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Large Sample Asymptotics for the Ensemble Kalman Filter

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Abstract: The ensemble Kalman filter (EnKF) has been proposed as a Monte Carlo, derivative–free, alternative to the extended Kalman filter, and is now widely used in sequential data assimilation, where state vectors of huge dimension (e.g. resulting from the discretization of pressure and velocity fields over a continent, as considered in meteorology) should be estimated from noisy measurements (e.g. collected at sparse in–situ stations). Even if the state and measurement equations are linear with additive Gaussian white noise, computing and storing the error covariance matrices involved in the Kalman filter is practically impossible, and it has been proposed to represent the filtering distribution with a sample (ensemble) of a few elements and to think of the corresponding empirical covariance matrix as an approximation of the intractable error covariance matrix. Extensions to nonlinear state equations have also been proposed.

Surprisingly, very little is known about the asymptotic behaviour of the EnKF, whereas on the other hand, the asymptotic behaviour of many different classes of particle filters is well understood, as the number of particles goes to infinity. Interpreting the ensemble elements as a population of particles with mean–field interactions (and not merely as an instrumental device producing the ensemble mean value as an estimate of the hidden state), we prove the convergence of the EnKF, with the classical rate $1/\sqrt{N}$, as the number N of ensemble elements increases to infinity. In the linear case, the limit of the empirical distribution of the ensemble elements is the usual (Gaussian distribution associated with the) Kalman filter, as expected, but in the more general case of a nonlinear state equation with linear observations, this limit differs from the usual Bayesian filter. To get the correct limit in this case, the mechanism that generates the elements in the EnKF should be interpreted as a proposal importance distribution, and appropriate importance weights should be assigned to the ensemble elements.

Key-words: sequential data assimilation, Kalman filter, ensemble Kalman filter (EnKF), Bayesian filter, particle filter, mean–field interaction, propagation of chaos.

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Comportement asymptotique du filtre de Kalman d'ensemble

Résumé : Le filtre de Kalman d'ensemble a été proposé à l'origine comme une alternative de type Monte–Carlo, sans dérivées, au filtre de Kalman étendu, et il est maintenant largement utilisé en assimilation de données séquentielle, où il s'agit d'estimer un vecteur d'état de très grande dimension (e.g. obtenu après discrétisation de champs de pression ou de vitesse à l'échelle d'un continent, comme en météorologie) à partir de mesures bruitées (e.g. recueillies au niveau de stations réparties au sol). Même si les équations d'état et d'observation sont linéaires, avec des bruits additifs gaussiens, calculer et conserver en mémoire les matrices de covariance d'erreur impliquées dans le filtre de Kalman est impossible en pratique, et il a été proposé de représenter la distribution de filtrage à l'aide d'un échantillon (ensemble) de quelques éléments et de voir dans la matrice de covariance empirique associée une approximation de la matrice de covariance d'erreur. Des extensions au cas d'équations d'état non–linéaires ont également été proposées.

Curieusement, très peu de résultats sont connus sur le comportement asymptotique du filtre de Kalman d'ensemble, alors que par ailleurs le comportement asymptotique de très nombreuses classes de filtres particulaires est bien compris, quand le nombre de particules croît vers l'infini. En interprétant les éléments d'ensemble comme une population de particules en interaction de champ moyen (et pas seulement comme un procédé instrumental produisant la valeur moyenne de l'ensemble comme estimation de l'état caché), nous montrons la convergence du filtre de Kalman d'ensemble, avec la vitesse classique en $1/\sqrt{N}$, quand le nombre N d'éléments d'ensemble croît vers l'infini. Dans le cas linéaire, la limite de la distribution empirique des éléments d'ensemble est (la distribution gaussienne associée avec) le filtre de Kalman, sans surprise, mais dans le cas plus général d'une équation d'état non–linéaire avec observations linéaires, cette limite ne coïncide pas avec le filtre bayésien. Pour obtenir la limite correcte dans ce cas, le mécanisme qui produit les éléments du filtre de Kalman d'ensemble doit être interprété comme une proposition de distribution d'importance, et les poids d'importance correspondants doivent être attribués aux éléments d'ensemble.

Mots-clés : assimilation de données séquentielle, filtre de Kalman, filtre de Kalman d'ensemble, filtre bayésien, filtre particulaire, interaction en champ moyen, propagation du chaos.

1 Introduction

The ensemble Kalman filter (EnKF) has been proposed [9] as a Monte Carlo, derivative–free, alternative to the extended Kalman filter, and is now widely used in sequential data assimilation [11], where state vectors of huge dimension (e.g. resulting from the discretization of pressure and velocity fields over a continent, as considered in meteorology) should be estimated from noisy measurements (e.g. collected at sparse in–situ stations). Even if the state and measurement equations are linear with additive Gaussian white noise, computing and storing the error covariance matrices involved in the Kalman filter is practically impossible, and it has been proposed to represent the filtering distribution with a sample (ensemble) of a few elements and to think of the corresponding empirical covariance matrix as an approximation of the intractable error covariance matrix. Extensions to nonlinear state equations have also been proposed.

A precise mathematical formulation of the EnKF is provided in [3, 2, 10] and comparisons with other Monte Carlo–based nonlinear filters can be found in [16, 1, 18, 14]. The EnKF has gained popularity because of its simple conceptual formulation and its ease of implementation, and in particular it requires no modification of the forecast model. Simulation studies have demonstrated the ability of the EnKF to efficiently handle strongly nonlinear dynamics and high–dimensional state spaces and it is now widely used in realistic applications with primitive equation models for the ocean and atmosphere.

Surprisingly, very little is known about the asymptotic behaviour of the EnKF, whereas on the other hand, the asymptotic behaviour of many different classes of particle filters is well understood, as the number of particles goes to infinity, see [5] or [4, 6, 7] and references therein. Interpreting the ensemble elements as a population of particles with mean–field interactions (and not merely as an instrumental device producing the ensemble mean value as an estimate of the hidden state), we prove the convergence of the EnKF, with the classical rate $1/\sqrt{N}$, as the number N of ensemble elements increases to infinity. In the linear case, the limit of the empirical distribution of the ensemble elements is the usual (Gaussian distribution associated with the) Kalman filter, as expected, but in the more general case of a nonlinear state equation with linear observations, this limit differs from the usual Bayesian filter. To get the correct limit in this case, the mechanism that generates the elements in the EnKF should be interpreted as a proposal importance distribution, and appropriate importance weights should be assigned to the ensemble elements.

1.1 EnKF as an implementation of the Kalman filter in high dimension

Consider the following linear Gaussian system

$$X_k = F_k \, X_{k-1} + W_k \qquad \text{with} \qquad W_k \sim \mathcal{N}(0, Q_k)$$

$$Y_k = H_k \, X_k + V_k \qquad \text{with} \qquad V_k \sim \mathcal{N}(0, R_k) \ ,$$

with additive Gaussian white noises and with Gaussian initial condition $X_0 \sim \mathcal{N}(m_0, \Sigma_0)$. It is assumed that the covariance matrix R_k is invertible. Clearly, the conditional probability distribution of the hidden state X_k given the past observations $Y_{0:k} = (Y_0, \cdots, Y_k)$ is Gaussian, with mean vector $\widehat{X}_k$ and covariance matrix P_k which satisfy the Kalman filter equations : in the prediction (forecast) step

$$\widehat{X}_k^- = F_k \, \widehat{X}_{k-1} \qquad \text{and} \qquad P_k^- = F_k \, P_{k-1} \, F_k^* + Q_k \ ,$$

in the correction (analysis) step

$$\widehat{X}_k = \widehat{X}_k^- + K_k \, (Y_k - H_k \, \widehat{X}_k^-) \qquad \text{and} \qquad P_k = (I - K_k \, H_k) \, P_k^- \ ,$$

with the Kalman gain matrix defined by

$$K_k = P_k^- \, H_k^* \, (H_k \, P_k^- \, H_k^* + R_k)^{-1} \ ,$$

and initially $\widehat{X}_0^- = m_0$ and $P_0^- = \Sigma_0$. If the dimension m of the hidden state is large, the covariance matrices P_{k-1} and P_k^- are very large $m \times m$ symmetric matrices, hence storing such matrices in memory (and storing the $m \times m$ matrix F_k as well) is almost impossible, and the matrix products in the prediction equation

$$P_k^- = F_k \, P_{k-1} \, F_k^* + Q_k \ ,$$

are even more problematic to work out. Usually, the dimension d of the observation is much less, and the matrix products in the expression of the Kalman gain matrix

$$K_k = P_k^- \; H_k^* \; (H_k \; P_k^- \; H_k^* + R_k)^{-1} \; ,$$

or in the correction equation

$$P_k = (I - K_k \; H_k) \; P_k^- = P_k^- - P_k^- \; H_k^* \; (H_k \; P_k^- \; H_k^* + R_k)^{-1} \; H_k \; P_k^- \; ,$$

are much less problematic to work out.

The idea [3, 2, 10] behind the ensemble Kalman filter (EnKF) is to use Monte Carlo samples and to use the corresponding empirical covariance matrix instead of the prediction covariance matrix. In practice, given an analysis ensemble $(X_{k-1}^{1,a}, \cdots, X_{k-1}^{N,a})$ of N elements, each ensemble element is propagated independently according to

$$X_k^{i,f} = F_k \; X_{k-1}^{i,a} + W_k^i \qquad \text{with} \qquad W_k^i \sim \mathcal{N}(0, Q_k) \; .$$

Notice that the i.i.d. random vectors $(W_k^1, \cdots, W_k^N)$ are *simulated* here, with the same statistics as the additive Gaussian noise W_k in the original state equation. The initial ensemble $(X_0^{1,f}, \cdots, X_0^{N,f})$ is *simulated* as i.i.d. Gaussian random vectors with mean m_0 and covariance matrix Σ_0, i.e. with the same statistics as the initial condition X_0. The empirical mean vector and covariance matrix of the forecast elements $(X_k^{1,f}, \cdots, X_k^{N,f})$ are defined as

$$m_k^N = \frac{1}{N} \sum_{i=1}^N X_k^{i,f} \qquad \text{and} \qquad P_k^N = \frac{1}{N} \sum_{i=1}^N (X_k^{i,f} - m_k^N) \; (X_k^{i,f} - m_k^N)^* \; ,$$

respectively. This empirical covariance matrix is then used in the correction step as follows

$$X_k^{i,a} = X_k^{i,f} + K_k^N \; (Y_k - H_k \; X_k^{i,f} - V_k^i) \qquad \text{with} \qquad V_k^i \sim \mathcal{N}(0, R_k) \; ,$$

with the empirical Kalman gain matrix defined by

$$K_k^N = P_k^N \; H_k^* \; (H_k \; P_k^N \; H_k^* + R_k)^{-1} \; .$$

Notice that the i.i.d. random vectors $(V_k^1, \cdots, V_k^N)$ are *simulated* here, with the same statistics as the additive Gaussian noise V_k in the original observation equation. The rationale behind the simulation of these i.i.d. random vectors is explained in Lemma 2.1. In practice however, the empirical covariance matrix P_k^N is never computed or stored : indeed, to evaluate the matrix–vector product $P_k^N \; u$ where u is a (column) vector of dimension m, only N scalar products need to be evaluated, since

$$P_k^N \; u = (\frac{1}{N} \sum_{i=1}^N (X_k^{i,f} - m_k^N) \; (X_k^{i,f} - m_k^N)^*) \; u = \frac{1}{N} \sum_{i=1}^N \lambda_k^i \; (X_k^{i,f} - m_k^N) \; ,$$

with $\lambda_k^i = (X_k^{i,f} - m_k^N)^* \; u$ for any $i = 1, \cdots, N$. In particular, H_k can be seen as a collection of d (row) vectors of dimension m, and to evaluate the matrix products $P_k^N \; H_k^*$ and $H_k \; P_k^N \; H_k^*$, only $N \times d$ scalar products need to be evaluated, since

$$P_k^N \; H_k^* = (\frac{1}{N} \sum_{i=1}^N (X_k^{i,f} - m_k^N) \; (X_k^{i,f} - m_k^N)^*) \; H_k^* = \frac{1}{N} \sum_{i=1}^N (X_k^{i,f} - m_k^N) \; (h_k^i)^* \; ,$$

and

$$H_k \; P_k^N \; H_k^* = H_k \; (\frac{1}{N} \sum_{i=1}^N (X_k^{i,f} - m_k^N) \; (X_k^{i,f} - m_k^N)^*) \; H_k^* = \frac{1}{N} \sum_{i=1}^N h_k^i \; (h_k^i)^* \; ,$$

with $h_k^i = H_k \; (X_k^{i,f} - m_k^N)$ for any $i = 1, \cdots, N$.

The question that naturally arises here is whether the empirical mean of forecast elements and analysis elements converge to the Kalman predictor and Kalman filter respectively, i.e. whether

$$\frac{1}{N} \sum_{i=1}^N X_k^{i,f} \longrightarrow \widehat{X}_k^- \qquad \text{and} \qquad \frac{1}{N} \sum_{i=1}^N X_k^{i,a} \longrightarrow \widehat{X}_k \; ,$$

in some sense, as $N \uparrow \infty$.

1.2 EnKF as a particle system with mean–field interactions

The ensemble Kalman filter idea has been extended to any system of the form

$$X_k = f_k(X_{k-1}) + W_k \qquad \text{with} \qquad W_k \sim \mathcal{N}(0, Q_k)$$

$$Y_k = H_k \, X_k + V_k \qquad \text{with} \qquad V_k \sim \mathcal{N}(0, R_k) \; ,$$

with additive Gaussian white noises and with non–necessarily Gaussian initial condition $X_0 \sim \eta_0$, see for instance [2, Section 4]. In practice, given an analysis ensemble $(X_{k-1}^{1,a}, \cdots, X_{k-1}^{N,a})$ of N elements, each ensemble element is propagated independently according to the following set of decoupled equations

$$X_k^{i,f} = f_k(X_{k-1}^{i,a}) + W_k^i \qquad \text{with} \qquad W_k^i \sim \mathcal{N}(0, Q_k) \; . \tag{1}$$

Notice that the i.i.d. random vectors $(W_k^1, \cdots, W_k^N)$ are *simulated* here, with the same statistics as the additive Gaussian noise W_k in the original state equation. The initial ensemble $(X_0^{1,f}, \cdots, X_0^{N,f})$ is *simulated* as i.i.d. random vectors with probability distribution η_0, i.e. with the same statistics as the initial condition X_0. The empirical mean vector and covariance matrix of the forecast elements $(X_k^{1,f}, \cdots, X_k^{N,f})$ are defined as

$$m_k^N = \frac{1}{N} \sum_{i=1}^N X_k^{i,f} \qquad \text{and} \qquad P_k^N = \frac{1}{N} \sum_{i=1}^N (X_k^{i,f} - m_k^N) \, (X_k^{i,f} - m_k^N)^* \; ,$$

respectively. This empirical covariance matrix is then used in the correction step to produce a new analysis ensemble $(X_k^{1,a}, \cdots, X_k^{N,a})$, according to the set of equations with mean–field interaction

$$X_k^{i,a} = X_k^{i,f} + K_k(P_k^N) \, (Y_k - H_k \, X_k^{i,f} - V_k^i) \qquad \text{with} \qquad V_k^i \sim \mathcal{N}(0, R_k) \; , \tag{2}$$

where the $m \times d$ matrix $K_k(P)$ is defined by

$$K_k(P) = P \, H_k^* \, (H_k \, P \, H_k^* + R_k)^{-1} \; , \tag{3}$$

for any $m \times m$ covariance matrix P. Notice that the i.i.d. random vectors $(V_k^1, \cdots, V_k^N)$ are *simulated* here, with the same statistics as the additive Gaussian noise V_k in the original observation equation.

The question that naturally arises here is whether the empirical probability distribution (uniform mixture of Dirac masses) of forecast elements and analysis elements converge to the Bayesian predictor and Bayesian filter respectively, i.e. whether

$$\mu_k^{N,f} = \frac{1}{N} \sum_{i=1}^N \delta_{X_k^{i,f}} \longrightarrow \mu_k^- \qquad \text{and} \qquad \mu_k^{N,a} = \frac{1}{N} \sum_{i=1}^N \delta_{X_k^{i,a}} \longrightarrow \mu_k \; ,$$

in some sense, as $N \uparrow \infty$, where

$$\mu_k^-(dx) = \mathbb{P}[X_k \in dx \mid Y_{0:k-1}] \qquad \text{and} \qquad \mu_k(dx) = \mathbb{P}[X_k \in dx \mid Y_{0:k}] \; ,$$

by definition. In view of the analysis equation, it is clear that each analysis element depends on the whole forecast ensemble $(X_k^{1,f}, \cdots, X_k^{N,f})$, which results in dependent analysis elements $(X_k^{1,a}, \cdots, X_k^{N,a})$. Notice however that dependence follows here from *mean–field interaction*, i.e. only through the empirical probability distribution $\mu_k^{N,f}$ of forecast elements, and even more explicitly, only through their empirical covariance matrix P_k^N. Intuitively, some law of large numbers should hold when the ensemble size N increases to infinity and if the empirical covariance matrix P_k^N would be replaced by its deterministic limit, then the ensemble elements would become independent at the limit : this phenomenon is known as *propagation of chaos* [15, 20]. To study the asymptotic behaviour of the empirical probability distributions

$$\mu_k^{N,f} = \frac{1}{N} \sum_{i=1}^N \delta_{X_k^{i,f}} \qquad \text{and} \qquad \mu_k^{N,a} = \frac{1}{N} \sum_{i=1}^N \delta_{X_k^{i,a}} \; ,$$

of the forecast elements and analysis elements, respectively, the idea [20, Section I.1] is to consider substitute i.i.d. random vectors. In practice, these vectors are propagated independently according to the following set of decoupled equations

$$\bar{X}_k^{i,f} = f_k(\bar{X}_{k-1}^{i,a}) + W_k^i \qquad \text{with} \qquad W_k^i \sim \mathcal{N}(0, Q_k) \; , \tag{4}$$

and

$$\bar{X}_k^{i,a} = \bar{X}_k^{i,f} + K_k(\bar{P}_k)\,(Y_k - H_k\,\bar{X}_k^{i,f} - V_k^i) \qquad \text{with} \qquad V_k^i \sim \mathcal{N}(0, R_k)\;, \tag{5}$$

where $\bar{P}_k$ denotes the covariance matrix of the random vector $\bar{X}_k^{i,f}$, and initially $\bar{X}_0^{i,f} = X_0^{i,f}$, i.e. the initial set of i.i.d. random vectors coincides exactly with the initial ensemble. By definition

$$\bar{m}_k = \mathbb{E}[\bar{X}_k^{i,f}] \qquad \text{and} \qquad \bar{P}_k = \mathbb{E}[(\bar{X}_k^{i,f} - \bar{m}_k)\,(\bar{X}_k^{i,f} - \bar{m}_k)^*]\;,$$

respectively. For later purposes, the empirical mean vector and covariance matrix of the i.i.d. random vectors $(\bar{X}_k^{1,f}, \cdots, \bar{X}_k^{N,f})$ are defined as

$$\bar{m}_k^N = \frac{1}{N}\sum_{i=1}^N \bar{X}_k^{i,f} \qquad \text{and} \qquad \bar{P}_k^N = \frac{1}{N}\sum_{i=1}^N (\bar{X}_k^{i,f} - \bar{m}_k^N)\,(\bar{X}_k^{i,f} - \bar{m}_k^N)^*\;,$$

respectively.

Intuitively, each random vector $\bar{X}_k^{i,f}$ or $\bar{X}_k^{i,a}$ individually is close (contiguous) to the corresponding element $X_k^{i,f}$ or $X_k^{i,a}$ in the ensemble Kalman filter, because it starts from exactly the same initial value $\bar{X}_0^{i,f} = X_0^{i,f}$ and it uses exactly the same i.i.d. random vectors $(W_0^i, \cdots, W_k^i)$ and $(V_0^i, \cdots, V_k^i)$ already *simulated* and used in the ensemble Kalman filter. Collectively, the large sample asymptotics of the substitute i.i.d. random vectors is simple to analyse, because of independance, but in counterpart the covariance matrix $\bar{P}_k$ is unknown in general, and so are the substitute i.i.d. random vectors themselves. In contrast, the elements in the ensemble Kalman filter are dependent, because they all contribute to the empirical covariance matrix P_k^N which results in mean–field interaction, but in counterpart this empirical covariance matrix is readily computable, and so are the elements in the ensemble Kalman filter. Let $\bar{\mu}_k^f$ and $\bar{\mu}_k^a$ be the probability distribution of the susbtitute i.i.d. random vectors $\bar{X}_k^{i,f}$ and $\bar{X}_k^{i,a}$ respectively. The contribution of this paper consists in the following answers to the questions raised above

- for linear systems with additive Gaussian white noises and with Gaussian initial condition, the probability distributions $\bar{\mu}_k^f$ and $\bar{\mu}_k^a$ coincide with the Gaussian distributions associated with the Kalman predictor and with the Kalman filter, respectively,
- however, for nonlinear systems with additive Gaussian white noises and with non–necessarily Gaussian initial condition, the probability distributions $\bar{\mu}_k^f$ and $\bar{\mu}_k^a$ differ from the Bayesian predictor μ_k^- and from the Bayesian filter μ_k, respectively,
- under suitable Lipschitz assumptions on the drift function f_k and moment conditions on the initial condition X_0, the empirical probability distribution $\mu_k^{N,f}$ of the forecast elements and the empirical probability distribution $\mu_k^{N,a}$ of the analysis elements converge to the probability distribution $\bar{\mu}_k^f$ and $\bar{\mu}_k^a$, respectively, i.e. for any ϕ in a large enough class of functions

$$\frac{1}{N}\sum_{i=1}^N \phi(X_k^{i,f}) = \int_{\mathbb{R}^m} \phi(x)\,\mu_k^{N,f}(dx) \longrightarrow \int_{\mathbb{R}^m} \phi(x)\,\bar{\mu}_k^f(dx)\;,$$

and

$$\frac{1}{N}\sum_{i=1}^N \phi(X_k^{i,a}) = \int_{\mathbb{R}^m} \phi(x)\,\mu_k^{N,a}(dx) \longrightarrow \int_{\mathbb{R}^m} \phi(x)\,\bar{\mu}_k^a(dx)\;,$$

almost surely and in $\mathbb{L}^p$, as $N \uparrow \infty$.

The paper is organized as follows. The limiting probability distributions are characterized in Section 2 in connection with the Kalman filter or the Bayesian filter, and sufficient conditions are given for the existence of moments. Preliminary estimates are obtained in Section 3, which establish the asymptotic behaviour of the empirical covariance matrices P_k^N and the corresponding gain matrices $K_k(P_k^N)$. Contiguity between elements in the ensemble Kalman filter and the corresponding substitute i.i.d. random vectors is proved in Section 4. These two sections are rather technical, and the impatient reader could jump to Section 5 directly, where convergence of the ensemble Kalman filter is deduced, with the classical rate $1/\sqrt{N}$. Concluding remarks and perspectives for future work are presented in Section 6, which include comparison with particle filters and with the weighted ensemble Kalman filter.

Throughout the paper, the observation sequence is considered as *fixed* and any estimate involving mathematical expectations or any almost sure statement does apply to the *simulated* random vectors only.

2 Identification of the limit, and a priori estimates

The limiting probability distributions $\bar{\mu}^f_k$ and $\bar{\mu}^a_k$ are defined as the probability distributions of the substitute i.i.d. random vectors $\bar{X}^{i,f}_k$ and $\bar{X}^{i,a}_k$ respectively, and are completely characterized by integrals of an arbitrary bounded measurable function ϕ. By definition

$$\int_{\mathbb{R}^m} \phi(x')\, \bar{\mu}^f_k(dx') = \mathbb{E}[\phi(\bar{X}^{i,f}_k)] \;=\; \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \phi(f_k(x)+w)\; p^W_k(dw)\; \bar{\mu}^a_{k-1}(dx) \;,$$

where $p^W_k(dw)$ denotes the Gaussian probability distribution with zero mean vector and covariance matrix Q_k, i.e. the probability distribution of the random vector W^i_k, which completely characterizes $\bar{\mu}^f_k$ in terms of $\bar{\mu}^a_{k-1}$. Sufficient conditions are given in Proposition 2.3 below, under which $\bar{\mu}^f_k$ has a finite second order moment, in which case the covariance matrix $\bar{P}_k$ is finite, and by definition

$$\int_{\mathbb{R}^m} \phi(x')\, \bar{\mu}^a_k(dx') = \mathbb{E}[\phi(\bar{X}^{i,a}_k)] \;=\; \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \phi(x + K_k(\bar{P}_k)\; (Y_k - H_k\, x - v))\; q^V_k(v)\, dv\; \bar{\mu}^f_k(dx) \;,$$

where $q^V_k(v)$ denotes the Gaussian density with zero mean vector and invertible covariance matrix R_k, i.e. the probability density of the random vector V^i_k, which completely characterizes $\bar{\mu}^a_k$ in terms of $\bar{\mu}^f_k$, and initially

$$\int_{\mathbb{R}^m} \phi(x)\; \bar{\mu}^f_0(dx) = \mathbb{E}[\phi(\bar{X}^{i,f}_0)] = \int_{\mathbb{R}^m} \phi(x)\; \eta_0(dx) \;,$$

which implies $\bar{\mu}^f_0 = \eta_0$. On the other hand, the Bayesian filter satisfies

$$\int_{\mathbb{R}^m} \phi(x')\; \mu^-_k(dx') = \mathbb{E}[\phi(X_k) \mid Y_{0:k-1}] = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \phi(f_k(x)+w)\; p^W_k(dw)\; \mu_{k-1}(dx) \;,$$

which completely characterizes μ^-_k in terms of μ_{k-1}, and

$$\int_{\mathbb{R}^m} \phi(x')\; \mu_k(dx') = \mathbb{E}[\phi(X_k) \mid Y_{0:k}] = \frac{\displaystyle\int_{\mathbb{R}^m} \phi(x')\; q^V_k(Y_k - H_k\, x')\; \mu^-_k(dx')}{\displaystyle\int_{\mathbb{R}^m} q^V_k(Y_k - H_k\, x')\; \mu^-_k(dx')} \;,$$

which completely characterizes μ_k in terms of μ^-_k, and initially

$$\int_{\mathbb{R}^m} \phi(x)\; \mu^-_0(dx) = \mathbb{E}[\phi(X_0)] = \int_{\mathbb{R}^m} \phi(x)\; \eta_0(dx) \;,$$

which implies $\mu^-_0 = \eta_0$.

2.1 Connection with the Kalman filter or the Bayesian filter

Consider the linear transformation T_k and the two nonlinear transformations $T^{\rm KF}_k(\cdot)$ and $T^{\rm BF}_k(\cdot)$ defined on the space of probability distributions, and completely characterized by integrals of an arbitrary boundary measurable function ϕ, as

$$\int_{\mathbb{R}^m} \phi(x)\; T_k\, \mu(dx) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \phi(f_k(x)+w)\; p^W_k(dw)\, \mu(dx) \;,$$

$$\int_{\mathbb{R}^m} \phi(x)\; T^{\rm KF}_k(\mu)(dx) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \phi(x + K_k(P(\mu))\, (Y_k - H_k\, x - v))\; q^V_k(v)\, dv\, \mu(dx) \;,$$

where $P(\mu)$ denotes the covariance matrix of the probability distribution μ, and

$$\int_{\mathbb{R}^m} \phi(x)\; T^{\rm BF}_k(\mu)(dx) = \frac{\displaystyle\int_{\mathbb{R}^m} \phi(x)\, q^V_k(Y_k - H_k\, x)\; \mu(dx)}{\displaystyle\int_{\mathbb{R}^m} q^V_k(Y_k - H_k\, x)\; \mu(dx)} \;,$$

respectively.

Lemma 2.1 *If $f_k(x) = F_k\, x$ and if the probability distribution μ is Gaussian with mean vector m and covariance matrix Σ, then the transformed probability distribution $T_k\, \mu$ is Gaussian, with mean vector $F_k\, m$ and covariance matrix $F_k\, \Sigma\, F_k^* + Q_k$.*

If the probability distribution μ is Gaussian, with mean vector m and covariance matrix Σ, then the two transformed probability distributions $T_k^{\rm KF}(\mu)$ and $T_k^{\rm BF}(\mu)$ are Gaussian, with the same mean vector $m + K_k(\Sigma)\,(Y_k - H_k\, m)$ and the same covariance matrix $(I - K_k(\Sigma)\, H_k)\, \Sigma$.

Proof. By definition, the transformed probability distribution $T_k\, \mu$ is the probability distribution of the random vector X' in the model

$$X' = f_k(X) + W_k \qquad \text{with} \qquad X \sim \mu \qquad \text{and} \qquad W_k \sim \mathcal{N}(0, Q_k)\ ,$$

where the random vectors X and W_k are independent. If $f_k(x) = F_k\, x$ and if the probability distribution μ is Gaussian, with mean vector m and covariance matrix Σ, then the random vector X' is Gaussian, with mean vector $F_k\, m$ and covariance matrix $F_k\, \Sigma\, F_k^* + Q_k$.

By definition, the transformed probability distribution $T_k^{\rm BF}(\mu)$ is the conditional probability distribution of the random vector X given Y_k in the model

$$Y_k = H_k\, X + V_k \qquad \text{with} \qquad X \sim \mu \qquad \text{and} \qquad V_k \sim \mathcal{N}(0, R_k)\ ,$$

where the random vectors X and V_k are independent. If the probability distribution μ is Gaussian, with mean vector m and covariance matrix Σ, then this conditional probability distribution is Gaussian, with mean vector $m + K_k(\Sigma)\,(Y_k - H_k\, m)$ and covariance matrix $(I - K_k(\Sigma)\, H_k)\, \Sigma$.

By definition, the transformed probability distribution $T_k^{\rm KF}(\mu)$ is the probability distribution of the random vector X' in the model

$$X' = X + K_k(P(\mu))\,(Y_k - H_k\, X - V_k) \qquad \text{with} \qquad X \sim \mu \qquad \text{and} \qquad V_k \sim \mathcal{N}(0, R_k)\ ,$$

where the random vectors X and V_k are independent, and where the observation Y_k is considered as *fixed*. If the probability distribution μ, whether it is Gaussian or not, has mean vector m and covariance matrix $P(\mu) = \Sigma$, then the random vector X' has mean vector $m + K_k(\Sigma)\,(Y_k - H_k\, m)$ and covariance matrix

$$(I - K_k(\Sigma)\, H_k)\, \Sigma\, (I - K_k(\Sigma)\, H_k)^* + K_k(\Sigma)\, R_k\, K_k^*(\Sigma) = (I - K_k(\Sigma)\, H_k)\, \Sigma\ ,$$

and if the probability distribution μ is Gaussian, then the random vector X' is Gaussian. □

If the probability distribution μ is not Gaussian, then the two transformed probability distributions $T_k^{\rm KF}(\mu)$ and $T_k^{\rm BF}(\mu)$ differ in general : consider for instance the case where μ is a finite mixture of Gaussian distributions, as illustrated in Example 2.2 below.

It follows from the above discussion that $\bar\mu_0^f = \mu_0^-$, and if $\bar\mu_{k-1}^a = \mu_{k-1}$ then necessarily $\bar\mu_k^f = T_k\, \bar\mu_{k-1}^a$ coincides with $\mu_k^- = T_k\, \mu_{k-1}$, but in general $\bar\mu_k^f = \mu_k^-$ does not necessarily imply that $\bar\mu_k^a = T_k^{\rm KF}(\bar\mu_k^f)$ coincides with $\mu_k = T_k^{\rm BF}(\mu_k^-)$, which means that in general the limiting probability distributions $\bar\mu_k^f$ and $\bar\mu_k^a$ do not coincide with the probability distributions μ_k^- and μ_k defining the Bayesian filter.

However, for linear systems of the form considered in Section 1.1, with additive Gaussian white noises and with Gaussian initial condition, the (probability distributions defining the) Bayesian filter coincide with the (Gaussian distributions associated with the) Kalman filter, i.e. the probability distribution μ_k^- is Gaussian, with mean vector $\widehat{X}_k^-$ and covariance matrix P_k^-, and the probability distribution μ_k is Gaussian, with mean vector $\widehat{X}_k$ and covariance matrix P_k. It is then easy to prove by induction that $\bar\mu_k^f = \mu_k^-$ and $\bar\mu_k^a = \mu_k$: indeed, it follows from the general case that $\bar\mu_0^f = \mu_0^-$, and it follows from the induction assumption that $\bar\mu_k^f = T_k\, \bar\mu_{k-1}^a$ coincides with $\mu_k^- = T_k\, \mu_{k-1}$, and since $\bar\mu_k^f = \mu_k^-$ is Gaussian, then it follows from Lemma 2.1 that $\bar\mu_k^a = T_k^{\rm KF}(\bar\mu_k^f)$ coincides with $\mu_k = T_k^{\rm BF}(\mu_k^-)$.

Example 2.2 If the probability distribution μ is a finite mixture of Gaussian distributions, with mean vectors $(x_i\, ,\, i \in I)$, all with the same covariance matrix Σ, and with positive mixture weights $(p_i\, ,\, i \in I)$, then its covariance matrix is defined by

$$P(\mu) = \Sigma + \sum_{i \in I} p_i\ (x_i - m)\,(x_i - m)^* \qquad \text{with} \qquad m = \sum_{i \in I} p_i\ x_i\ .$$

The transformed probability distribution $T_k^{\rm KF}(\mu)$ is a finite mixture of Gaussian distributions, with mean vectors $(x_i^{\rm KF}\,,\, i \in I)$, all with the same covariance matrix

$$P^{\rm KF} = (I - K_k(P(\mu))\, H_k)\, \Sigma\, (I - K_k(P(\mu))\, H_k)^* + K_k(P(\mu))\, R_k\, K_k^*(P(\mu)) \ ,$$

and with unchanged mixture weights $(p_i^{\rm KF}\,,\, i \in I)$, defined by

$$x_i^{\rm KF} = x_i + K_k(P(\mu))\; (Y_k - H_k\, x_i) \qquad \text{and} \qquad p_i^{\rm KF} = p_i \qquad \text{for any } i \in I.$$

On the other hand, the transformed probability distribution $T_k^{\rm BF}(\mu)$ is a finite mixture of Gaussian distributions, with mean vectors $(x_i^{\rm BF}\,,\, i \in I)$, all with the same covariance matrix $P^{\rm BF} = (I - K_k(\Sigma)\, H_k)\, \Sigma$, and with updated mixture weights $(p_i^{\rm BF}\,,\, i \in I)$, defined by

$$x_i^{\rm BF} = x_i + K_k(\Sigma)\; (Y_k - H_k\, x_i) \qquad \text{and} \qquad p_i^{\rm BF} = \frac{p_i\, w_i}{\displaystyle\sum_{j \in I} p_j\, w_j} \qquad \text{for any } i \in I,$$

in terms of the likelihood weights

$$w_i = \exp\{-\tfrac{1}{2}\, (Y_k - H_k\, x_i)^*\; \Xi_k^{-1}\; (Y_k - H_k\, x_i)\} \qquad \text{for any } i \in I \qquad \text{where} \qquad \Xi_k = H_k\, \Sigma\, H_k^* + R_k \ .$$

In Figures 1 and 2, the prior density (in black), the limiting ensemble Kalman filter (EnKF) density (in blue) and the Bayesian filter density (in red) are displayed for a one–dimensional example where

- the bi–modal prior density has a heavy mode located at $x_1 = +2$ with weight $p_1 = 0.8$, and a light mode located at $x_2 = -2$ with weight $p_2 = 0.2$, respectively, both with the same variance $\Sigma = 0.25$,
- the observation coefficient is $H_k = 1$ and the observation noise variance is $R_k = 1$.

Figure 1 corresponds to an observed value $Y_k = +0.5$ in the heavy mode, while Figure 2 corresponds to an observed value $Y_k = -1.5$ in the light mode. Clearly, the EnKF and the Bayesian filter have different strategies to combine the information brought by the bi–modal prior density and the information brought by an observed value in either one mode or the other. In particular, the two Gaussian components contributing to the EnKF density have a variance so large that the EnKF density has only one mode and an even larger variance.

2.2 A priori estimates (existence of moments)

Two different assumptions are introduced for the drift function : Assumption A is sufficient to handle the linear case, whereas Assumption B allows to handle more general cases, including the (discretized) Lorenz model for instance.

Assumption A The drift function is globally Lipschitz continuous, i.e.

$$|f_k(x) - f_k(x')| \leq L\, |x - x'| \ ,$$

for any $x, x' \in \mathbb{R}^m$.

Assumption B The drift function is locally Lipschitz continuous, with at most polynomial growth at infinity, i.e.

$$|f_k(x) - f_k(x')| \leq L\, |x - x'|\; (1 + |x|^s + |x'|^s\,) \ ,$$

for any $x, x' \in \mathbb{R}^m$ and for some $s \geq 0$.

Clearly, Assumption A is a special case of Assumption B, for $s = 0$. Notice that under Assumption A, the drift function has at most linear growth at infinity, i.e.

$$|f_k(x)| \leq M\; (1 + |x|\,) \ ,$$

for any $x \in \mathbb{R}^m$, whereas under Assumption B, the drift function has at most polynomial growth at infinity, i.e.

$$|f_k(x)| \leq M\; (1 + |x|^{s+1}\,) \ ,$$

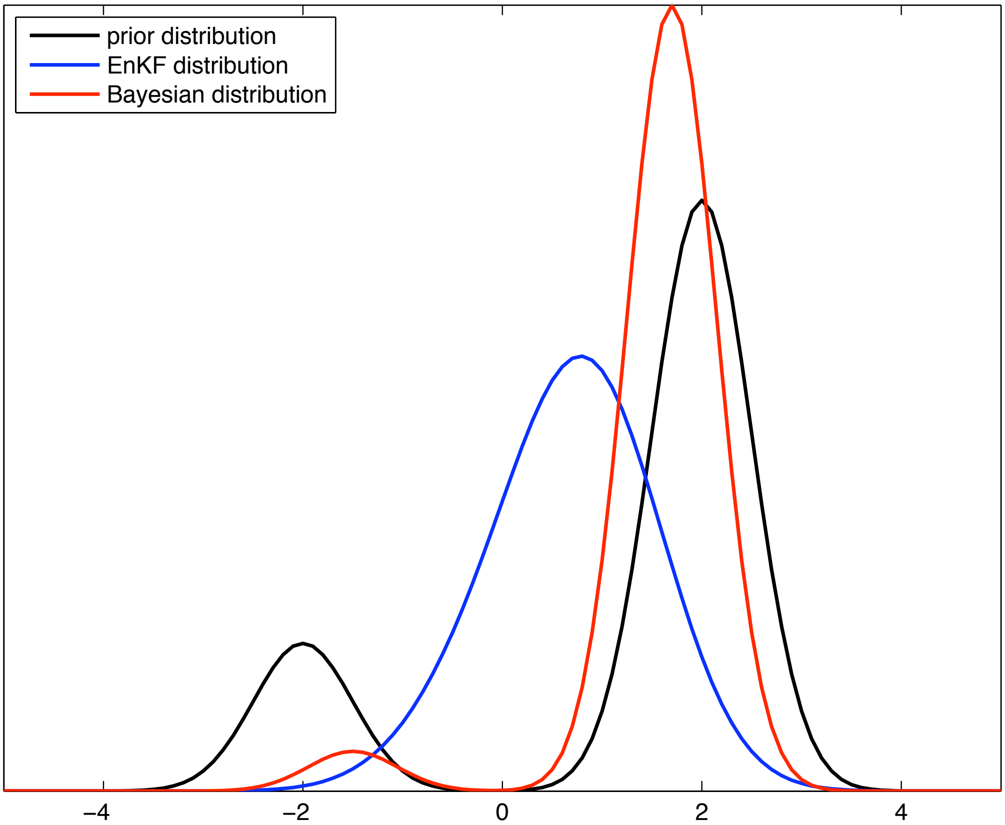

Figure 1: Observed value $Y_k = +0.5$ in the heavy mode

for any $x \in \mathbb{R}^m$, and using the triangle inequality yields the following asymmetric form of the local Lipschitz condition

$$|f_k(x) - f_k(x')| \leq L\, |x - x'|\, (1 + |x|^s\,) + L\, |x - x'|^{s+1} \ ,$$

for any $x, x' \in \mathbb{R}^m$, with another constant L. Introduce the following notations for the moments

$$\bar{M}_k^{p,f} = (\mathbb{E}|\bar{X}_k^{i,f}|^p)^{1/p} = (\int_{\mathbb{R}^m} |x|^p\ \bar{\mu}_k^f(dx)\,)^{1/p} \ ,$$

and

$$\bar{M}_k^{p,a} = (\mathbb{E}|\bar{X}_k^{i,a}|^p)^{1/p} = (\int_{\mathbb{R}^m} |x|^p\ \bar{\mu}_k^a(dx)\,)^{1/p} \ ,$$

and similarly for the moments of the perturbed residuals

$$\bar{R}_k^p = (\mathbb{E}|Y_k - H_k\, \bar{X}_k^{i,f} - V_k^i|^p)^{1/p} \ ,$$

and notice that the triangle inequality yields

$$\bar{R}_k^p \leq (\int_{\mathbb{R}^m} |Y_k - H_k\, x|^p\ \bar{\mu}_k^f(dx)\,)^{1/p} + (\mathbb{E}|V_k^i|^p)^{1/p} \leq |Y_k| + \|H_k\|\ \bar{M}_k^{p,f} + c_p\ \lambda_{\max}^{1/2}(R_k) \ ,$$

where $\lambda_{\max}(R_k)$ denotes the largest eigenvalue of the covariance matrix R_k.

Proposition 2.3 *If Assumption A holds, and if the random vector X_0, or equivalently the probability distribution η_0, has finite moment of order p for some $p \geq 2$, then the random vectors $\bar{X}_k^{i,f}$ and $\bar{X}_k^{i,a}$, or equivalently the probability distributions $\bar{\mu}_k^f$ and $\bar{\mu}_k^a$, have finite moments of the same order p, and in particular the covariance matrix $\bar{P}_k$ is finite.*

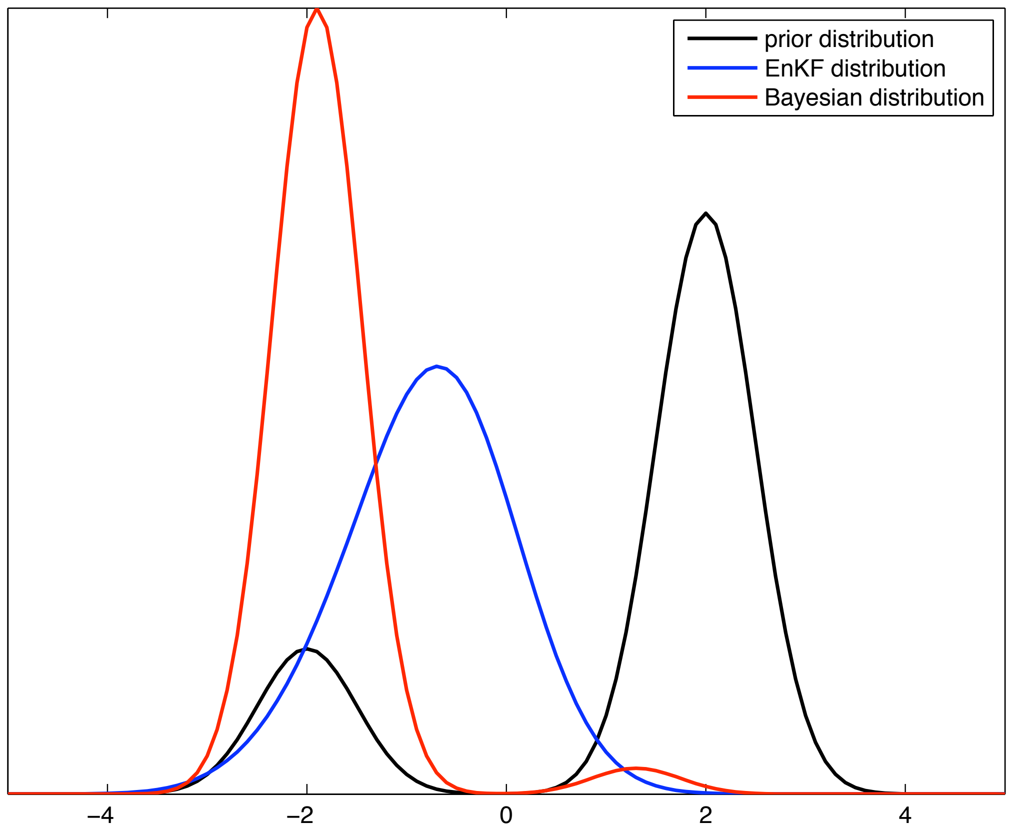

Figure 2: Observed value $Y_k = -1.5$ in the light mode

If Assumption B holds, and if the random vector X_0, or equivalently the probability distribution η_0, has finite moments of any order, then the random vectors $\bar{X}_k^{i,f}$ and $\bar{X}_k^{i,a}$, or equivalently the probability distributions $\bar{\mu}_k^f$ and $\bar{\mu}_k^a$, have finite moments of any order, and in particular the covariance matrix $\bar{P}_k$ is finite.

Proof (by induction). If Assumption A holds, then

$$|\bar{X}_k^{i,f}| \leq M\ (1 + |\bar{X}_{k-1}^{i,a}|) + |W_k^i|\ ,$$

and using the triangle inequality yields

$$(\,\mathbb{E}|\bar{X}_k^{i,f}|^p\,)^{1/p} \leq M\ (1 + (\,\mathbb{E}|\bar{X}_{k-1}^{i,a}|^p\,)^{1/p}) + (\,\mathbb{E}|W_k^i|^p\,)^{1/p}\ ,$$

hence

$$\bar{M}_k^{p,f} \leq M\ (1 + \bar{M}_{k-1}^{p,a}) + c_p\ \lambda_{\max}^{1/2}(Q_k)\ ,$$

whereas if Assumption B holds, then

$$|\bar{X}_k^{i,f}| \leq M\ (1 + |\bar{X}_{k-1}^{i,a}|^{s+1}) + |W_k^i|\ ,$$

and using the triangle inequality yields

$$(\,\mathbb{E}|\bar{X}_k^{i,f}|^p\,)^{1/p} \leq M\ (1 + (\,\mathbb{E}|\bar{X}_{k-1}^{i,a}|^{p\,(s+1)}\,)^{1/p}) + (\,\mathbb{E}|W_k^i|^p\,)^{1/p}\ ,$$

hence

$$\bar{M}_k^{p,f} \leq M\ (1 + (\bar{M}_{k-1}^{p\,(s+1),a})^{s+1}) + c_p\ \lambda_{\max}^{1/2}(Q_k)\ ,$$

where $\lambda_{\max}(Q_k)$ denotes the largest eigenvalue of the covariance matrix Q_k. In particular

$$u^*\,\bar{P}_k\,u \leq \int_{\mathbb{R}^m} |u^*\,x|^2\ \bar{\mu}_k^f(dx) \leq |u|^2\ \int_{\mathbb{R}^m} |x|^2\ \bar{\mu}_k^f(dx)\ ,$$

hence
$$\|\bar{P}_k\| = \sup_{u \neq 0} \frac{u^* \, \bar{P}_k \, u}{|u|^2} \leq \int_{\mathbb{R}^m} |x|^2 \; \bar{\mu}_k^f(dx) = (\bar{M}_k^{2,f})^2 < \infty \; .$$
In any case
$$|\bar{X}_k^{i,a}| \leq |\bar{X}_k^{i,f}| + \|K_k(\bar{P}_k)\| \; |Y_k - H_k \, \bar{X}_k^{i,f} + V_k^i| \; ,$$
whether Assumption A or Assumption B holds or not, and using the triangle inequality yields
$$(\, \mathbb{E}|\bar{X}_k^{i,a}|^p \,)^{1/p} \leq (\, \mathbb{E}|\bar{X}_k^{i,f}|^p \,)^{1/p} + \|K_k(\bar{P}_k)\| \; (\, \mathbb{E}|Y_k - H_k \, \bar{X}_k^{i,f} + V_k^i|^p \,)^{1/p} \; ,$$
hence
$$\bar{M}_k^{p,a} \leq \bar{M}_k^{p,f} + \|K_k(\bar{P}_k)\| \; \bar{R}_k^p \leq \bar{M}_k^{p,f} + \|K_k(\bar{P}_k)\| \; (\, |Y_k| + \|H_k\| \; \bar{M}_k^{p,f} + c_p \; \lambda_{\max}^{1/2}(R_k)) \; ,$$
where $\lambda_{\max}(R_k)$ denotes the largest eigenvalue of the covariance matrix R_k. □

Introduce the following notation for the empirical moments
$$\bar{M}_k^{N,p,f} = (\frac{1}{N} \sum_{i=1}^N |\bar{X}_k^{i,f}|^p)^{1/p} \qquad \text{and} \qquad \bar{M}_k^{N,p,a} = (\frac{1}{N} \sum_{i=1}^N |\bar{X}_k^{i,a}|^p)^{1/p} \; ,$$
and similarly for the empirical moments of the perturbed residuals
$$\bar{R}_k^{N,p} = (\frac{1}{N} \sum_{i=1}^N |Y_k - H_k \, \bar{X}_k^{i,f} - V_k^i|^p)^{1/p} \; .$$
It follows from the strong law of large numbers that : if the moment $\bar{M}_k^{p,f}$ is finite, then
$$\bar{M}_k^{N,p,f} \longrightarrow \bar{M}_k^{p,f} \qquad \text{and} \qquad \bar{R}_k^{N,p} \longrightarrow \bar{R}_k^p \; ,$$
almost surely as $N \uparrow \infty$, and if the moment $\bar{M}_k^{p,a}$ is finite, then
$$\bar{M}_k^{N,p,a} \longrightarrow \bar{M}_k^{p,a} \; ,$$
almost surely as $N \uparrow \infty$,

3 Control of the Kalman gain matrix

There is only one visible difference between equations (1) and (2) for the elements in the ensemble Kalman filter, and equations (4) and (5) for the corresponding substitute i.i.d. random vectors : indeed, the coefficients are the same and the same random input vectors are used, however the Kalman gain matrix $K_k(P_k^N)$ in equation (2) is based on the empirical covariance matrix P_k^N, which is responsible for mean–field interaction and dependence, whereas the Kalman gain matrix $K_k(\bar{P}_k)$ in equation (5) is based on the deterministic limiting covariance matrix $\bar{P}_k$, which is responsible for decoupling and independence. To assess the impact of this substitution, notice that if Assumption A holds, then by difference
$$|X_k^{i,f} - \bar{X}_k^{i,f}| \leq L \, |X_{k-1}^{i,a} - \bar{X}_{k-1}^{i,a}| \; , \tag{6}$$
whereas if Assumption B holds, then by difference
$$|X_k^{i,f} - \bar{X}_k^{i,f}| \leq L \, |X_{k-1}^{i,a} - \bar{X}_{k-1}^{i,a}| \; (1 + |\bar{X}_{k-1}^{i,a}|^s) + L \, |X_{k-1}^{i,a} - \bar{X}_{k-1}^{i,a}|^{s+1} \; . \tag{7}$$
Similarly, by difference
$$\begin{aligned} X_k^{i,a} - \bar{X}_k^{i,a} \; &= \; X_k^{i,f} + K_k(P_k^N) \; (Y_k - H_k \; X_k^{i,f} - V_k^i) - \bar{X}_k^{i,f} - K_k(\bar{P}_k) \; (Y_k - H_k \; \bar{X}_k^{i,f} - V_k^i) \\ &= \; (I - K_k(\bar{P}_k) \, H_k) \; (X_k^{i,f} - \bar{X}_k^{i,f}) - (K_k(P_k^N) - K_k(\bar{P}_k)) \; H_k \; (X_k^{i,f} - \bar{X}_k^{i,f}) \\ &\quad + (K_k(P_k^N) - K_k(\bar{P}_k)) \; (Y_k - H_k \; \bar{X}_k^{i,f} + V_k^i) \; , \end{aligned}$$

hence

$$
\begin{aligned}
|X_k^{i,a} - \bar{X}_k^{i,a}| \;\leq\; & \|I - K_k(\bar{P}_k)\, H_k\| \; |X_k^{i,f} - \bar{X}_k^{i,f}| \\
& + \|K_k(P_k^N) - K_k(\bar{P}_k)\| \; \|H_k\| \; |X_k^{i,f} - \bar{X}_k^{i,f}| \\
& + \|K_k(P_k^N) - K_k(\bar{P}_k)\| \; |Y_k - H_k\, \bar{X}_k^{i,f} + V_k^i| \; .
\end{aligned} \tag{8}
$$

The first step addressed in Proposition 3.1 is to prove some local Lipschitz continuity of the mapping $P \mapsto K_k(P)$ in order to control the difference $K_k(P_k^N) - K_k(\bar{P}_k)$ in terms of the difference $P_k^N - \bar{P}_k$, which is then decomposed as $(P_k^N - \bar{P}_k^N) + (\bar{P}_k^N - \bar{P}_k)$. The second step addressed in Lemma 3.2 is to study the contiguity of the empirical covariance matrices : the difference $P_k^N - \bar{P}_k^N$ can be controlled in terms of

$$
\Delta_k^{N,2,f} = (\frac{1}{N} \sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|^2)^{1/2} \; ,
$$

which is a measure of contiguity between elements in the ensemble Kalman filter and the corresponding substitute i.i.d. random vectors. The convergence of $\Delta_k^{N,2,f}$ to zero almost surely and in $\mathbb{L}^p$ as $N \uparrow \infty$, is proved in Propositions 4.2 and 4.4, respectively. The third and last step addressed in Lemma 3.3 is to prove the consistency of the empirical covariance matrix for the substitute i.i.d. random vectors : because of independence, the difference $\bar{P}_k^N - \bar{P}_k$ goes to zero almost surely and in $\mathbb{L}^p$ as $N \uparrow \infty$, by the strong law of large numbers and by the Marcinkiewicz–Zygmund inequality [13, Chapter 3, Section 8], respectively.

3.1 Local Lipschitz continuity of the Kalman gain matrix

If the $d \times d$ covariance matrix R_k is invertible, then its smallest eigenvalue $\lambda_{\min}(R_k)$ is positive, and it is possible to prove that the mapping $P \mapsto K_k(P)$ defined in (3) has at most linear growth and is locally Lipschitz continuous, as follows.

Proposition 3.1 *If the $d \times d$ covariance matrix R_k is invertible, then*

$$
\|K_k(P)\| \leq \frac{\|H_k\|}{\lambda_{\min}(R_k)} \; \|P\| \; ,
$$

and

$$
\|K_k(P) - K_k(P')\| \leq \|I - K_k(P')\; H_k\| \; \frac{\|H_k\|}{\lambda_{\min}(R_k)} \; \|P - P'\| \; , \tag{9}
$$

for any (not necessarily invertible) $m \times m$ covariance matrices P and P'.

Proof. Since $H_k\; P\; H_k^* + R_k$ is a symmetric matrix, there exist an orthogonal matrix O and a diagonal matrix D such that $H_k\; P\; H_k^* + R_k = O\, D\, O^*$, hence $(H_k\; P\; H_k^* + R_k)^{-1} = O\, D^{-1}\, O^* = O\, D^{-1/2}\, O^*\; O\, D^{-1/2}\, O^* = T^*\; T$, with $T = O\, D^{-1/2}\, O^*$ by definition, and

$$
\|(H_k\; P\; H_k^* + R_k)^{-1}\| = \sup_{u \neq 0} \frac{u^*\; (H_k\; P\; H_k^* + R_k)^{-1}\; u}{|u|^2} = \sup_{u \neq 0} \frac{|T\; u|^2}{|u|^2} = \sup_{v \neq 0} \frac{|v|^2}{|T^{-1}\; v|^2} \; .
$$

Clearly $T^{-1} = O\, D^{1/2}\, O^*$, hence

$$
|T^{-1}\; v|^2 = v^*\; O\, D^{1/2}\, O^*\; O\, D^{1/2}\, O^*\; v = v^*\; O\, D\, O^*\; v = v^*\; (H_k\; P\; H_k^* + R_k)\; v \; ,
$$

and

$$
\|(H_k\; P\; H_k^* + R_k)^{-1}\| = \sup_{v \neq 0} \frac{|v|^2}{v^*\; (H_k\; P\; H_k^* + R_k)\; v} \leq \sup_{v \neq 0} \frac{|v|^2}{v^*\; R_k\; v} = \frac{1}{\lambda_{\min}(R_k)} \; .
$$

Therefore

$$
\|K_k(P)\| \leq \frac{\|H_k\|}{\lambda_{\min}(R_k)} \; \|P\| \; ,
$$

since $\|H_k^*\| = \|H_k\|$ for the norm matrix associated with the Euclidean norm, and by difference

$$\begin{aligned}
K_k(P) - K_k(P') &= P\, H_k^*\, (H_k\, P\, H_k^* + R_k)^{-1} - P'\, H_k^*\, (H_k\, P'\, H_k^* + R_k)^{-1} \\
&= P\, H_k^*\, (H_k\, P\, H_k^* + R_k)^{-1} - P'\, H_k^*\, (H_k\, P\, H_k^* + R_k)^{-1} \\
&\qquad + P'\, H_k^*\, ((H_k\, P\, H_k^* + R_k)^{-1} - (H_k\, P'\, H_k^* + R_k)^{-1}) \\
&= (P - P')\, H_k^*\, (H_k\, P\, H_k^* + R_k)^{-1} \\
&\qquad - P'\, H_k^*\, (H_k\, P'\, H_k^* + R_k)^{-1}\, H_k\, (P - P')\, H_k^*\, (H_k\, P\, H_k^* + R_k)^{-1} \\
&= (P - P')\, H_k^*\, (H_k\, P\, H_k^* + R_k)^{-1} \\
&\qquad - K_k(P')\, H_k\, (P - P')\, H_k^*\, (H_k\, P\, H_k^* + R_k)^{-1} \\
&= (I - K_k(P')\, H_k)\, (P - P')\, H_k^*\, (H_k\, P\, H_k^* + R_k)^{-1}\ ,
\end{aligned}$$

hence

$$\|K_k(P) - K_k(P')\| \leq \|I - K_k(P')\, H_k\|\, \frac{\|H_k\|}{\lambda_{\min}(R_k)}\, \|P - P'\|\ . \qquad \Box$$

3.2 Contiguity control of the empirical covariance matrices

Lemma 3.2 *The difference between the two empirical covariance matrices satisfies*

$$\|P_k^N - \bar{P}_k^N\| \leq 2\, |\Delta_k^{N,2,f}|^2 + 4\, \bar{M}_k^{N,2,f}\ \Delta_k^{N,2,f}\ . \tag{10}$$

Proof. Notice that

$$P_k^N \;=\; \frac{1}{N} \sum_{i=1}^N X_k^{i,f}\, (X_k^{i,f})^* - (\frac{1}{N} \sum_{i=1}^N X_k^{i,f})\, (\frac{1}{N} \sum_{i=1}^N X_k^{i,f})^*\ ,$$

and similarly

$$\bar{P}_k^N \;=\; \frac{1}{N} \sum_{i=1}^N \bar{X}_k^{i,f}\, (\bar{X}_k^{i,f})^* - (\frac{1}{N} \sum_{i=1}^N \bar{X}_k^{i,f})\, (\frac{1}{N} \sum_{i=1}^N \bar{X}_k^{i,f})^*\ .$$

Using the identity $a^2 - b^2 = (a-b)^2 + 2\, b\, (a-b)$ yields

$$\begin{aligned}
&u^*\, (P_k^N - \bar{P}_k^N)\, u = \\
&= \frac{1}{N} \sum_{i=1}^N |u^*\, X_k^{i,f}|^2 - \frac{1}{N} \sum_{i=1}^N |u^*\, \bar{X}_k^{i,f}|^2 - |\frac{1}{N} \sum_{i=1}^N u^*\, X_k^{i,f}|^2 + |\frac{1}{N} \sum_{i=1}^N u^*\, \bar{X}_k^{i,f}|^2 \\
&= \frac{1}{N} \sum_{i=1}^N |u^*\, (X_k^{i,f} - \bar{X}_k^{i,f})|^2 + 2\, \frac{1}{N} \sum_{i=1}^N (u^*\, \bar{X}_k^{i,f})\, (u^*\, (X_k^{i,f} - \bar{X}_k^{i,f})) \\
&\qquad - |\frac{1}{N} \sum_{i=1}^N u^*\, (X_k^{i,f} - \bar{X}_k^{i,f})|^2 - 2\, (\frac{1}{N} \sum_{i=1}^N u^*\, \bar{X}_k^{i,f})\, (\frac{1}{N} \sum_{i=1}^N u^*\, (X_k^{i,f} - \bar{X}_k^{i,f}))\ ,
\end{aligned}$$

for any u, hence

$$\begin{aligned}
\|P_k^N - \bar{P}_k^N\| &= \sup_{u \neq 0} \frac{|u^* \, (P_k^N - \bar{P}_k^N) \, u|}{|u|^2} \\
&\leq \frac{1}{N} \sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|^2 + 2 \, \frac{1}{N} \sum_{i=1}^N |\bar{X}_k^{i,f}| \, |X_k^{i,f} - \bar{X}_k^{i,f}| \\
&\quad + (\frac{1}{N} \sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|)^2 + 2 \, (\frac{1}{N} \sum_{i=1}^N |\bar{X}_k^{i,f}|) \, (\frac{1}{N} \sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|) \\
&\leq 2 \, \frac{1}{N} \sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|^2 + 4 \, (\frac{1}{N} \sum_{i=1}^N |\bar{X}_k^{i,f}|^2)^{1/2} \, (\frac{1}{N} \sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|^2)^{1/2} \; ,
\end{aligned}$$

or in other words

$$\|P_k^N - \bar{P}_k^N\| \leq 2 \, |\Delta_k^{N,2,f}|^2 + 4 \, \bar{M}_k^{N,2,f} \; \Delta_k^{N,2,f} \; . \qquad \square$$

3.3 Consistency of the empirical covariance matrix

Lemma 3.3 *The difference between the covariance matrix of the substitute i.i.d. random vectors and their empirical covariance matrix satisfies*

$$\varepsilon_k^N = \|\bar{P}_k^N - \bar{P}_k\| \longrightarrow 0 \; ,$$

almost surely as $N \uparrow \infty$, and

$$\sup_{N \geq 1} \sqrt{N} \; (\mathbb{E}|\varepsilon_k^N|^p \,)^{1/p} < \infty \; ,$$

for any order $p \geq 2$.

Proof. The following decomposition holds

$$\begin{aligned}
\bar{P}_k^N &= \frac{1}{N} \sum_{i=1}^N (\bar{X}_k^{i,f} - \bar{m}_k^N) \, (\bar{X}_k^{i,f} - \bar{m}_k^N)^* \\
&= \frac{1}{N} \sum_{i=1}^N (\bar{X}_k^{i,f} - \bar{m}_k - (\bar{m}_k^N - \bar{m}_k)) \, (\bar{X}_k^{i,f} - \bar{m}_k - (\bar{m}_k^N - \bar{m}_k))^* \\
&= \frac{1}{N} \sum_{i=1}^N (\bar{X}_k^{i,f} - \bar{m}_k) \, (\bar{X}_k^{i,f} - \bar{m}_k)^* - (\frac{1}{N} \sum_{i=1}^N \bar{X}_k^{i,f} - \bar{m}_k) \, (\bar{m}_k^N - \bar{m}_k)^* \\
&\quad - (\bar{m}_k^N - \bar{m}_k) \, (\frac{1}{N} \sum_{i=1}^N \bar{X}_k^{i,f} - \bar{m}_k)^* + (\bar{m}_k^N - \bar{m}_k)(\bar{m}_k^N - \bar{m}_k)^* \\
&= \frac{1}{N} \sum_{i=1}^N (\bar{X}_k^{i,f} - \bar{m}_k) \, (\bar{X}_k^{i,f} - \bar{m}_k)^* - (\bar{m}_k^N - \bar{m}_k)(\bar{m}_k^N - \bar{m}_k)^* \; ,
\end{aligned}$$

hence

$$u^* \, (\bar{P}_k^N - \bar{P}_k) \, u = u^* \, (\frac{1}{N} \sum_{i=1}^N (\bar{X}_k^{i,f} - \bar{m}_k) \, (\bar{X}_k^{i,f} - \bar{m}_k)^* - \bar{P}_k) \, u - |u^* \, (\frac{1}{N} \sum_{i=1}^N \bar{X}_k^{i,f} - \bar{m}_k)|^2 \; ,$$

for any u, and

$$\begin{aligned}
\varepsilon_k^N = \|\bar{P}_k^N - \bar{P}_k\| &= \sup_{u \neq 0} \frac{|u^* \, (\bar{P}_k^N - \bar{P}_k) \, u|}{|u|^2} \\
&\leq \|\frac{1}{N} \sum_{i=1}^N (\bar{X}_k^{i,f} - \bar{m}_k) \, (\bar{X}_k^{i,f} - \bar{m}_k)^* - \bar{P}_k\| + |\frac{1}{N} \sum_{i=1}^N \bar{X}_k^{i,f} - \bar{m}_k \,|^2 \; .
\end{aligned}$$

Finally, notice that

$$|u^* \, M \, u| = | \sum_{j,j'=1}^{m} u_j \, M_{jj'} \, u_{j'} | \leq \max_{j=1,\cdots,m} |u_j|^2 \sum_{j,j'=1}^{m} |M_{jj'}| \leq |u|^2 \sum_{j,j'=1}^{m} |M_{jj'}|$$

hence

$$\|M\| = \sup_{u \neq 0} \frac{|u^* \, M \, u|}{|u|^2} \leq \sum_{j,j'=1}^{m} |M_{jj'}| \; ,$$

for any symmetric $m \times m$ matrix M. Taking

$$M = \frac{1}{N} \sum_{i=1}^{N} (\bar{X}_k^{i,f} - \bar{m}_k) \, (\bar{X}_k^{i,f} - \bar{m}_k)^* - \bar{P}_k \; ,$$

gives

$$M_{jj'} = \frac{1}{N} \sum_{i=1}^{N} (\bar{X}_k^{i,j,f} - \bar{m}_k^j) \, (\bar{X}_k^{i,j',f} - \bar{m}_k^{j'}) - \bar{P}_k^{j,j'} \; ,$$

and it follows from the law of large numbers that

$$\varepsilon_k^N \; \leq \; \sum_{j,j'=1}^{m} | \, \frac{1}{N} \sum_{i=1}^{N} (\bar{X}_k^{i,j,f} - \bar{m}_k^j) \, (\bar{X}_k^{i,j',f} - \bar{m}_k^{j'}) - \bar{P}_k^{j,j'} \, | + \sum_{j=1}^{m} | \frac{1}{N} \sum_{i=1}^{N} \bar{X}_k^{i,j,f} - \bar{m}_k^j \, |^2 \longrightarrow 0 \; ,$$

almost surely as $N \uparrow \infty$, and using the triangle inequality and then the Marcinkiewicz–Zygmund inequality, yields

$$\begin{aligned}
(\, \mathbb{E}|\varepsilon_k^N|^p \,)^{1/p} \; &\leq \; \sum_{j,j'=1}^{m} (\, \mathbb{E}| \, \frac{1}{N} \sum_{i=1}^{N} (\bar{X}_k^{i,j,f} - \bar{m}_k^j) \, (\bar{X}_k^{i,j',f} - \bar{m}_k^{j'}) - \bar{P}_k^{j,j'} \, |^p \,)^{1/p} \\
&\quad + \sum_{j=1}^{m} (\, \mathbb{E}| \frac{1}{N} \sum_{i=1}^{N} \bar{X}_k^{i,j,f} - \bar{m}_k^j \, |^{2\,p} \,)^{1/p} \\
&\leq \; \frac{c_p}{\sqrt{N}} \sum_{j,j'=1}^{m} (\, \mathbb{E}|(\bar{X}_k^{i,j,f} - \bar{m}_k^j) \, (\bar{X}_k^{i,j',f} - \bar{m}_k^{j'}) - \bar{P}_k^{j,j'} \, |^p \,)^{1/p} \\
&\quad + \frac{c_{2p}^2}{N} \sum_{j=1}^{m} (\, \mathbb{E}|\bar{X}_k^{i,j,f} - \bar{m}_k^j|^{2\,p} \,)^{1/p} \; ,
\end{aligned}$$

for any order $p \geq 2$. □

4 Contiguity of the elements

Introduce the following definitions

$$\Delta_k^{N,p,f} = (\frac{1}{N} \sum_{i=1}^{N} |X_k^{i,f} - \bar{X}_k^{i,f}|^p)^{1/p} \qquad \text{and} \qquad \Delta_k^{N,p,a} = (\frac{1}{N} \sum_{i=1}^{N} |X_k^{i,a} - \bar{X}_k^{i,a}|^p)^{1/p} \; ,$$

as measures of contiguity between elements in the ensemble Kalman filter and the corresponding substitute i.i.d. random vectors. These quantities satisfy the following recurrence inequalities, which are then used in Propositions 4.2 and 4.4 to prove almost sure contiguity and $\mathbb{L}^p$ contiguity, respectively.

Lemma 4.1 *If Assumption A holds, then*

$$\Delta_k^{N,p,f} \leq L \, \Delta_{k-1}^{N,p,a} \; , \tag{11}$$

whereas if Assumption B holds, then

$$\Delta_k^{N,p,f} \le L\, \Delta_{k-1}^{N,p\,r,a}\, (1 + |\bar{M}_{k-1}^{N,s\,p\,r',a}|^s) + L\, |\Delta_{k-1}^{N,p\,(s+1),a}|^{s+1}\ , \tag{12}$$

where r, r' are conjugate exponents, i.e. $1/r + 1/r' = 1$, and in any case

$$\Delta_k^{N,p,a} \le C_k\, (\Delta_k^{N,p,f} + (2\, |\Delta_k^{N,2,f}|^2 + 4\, \bar{M}_k^{N,2,f}\ \Delta_k^{N,2,f} + \varepsilon_k^N)\ (\Delta_k^{N,p,f} + \bar{R}_k^{N,p}))\ , \tag{13}$$

whether Assumption A or Assumption B holds or not.

PROOF. If Assumption A holds, then it follows from (6) that

$$(\frac{1}{N} \sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|^p\,)^{1/p} \le L\, (\frac{1}{N} \sum_{i=1}^N |X_{k-1}^{i,a} - \bar{X}_{k-1}^{i,a}|^p\,)^{1/p}\ ,$$

or in other words

$$\Delta_k^{N,p,f} \le L\ \Delta_{k-1}^{N,p,a}\ ,$$

whereas if Assumption B holds, then using the triangle inequality it follows from (7) that

$$\begin{aligned}(\frac{1}{N} \sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|^p)^{1/p}\ &\le\ L\, (\frac{1}{N} \sum_{i=1}^N |X_{k-1}^{i,a} - \bar{X}_{k-1}^{i,a}|^p\ (1 + |\bar{X}_{k-1}^{i,a}|^s)^p)^{1/p} \\ &\quad + L\, (\frac{1}{N} \sum_{i=1}^N |X_{k-1}^{i,a} - \bar{X}_{k-1}^{i,a}|^{p\,(s+1)}\,)^{1/p}\ ,\end{aligned}$$

and using the Hölder inequality and then the triangle inequality again to control the first term, yields

$$\begin{aligned}&(\frac{1}{N} \sum_{i=1}^N |X_{k-1}^{i,a} - \bar{X}_{k-1}^{i,a}|^p\ (1 + |\bar{X}_{k-1}^{i,a}|^s)^p)^{1/p} \\ &\le (\frac{1}{N} \sum_{i=1}^N |X_{k-1}^{i,a} - \bar{X}_{k-1}^{i,a}|^{p\,r})^{1/p\,r}\ (\frac{1}{N} \sum_{i=1}^N (1 + |\bar{X}_{k-1}^{i,a}|^s)^{p\,r'})^{1/p\,r'} \\ &\le (\frac{1}{N} \sum_{i=1}^N |X_{k-1}^{i,a} - \bar{X}_{k-1}^{i,a}|^{p\,r})^{1/p\,r}\ (1 + (\frac{1}{N} \sum_{i=1}^N |\bar{X}_{k-1}^{i,a}|^{s\,p\,r'})^{1/p\,r'})\ ,\end{aligned}$$

where r, r' are conjugate exponents, i.e. $1/r + 1/r' = 1$, or in other words

$$\Delta_k^{N,p,f} \le L\, \Delta_{k-1}^{N,p\,r,a}\, (1 + |\bar{M}_{k-1}^{N,s\,p\,r',a}|^s) + L\, |\Delta_{k-1}^{N,p\,(s+1),a}|^{s+1}\ .$$

Similarly, it follows from (8) and (9) that in any case

$$\begin{aligned}|X_k^{i,a} - \bar{X}_k^{i,a}|\ &\le\ \|I - K_k(\bar{P}_k)\ H_k\|\ |X_k^{i,f} - \bar{X}_k^{i,f}| \\ &\quad + \|K_k(P_k^N) - K_k(\bar{P}_k)\|\ \|H_k\|\ |X_k^{i,f} - \bar{X}_k^{i,f}| \\ &\quad + \|K_k(P_k^N) - K_k(\bar{P}_k)\|\ |Y_k - H_k\ \bar{X}_k^{i,f} + V_k^i| \\ &\le\ \|I - K_k(\bar{P}_k)\ H_k\|\ |X_k^{i,f} - \bar{X}_k^{i,f}| \\ &\quad + \|I - K_k(\bar{P}_k)\ H_k\|\ \frac{\|H_k\|^2}{\lambda_{\min}(R_k)}\ \|P_k^N - \bar{P}_k\|\ |X_k^{i,f} - \bar{X}_k^{i,f}| \\ &\quad + \|I - K_k(\bar{P}_k)\ H_k\|\ \frac{\|H_k\|}{\lambda_{\min}(R_k)}\ \|P_k^N - \bar{P}_k\|\ |Y_k - H_k\ \bar{X}_k^{i,f} - V_k^i| \\ &\le\ C_k\ (\,|X_k^{i,f} - \bar{X}_k^{i,f}| + \|P_k^N - \bar{P}_k\|\ (\,|X_k^{i,f} - \bar{X}_k^{i,f}| + |Y_k - H_k\ \bar{X}_k^{i,f} - V_k^i|\,))\ ,\end{aligned}$$

whether Assumption A or Assumption B holds or not, where

$$C_k = \|I - K_k(\bar{P}_k)\; H_k\| \;\max(1, \frac{\|H_k\|}{\lambda_{\min}(R_k)}, \frac{\|H_k\|^2}{\lambda_{\min}(R_k)}) \;,$$

and using the triangle inequality yields

$$\begin{aligned}
(\frac{1}{N}\sum_{i=1}^N |X_k^{i,a} - \bar{X}_k^{i,a}|^p)^{1/p} \;\leq\; C_k \;(\,(\frac{1}{N}\sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|^p)^{1/p} \\
+ \|P_k^N - \bar{P}_k\| \;(\frac{1}{N}\sum_{i=1}^N |X_k^{i,f} - \bar{X}_k^{i,f}|^p)^{1/p} \\
+ \|P_k^N - \bar{P}_k\| \;(\frac{1}{N}\sum_{i=1}^N |Y_k - H_k\; \bar{X}_k^{i,f} - V_k^i|^p)^{1/p}\;) \;,
\end{aligned}$$

or in other words

$$\Delta_k^{N,p,a} \leq C_k\; (\Delta_k^{N,p,f} + \|P_k^N - \bar{P}_k\|\; (\Delta_k^{N,p,f} + \bar{R}_k^{N,p})) \;,$$

and it follows from the triangle inequality and from (10) that

$$\|P_k^N - \bar{P}_k\| \leq \|P_k^N - \bar{P}_k^N\| + \|\bar{P}_k^N - \bar{P}_k\| \leq 2\,|\Delta_k^{N,2,f}|^2 + 4\,\bar{M}_k^{N,2,f}\;\Delta_k^{N,2,f} + \varepsilon_k^N \;,$$

hence

$$\Delta_k^{N,p,a} \leq C_k\; (\Delta_k^{N,p,f} + (2\,|\Delta_k^{N,2,f}|^2 + 4\,\bar{M}_k^{N,2,f}\;\Delta_k^{N,2,f} + \varepsilon_k^N)\; (\Delta_k^{N,p,f} + \bar{R}_k^{N,p})) \;. \qquad \square$$

4.1 Almost sure contiguity of the elements

Proposition 4.2 *If Assumption A holds, and if the random vector X_0 has finite moment of order p for some $p \geq 2$, then*

$$\Delta_k^{N,p,f} \longrightarrow 0 \qquad \textit{and} \qquad \Delta_k^{N,p,a} \longrightarrow 0 \;,$$

for the same order p, almost surely as $N \uparrow \infty$.

If Assumption B holds, and if the random vector X_0 has finite moments of any order, then

$$\Delta_k^{N,p,f} \longrightarrow 0 \qquad \textit{and} \qquad \Delta_k^{N,p,a} \longrightarrow 0 \;,$$

for any order p, almost surely as $N \uparrow \infty$.

Proof (by induction). Initially

$$\Delta_0^{N,p,f} = (\frac{1}{N}\sum_{i=1}^N |X_0^{i,f} - \bar{X}_0^{i,f}|^p)^{1/p} = 0 \;.$$

If Assumption A holds, then estimate (11) holds, i.e.

$$\Delta_k^{N,p,f} \leq L\; \Delta_{k-1}^{N,p,a} \;,$$

and it follows from the induction assumption that $\Delta_k^{N,p,f} \longrightarrow 0$ almost surely as $N \uparrow \infty$, whereas if Assumption B holds, then estimate (12) holds, i.e.

$$\Delta_k^{N,p,f} \leq L\; \Delta_{k-1}^{N,p\,r,a}\; (1 + |\bar{M}_{k-1}^{N,s\,p\,r',a}|^s) + L\; |\Delta_{k-1}^{N,p\,(s+1),a}|^{s+1} \;,$$

and since $\bar{M}_{k-1}^{N,s\,p\,r',a} \longrightarrow \bar{M}_{k-1}^{s\,p\,r',a}$ almost surely as $N \uparrow \infty$, with a finite limit $\bar{M}_{k-1}^{s\,p\,r',a}$, then it follows from the induction assumption that $\Delta_k^{N,p,f} \longrightarrow 0$ almost surely as $N \uparrow \infty$. In any case estimate (13) holds, i.e.

$$\Delta_k^{N,p,a} \leq C_k\; (\Delta_k^{N,p,f} + (2\,|\Delta_k^{N,2,f}|^2 + 4\,\bar{M}_k^{N,2,f}\;\Delta_k^{N,2,f} + \varepsilon_k^N)\; (\Delta_k^{N,p,f} + \bar{R}_k^{N,p})) \;,$$

whether Assumption A or Assumption B holds or not, and since $\bar{M}_k^{N,2,f} \longrightarrow \bar{M}_k^{2,f}$ and $\bar{R}_k^{N,p} \longrightarrow \bar{R}_k^p$ almost surely as $N \uparrow \infty$, with finite limits $\bar{M}_k^{2,f}$ and $\bar{R}_k^p$, and since $\varepsilon_k^N \longrightarrow 0$ almost surely as $N \uparrow \infty$ in view of Lemma 3.3, then it follows from the induction assumption that $\Delta_k^{N,p,a} \longrightarrow 0$ almost surely as $N \uparrow \infty$. $\square$

4.2 $\mathbb{L}^p$–contiguity of the elements

The quantities

$$D_k^{N,p,f} = (\,\mathbb{E}|X_k^{i,f} - \bar{X}_k^{i,f}|^p\,)^{1/p} \qquad \text{and} \qquad D_k^{N,p,a} = (\,\mathbb{E}|X_k^{i,a} - \bar{X}_k^{i,a}|^p\,)^{1/p} \ ,$$

which still depends on N, since the probability distribution of an element in the ensemble Kalman filter depends on the ensemble size, are used to control the moments of the measures of contiguity.

Lemma 4.3

$$(\,\mathbb{E}|\Delta_k^{N,p,f}|^q\,)^{1/q} \leq D_k^{N,p\vee q,f} \qquad \textit{and} \qquad (\,\mathbb{E}|\Delta_k^{N,p,a}|^q\,)^{1/q} \leq D_k^{N,p\vee q,a} \ ,$$

and

$$(\,\mathbb{E}|\bar{M}_k^{N,p,f}|^q\,)^{1/q} \leq \bar{M}_k^{p\vee q,f} \qquad \textit{and} \qquad (\,\mathbb{E}|\bar{M}_k^{N,p,a}|^q\,)^{1/q} \leq \bar{M}_k^{p\vee q,a} \ ,$$

whereas equalities hold if $p = q$.

Proof. Throughout the proof, let the superscript $\bullet$ index either forecast–related or analysis–related quantities. Clearly

$$\mathbb{E}|\Delta_k^{N,p,\bullet}|^p = \mathbb{E}(\frac{1}{N}\sum_{i=1}^N |X_k^{i,\bullet} - \bar{X}_k^{i,\bullet}|^p) = \mathbb{E}|X_k^{i,\bullet} - \bar{X}_k^{i,\bullet}|^p = |D_k^{N,p,\bullet}|^p \ ,$$

since $(X_k^{i,\bullet}, \bar{X}_k^{i,\bullet})$ have the same joint probability distribution for any $i = 1, \cdots, N$, by symmetry, hence

$$(\,\mathbb{E}|\Delta_k^{N,p,\bullet}|^p\,)^{1/p} = D_k^{N,p,\bullet} \ ,$$

which proves the result for $q = p$. If $q \geq p$, then the mapping $x \mapsto x^{q/p}$ is convex and it follows from the Jensen inequality that

$$|\Delta_k^{N,p,\bullet}|^q = (\frac{1}{N}\sum_{i=1}^N |X_k^{i,\bullet} - \bar{X}_k^{i,\bullet}|^p)^{q/p} \leq (\frac{1}{N}\sum_{i=1}^N |X_k^{i,\bullet} - \bar{X}_k^{i,\bullet}|^q) = |\Delta_k^{N,q,\bullet}|^q \ ,$$

hence

$$(\,\mathbb{E}|\Delta_k^{N,p,\bullet}|^q\,)^{1/q} \leq (\,\mathbb{E}|\Delta_k^{N,q,\bullet}|^q\,)^{1/q} = D_k^{N,q,\bullet} \ .$$

If $q \leq p$, then the mapping $x \mapsto x^{q/p}$ is concave and it follows from the Jensen inequality that

$$\mathbb{E}|\Delta_k^{N,p,\bullet}|^q = \mathbb{E}(|\Delta_k^{N,p,\bullet}|^p)^{q/p} \leq (\,\mathbb{E}|\Delta_k^{N,p,\bullet}|^p\,)^{q/p} = |D_k^{N,p,\bullet}|^q \ ,$$

hence

$$(\,\mathbb{E}|\Delta_k^{N,p,\bullet}|^q\,)^{1/q} \leq D_k^{N,p,\bullet} \ .$$

Similarly

$$\mathbb{E}|\bar{M}_k^{N,p,\bullet}|^p = \mathbb{E}(\frac{1}{N}\sum_{i=1}^N |\bar{X}_k^{i,\bullet}|^p) = \mathbb{E}|\bar{X}_k^{i,\bullet}|^p = |\bar{M}_k^{p,\bullet}|^p \ ,$$

since $\bar{X}_k^{i,\bullet}$ has the same distribution for any $i = 1, \cdots, N$, hence

$$(\,\mathbb{E}|\bar{M}_k^{N,p,\bullet}|^p\,)^{1/p} = \bar{M}_k^{p,\bullet} \ ,$$

which proves the result for $q = p$, and the result for the two cases $q \geq p$ and $q \leq p$ can again be obtained by simple convexity arguments. □

Proposition 4.4 *If Assumption B holds, and if the random vector X_0 has finite moments of any order, then*

$$\sup_{N \geq 1} \sqrt{N}\, D_k^{N,p,f} < \infty \qquad \textit{and} \qquad \sup_{N \geq 1} \sqrt{N}\, D_k^{N,p,a} < \infty \ ,$$

for any order p.

PROOF (by induction). Initially

$$\Delta_0^{N,p,f} = (\frac{1}{N} \sum_{i=1}^N |X_0^{i,f} - \bar{X}_0^{i,f}|^p)^{1/p} = 0 \; .$$

If Assumption A holds, then estimate (11) holds, i.e.

$$\Delta_k^{N,p,f} \leq L \; \Delta_{k-1}^{N,p,a} \; ,$$

hence

$$(\, \mathbb{E}|\Delta_k^{N,p,f}|^p \,)^{1/p} \leq L \; (\, \mathbb{E}|\Delta_{k-1}^{N,p,a}|^p \,)^{1/p} \; ,$$

or in other words

$$D_k^{N,p,f} \leq L \; D_{k-1}^{N,p,a} \; ,$$

in view of Lemma 4.3, and it follows from the induction assumption that $\sqrt{N} \, D_k^{N,p,f}$ is bounded uniformly w.r.t. N, whereas if Assumption B holds, then estimate (12) holds, i.e.

$$\Delta_k^{N,p,f} \leq L \; \Delta_{k-1}^{N,p\,r,a} \; (1 + |\bar{M}_{k-1}^{N,s\,p\,r',a}|^s) + L \; |\Delta_{k-1}^{N,p\,(s+1),a}|^{s+1} \; ,$$

and using the triangle inequality yields

$$(\, \mathbb{E}|\Delta_k^{N,p,f}|^p \,)^{1/p} \; \leq \; L \; (\, \mathbb{E}(|\Delta_{k-1}^{N,p\,r,a}|^p \; (1 + |\bar{M}_{k-1}^{N,s\,p\,r',a}|^s)^p) \,)^{1/p} + L \; (\, \mathbb{E}|\Delta_{k-1}^{N,p\,(s+1),a}|^{p(s+1)} \,)^{1/p} \; ,$$

and using the Hölder inequality and then the triangle inequality again to control the first term, yields

$$\begin{aligned} &(\, \mathbb{E}(|\Delta_{k-1}^{N,p\,r,a}|^p \; (1 + |\bar{M}_{k-1}^{N,s\,p\,r',a}|^s)^p) \,)^{1/p} \\ &\qquad \leq (\, \mathbb{E}|\Delta_{k-1}^{N,p\,r,a}|^{p\,r})^{1/p\,r} \; (\, \mathbb{E}(1 + |\bar{M}_{k-1}^{N,s\,p\,r',a}|^s)^{p\,r'} \,)^{1/p\,r'} \\ &\qquad \leq (\, \mathbb{E}|\Delta_{k-1}^{N,p\,r,a}|^{p\,r})^{1/p\,r} \; (1 + (\, \mathbb{E}|\bar{M}_{k-1}^{N,s\,p\,r',a}|^{s\,p\,r'} \,)^{1/p\,r'}) \; , \end{aligned}$$

where r, r' are conjugate exponents, i.e. $1/r + 1/r' = 1$, or in other words

$$D_k^{N,p,f} \; \leq \; L \; D_{k-1}^{N,p\,r,a} \; (1 + |\bar{M}_{k-1}^{s\,p\,r',a}|^s) + L \; |D_{k-1}^{N,p\,(s+1),a}|^{s+1} \; ,$$

in view of Lemma 4.3, and since $\bar{M}_{k-1}^{s\,p\,r',a}$ is finite, then it follows from the induction assumption that $\sqrt{N} \, D_k^{N,p,f}$ is bounded uniformly w.r.t. N. In any case estimate (13) holds, i.e.

$$\begin{aligned} \Delta_k^{N,p,a} \; &\leq \; C_k \; (\Delta_k^{N,p,f} + (2 \, |\Delta_k^{N,2,f}|^2 + 4 \, \bar{M}_k^{N,2,f} \; \Delta_k^{N,2,f} + \varepsilon_k^N) \; (\Delta_k^{N,p,f} + \bar{R}_k^{N,p})) \\ &\leq \; C_k \; \Delta_k^{N,p,f} + 2 \, C_k \; \Delta_k^{N,2,f} \; (\Delta_k^{N,2,f} + 2 \, \bar{M}_k^{N,2,f}) \; (\Delta_k^{N,p,f} + \bar{R}_k^{N,p}) \\ &\qquad + C_k \; \varepsilon_k^N \; (\Delta_k^{N,p,f} + \bar{R}_k^{N,p}) \; , \end{aligned}$$

whether Assumption A or Assumption B holds or not, and using the triangle inequality yields

$$\begin{aligned} (\, \mathbb{E}|\Delta_k^{N,p,a}|^p \,)^{1/p} \; &\leq \; C_k \; (\, \mathbb{E}|\Delta_k^{N,p,f}|^p \,)^{1/p} \\ &\qquad + 2 \, C_k \; (\, \mathbb{E}(|\Delta_k^{N,2,f}|^p \; |\Delta_k^{N,2,f} + 2 \, \bar{M}_k^{N,2,f}|^p \; |\Delta_k^{N,p,f} + \bar{R}_k^{N,p}|^p) \,)^{1/p} \\ &\qquad + C_k \; (\, \mathbb{E}(|\varepsilon_k^N|^p \; |\Delta_k^{N,p,f} + \bar{R}_k^{N,p}|^p) \,)^{1/p} \; , \end{aligned}$$

and using the Hölder inequality and then the triangle inequality again to control the second and third terms, yields

$$\begin{aligned}
&(\mathbb{E}(|\Delta_k^{N,2,f}|^p\, |\Delta_k^{N,2,f} + 2\,\bar{M}_k^{N,2,f}|^p\, |\Delta_k^{N,p,f} + \bar{R}_k^{N,p}|^p))^{1/p} \\
&\le (\mathbb{E}|\Delta_k^{N,2,f}|^{p\,r})^{1/p\,r}\, (\mathbb{E}|\Delta_k^{N,2,f} + 2\,\bar{M}_k^{N,2,f}|^{p\,r'})^{1/p\,r'}\, (\mathbb{E}|\Delta_k^{N,p,f} + \bar{R}_k^{N,p}|^{p\,r''})^{1/p\,r''} \\
&\le (\mathbb{E}|\Delta_k^{N,2,f}|^{p\,r})^{1/p\,r}\, ((\mathbb{E}|\Delta_k^{N,2,f}|^{p\,r'})^{1/p\,r'} + 2\,(\mathbb{E}|\bar{M}_k^{N,2,f}|^{p\,r'})^{1/p\,r'}) \\
&\qquad ((\mathbb{E}|\Delta_k^{N,p,f}|^{p\,r''})^{1/p\,r''} + (\mathbb{E}|\bar{R}_k^{N,p}|^{p\,r''})^{1/p\,r''})\;,
\end{aligned}$$

where r, r', r'' are conjugate exponents, i.e. $1/r + 1/r' + 1/r'' = 1$, and

$$\begin{aligned}
&(\mathbb{E}(|\varepsilon_k^N|^p\, |\Delta_k^{N,p,f} + \bar{R}_k^{N,p}|^p))^{1/p} \\
&\le (\mathbb{E}|\varepsilon_k^N|^{p\,q})^{1/p\,q}\, (\mathbb{E}|\Delta_k^{N,p,f} + \bar{R}_k^{N,p}|^{p\,q'})^{1/p\,q'} \\
&\le (\mathbb{E}|\varepsilon_k^N|^{p\,q})^{1/p\,q}\, ((\mathbb{E}|\Delta_k^{N,p,f}|^{p\,q'})^{1/p\,q'} + (\mathbb{E}|\bar{R}_k^{N,p}|^{p\,q'})^{1/p\,q'})\;,
\end{aligned}$$

where q, q' are conjugate exponents, i.e. $1/q + 1/q' = 1$, or in other words

$$\begin{aligned}
D_k^{N,p,a} \;\le\; & C_k\, D_k^{N,p,f} + 2\,C_k\, D_k^{N,2\vee(p\,r),f}\, (D_k^{N,2\vee(p\,r'),f} + 2\,\bar{M}_k^{2\vee(p\,r'),f})\, (D_k^{N,p\,r'',f} + \bar{R}_k^{p\,r''}) \\
&+ C_k\, (\mathbb{E}|\varepsilon_k^N|^{p\,q})^{1/p\,q}\, (D_k^{N,p\,q',f} + \bar{R}_k^{p\,q'})\;,
\end{aligned}$$

in view of Lemma 4.3, and since $\bar{M}_k^{2\vee(p\,r'),f}$, $\bar{R}_k^{p\,r''}$ and $\bar{R}_k^{p\,q'}$ are finite, and since $\sqrt{N}\,\varepsilon_k^N$ is bounded in $\mathbb{L}^{p\,q}$ uniformly w.r.t. N in view of Lemma 3.3, then it follows from the induction assumption that $\sqrt{N}\,D_k^{N,p,a}$ is bounded uniformly w.r.t. N. □

5 Convergence of the ensemble Kalman filter

In view of the decomposition

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \phi(X_k^{i,\bullet}) - \int_{\mathbb{R}^m} \phi(x)\, \bar{\mu}_k^\bullet(dx) \\
&= \frac{1}{N}\sum_{i=1}^N (\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet})) + (\frac{1}{N}\sum_{i=1}^N \phi(\bar{X}_k^{i,\bullet}) - \int_{\mathbb{R}^m} \phi(x)\, \bar{\mu}_k^\bullet(dx))\;,
\end{aligned}$$

where the superscript $^\bullet$ indexes either forecast–related or analysis–related quantities, it is intuitively clear that almost sure contiguity of the elements, proved in Proposition 4.2, and the strong law of large numbers for the substitute i.i.d. random vectors could be used to prove weak convergence of the empirical probability distribution associated with the elements of the ensemble Kalman filter, almost surely. Similarly, $\mathbb{L}^p$–contiguity of the elements, proved in Proposition 4.4, and the Marcinkiewicz–Zygmund inequality [13, Chapter 3, Section 8] for the substitute i.i.d. random vectors could be used to prove weak convergence of the empirical probability distribution associated with the elements of the ensemble Kalman filter, in $\mathbb{L}^p$–mean. These statements are proved in Theorems 5.1 and 5.2, respectively.

5.1 Almost sure convergence

Theorem 5.1 *Let ϕ be a locally Lipschitz continuous function, with at most polynomial growth at infinity, i.e.*

$$|\phi(x) - \phi(x')| \le L\, |x - x'|\, (1 + |x|^\sigma + |x'|^\sigma)\;,$$

for any $x, x' \in \mathbb{R}^m$ and for some $\sigma \ge 0$.

If Assumption A holds, and if the random vector X_0 has finite moment of order p for some $p \geq \max(2, \sigma+1)$, then

$$\frac{1}{N} \sum_{i=1}^N \phi(X_k^{i,f}) \longrightarrow \int_{\mathbb{R}^m} \phi(x) \; \bar{\mu}_k^f(dx) \qquad \text{and} \qquad \frac{1}{N} \sum_{i=1}^N \phi(X_k^{i,a}) \longrightarrow \int_{\mathbb{R}^m} \phi(x) \; \bar{\mu}_k^a(dx) \; ,$$

almost surely as $N \uparrow \infty$.

If Assumption B holds, and if the random vector X_0 has finite moments of any order, then

$$\frac{1}{N} \sum_{i=1}^N \phi(X_k^{i,f}) \longrightarrow \int_{\mathbb{R}^m} \phi(x) \; \bar{\mu}_k^f(dx) \qquad \text{and} \qquad \frac{1}{N} \sum_{i=1}^N \phi(X_k^{i,a}) \longrightarrow \int_{\mathbb{R}^m} \phi(x) \; \bar{\mu}_k^a(dx) \; ,$$

almost surely as $N \uparrow \infty$.

Proof. Throughout the proof, let the superscript $\bullet$ index either forecast–related or analysis–related quantities. Clearly

$$|\phi(x)| \leq M \; (1 + |x|^{\sigma+1}) \; ,$$

for any $x \in \mathbb{R}^m$, and it follows from Proposition 2.3 that

$$\int_{\mathbb{R}^m} |\phi(x)| \; \bar{\mu}_k^\bullet(dx) \leq M \; (1 + \int_{\mathbb{R}^m} |x|^{\sigma+1} \; \bar{\mu}_k^\bullet(dx)) = M \; (1 + |\bar{M}_k^{\sigma+1,\bullet}|^{\sigma+1}) < \infty \; ,$$

hence it follows from the strong law of large numbers that

$$\frac{1}{N} \sum_{i=1}^N \phi(\bar{X}_k^{i,\bullet}) \longrightarrow \int_{\mathbb{R}^m} \phi(x) \; \bar{\mu}_k^\bullet(dx) \; , \tag{14}$$

almost surely as $N \uparrow \infty$. Clearly

$$|\phi(x) - \phi(x')| \leq L \; |x - x'| \; (1 + |x|^\sigma) + L \; |x - x'|^{\sigma+1} \; ,$$

for any $x, x' \in \mathbb{R}^m$, with another constant L, and using the triangle inequality yields

$$\begin{aligned} &| \frac{1}{N} \sum_{i=1}^N (\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet})) \, | \\ &\leq \frac{1}{N} \sum_{i=1}^N |\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet})| \\ &\leq L \; \frac{1}{N} \sum_{i=1}^N |X_k^{i,\bullet} - \bar{X}_k^{i,\bullet}| \; (1 + |\bar{X}_k^{i,\bullet}|^\sigma) + L \; \frac{1}{N} \sum_{i=1}^N |X_k^{i,\bullet} - \bar{X}_k^{i,\bullet}|^{\sigma+1} \; , \end{aligned}$$

and using the Hölder inequality and then the triangle inequality again to control the first term, yields

$$\begin{aligned} &\frac{1}{N} \sum_{i=1}^N |X_k^{i,\bullet} - \bar{X}_k^{i,\bullet}| \; (1 + |\bar{X}_k^{i,\bullet}|^\sigma) \\ &\leq (\frac{1}{N} \sum_{i=1}^N |X_k^{i,\bullet} - \bar{X}_k^{i,\bullet}|^r \,)^{1/r} \; (\frac{1}{N} \sum_{i=1}^N (1 + |\bar{X}_k^{i,\bullet}|^\sigma)^{r'} \,)^{1/r'} \\ &\leq (\frac{1}{N} \sum_{i=1}^N |X_k^{i,\bullet} - \bar{X}_k^{i,\bullet}|^r \,)^{1/r} \; (1 + (\frac{1}{N} \sum_{i=1}^N |\bar{X}_k^{i,\bullet}|^{\sigma \, r'} \,)^{1/r'}) \; , \end{aligned}$$

where r, r' are conjugate exponents, i.e. $1/r + 1/r' = 1$, or in other words

$$| \frac{1}{N} \sum_{i=1}^N (\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet})) \, | \leq L \; \Delta_k^{N,r,\bullet} \; (1 + |\bar{M}_k^{N,\sigma \, r',\bullet}|^\sigma) + L \; |\Delta_k^{N,\sigma+1,\bullet}|^{\sigma+1} \; . \tag{15}$$

Taking $r = \sigma\, r' = \sigma + 1$ yields

$$| \frac{1}{N} \sum_{i=1}^{N} (\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet})) \,| \leq L\, \Delta_k^{N,\sigma+1,\bullet}\, (1 + |\bar{M}_k^{N,\sigma+1,\bullet}|^{\sigma}) + L\, |\Delta_k^{N,\sigma+1,\bullet}|^{\sigma+1} ,$$

and since $\bar{M}_k^{N,\sigma+1,\bullet} \longrightarrow \bar{M}_k^{\sigma+1,\bullet}$ almost surely as $N \uparrow \infty$, with a finite limit $\bar{M}_k^{\sigma+1,\bullet}$, then it follows from Proposition 4.2 that

$$| \frac{1}{N} \sum_{i=1}^{N} (\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet})) \,| \longrightarrow 0 , \tag{16}$$

almost surely as $N \uparrow \infty$. Combining (14) and (16) with the decomposition

$$\begin{aligned} &\frac{1}{N} \sum_{i=1}^{N} \phi(X_k^{i,\bullet}) - \int_{\mathbb{R}^m} \phi(x)\, \bar{\mu}_k^{\bullet}(dx) \\ &= \frac{1}{N} \sum_{i=1}^{N} (\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet})) + (\, \frac{1}{N} \sum_{i=1}^{N} \phi(\bar{X}_k^{i,\bullet}) - \int_{\mathbb{R}^m} \phi(x)\, \bar{\mu}_k^{\bullet}(dx)\,) , \end{aligned}$$

finishes the proof. □

5.2 $\mathbb{L}^p$–convergence and rate of convergence

Theorem 5.2 *Let ϕ be a locally Lipschitz continuous function, with at most polynomial growth at infinity, i.e.*

$$|\phi(x) - \phi(x')| \leq L\, |x - x'|\, (1 + |x|^{\sigma} + |x'|^{\sigma}) ,$$

for any $x, x' \in \mathbb{R}^m$ and for some $\sigma \geq 0$.

If Assumption B holds, and if the random vector X_0 has finite moments of any order, then

$$\sup_{N \geq 1} \sqrt{N}\, (\,\mathbb{E}| \frac{1}{N} \sum_{i=1}^{N} \phi(X_k^{i,f}) - \int_{\mathbb{R}^m} \phi(x)\, \bar{\mu}_k^{f}(dx)\, |^p\,)^{1/p} < \infty ,$$

and

$$\sup_{N \geq 1} \sqrt{N}\, (\,\mathbb{E}| \frac{1}{N} \sum_{i=1}^{N} \phi(X_k^{i,a}) - \int_{\mathbb{R}^m} \phi(x)\, \bar{\mu}_k^{a}(dx)\, |^p\,)^{1/p} < \infty ,$$

for any order p.

Proof. Throughout the proof, let the superscript $\bullet$ index either forecast–related or analysis–related quantities. Clearly

$$|\phi(x)| \leq M\, (1 + |x|^{\sigma+1}) ,$$

for any $x \in \mathbb{R}^m$, and using the triangle inequality, it follows from Proposition 2.3 that

$$\begin{aligned} (\int_{\mathbb{R}^m} |\phi(x)|^p\, \bar{\mu}_k^{\bullet}(dx)\,)^{1/p} \;&\leq\; M\, (\int_{\mathbb{R}^m} (1 + |x|^{\sigma+1})^p\, \bar{\mu}_k^{\bullet}(dx))\,)^{1/p} \\ &\leq\; M\, (1 + (\int_{\mathbb{R}^m} |x|^{p(\sigma+1)}\, \bar{\mu}_k^{\bullet}(dx))^{1/p}\,) \\ &=\; M\, (1 + |\bar{M}_k^{p(\sigma+1),\bullet}|^{\sigma+1}) < \infty , \end{aligned}$$

hence it follows from the Marcinkiewicz–Zygmund inequality that

$$(\,\mathbb{E}| \frac{1}{N} \sum_{i=1}^{N} \phi(\bar{X}_k^{i,\bullet}) - \int_{\mathbb{R}^m} \phi(x)\, \bar{\mu}_k^{\bullet}(dx)|^p\,)^{1/p} \leq \frac{c_p}{\sqrt{N}}\, (\,\mathbb{E}|\phi(\bar{X}_k^{i,\bullet}) - \int_{\mathbb{R}^m} \phi(x)\, \bar{\mu}_k^{\bullet}(dx)|^p\,)^{1/p} ,$$

or in other words

$$\sup_{N \geq 1} \sqrt{N}\, (\,\mathbb{E}| \frac{1}{N} \sum_{i=1}^{N} \phi(\bar{X}_k^{i,\bullet}) - \int_{\mathbb{R}^m} \phi(x)\, \bar{\mu}_k^{\bullet}(dx)|^p\,)^{1/p} < \infty . \tag{17}$$

Using the triangle inequality, it follows from (15) that

$$(\,\mathbb{E}|\,\frac{1}{N}\,\sum_{i=1}^N(\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet}))\,|^p\,)^{1/p} \,\leq\, L\,(\,\mathbb{E}(|\Delta_k^{N,r,\bullet}|^p\,(1+|\bar{M}_k^{N,\sigma\,r',\bullet}|^\sigma)^p)\,)^{1/p}$$

$$+\,L\,(\,\mathbb{E}|\Delta_k^{N,\sigma+1,\bullet}|^{p(\sigma+1)}\,)^{1/p}\ ,$$

and using the Hölder inequality and then the triangle inequality again to control the first term, yields

$$\begin{aligned}&(\,\mathbb{E}(|\Delta_k^{N,r,\bullet}|^p\,(1+|\bar{M}_k^{N,\sigma\,r',\bullet}|^\sigma)^p)\,)^{1/p}\\ &\quad\leq(\,\mathbb{E}|\Delta_k^{N,r,\bullet}|^{p\,r}\,)^{1/p\,r}\,(\,\mathbb{E}(1+|\bar{M}_k^{N,\sigma\,r',\bullet}|^\sigma)^{p\,r'}\,)^{1/p\,r'}\\ &\quad\leq(\,\mathbb{E}|\Delta_k^{N,r,\bullet}|^{p\,r}\,)^{1/p\,r}\,(1+(\,\mathbb{E}|\bar{M}_k^{N,\sigma\,r',\bullet}|^{\sigma\,p\,r'}\,)^{1/p\,r'})\ ,\end{aligned}$$

where r, r' are conjugate exponents, i.e. $1/r + 1/r' = 1$, or in other words

$$(\,\mathbb{E}|\,\frac{1}{N}\,\sum_{i=1}^N(\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet}))\,|^p\,)^{1/p} \,\leq\, L\,D_k^{N,p\,r,\bullet}\,(1+|\bar{M}_k^{\sigma\,p\,r',\bullet}|^\sigma) + L\,|D_k^{N,p\,(\sigma+1)}|^{\sigma+1}\ ,$$

in view of Lemma 4.3, and since $\bar{M}_k^{\sigma\,p\,r',\bullet}$ is finite, then it follows from Proposition 4.4 that

$$\sup_{N\geq 1}\,\sqrt{N}\,(\,\mathbb{E}|\,\frac{1}{N}\,\sum_{i=1}^N(\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet}))\,|^p\,)^{1/p} < \infty\ . \tag{18}$$

Combining (17) and (18) with the decomposition

$$\begin{aligned}&(\,\mathbb{E}|\,\frac{1}{N}\,\sum_{i=1}^N\phi(X_k^{i,\bullet}) - \int_{\mathbb{R}^m}\phi(x)\,\bar{\mu}_k^\bullet(dx)|^p\,)^{1/p}\\ &\leq(\,\mathbb{E}|\,\frac{1}{N}\,\sum_{i=1}^N(\phi(X_k^{i,\bullet}) - \phi(\bar{X}_k^{i,\bullet}))\,|^p\,)^{1/p} + (\,\mathbb{E}|\,\frac{1}{N}\,\sum_{i=1}^N\phi(\bar{X}_k^{i,\bullet}) - \int_{\mathbb{R}^m}\phi(x)\,\bar{\mu}_k^\bullet(dx)\,|^p\,)^{1/p}\ ,\end{aligned}$$

finishes the proof. □

6 Conclusion and perspectives

The results of the previous sections show that

- for linear systems of the form considered in Section 1.1, with additive Gaussian white noises and with Gaussian initial condition, the empirical mean of the ensemble elements converges to the Kalman filter,
- however, for nonlinear systems of the form introduced in Section 1.2, with additive Gaussian white noises and with non–necessarily Gaussian initial condition, the empirical probability distribution of the ensemble elements converges to the wrong limit, i.e. the limiting probability distribution differs from the usual Bayesian filter.

This may be seen as a negative result, and the question that naturally arise is whether it is possible to improve the ensemble Kalman filter in some way. Indeed, there exists another class of Monte Carlo–based approximations to the Bayesian filter, which have been extensively studied both practically and theoretically [4, 6, 7], and which could be applied to a much broader class of nonlinear models or even more general hidden Markov models. These particle filters provide an approximation of the Bayesian filter in terms of the weighted empirical probability distribution

$$\mu_k^N = \sum_{i=1}^N w_k^i\,\delta_{\xi_k^i} \qquad \text{with} \qquad \sum_{i=1}^N w_k^i = 1\ ,$$

associated with a population of N particles, characterized by their positions $(\xi_k^1, \cdots, \xi_k^N)$ and their nonnegative weights $(w_k^1, \cdots, w_k^N)$. Many different variants of particle filters have been proposed, depending on the answers to issues such as : how to initialize the particle system, how to move particle positions, how to update particle weights, how to exploit particle weights, etc. Many convergence results hold generically as the population size N goes to infinity, with the Bayesian filter as the limit, e.g. convergence in $\mathbb{L}^p$–mean

$$(\, \mathbb{E}| \sum_{i=1}^N w_k^i \; \phi(\xi_k^i) - \int_{\mathbb{R}^m} \phi(x) \; \mu_k(dx) \, |^p \,)^{1/p} \longrightarrow 0 \; ,$$

for any order p as $N \uparrow \infty$, and central limit theorem

$$\sqrt{N} \; (\sum_{i=1}^N w_k^i \; \phi(\xi_k^i) - \int_{\mathbb{R}^m} \phi(x) \; \mu_k(dx)) \Longrightarrow \mathcal{N}(0, v_k(\phi)) \; ,$$

in distribution as $N \uparrow \infty$, with a more or less explicit expression for the asymptotic variance $v_k(\phi)$, depending on the implementation.

For nonlinear systems precisely of the form

$$X_k \;=\; f_k(X_{k-1}) + W_k \qquad \text{with} \qquad W_k \sim \mathcal{N}(0, Q_k)$$

$$Y_k \;=\; H_k \; X_k + V_k \qquad \text{with} \qquad V_k \sim \mathcal{N}(0, R_k) \; ,$$

introduced in Section 1.2, with additive Gaussian white noises and with non–necessarily Gaussian initial condition $X_0 \sim \eta_0$, the favorite particle filter implementation [8, Section II.D] consists in sampling particle positions according to the probability distribution of X_k given X_{k-1} and Y_k, and assigning weights proportional to the probability density of Y_k given X_{k-1}. This results in the following algorithm : in the first part of the mutation step, given a population $(\xi_{k-1}^1, \cdots, \xi_{k-1}^N)$ of N particles, each particle is propagated independently according to

$$\xi_k^{i,-} = f_k(\xi_{k-1}^i) + W_k^i \qquad \text{with} \qquad W_k^i \sim \mathcal{N}(0, Q_k) \; . \tag{19}$$

Notice that the i.i.d. random vectors $(W_k^1, \ldots, W_k^N)$ are *simulated* here, with the same statistics as the additive Gaussian noise W_k in the original state equation. The initial population $(\xi_0^{1,-}, \cdots, \xi_0^{N,-})$ is *simulated* as i.i.d. random vectors with probability distribution η_0, i.e. with the same statistics as the initial condition X_0. In the second part of the mutation step, each particle is propagated independently according to

$$\xi_k^i = \xi_k^{i,-} + K_k(Q_k) \; (Y_k - H_k \, \xi_k^{i,-} - V_k^i) \qquad \text{with} \qquad V_k^i \sim \mathcal{N}(0, R_k) \; , \tag{20}$$

with Kalman gain matrix

$$K_k(Q_k) = Q_k \, H_k^* \; (H_k \, Q_k \, H_k^* + R_k)^{-1} \; .$$

Notice that the i.i.d. random vectors $(V_k^1, \ldots, V_k^N)$ are *simulated* here, with the same statistics as the additive Gaussian noise V_k in the original observation equation. Combining the two mutation steps together yields

$$\begin{aligned} \xi_k^i \;&=\; (f_k(\xi_{k-1}^i) + W_k^i) + K_k(Q_k) \; (Y_k - H_k \, (f_k(\xi_{k-1}^i) + W_k^i) - V_k^i) \\ &=\; f_k(\xi_{k-1}^i) + K_k(Q_k) \; (Y_k - H_k \, f_k(\xi_{k-1}^i)) + (I - K_k(Q_k) \, H_k) \; W_k^i - K_k(Q_k) \; V_k^i \; , \end{aligned}$$

so that, conditionally w.r.t. $\xi_{k-1}^i = x$, the random vector ξ_k^i is Gaussian with mean vector

$$m_k(x) = f_k(x) + K_k(Q_k) \; (Y_k - H_k \, f_k(x)) \; ,$$

and covariance matrix

$$(I - K_k(Q_k) \, H_k) \; Q_k \; (I - K_k(Q_k) \, H_k)^* + K_k(Q_k) \; R_k \; (K_k(Q_k))^* = (I - K_k(Q_k) \, H_k) \; Q_k \; .$$

In the weighting step, each weight is updated according to

$$w_k^i \propto w_{k-1}^i \; \exp\{-\tfrac{1}{2} \, (Y_k - H_k \, f_k(\xi_{k-1}^i))^* \; \Xi_k^{-1} \; (Y_k - H_k \, f_k(\xi_{k-1}^i)) \, \} \; ,$$

with covariance matrix $\Xi_k = H_k\, Q_k\, H_k^* + R_k$.

Even with this optimal choice of the importance distribution, it may happen that the particle weights $(w_k^1, \cdots, w_k^N)$ degenerate, i.e. depart significantly from equidistribution, in the sense that a few particles only, or even a single particle, get most of the weight. In this case, it is a good idea to resample the particle positions $(\xi_k^1, \cdots, \xi_k^N)$ according to their respective weights, so that particles with low weights are discarded, whereas particles with high weights are replicated. There are many different ways to perform the resampling step, and adaptive rules have also been proposed to decide on–line when to resample.

It has recently been raised [12, 19] however, that particle filters may collapse in high dimension : indeed, the usual adaptation techniques, such as using a better importance distribution or resampling, are no longer efficient in high dimension, and this challenging issue would deserve further study. By construction, the EnKF is not exposed to this degeneracy problem, just because there are no weights attached to elements.

There is again a visible difference between equations (1) and (2) for the elements in the ensemble Kalman filter, and equations (19) and (20) for the particle filter with optimal importance distribution : the Kalman gain matrix $K_k(P_k^N)$ in equation (2) is based on the empirical covariance matrix P_k^N, which is responsible for mean–field interaction and dependence, whereas the Kalman gain matrix $K_k(Q_k)$ in equation (20) is based on the deterministic covariance matrix Q_k of the additive Gaussian noise W_k in the original state equation, which is responsible for decoupling and independence. Notice that the covariance matrix Q_k is available, and the Kalman gain matrix $K_k(Q_k)$ is readily computable. If necessary, an independent sample can be simulated in order to approximate, without mean–field interaction, the covariance matrix Q_k in terms of an empirical covariance matrix.

The other difference with the ensemble Kalman filter is the existence of weights attached to particles, which can also be exploited to resample the population, if needed, and which are responsible for the convergence of the particle filter to the Bayesian filter. Quite naturally, it has recently been suggested [17, Chapitre 2] to interpret equations (1) and (2) for the elements in the ensemble Kalman filter, as defining an importance distribution with mean–field interaction, and to attach the corresponding importance weights to the ensemble elements. Numerical evidence has already been provided about the practical improvement obtained with this modification, and the resulting weighted ensemble Kalman filter should now converge to the Bayesian filter, as the ensemble size goes to infinity. This convergence issue deserves further investigation, including the proof of a central limit theorem, which would make it possible to compare the weighted ensemble Kalman filter and the particle filter with optimal importance distribution, on the basis of their respective asymptotic variance.

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