}}{(1+K(\theta))^{p/\gamma} N_p(x)^{p/\gamma} H_\gamma(f)^{p/\gamma}} = \sum_{i=0}^{N} \omega(T_i, T_{i+1}) \leqslant \omega(0, T_{N+1}).$$

Thus, if $T \leqslant T_{N+1}$, one has $\omega(0,T) \leqslant \omega(0, T_{N+1})$ and then

$$N+1 \geqslant \frac{\omega(0,T)(1+K(\theta))^{p/\gamma} N_p(x)^{p/\gamma} H_\gamma(f)^{p/\gamma}}{\varepsilon^{p/\gamma}}.$$

Thus,

$$\varepsilon N + \varepsilon \geqslant \frac{\omega(0,T)(1+K(\theta))^{p/\gamma} N_p(x)^{p/\gamma} H_\gamma(f)^{p/\gamma}}{\varepsilon^{p/\gamma-1}}.$$

We are willing to choose N as small as possible. We may then choose ε small enough such that

$$\varepsilon N \leqslant 2\frac{\omega(0,T)(1+K(\theta))^{p/\gamma} N_p(x)^{p/\gamma} H_\gamma(f)^{p/\gamma}}{\varepsilon^{p/\gamma-1}}.$$

Now, with Lemma 4 in Section A, we have for some constants C_6, C_7 and C_8 depending only on ε and C_5,

$$\sup_{t\in[0,T]} |y_t| \leqslant C_6 \exp(\varepsilon N C_5)\left(|a| + C_7 + C_8\frac{|f(0)|}{H_\gamma(f)}\right).$$

With (5), and this inequality, one gets a bound on $N_p(y)$ that depends only on $\omega(0,T)$, $N_p(x)$, $H_\gamma(f)$, $f(0)$, a, ε, p and γ.

Case of f bounded. The case where f is bounded is simpler, since for $T = \tau(0)$,

$$N_p(y) \leqslant \varepsilon N_p(y)^\gamma + \|f\|_\infty N_p(x).$$

As previously, if $\varepsilon \leqslant 1/2$ and $\omega(0,\tau(0)) \leqslant 1$,

$$N_p(y) \leqslant 1 + 2\|f\|_\infty N_p(x)$$

and

$$\begin{aligned}\sup_{t\in[0,T]} |y_t - a| &\leqslant \omega(0,\tau(0))^{1/p} + 2\|f\|_\infty N_p(x)\omega(0,\tau(0))^{1/p} \\ &\leqslant \omega(0,\tau(0))^{\gamma/p} + C_5\|f\|_\infty H_\gamma(f)^{-1}\varepsilon \\ &\leqslant \frac{\varepsilon}{H_\gamma(f)N_p(x)(1+K(\theta))} + C_5\|f\|_\infty H_\gamma(f)^{-1}\varepsilon.\end{aligned}$$

Note that if $\gamma = 1$, we get the simpler inequality $N_p(y) \leqslant 2\|f\|_\infty N_p(x)$.

Thus, for any $T > 0$, with N as above,

$$\sup_{t\in[0,T]} |y_t - a| \leqslant N\varepsilon C_9 \text{ with } C_9 := \|f\|_\infty H_\gamma(f)^{-1} + H_\gamma(f)^{-1}N_p(x)^{-1}(1+K(\theta))^{-1}.$$

We then obtain the estimate (11) on $\sup_{t\in[0,T]} |y_t - a|$. □

3.2 An *a priori* estimate for discrete approximations

We now provide an inequality similar to (13) for discrete approximations.

Let us consider a partition $\Pi = \{t_i\}_{i=0,\dots,n}$ of $[0,T]$ as well as families $\{y_i\}_{i=0,\dots,n}$ and $\{\varepsilon_i\}_{i=0,\dots,n}$ of elements of V satisfying the relation

$$y_{i+1} = y_i + f(y_i)x_{t_i,t_{i+1}} + \varepsilon_i,\ y_0 = a.$$

The equivalent of the p-variation norm along a partition is defined by

$$N_p(y|\Pi) := \sup_{0\leqslant \ell<k\leqslant n} \frac{|y_k - y_\ell|}{\omega(t_k,t_\ell)^{1/p}}.$$

Proposition 2. *For y and ε as above with $N_p(\varepsilon|\Pi) < +\infty$, $N_p(y|\Pi)$ is bounded by a constant that depends only on T, p, γ, $N_p(x)$, $H_\gamma(f)$, $f(a)$ and $N_p(\varepsilon|\Pi)$.*

Proof. Given $0 \leqslant \ell < k \leqslant n$, let us fix a family of partitions $(\Pi^i)_{i=1,\dots,m}$ of in Section 2 such that $\Pi^1 = \{t_\ell, t_k\}$ and $\Pi^m = \Pi \cap [t_\ell, t_k]$. Then as in Section 2,

$$\left|\sum_{i=\ell}^{k-1} f(y_i)x_{t_i,t_{i+1}} - f(y_\ell)x_{t_\ell,t_k}\right| \leqslant K(\theta)N_p(y|\Pi)^\gamma H_\gamma(f)N_p(x|\Pi)(t_k - t_\ell)^\theta$$

and then, using the fact that f if γ-Hölder

$$N_p(y|\Pi) \leqslant (1+K(\theta))N_p(y|\Pi)^\gamma H_\gamma(f)N_p(x) + |f(a)|N_p(x) + N_p(\varepsilon|\Pi).$$

The end of the proof is similar to the one of Proposition 1. □

3.3 Existence of a solution

It is now possible to prove the existence of a solution to (1), which we prove by two different ways.

Proposition 3. *Let V be a finite-dimensional space. For any $T > 0$, x in $\Omega_p([0,T];V)$ and f a* $\mathrm{Lip}(\gamma)$*-vector field with $\gamma+1 > p$, there exists a solution to* (1).

Remark 5. The proofs proposed here may easily be adapted to show existence of solutions to the perturbed controlled differential equation

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s + h_t$$

where h is a path in $\Omega_p([0,T],V)$.

Proof of Proposition 3 Using a Picard scheme. Let us take y^0 in $\Omega_{p,\omega}([0,T],V)$ and define recursively

$$y_t^{n+1} = a + \int_0^t f(y_s^n)\,\mathrm{d}x_s.$$

Since $N_p(y^{n+1}) = N_p\left(\int_0^\cdot f(y_r^n)\,\mathrm{d}x_r\right)$, it follows from a slight modification of (12)-(13) that

$$N_p(y^{n+1}) \leqslant C_{10}N_p(y^n)^\gamma \omega(0,T)^{\gamma/p} + C_{11}$$

with $C_{10} = (1+K(\theta))H_\gamma(f)N_p(x)$ and $C_{11} = |f(a)|N_p(x)$.

Hence, if R is such that $N_p(y^n) \leqslant R$ and T is such that

$$C_{10}R^\gamma\omega(0,T)^{\gamma/p} + C_{11} \leqslant R,$$

it follows that $N_p(y^{n+1}) \leqslant R$ and then the sequence $\{y^n\}_{n\in\mathbb{N}}$ is bounded in $\Omega_{p,\omega}([0,T],V)$.

It follows from an application of the Ascoli-Arzelà theorem and the inequality $N_r(x)^r \leqslant 2^{r-p}\|x\|_\infty^{r-p}N_p(x)^p$ that there exists a convergent subsequence $\{y^{n_k}\}_{k\geqslant 0}$ in $\Omega_{r,\omega}([0,T],V)$ for any $r < p$ to some element y in $\Omega_{p,\omega}([0,T],V)$. Since $f(y^{n_k})$ converges to $f(y)$ in $\Omega_{r/\gamma}([0,T],L(V,U))$, any limit y is solution to (1) when T is small enough. As the choice of T depends only on $H_\gamma(f)$, $N_p(x)$, p and γ, this is also true for any time T by solving recursively (1) on a finite number of time intervals $[T_i,T_{i+1}]$ with $\omega(T_i,T_{i+1})$ satisfying $C_{10}R^\gamma\omega(T_i,T_{i+1})^{\gamma/p} + C_{11} \leqslant R$. □

We give now another proof that relies on the Euler scheme. For this, we use a supplementary hyothesis (16). Note that however, for an appropriate control defined as in Lemma 1, one may skip this condition by studying the problem with a $1/p$-Hölder continuous path and using a time change.

Proof of Proposition 3 using the Euler scheme. Here, we provided an argument using the Euler scheme. For this, we need an additional assumption on the control ω. We then assume that there exists an increasing continuous function ψ and a constant δ such that

$$\sup_{|t-s|<\delta, s\neq t} \max\left\{\frac{\omega(s,t)}{\psi(t)-\psi(s)}, \frac{\psi(t)-\psi(s)}{\omega(s,t)}\right\} < +\infty. \tag{16}$$

Let $\Pi = \{t_i\}_{i=0,\dots,n}$ be a partition of $[0,T]$ and consider the family $\{y_i\}_{i=0,\dots,n}$ constructed recursively by

$$y_{i+1} = y_i + f(y_i)x_{t_i,t_{i+1}},\ y_0 = a.$$

Proposition 2 gives a control on $N_p(y|\Pi)$. Let us extend y to a continuous path on $[0,T]$ by

$$y_t = y_i + \frac{\psi(t)-\psi(t_i)}{\psi(t_{i+1})-\psi(t_i)}(y_{i+1}-y_i),\ t\in[t_i,t_{i+1}].$$

If $t_i \leqslant s < t \leqslant t_{i+1}$, then

$$|y_t - y_s| \leqslant N_p(y|\Pi)\omega(t_i,t_{i+1})^{1/p}\frac{\psi(t)-\psi(s)}{\psi(t_{i+1})-\psi(t_i)}.$$

If $\sup_{i=0,\dots,n-1} t_{i+1}-t_i \leqslant \delta$, with (16), one easily get that

$$|y_t - y_s| \leqslant C_{12}N_p(y|\Pi)\omega(s,t)^{1/p}.$$

If $s \leqslant t_\ell \leqslant t_k \leqslant t$,

$$\begin{aligned}
|y_t - y_s| &\leqslant |y_s - y_{t_\ell}| + |y_{t_\ell} - y_{t_k}| + |y_{t_k} - y_t| \\
&\leqslant N_p(y|\Pi)\big(C_{12}\omega(s,t_\ell)^{1/p} + \omega(t_\ell,t_k)^{1/p} + C_{12}\omega(t_k,t)^{1/p}\big) \\
&\leqslant N_p(y|\Pi)\max\{C_{12},1\}3^{p-1}\omega(s,t)^{1/p}.
\end{aligned}$$

It follows that

$$N_p(y) \leqslant \max\{C_{12},1\}3^{p-1}N_p(y|\Pi)$$

for some constant C_{12} that does not depend on Π.

Now, let us consider an increasing family of partitions $\Pi^n := \{t_i^n\}_{i=0,\dots,n}$ whose meshes decrease to 0 and let y^n be the corresponding solution of the Euler scheme. Then there exists a subsequence $\{y^{n_k}\}$ which converges in the r-variation topology for any $r < p$ to some element y in $\Omega_p([0,T], V)$.

Since

$$y_{i+1}^n = y_i^n + \int_{t_i^n}^{t_{i+1}^n} f(y_s^n)\,\mathrm{d}x_s + \int_{t_i^n}^{t_{i+1}^n} (f(y_s^n) - f(y_{t_i^n}^n))\,\mathrm{d}x_s,$$

it follows that for any $0 \leqslant \ell < k \leqslant n$,

$$\left| y_{t_k^n}^n - y_{t_\ell^n}^n - \int_{t_\ell^n}^{t_k^n} f(y_s^n)\,\mathrm{d}x_s \right| \leqslant N_p(y^n|\Pi^n)^\gamma H_\gamma(f) N_p(x) \sum_{i=\ell}^{k-1} (t_{i+1}^n - t_i^n)^\theta.$$

Since $\theta > 1$, for any $s, t \in \cap_{n\in\mathbb{N}} \Pi^n$, $s \leqslant t$ one deduces that

$$\left| y_t^n - y_s^n - \int_s^t f(y_r^n)\,\mathrm{d}x_r \right| \leqslant C \sup_{i=0,\dots,n-1} \omega(t_i, t_{i+1})^{\theta-1} \omega(0,T).$$

Since $f(y^{n_k})$ converges to $f(y)$ in $\Omega_{\gamma/r}([0,T]; V)$ for any $r < p$, it follows that

$$y_t = y_s + \int_s^t f(y_s)\,\mathrm{d}x_s$$

and then that any limit of $\{y^n\}$ is a solution to (1). $\square$

4 Continuity and uniqueness

Up to now, we have proved only existence of solutions to (1), but nothing ensures their uniqueness. In general, it is hopeless to get the uniqueness, as there are an infinite number of solutions to

$$y_t = \int_0^t f(y_s)\,\mathrm{d}s \text{ with } f(y) = \sqrt{y} \in \mathrm{Lip}(1/2),$$

since this equation is equivalent to $y' = \sqrt{y}$, $y_0 = 0$. For this equation, it is well known that for any $C > 0$, $y_t = ((t-C)/2)_+^2$ is solution to this equation.

Indeed, uniqueness will be granted under a stronger regularity assumption on f. Let us assume that f is continuously differentiable and its derivative is a γ-Hölder continuous function from V to $L(V \otimes U, V)$. We then say that f is a $\mathrm{Lip}(1+\gamma)$-vector field, and we still assume (2). We still consider that $1 + \gamma > p$.

In [7], A.M. Davie gives also a counterexample to the uniqueness of (1) when $f \in \mathrm{Lip}(1+\gamma)$ with $1+\gamma < p$, which means that the condition $1+\gamma > p$ is sharp (if we exclude the case $1+\gamma = p$ where an approach by Besov spaces may be useful).

Proposition 4. *If f is a $\mathrm{Lip}(1+\gamma)$-vector field with $(\gamma+1) > p$ and $x \in \Omega_p([0,T],U)$, then the solution to* (1) *is unique. In addition, the map $x \mapsto y$, called the* Itô map*, is locally Lipschitz continuous with respect to (a,f,x). More precisely, let y (resp. $\widehat{y}$) be the solution to $y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s$ (resp. $\widehat{y}_t = \widehat{a} + \int_0^t \widehat{f}(\widehat{y}_s)\,\mathrm{d}s$). We assume that $\|y\|_\infty \leqslant R$, $N_p(y) \leqslant R$, $|a| \leqslant R$, $H_\gamma(\nabla f) \leqslant R$, $\|\nabla f\|_\infty \leqslant R$, $N_p(x) \leqslant R$ and the same holds true when (a,f,x,y) is replaced by $(\widehat{a},\widehat{f},\widehat{x},\widehat{y})$. Then there exists a constant C_{13} depending on $\omega(0,T)$ and R such that*

$$N_p(y-\widehat{y}) \leqslant C_{13}(N_p(x-\widehat{x}) + \|f-\widehat{f}\|_{\infty,B(0,R)} + \|\nabla f - \nabla \widehat{f}\|_{\infty,B(0,R)} + |a-\widehat{a}|),$$

where for a function g, $\|g\|_{\infty,B(0,R)} := \sup_{|x|\leqslant R} |g(x)|$.

A practical importance is the following: any Rough Differential Equation may be approximated by an ordinary differential equation controlled by a piecewise smooth path, by using Lemma 1.

Remark 6. Let us note that similar computations may be carried to estimate the distance between two Euler approximations. However, we skip the computations for the sake of simplicity.

Proof. Let y and $\widehat{y}$ be two paths in $\Omega_p([0,T],V)$. Let Π a partition of $[0,T]$ and $s<t$ two points of Π. Let us note that

$$\begin{aligned}
&|f(y_t) - f(\widehat{y}_t) - f(y_s) + f(\widehat{y}_s)| \\
&= \left| \int_0^1 \nabla f(\widehat{y}_t + \tau(y_t - \widehat{y}_t))(y_t - \widehat{y}_t)\,\mathrm{d}\tau - \int_0^1 \nabla f(\widehat{y}_s + \tau(y_s - \widehat{y}_s))(y_s - \widehat{y}_s)\,\mathrm{d}\tau \right| \\
&\qquad\leqslant \left| \int_0^1 \nabla f(\widehat{y}_t + \tau(y_t - \widehat{y}_t))(y_t - \widehat{y}_t - y_s + \widehat{y}_s)\,\mathrm{d}\tau \right| \\
&\quad + \left| \int_0^1 \big(\nabla f(\widehat{y}_s + \tau(y_s - \widehat{y}_s))(y_s - \widehat{y}_s) - \nabla f(\widehat{y}_t + \tau(y_t - \widehat{y}_t))\big)(y_s - \widehat{y}_s)\,\mathrm{d}\tau \right| \\
&\leqslant \|\nabla f\|_\infty N_p(y-\widehat{y}|\Pi)\omega(s,t)^{1/p} + \int_0^t H_\gamma(\nabla f)|\tau(\widehat{y}_s - \widehat{y}_t) + (1-\tau)(y_s - y_t)|^\gamma (y_s - \widehat{y}_s)\,\mathrm{d}\tau \\
&\leqslant \|\nabla f\|_\infty N_p(y-\widehat{y}|\Pi)\omega(s,t)^{1/p} + \omega(s,t)^{\gamma/p} H_\gamma(\nabla f)(N_p(y|\Pi)^\gamma + N_p(\widehat{y}|\Pi)^\gamma)|y_s - \widehat{y}_s|.
\end{aligned}$$

As

$$|y_s - \widehat{y}_s| \leqslant |y_s - y_0 - \widehat{y}_s - \widehat{y}_0| + |a-b| \leqslant N_p(y-\widehat{y}|\Pi)\omega(0,T)^{1/p} + |a-b|,$$

it follows that for $F(t) = f(y_t) - f(\widehat{y}_t)$,

$$N_{p/\gamma}(F|\Pi) \leqslant C_{14} N_p(y - \widehat{y}|\Pi) + C_{15}|b - a|, \tag{17}$$

where C_{14} and C_{15} are constants that depend only on $H_\gamma(\nabla f)$, $\|\nabla f\|_\infty$, T, p and γ. This is also true for any $s, t \in [0, T]$ and then

$$N_{p/\gamma}(F) \leqslant C_{14} N_p(y - \widehat{y}) + C_{15}|b - a|. \tag{18}$$

For x and $\widehat{x}$ in $\Omega_p([0, T], U)$ and $\mathrm{Lip}(1 + \gamma)$-vector fields f, $\widehat{f}$, let us consider the solutions to

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s \text{ and } \widehat{y}_t = \widehat{a} + \int_0^t \widehat{f}(\widehat{y}_s)\,\mathrm{d}\widehat{x}_s.$$

By linearity of the Young integral and for F as above,

$$y_t - \widehat{y}_t = a - \widehat{a} + \int_0^t F(s)\,\mathrm{d}x_s + \int_0^t g(\widehat{y}_s)\,\mathrm{d}x_s + \int_0^t \widehat{f}(\widehat{y}_s)\,\mathrm{d}u_s$$

with $u_t = x_t - \widehat{x}_t$ and $g = f - \widehat{f}$. Hence,

$$\begin{aligned} &N_p(y - \widehat{y}) \\ &\quad\leqslant (1 + K(\theta)) N_{\gamma/p}(F) N_p(x) T^{\gamma/p} + (1 + K(2)) H_1(g) N_p(\widehat{y}) H_p(x) T^{1/p} \\ &\qquad + H_1(\widehat{f}) N_p(\widehat{y}) H_p(u) T^{1/p} + |g(a)| N_p(x) + |\widehat{f}(a)| N_p(u) \\ &\qquad\qquad + |f(a) - \widehat{f}(\widehat{a})| H_p(x). \end{aligned}$$

Let us assume that all the values $\|\nabla f\|_\infty$, $H_\gamma(\nabla f)$, $\|\nabla \widehat{f}\|_\infty$, $H_\gamma(\widehat{f})$, $|a|$, $\widehat{a}$, $H_p(x)$ and $H_p(\widehat{x})$ are smaller than a given value R. Let us note that

$$|f(a) - \widehat{f}(\widehat{a})| \leqslant \|\nabla \widehat{f}\|_\infty |a - \widehat{a}| + |f(a) - \widehat{f}(a)|$$

With (18), there exists a time T small enough and a constant K, depending only on R, p, γ, such that

$$N_p(y - \widehat{y}) \leqslant C_{16}(|a - \widehat{a}| + \|f - \widehat{f}\|_{\infty, B(0,R)} + H_1(g) + N_p(u)). \tag{19}$$

Using the usual argument, this may be extended to any time T, up to changing the constant C_{16}. Let us note that if $a = \widehat{a}$, $f = \widehat{f}$ and $x = \widehat{x}$, then $y = \widehat{y}$ and the solution to (1) is necessarily unique. □

5 Convergence of the Euler scheme

If f is a Lip$(1+\gamma)$-vector field, we may now study the distance between the Euler scheme along a partition $\Pi = \{t_i\}_{i=0,\dots,n}$ and the unique solution to (1). For this, let us denote by $\{y_i\}$ the family given by

$$y_{i+1} = y_i + f(y_i)x_{t_i,t_{i+1}},\ y_0 = a,$$

and by z the solution to $z_t = \widehat{a} + \int_0^t f(z_s)\,\mathrm{d}x_s$ for some $\widehat{a}$. With $z_i = z_{t_i}$, the family $\{z_i\}$ is solution to

$$z_{i+1} = z_i + f(z_i)x_{t_i,t_{i+1}} + \varepsilon_i \text{ with } \varepsilon_i := \int_{t_i}^{t_{i+1}} (f(z_s) - f(z_i))\,\mathrm{d}x_r.$$

From y, we construct a continuous path in $\Omega_p([0,T],V)$.

Let us set

$$\delta := \sup_{i=0,\dots,n-1} \omega(t_i,t_{i+1}).$$

Proposition 5. *With δ as above, there exists a constant C_{17} such that*

$$\sup_{t\in[0,T]} |y_t - z_t| \leqslant C_{17}\delta^{2/p-1},$$

where C_{17} is a constant that depends only on $\|\nabla f\|_\infty$, $H_\gamma(f)$, $\omega(0,T)$, $N_p(x)$, p and γ.

Remark 7. When $p=1$, which means that x is of finite variation, then the order of convergence is equal to 1. When p converges to 2, the order of convergence tends to 0. For stochastic differential equations driven by the fractional Brownian motion, the order of convergence is the same as the one found in [24, 30].

Proof. Hence,

$$y_{i+1} - z_{i+1} = y_i - z_i + F(t_i)x_{t_i,t_{i+1}} - \varepsilon_i.$$

Using the same computations as in the proof of Proposition 2, for any $0 \leqslant \ell < k \leqslant n$, one has

$$\begin{aligned}&|y_k - z_k - y_\ell - z_\ell|\\ &\quad\leqslant N_{\gamma/p}(F|\Pi)N_p(x)\omega(t_\ell,t_k)^{(\gamma+1)/p} + |F(t_k)|N_p(x)\omega(t_\ell,t_k)^{1/p} + \left|\sum_{i=\ell}^{k-1}\varepsilon_i\right|.\end{aligned}$$

On the other hand,

$$\left|\int_{t_i}^{t_{i+1}} (f(z_s) - f(z_i))\,\mathrm{d}x_s\right| \leqslant C(2/p)H_1(f)N_p(z)N_p(x)\omega(t_i, t_{i+1})^{2/p}.$$

It follows that

$$\sum_{i=\ell}^{k-1} \omega(t_i, t_{i+1})^{2/p} \leqslant \omega(t_\ell, t_k)\delta^{2/p-1}$$

and then for some constant C_{18} that depends only on $\|\nabla f\|_\infty$, $N_p(x)$, p and γ, it follows that

$$\left|\sum_{i=\ell}^{k-1} \varepsilon_i\right| \leqslant C_{18}\delta^{2/p-1}\omega(0,T)^{1-1/p}\omega(t_\ell, t_k)^{1/p},$$

so that, with (17), since $|F(t_k)| \leqslant C_{19}\omega(0,t_k)^{\gamma/p}N_p(y-z|\Pi) + |F(0)|$,

$$N_p(y-z|\Pi) \leqslant C_{20}N_p(y-z|\Pi)\omega(0,T)^{\gamma/p} + C_{21}\delta^{2/p} + C_{22}|F(0)|.$$

Thus, for T small enough so that

$$C_{20}\omega(0,T)^{\gamma/p} \leqslant \frac{1}{2},$$

it follows that

$$N_p(y-z|\Pi) \leqslant 2C_{21}\delta^{2/p-1} + 2C_{22}|F(0)|. \tag{20}$$

Using the usual argument, by considering that T is a point of Π, it is possible to consider the Euler scheme starting from y_T and the solution starting from z_T to get again an inequality of type (20) on time intervals $[T_i, T_{i+1}]$, so that up to changing the constants, (20) is true for any time interval. Since $F(0) = 0$ when $\widehat{a} = a$ and $F(T_i) \leqslant C_{23}N_p(y-z|\Pi \cap [0,T_i])$, it follows that $N_p(y-z|\Pi) \leqslant C_{24}\delta^{2/p-1}$. On the other hand, for $t_i \leqslant t < t_{i+1}$,

$$\begin{aligned}|y_t - z_t| &\leqslant |y_t - y_{t_i}| + |z_{t_i} - y_{t_i}| + |z_t - z_{t_i}|\\ &\leqslant N_p(y)\delta^{1/p} + N_p(y-z|\Pi)\omega(0,T)^{1/p} + N_p(z)\delta^{1/p}.\end{aligned}$$

Since $\delta^{1/p} \leqslant \delta^{2/p-1}$ for $\delta \leqslant 1$, it follows that the Euler scheme converges at rate $2/p-1$. □

6 Flow property and differentiability

6.1 Flow of homeomorphisms

Let us start with a small lemma regarding the time inversion.

Lemma 2. *Given y in $\Omega_q([0,T],L(U,V))$ and x in $\Omega_p([0,T],U)$ with $p^{-1}+q^{-1}>1$. Then*

$$\int_0^t y_s\,\mathrm{d}x_s = -\int_{T-t}^T y_{T-s}\,\mathrm{d}x_{T-s}.$$

Proof. the Young integral $\int_0^t y_s\,\mathrm{d}x_s$ is define as a limit of the Riemann sum

$$J_n(0,t)=\sum_{i=0}^{m_n} y_{t_i^n} x_{t_i^n,t_{i+1}^n}$$

along a family of partitions $\Pi^n=\{t_i^n\}_{i=0,\dots,m_n}$ such that $\sup_{i=0,\dots,m_n}\omega(t_i^n,t_{i+1}^n)$ decreases to 0.

Hence, for $T>t$,

$$J_n(0,t)=\sum_{i=0}^{m_n} y_{t_{i+1}^n} x_{t_i^n,t_{i+1}^n}-\sum_{i=0}^{m_n} y_{t_i^n,t_{i+1}^n} x_{t_i^n,t_{i+1}^n}$$

But

$$\sum_{i=0}^{m_n} y_{t_i^n,t_{i+1}^n} x_{t_i^n,t_{i+1}^n}\leqslant N_q(y)N_p(x)\sum_{i=0}^{m_n}\omega(t_i,t_{i+1})^{1/p+1/q}\xrightarrow[n\to\infty]{}0.$$

Setting $u_t=y_{T-t}$ and $v_t=x_{T-t}$, one obtains that

$$J_n(0,t)=-\sum_{i=0}^{m_n} u_{T-t_{i+1}^n} v_{T-t_i^n,T-t_{i+1}}+\mathrm{o}(1)$$

which converges to the Young integral $\int_{T-t}^T u_s\,\mathrm{d}v_s$, since $\{T-t_i^n\}_{i=m_n,\dots,0}$ is a partition of $[T-t,T]$. □

For a in V, let us set

$$y_t(a)=a+\int_0^t f(y_s(a))\,\mathrm{d}x_s,\ t\in[0,T] \tag{21}$$

and

$$z_t(a)=a+\int_0^t f(z_s(a))\,\mathrm{d}x_{T-s}. \tag{22}$$

Using the time inversion property,

$$y_{T-t}(a) = a - \int_{T-t}^{T} f(y_{T-s}(a))\,\mathrm{d}x_{T-s} = y_T(a) + \int_0^t f(y_{T-s})\,\mathrm{d}x_{T-s}.$$

From the uniqueness of the solution to (22), $y_{T-t}(a) = z_t(y_T(a))$, and $y_0(a) = a$. Similary, $z_{T-t}(y_T(a)) = z_{T-t}(a)$ and then $a = y_T(z_T(a)) = z_T(y_T(a))$.

Proposition 6. *Let f be in $\mathrm{Lip}(1+\gamma)$ and $x \in \Omega_p([0,T],U)$ with $1+\gamma > p$. For any $T > 0$, the map $a \mapsto y_T(a)$ defines an homeomorphism from V to $y_T(V)$ and its inverse is $a \mapsto z_T(a)$.*

Indeed, not only it is a homeomorphims, but also a diffeomorphism (See 6.3 below).

6.2 Linear equation

Among the equations of interest are the ones of type

$$u_t = a + \int_0^t g(s)u_s\,\mathrm{d}x_s + h(t), \tag{23}$$

where g belongs to $\Omega_{p/\gamma}([0,T], L(W \otimes U, W))$, and h belongs to $\Omega_p([0,T], W)$ for a Banach space W.

For u in $\Omega_p([0,T], W)$, it follows from standard computations that $G(s) := g(s)u_s$ belongs to $\Omega_{p/\gamma}([0,T], L(U,W))$ with

$$\begin{aligned} N_{p/\gamma}(G|\Pi) &\leqslant N_{p/\gamma}(g|\Pi) \sup_{t\in\Pi} |u_t| + N_p(u|\Pi)\|g\|_\infty \\ &\leqslant N_{p/\gamma}(g)|u_0| + N_{p/\gamma}(g)N_p(u|\Pi)\omega(0,T)^{1/p} + N_p(u|\Pi)\|g\|_\infty \end{aligned}$$

for any partition Π. Hence, there exists constants C_{25} and C_{26} depending only on $N_{p/\gamma}(g)$ and $g(0)$ (since $\|g\|_\infty$ depends on these constants) such that

$$N_{p/\gamma}(G|\Pi) \leqslant C_{25}|a| + C_{26}N_p(u|\Pi).$$

This is also true when $N_\cdot(\cdot|\Pi)$ is replaced by $N_\cdot(\cdot)$.

Hence, any solution to (23) satisfies

$$N_p(u) \leqslant C_{27}N_p(x)(C_{25}|a| + C_{26}N_p(u))\omega(0,T)^{\gamma/p} + |a|\cdot|g(0)|N_p(x) + N_p(h).$$

It follows that for T small enough and then for any T that

$$N_p(u) \leqslant C_{28}(|a| + N_p(h)). \tag{24}$$

In particular, as the Young integral is linear, (23) is linear and there exists at most one solution to (23).

We do not deal with existence, which may be proved as previously.

Proposition 7. *There exists a unique solution in $\Omega_p([0,T],W)$ to* (23) *which satisfies* (24).

6.3 Differentiabilty of the Itô map

For x and h in $\Omega_p([0,T],U)$ and f in $\mathrm{Lip}(1+\gamma)$ with $1+\gamma > p$, let us consider the solution to

$$y_t(a,h) = a + \int_0^t f(y_s(a,h))\,\mathrm{d}x_s + \int_0^t f(y_s(a,h))\,\mathrm{d}h_s.$$

We have

$$z_t := y_t(a',h) - y_t(a,0) = a' - a + \int_0^t F(s)\,\mathrm{d}x_s + \int_0^t f(y_s(a',h))\,\mathrm{d}h_s$$

where

$$\begin{aligned} F(s) = f(y_s(a',h)) - f(y_s(a,0)) &= \int_0^1 \nabla f(y_s(a,0) + \tau z_s) z_s\,\mathrm{d}\tau \\ &= H(s) + \nabla f(y_s(a,0)) z_s \end{aligned}$$

with

$$H(s) = \int_0^1 (\nabla f(y_s(a,0) + \tau z_s) - \nabla f(y_s(a,0))) z_s\,\mathrm{d}\tau.$$

Since $y(a,0)$ belongs to $\Omega_p([0,T],V)$, it follows that $g(s,a) := \nabla f(y_s(a,0))$ belongs to $\Omega_{p/\gamma}([0,T],L(V,V))$ and then z is solution to

$$\begin{aligned} z_t = a' - a + \int_0^t g(s,a) z_s\,\mathrm{d}x_s + \int_0^t g(s,a) z_s\,\mathrm{d}h_s \\ + \int_0^t H(s)\,\mathrm{d}x_s + \int_0^t H(s)\,\mathrm{d}h_s + \int_0^t f(y_s(a,0))\,\mathrm{d}h_s. \end{aligned}$$

Let us consider then the solution to

$$u_t(\varepsilon,h) = \varepsilon + \int_0^t g(s,a) u_s(\varepsilon,h)\,\mathrm{d}x_s + \int_0^t f(y_s(a,0))\,\mathrm{d}h_s.$$

From Proposition 7, then

$$N_p(z - u(a'-a,h)) \leqslant C_{28} N_p\left(\int_0^\cdot H(s)\,\mathrm{d}x_s\right) + C_{28} N_p\left(\int_0^\cdot H(s)\,\mathrm{d}h_s\right).$$

On the other hand, for $\tau \in [0,1]$,

$$\begin{aligned} \nabla f(y_s(a,0) + \tau z_s) - \nabla f(y_s(a,0)) - (\nabla f(y_t(a,0) + \tau z_t) - \nabla f(y_t(a,0))) \\ \leqslant \begin{cases} \|z\|_\infty^\gamma H_\gamma(\nabla f), \\ (N_p(y(a,0))^\gamma + N_p(y(a',h))^\gamma)\omega(s,t)^{\gamma/p} \leqslant C_{29}\omega(s,t)^{\gamma/p}. \end{cases} \end{aligned}$$

Thus, for $\kappa \in [0,1]$,

$$|H(s) - H(t)| \leqslant \|z\|_\infty^\gamma H_\gamma(f) H_p(z)\omega(s,t)^{1/p} + \|z\|_\infty^{1+\gamma(1-\kappa)} C_{29}\omega(s,t)^{\gamma\kappa/p}.$$

Choosing κ such that $\kappa\gamma + 1 > p$ and using the fact that $|H(0)| \leqslant |a' - a|^{1+\gamma}$ it follows that

$$N_p\left(\int_0^\cdot H(s)\,\mathrm{d}x_s\right) \leqslant C_{30}\big(\|z\|_\infty^\gamma H_p(z) + \|z\|_\infty^{1+\gamma(1-\kappa)}\big) + C_{31}|a' - a|^{1+\gamma}$$

and

$$N_p\left(\int_0^\cdot H(s)\,\mathrm{d}h_s\right) \leqslant C_{32}\big(\|z\|_\infty^\gamma H_p(z) + \|z\|_\infty^{1+\gamma(1-\kappa)}\big) N_p(h) + C_{33} N_p(h)|a' - a|^{1+\gamma}.$$

Let us assume that $|a' - a| \leqslant \varepsilon$ with ε small enough and $N_p(h) \leqslant \varepsilon$. Then using the expression of z and Proposition 1,

$$N_p(z) \leqslant C_{34}\big(\|z\|_\infty^\gamma H_p(z) + \|z\|_\infty^{1+\gamma(1-\kappa)}\big)(1+\varepsilon) + C_{35}\varepsilon.$$

Since $\|z\|_\infty \leqslant N_p(z)\omega(0,T)^{1/p} + \varepsilon$, it follows that

$$N_p(z) \leqslant C_{36}(N_p(z)\omega(0,T)^{1/p} + \varepsilon).$$

Hence, $N_p(z) \leqslant C_{37}\varepsilon$ and $\|z\|_\infty \leqslant \varepsilon$. Then,

$$N_p(z - u(a' - a, h)) \leqslant C\varepsilon^{1+\gamma(1-\kappa)}.$$

Let us consider the Itô map $\mathfrak{I}$ from $U \times \Omega_{p,\omega}([0,T],U)$ to $\Omega_{p,\omega}([0,T],V)$ which maps (a,x) to the solution to $y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s$.

Proposition 8. *Let $\mathfrak{I}$ be the Itô map defined for a* $\mathrm{Lip}(1+\gamma)$*-vector field with $\gamma+1 > p$. Let κ such that $\gamma\kappa + 1 > p$. Then $\mathfrak{I}$ is locally Fréchet differentiable and its derivative* $\mathrm{d}\mathfrak{I}$ *is given by the solution of the equation*

$$\mathrm{d}\mathfrak{I}(a,x)\cdot(\alpha,h) = \alpha + \int_0^t \nabla f(\mathfrak{I}_s(a,x))\mathrm{d}\mathfrak{I}_s(a,x)\cdot(\alpha,h)\,\mathrm{d}x_s + \int_0^t f(\mathfrak{I}_s(a,x))\,\mathrm{d}h_s.$$

In addition,

$$N_p(\mathfrak{I}(a+\varepsilon\alpha, x+\varepsilon h) - \mathfrak{I}(a,x) - \varepsilon\mathrm{d}\mathfrak{I}(a,x)\cdot(\alpha,h)) \leqslant C\varepsilon^{1+\gamma(1-\kappa)}.$$

If J and K are the matrix-valued solution to

$$J_t = \mathrm{Id} + \int_0^t \nabla f(y_s) J_s\,\mathrm{d}x_s \text{ and } K_t = \mathrm{Id} - \int_0^t K_s \nabla f(y_s)\,\mathrm{d}x_s$$

with $y_t = \mathfrak{I}_t(a,x)$, then using first for x a smooth path and passing to the limit, $K_t J_t = J_t K_t = \mathrm{Id}$ for any $t \in [0,T]$, which means that J_t is invertible. Since $J_t \alpha = \mathrm{d}\mathfrak{I}(a,x)\cdot(\alpha,0)$, it follows that $t \mapsto \mathfrak{I}_t(a,x)$ is a flow of *diffeomorphisms*. Of course, still using smooth paths for x and h and passing to the limit, one has the classical formula,

$$\mathrm{d}\mathfrak{I}(a,x)\cdot(0,h)_t = J_t \int_0^t K_s f(\mathfrak{I}_s(a,x))\,\mathrm{d}h_s.$$

Assuming higher-order differentiability of f, it is possible to get a higher-order development of $\mathfrak{I}(a+\varepsilon\alpha, x+\varepsilon h) - \mathfrak{I}(a,x)$, as in [19].

In addition, this result may serve as a base for dealing with Malliavin calculus for SDE driven by fractional Brownian motion [2, 32].

7 Case of a differential equation with drift

We now consider

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s + \int_0^t g(y_s)\,\mathrm{d}s \tag{25}$$

under the hypothesis that g is Lipschitz continuous with constant L, x is α-Hölder, $\alpha > 1/2$ and f is in $\mathrm{Lip}(1+\gamma)$ with $\alpha(1+\gamma) > 1$.

This kind of equation is motivated by Stochastic Differential Equations driven by fractional Brownian motion, and this result has to be compared with the one in [31].

With the remark at the end of the introduction, the condition on g is sufficient to ensure the existence of a solution to (25), but not its uniqueness because we need g in $\mathrm{Lip}(1+\varepsilon)$ for some $\varepsilon > 0$.

Proposition 9. *Under the above conditions on f, g and x, there exists a unique solution to* (25) *which is α-Hölder continuous.*

Proof. Yet if y and z are two α-Hölder continuous paths with $y_0 = z_0 = a$,

$$\left|\int_s^t (g(y_r) - g(z_r))\,\mathrm{d}s_r\right| \leqslant L\|y-z\|_\infty (t-s) \leqslant L H_\alpha(y-z)(t-s)^\alpha T^{1-\alpha}.$$

It follows that if y and z are two solutions to (25),

$$H_\alpha(y-z) \leqslant C_{38} H_\alpha(y-z) T^{2\alpha-1} + L H_\alpha(y-z) T^{1-\alpha}$$

and then, for T small enough, $H_\alpha(y-z) = 0$ and $y = z$. $\square$

A A useful inequality

We give a simple inequality which we used in the proof of Proposition 1. This follows from the discrete Gronwall inequality.

Lemma 3. *Let L be a positive constant and K be a non-negative constant. If x_i satisfies*

$$x_{i+1} \leqslant (1+L)x_i + K \text{ and } x_i \geqslant 0,$$

then for any integer $N \geqslant 1$,

$$x_N \leqslant \left\{(e^L-1)^{-1}K + (1+L(e^L-1)^{-1})x_0\right\}\exp(NL). \tag{26}$$

Proof. Set $y_i = x_{i+1} - x_i$. Then

$$y_i \leqslant L\sum_{j=0}^{i-1} y_j + Lx_0 + K.$$

From the discrete Gronwall inequality,

$$y_i \leqslant (K + Lx_0)\exp(Li)$$

and then

$$x_{i+1} \leqslant (K+Lx_0)\frac{\exp(L(i+1))-1}{\exp(L)-1} + x_0.$$

Hence the result. □

Lemma 4. *Let x be a continuous path such that*

$$\sup_{t\in[T_i,T_{i+1}]} |x_t| \leqslant (1+L)|x_{T_i}| + K.$$

Then for some constant C_{39} depending only on L and C_{40} depending on L and K,

$$\sup_{t\in[0,T_N]} |y_t| \leqslant \exp(NL)(C_{39}|x_0| + C_{40}).$$

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