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A Backward Particle Interpretation of Feynman-Kac Formulae

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Abstract: We design a particle interpretation of Feynman-Kac measures on path spaces based on a backward Markovian representation combined with a traditional mean field particle interpretation of the flow of their final time marginals. In contrast to traditional genealogical tree based models, these new particle algorithms can be used to compute normalized additive functionals "on-the-fly" as well as their limiting occupation measures with a given precision degree that does not depend on the final time horizon.

We provide uniform convergence results w.r.t. the time horizon parameter as well as functional central limit theorems and exponential concentration estimates, yielding what seems to be the first results of this type for this class of models. We also illustrate these results in the context of computational physics and imaginary time Schroedinger type partial differential equations, with a special interest in the numerical approximation of the invariant measure associated to h-processes.

Key-words: Feynman-Kac models, mean field particle algorithms, functional central limit theorems, exponential concentration, non asymptotic estimates.

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A Backward Particle Interpretation of Feynman-Kac Formulae

Résumé : Nous présentons de nouvelles interprétations particulaires de mesures de Feynman-Kac trajectorielles fondées sur une représentation markovienne à rebours de ces modèles, couplée avec les interprétations particulaires de type champ moyen classiques du flot des mesures marginales par rapport aux temps terminaux. A la différence des algorithmes particulaires fondés sur des évolutions d'arbres généalogiques, ces nouvelles techniques permettent de calculer récursivement des fonctionnelles additives normalisées et leur mesures limites avec un degré de précision uniforme par rapport à l'horizon temporel considéré.

Nous proposons des résultats de convergence uniformes par rapport à l'horizon temporel, ainsi que des théorèmes de la limite centrale fonctionnels et des inégalités de concentration exponentielles. Ces résultats semblent être les premiers de ce type pour cette classe d'algorithmes particulaires. Nous illustrons ces résultats en physique numérique avec des approximations particulaires d'équations aux dérivées partielles de type Schroedinger et le calcul effectif des mesures stationnaires associées aux h-processus.

Mots-clés : Modèles de Feynman-Kac, algorithmes stochastiques de type champ moyen, théorème de la limite centrale fonctionnels, inégalités de concentration exponentielle, estimations non asymptotiques

1 Introduction

Let $(E_n)_{n\geq 0}$ be a sequence of measurable spaces equipped with some σ-fields $(\mathcal{E}_n)_{n\geq 0}$, and we let $\mathcal{P}(E_n)$ be the set of all probability measures over the set E_n. We let X_n be a Markov chain with Markov transition M_n on E_n, and we consider a sequence of $(0,1]$-valued potential functions G_n on the set E_n. The Feynman-Kac path measure associated with the pairs (M_n, G_n) is the probability measure $\mathbb{Q}_n$ on the product state space $E_{[0,n]} := (E_0 \times \ldots \times E_n)$ defined by the following formula

$$d\mathbb{Q}_n := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0\leq p<n} G_p(X_p) \right\} d\mathbb{P}_n \tag{1.1}$$

where $\mathcal{Z}_n$ is a normalizing constant and $\mathbb{P}_n$ is the distribution of the random paths $(X_p)_{0\leq p\leq n}$ of the Markov process X_p from the origin $p=0$, up to the current time $p=n$. We also denote by $\Gamma_n = \mathcal{Z}_n\ \mathbb{Q}_n$ its unnormalized version.

These distributions arise in a variety of application areas, including filtering, Bayesian inference, branching processes in biology, particle absorption problems in physics and many other instances. We refer the reader to the pair of books [4, 10] and references therein. Feynman-Kac models also play a central role in the numerical analysis of certain partial differential equations, offering a natural way to solve these functional integral models by simulating random paths of stochastic processes. These Feynman-Kac models were originally presented by Mark Kac in 1949 [12] for continuous time processes. These continuous time models are used in molecular chemistry and computational physics to calculate the ground state energy of some Hamiltonian operators associated with some potential function V describing the energy of a molecular configuration (see for instance [1, 5, 15, 19], and references therein).

To better connect these partial differential equation models with (1.1), let us assume that $M_n(x_{n-1}, dx_n)$ is the Markov probability transition $X_n = x_n \rightsquigarrow X_{n+1} = x_{n+1}$ coming from a discretization in time $X_n = X'_{t_n}$ of a continuous time E-valued Markov process X'_t on a given time mesh $(t_n)_{n\geq 0}$ with a given time step $(t_n - t_{n-1}) = \Delta t$. For potential functions of the form $G_n = e^{-V\Delta t}$, the measures $\mathbb{Q}_n \simeq_{\Delta t\to 0} \mathbb{Q}_{t_n}$ represents the time discretization of the following distribution:

$$d\mathbb{Q}_t = \frac{1}{\mathcal{Z}_t}\ \exp\left(-\int_0^t\ V(X'_s)\ ds\right)\ d\mathbb{P}_t^{X'}$$

where $\mathbb{P}_t^{X'}$ stands for the distribution of the random paths $(X'_s)_{0\leq s\leq t}$ with a given infinitesimal generator L. The marginal distributions γ_t at time t of the unnormalized measures $\mathcal{Z}_t\ d\mathbb{Q}_t$ are the solution of the so-called imaginary time Schroedinger equation, given in weak formulation on every sufficiently regular function f by

$$\frac{d}{dt}\ \gamma_t(f) := \gamma_t(L^V(f)) \quad \text{with} \quad L^V = L - V$$

The errors introduced by the discretization of the time are well understood for regular models, we refer the interested reader to [6, 9, 14, 16] in the context of nonlinear filtering.

In this article, we design a numerical approximation scheme for the distributions $\mathbb{Q}_n$ based on the simulation of a sequence of mean field interacting particle systems. In molecular chemistry, these evolutionary type models are often interpreted as a quantum or diffusion Monte Carlo model. In this context, particles often are referred as walkers, to distinguish the virtual particle-like objects to physical particles, like electrons of atoms. In contrast to traditional genealogical tree based approximations (see for instance [4]), the particle model presented in this article can approximate additive functionals of the form

$$\overline{F}_n(x_0, \ldots, x_n) = \frac{1}{(n+1)} \sum_{0\leq p\leq n} f_p(x_p) \tag{1.2}$$

"uniformly well" with respect to the time horizon. Moreover this computation can be done "on-the-fly". To give a flavor of the impact of these results, we recall that the precision of the algorithm corresponds to the size N of the particle system. If $\mathbb{Q}_n^N$ stands for the N-particle approximation of $\mathbb{Q}_n$, under some appropriate regularity properties, we shall prove the following uniform and non asymptotic Gaussian concentration estimates[1]:

$$\frac{1}{N}\log\sup_{n\geq 0}\mathbb{P}\left(\left|[\mathbb{Q}_n^N-\mathbb{Q}_n](\overline{F}_n)\right|\geq\frac{b}{\sqrt{N}}+\epsilon\right)\leq -\epsilon^2/(2b^2)$$

for any $\epsilon > 0$, and for some finite constant $b < \infty$. In the filtering context, $\mathbb{Q}_n^N$ corresponds to the sequential Monte Carlo approximation of the forward filtering backward smoothing recursion. Recently, a theoretical study of this problem was undertaken by [8]. Our results complement theirs and we present functional central limit theorems as well as non-asymptotic variance bounds. Additionally, we show how the forward filtering backward smoothing estimates of additive functionals can be computed using a forward only recursion. This has applications to online parameter estimation for non-linear non-Gaussian state-space models.

For time homogeneous models $(M_n, f_n, G_n) = (M, f, G)$ with a lower bounded potential function $G > \delta$, and a M-reversible transition w.r.t. to some probability measure μ s.t. $M(x, .) \sim \mu$ and $(M(x, .)/d\mu) \in \mathbb{L}_2(\mu)$, it can be established that $\mathbb{Q}_n(F_n)$ converges to $\mu_h(f)$, as $n \to \infty$, with the measure μ_h defined below

$$\mu_h(dx) := \frac{1}{\mu(hM(h))}\ h(x)\ M(h)(x)\ \mu(dx)$$

In the above display, h is a positive eigenmeasure associated with the top eigenvalue of the integral operator $Q(x, dy) = G(x)M(x, dy)$ on $\mathbb{L}_2(\mu)$ (see for instance section 12.4 in [4]). This measure μ_h is in fact the invariant measure of the h-process defined as the Markov chain X^h with elementary Markov transitions $M_h(x, dy) \propto M(x, dy)h(y)$. As the initiated reader would have certainly noticed, the above convergence result is only valid under some appropriate mixing conditions on the h-process. The long time behavior of these h-processes and their connections to various applications areas of probability, analysis, geometry and partial differential equations, has been the subject of countless papers for many years in applied probability. In our framework, using elementary manipulations, the Gaussian estimate given above can be used to calibrate the convergence of the particle estimate $\mathbb{Q}_n^N(F_n)$ towards $\mu_h(f)$, as the pair of parameters N and $n \to \infty$.

The rest of this article is organized as follows:

In section 2, we describe the mean field particle models used to design the particle approximation measures $\mathbb{Q}_n^N$. In section 3, we state the main results presented in this article, including a functional central limit theorem, and non asymptotic mean error bounds. Section 4 is dedicated to a key backward Markov chain representation of the measures $\mathbb{Q}_n$. The analysis of our particle approximations is provided in section 5. The next two sections, section 6 and section 7, are mainly concerned with the proof of the two main theorems presented in section 3. In the final section, section 8, we provide some comparisons between the backward particle model discussed in this article and the more traditional genealogical tree based particle model.

For the convenience of the reader, we end this introduction with some notation used in the present article. We denote respectively by $\mathcal{M}(E)$, and $\mathcal{B}(E)$, the set of all finite signed measures on some measurable space $(E, \mathcal{E})$, and the Banach space of all bounded and measurable functions f equipped with the uniform norm $\|f\|$. We let $\mu(f) = \int\ \mu(dx)\ f(x)$, be the Lebesgue integral of a function $f \in \mathcal{B}(E)$, with respect to a measure $\mu \in \mathcal{M}(E)$. We recall that a bounded integral kernel $M(x, dy)$ from a measurable space $(E, \mathcal{E})$ into an auxiliary measurable space $(E', \mathcal{E}')$ is an operator $f \mapsto M(f)$ from $\mathcal{B}(E')$ into $\mathcal{B}(E)$ such that the functions

$$x \mapsto M(f)(x) := \int_{E'} M(x, dy)f(y)$$

[1]Consult the last paragraph of this section for a statement of the notation used in this article.

are $\mathcal{E}$-measurable and bounded, for any $f \in \mathcal{B}(E')$. In the above displayed formulae, dy stands for an infinitesimal neighborhood of a point y in E'. The kernel M also generates a dual operator $\mu \mapsto \mu M$ from $\mathcal{M}(E)$ into $\mathcal{M}(E')$ defined by $(\mu M)(f) := \mu(M(f))$. A Markov kernel is a positive and bounded integral operator M with $M(1) = 1$. Given a pair of bounded integral operators (M_1, M_2), we let $(M_1 M_2)$ the composition operator defined by $(M_1 M_2)(f) = M_1(M_2(f))$. For time homogenous state spaces, we denote by $M^m = M^{m-1}M = MM^{m-1}$ the m-th composition of a given bounded integral operator M, with $m \geq 1$. Given a positive function G on E, we let $\Psi_G \,:\, \eta \in \mathcal{P}(E) \mapsto \Psi_G(\eta) \in \mathcal{P}(E)$, be the Boltzmann-Gibbs transformation defined by

$$\Psi_G(\eta)(dx) := \frac{1}{\eta(G)}\; G(x)\; \eta(dx)$$

2 Description of the models

The numerical approximation of the path-space distributions (1.1) requires extensive calculations. The mean field particle interpretation of these models are based on the fact that the flow of the n-th time marginals η_n of the measures $\mathbb{Q}_n$ satisfy a non linear evolution equation of the following form

$$\eta_{n+1}(dy) = \int \eta_n(dx) K_{n+1,\eta_n}(x, dy) \tag{2.1}$$

for some collection of Markov transitions $K_{n+1,\eta}$, indexed by the time parameter $n \geq 0$ and the set of probability measures $\mathcal{P}(E_n)$. The mean field particle interpretation of the nonlinear measure valued model (2.1) is the E_n^N-valued Markov chain

$$\xi_n = \left(\xi_n^1, \xi_n^2, \ldots, \xi_n^N\right) \in E_n^N$$

with elementary transitions defined as

$$\mathbb{P}\left(\xi_{n+1} \in dx \mid \xi_n\right) = \prod_{i=1}^{N} K_{n+1,\eta_n^N}(\xi_n^i, dx^i) \quad \text{with} \quad \eta_n^N := \frac{1}{N}\sum_{j=1}^{N} \delta_{\xi_n^j} \tag{2.2}$$

In the above displayed formula, dx stands for an infinitesimal neighborhood of the point $x = (x^1, \ldots, x^N) \in E_{n+1}^N$. The initial system ξ_0 consists of N independent and identically distributed random variables with common law η_0. We let $\mathcal{F}_n^N := \sigma\left(\xi_0, \ldots, \xi_n\right)$ be the natural filtration associated with the N-particle approximation model defined above. The resulting particle model coincides with a genetic type stochastic algorithm $\xi_n \rightsquigarrow \widehat{\xi}_n \rightsquigarrow \xi_{n+1}$ with selection transitions $\xi_n \rightsquigarrow \widehat{\xi}_n$ and mutation transitions $\widehat{\xi}_n \rightsquigarrow \xi_{n+1}$ dictated by the potential (or fitness) functions G_n and the Markov transitions M_{n+1}.

During the selection stage $\xi_n \rightsquigarrow \widehat{\xi}_n$, for every index i, with a probability $\epsilon_n G_n(\xi_n^i)$, we set $\widehat{\xi}_n^i = \xi_n^i$, otherwise we replace ξ_n^i with a new individual $\widehat{\xi}_n^i = \xi_n^j$ randomly chosen from the whole population with a probability proportional to $G_n(\xi_n^j)$. The parameter $\epsilon_n \geq 0$ is a tuning parameter that must satisfy the constraint $\epsilon_n G_n(\xi_n^i) \leq 1$, for every $1 \leq i \leq N$. For $\epsilon_n = 0$, the resulting proportional selection transition corresponds to the so-called simple genetic model. During the mutation stage, the selected particles $\widehat{\xi}_n^i \rightsquigarrow \xi_{n+1}^i$ evolve independently according to the Markov transitions M_{n+1}.

If we interpret the selection transition as a birth and death process, then arises the important notion of the ancestral line of a current individual. More precisely, when a particle $\widehat{\xi}_{n-1}^i \longrightarrow \xi_n^i$ evolves to a new location ξ_n^i, we can interpret $\widehat{\xi}_{n-1}^i$ as the parent of ξ_n^i. Looking backwards in time and recalling that the particle $\widehat{\xi}_{n-1}^i$ has selected a site ξ_{n-1}^j in the configuration at time $(n-1)$, we can interpret this site ξ_{n-1}^j as the parent of $\widehat{\xi}_{n-1}^i$ and therefore as the ancestor denoted $\xi_{n-1,n}^i$ at level $(n-1)$ of ξ_n^i. Running backwards in time we may trace the whole ancestral line

$$\xi_{0,n}^i \longleftarrow \xi_{1,n}^i \longleftarrow \ldots \longleftarrow \xi_{n-1,n}^i \longleftarrow \xi_{n,n}^i = \xi_n^i \tag{2.3}$$

More interestingly, the occupation measure of the corresponding N-genealogical tree model converges as $N \to \infty$ to the conditional distribution $\mathbb{Q}_n$. For any function F_n on the path space $E_{[0,n]}$, we have the following convergence (to be stated precisely later) as $N \to \infty$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} F_n(\xi^i_{0,n}, \xi^i_{1,n}, \ldots, \xi^i_{n,n}) = \int \mathbb{Q}_n(d(x_0,\ldots,x_n))\ F_n(x_0,\ldots,x_n) \tag{2.4}$$

This convergence result can be refined in various directions. Nevertheless, the asymptotic variance $\sigma^2_n(F_n)$ of the above occupation measure around $\mathbb{Q}_n$ increases quadratically with the final time horizon n for additive functions of the form

$$F_n(x_0,\ldots,x_n) = \sum_{0\le p\le n} f_p(x_p) \Rightarrow \sigma^2_n(F_n) \simeq n^2 \tag{2.5}$$

comprised of some collection of non negative functions f_p on E_p. To be more precise, let us examine a time homogeneous model $(E_n, f_n, G_n, M_n) = (E, f, G, M)$ with constant potential functions $G_n = 1$ and mutation transitions M s.t. $\eta_0 M = \eta_0$. For the choice of the tuning parameter $\epsilon = 0$, using the asymptotic variance formulae in [4, eqn. (9.13), page 304], for any function f s.t. $\eta_0(f) = 0$ and $\eta_0(f^2) = 1$ we prove that

$$\sigma^2_n(F_n) = \sum_{0\le p\le n} \mathbb{E}\left(\left[\sum_{0\le q\le n} M^{(q-p)_+}(f)(X_q)\right]^2\right)$$

with the positive part $a_+ = \max(a,0)$ and the convention $M^0 = Id$, the identity transition. For $M(x,dy) = \eta_0(dy)$, we find that

$$\sigma^2_n(F_n) = \sum_{0\le p\le n} \mathbb{E}\left(\left[\sum_{0\le q\le p} f(X_q)\right]^2\right) = (n+1)(n+2)/2 \tag{2.6}$$

We further assume that the Markov transitions $M_n(x_{n-1}, dx_n)$ are absolutely continuous with respect to some measures $\lambda_n(dx_n)$ on E_n and we have

$$(H) \qquad \forall (x_{n-1}, x_n) \in (E_{n-1}\times E_n) \qquad H_n(x_{n-1}, x_n) = \frac{dM_n(x_{n-1}, .)}{d\lambda_n}(x_n) > 0$$

In this situation, we have the backward decomposition formula

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \eta_n(dx_n)\ \mathcal{M}_n(x_n, d(x_0,\ldots,x_{n-1})) \tag{2.7}$$

with the Markov transitions $\mathcal{M}_n$ defined below

$$\mathcal{M}_n(x_n, d(x_0,\ldots,x_{n-1})) := \prod_{q=1}^{n} M_{q,\eta_{q-1}}(x_q, dx_{q-1})$$

In the above display, $M_{n+1,\eta}$ is the collection of Markov transitions defined for any $n \ge 0$ and $\eta \in \mathcal{P}(E_n)$ by

$$M_{n+1,\eta}(x, dy) = \frac{1}{\eta\left(G_n H_{n+1}(.,x)\right)}\ G_n(y)\ H_{n+1}(y,x)\ \eta(dy) \tag{2.8}$$

A detailed proof of this formula and its extended version is provided in section 4.

Using the representation in (2.7), one natural way to approximate $\mathbb{Q}_n$ is to replace the measures η_n with their N-particle approximations η^N_n. The resulting particle approximation measures, $\mathbb{Q}^N_n$, is then

$$\mathbb{Q}^N_n(d(x_0,\ldots,x_n)) := \eta^N_n(dx_n)\ \mathcal{M}^N_n(x_n, d(x_0,\ldots,x_{n-1})) \tag{2.9}$$

with the random transitions

$$\mathcal{M}_n^N(x_n, d(x_0, \ldots, x_{n-1})) := \prod_{q=1}^{n} M_{q,\eta_{q-1}^N}(x_q, dx_{q-1}) \tag{2.10}$$

At this point, it is convenient to recall that for any bounded measurable function f_n on E_n, the measures η_n can be written as follows:

$$\eta_n(f_n) := \frac{\gamma_n(f_n)}{\gamma_n(1)} \quad \text{with} \quad \gamma_n(f_n) := \mathbb{E}\left(f_n(X_n) \prod_{0\le p<n} G_p(X_p)\right) = \eta_n(f_n) \prod_{0\le p<n} \eta_p(G_p) \tag{2.11}$$

The multiplicative formula in the r.h.s. of (2.11) is easily checked using the fact that $\gamma_{n+1}(1) = \gamma_n(G_n) = \eta_n(G_n)\ \gamma_n(1)$. Mimicking the above formulae, we set

$$\Gamma_n^N = \gamma_n^N(1) \times \mathbb{Q}_n^N \quad \text{with} \quad \gamma_n^N(1) := \prod_{0\le p<n} \eta_p^N(G_p) \quad \text{and} \quad \gamma_n^N(dx) = \gamma_n^N(1) \times \eta_n^N(dx)$$

Notice that the N-particle approximation measures $\mathbb{Q}_n^N$ can be computed recursively with respect to the time parameter. For instance, for linear functionals of the form (2.5), we have

$$\mathbb{Q}_n^N(F_n) = \eta_n^N(F_n^N)$$

with a sequence of random functions F_n^N on E_n that can be computed "on-the-fly" according to the following recursion

$$F_n^N = \sum_{0\le p\le n} \left[M_{n,\eta_{n-1}^N} \ldots M_{p+1,\eta_p^N}\right](f_p) = f_n + M_{n,\eta_{n-1}^N}(F_{n-1}^N)$$

with the initial value $F_0^N = f_0$. In contrast to the genealogical tree based particle model (2.4), this new particle algorithm requires N^2 computations instead of N, in the sense that:

$$\forall 1 \le j \le N \qquad F_n^N(\xi_n^j) = f_n(\xi_n^j) + \sum_{1\le i\le N} \frac{G_{n-1}(\xi_{n-1}^i) H_n(\xi_{n-1}^i, \xi_n^j)}{\sum_{1\le i'\le N} G_{n-1}(\xi_{n-1}^{i'}) H_n(\xi_{n-1}^{i'}, \xi_n^j)}\ F_{n-1}^N(\xi_{n-1}^i)$$

An important application of this recursion is to parameter estimation for non-linear non-Gaussian state-space models. For instance, it may be used to implement an on-line version of the Expectation-Maximization algorithm as detailed in [13, Section 3.2]. In a different approach to recursive parameter estimation, an online particle algorithm is presented in [17] to compute the score for non-linear non-Gaussian state-space models. In fact, the algorithm of [17] is actually implementing a special case of the above recursion and may be reinterpreted as an "on-the-fly" computation of the forward filtering backward smoothing estimate of an additive functional derived from Fisher's identity.

The convergence analysis of the N-particle measures $\mathbb{Q}_n^N$ towards their limiting value $\mathbb{Q}_n$, as $N \to \infty$, is intimately related to the convergence of the flow of particle measures $(\eta_p^N)_{0\le p\le n}$ towards their limiting measures $(\eta_p)_{0\le p\le n}$. Several estimates can be easily derived more or less directly from the convergence analysis of the particle occupation measures η_n^N developed in [4], including $\mathbb{L}_p$-mean error bounds and exponential deviation estimates. It is clearly out of the scope of the present work to review all these consequences. One of the central objects in this analysis is the local sampling errors V_n^N induced by the mean field particle transitions and defined by the following stochastic perturbation formula

$$\eta_n^N = \eta_{n-1}^N K_{n,\eta_{n-1}^N} + \frac{1}{\sqrt{N}}\ V_n^N \tag{2.12}$$

The fluctuation and the deviations of these centered random measures V_n^N can be estimated using non asymptotic Kintchine's type $\mathbb{L}_r$-inequalities, as well as Hoeffding's or Bernstein's type exponential deviations [4, 7]. We also proved in [3] that these random perturbations behave asymptotically as Gaussian random perturbations. More precisely, for any fixed time horizon $n \geq 0$, the sequence of random fields V_n^N converges in law, as the number of particles N tends to infinity, to a sequence of independent, Gaussian and centered random fields V_n ; with, for any bounded function f on E_n, and $n \geq 0$,

$$\mathbb{E}(V_n(f)^2) = \int \eta_{n-1}(dx) K_{n,\eta_{n-1}}(x, dy) \left(f(y) - K_{n,\eta_{n-1}}(f)(x)\right)^2 \tag{2.13}$$

In section 5, we provide some key decompositions expressing the deviation of the particle measures $(\Gamma_n^N, \mathbb{Q}_n^N)$ around their limiting values $(\Gamma_n, \mathbb{Q}_n)$ in terms of these local random fields models. These decomposition can be used to derive almost directly some exponential and $\mathbb{L}_p$-mean error bounds using the stochastic analysis developed in [4]. We shall use these functional central limit theorems and some of their variations in various places in the present article.

3 Statement of some results

In the present article, we have chosen to concentrate on functional central limit theorems, as well as on non asymptotic variance theorems in terms of the time horizon. To describe our results, it is necessary to introduce the following notation. Let $\beta(M)$ denote the Dobrushin coefficient of a Markov transition M from a measurable space E into another measurable space E' which defined by the following formula

$$\beta(M) := \sup \; \{\mathrm{osc}(M(f)) \; ; \; f \in \mathrm{Osc}_1(E')\}$$

where $\mathrm{Osc}_1(E')$ stands the set of $\mathcal{E}'$-measurable functions f with oscillation, denoted $\mathrm{osc}(f) = \sup\{[f(x) - f(y)] \; ; \; x, y \in E'\}$, less than or equal to 1. Some stochastic models discussed in the present article are based on sequences of random Markov transitions M^N that depend on some mean field particle model with N random particles. In this case, $\beta(M^N)$ may fail to be measurable. For this type of models we shall use outer probability measures to integrate these quantities. For instance, the mean value $\mathbb{E}\left(\beta(M^N)\right)$ is to be understood as the infimum of the quantities $\mathbb{E}(B^N)$ where $B^N \geq \beta(M^N)$ are measurable dominating functions. We also recall that γ_n satisfy the linear recursive equation

$$\gamma_n = \gamma_p Q_{p,n} \quad \text{with} \quad Q_{p,n} = Q_{p+1} Q_{p+2} \ldots Q_n \quad \text{and} \quad Q_n(x, dy) = G_{n-1}(x) \; M_n(x, dy)$$

for any $0 \leq p \leq n$. Using elementary manipulations, we also check that

$$\Gamma_n(F_n) = \gamma_p D_{p,n}(F_n)$$

with the bounded integral operators $D_{p,n}$ from E_p into $E_{[0,n]}$ defined below

$$D_{p,n}(F_n)(x_p) := \int \mathcal{M}_p(x_p, d(x_0, \ldots, x_{p-1})) \mathcal{Q}_{p,n}(x_p, d(x_{p+1}, \ldots, x_n)) \; F_n(x_0, \ldots, x_n) \tag{3.1}$$

with

$$\mathcal{Q}_{p,n}(x_p, d(x_{p+1}, \ldots, x_n)) := \prod_{p \leq q < n} Q_{q+1}(x_q, dx_{q+1})$$

We also let $(G_{p,n}, P_{p,n})$ be the pair of potential functions and Markov transitions defined below

$$G_{p,n} = Q_{p,n}(1)/\eta_p Q_{p,n}(1) \quad \text{and} \quad P_{p,n}(F_n) = D_{p,n}(F_n)/D_{p,n}(1) \tag{3.2}$$

Let the mapping $\Phi_{p,n} : \mathcal{P}(E_p) \to \mathcal{P}(E_n)$, $0 \leq p \leq n$, be defined as follows

$$\Phi_{p,n}(\mu_p) = \frac{\mu_p Q_{p,n}}{\mu_p Q_{p,n}(1)}$$

Our first main result is a functional central limit theorem for the pair of random fields on $\mathcal{B}(E_{[0,n]})$ defined below

$$W_n^{\Gamma,N} := \sqrt{N}\left(\Gamma_n^N - \Gamma_n\right) \quad \text{and} \quad W_n^{\mathbb{Q},N} := \sqrt{N}\ [\mathbb{Q}_n^N - \mathbb{Q}_n]$$

$W_n^{\Gamma,N}$ is centered in the sense that $\mathbb{E}\left(W_n^{\Gamma,N}(F_n)\right) = 0$ for any $F_n \in \mathcal{B}(E_{[0,n]})$. The proof of this surprising unbiasedness property can be found in corollary 5.3, in section 5.

The first main result of this article is the following multivariate fluctuation theorem.

Theorem 3.1 *We suppose that the following regularity condition is met for any $n \geq 1$ and for any pair of states $(x,y) \in (E_{n-1}, E_n)$*

$$(H^+) \qquad h_n^-(y) \leq H_n(x,y) \leq h_n^+(y) \quad \textit{with} \quad (h_n^+/h_n^-) \in \mathbb{L}_4(\eta_n) \quad \textit{and} \quad h_n^+ \in \mathbb{L}_1(\lambda_n) \tag{3.3}$$

In this situation, the sequence of random fields $W_n^{\Gamma,N}$, resp. $W_n^{\mathbb{Q},N}$, converge in law, as $N \to \infty$, to the centered Gaussian fields W_n^{Γ}, resp. $W_n^{\mathbb{Q}}$, defined for any $F_n \in \mathcal{B}(E_{[0,n]})$ by

$$\begin{aligned} W_n^{\Gamma}(F_n) &= \sum_{p=0}^{n} \gamma_p(1)\ V_p\left(D_{p,n}(F_n)\right) \\ W_n^{\mathbb{Q}}(F_n) &= \sum_{p=0}^{n} V_p\left(G_{p,n}\ P_{p,n}(F_n - \mathbb{Q}_n(F_n))\right) \end{aligned}$$

A interpretation of the corresponding limiting variances in terms of conditional distributions of $\mathbb{Q}_n$ w.r.t. to the time marginal coordinates is provided in section 8.1.

The second main result of the article is the following non asymptotic theorem.

Theorem 3.2 *For any $r \geq 1$, $n \geq 0$, $F_n \in \mathrm{Osc}_1(E_{[0,n]})$ we have the non asymptotic estimates*

$$\sqrt{N}\ \mathbb{E}\left(\left|[\mathbb{Q}_n^N - \mathbb{Q}_n](F_n)\right|^r\right)^{\frac{1}{r}} \leq a_r \sum_{0 \leq p \leq n} b_{p,n}^2\ c_{p,n}^N \tag{3.4}$$

for some finite constants $a_r < \infty$ whose values only depend on the parameter r, and a pair of constants $(b_{p,n}, c_{p,n}^N)$ such that

$$b_{p,n} \leq \sup_{x,y}\left(Q_{p,n}(1)(x)/Q_{p,n}(1)(y)\right) \quad \textit{and} \quad c_{p,n}^N \leq \mathbb{E}\left(\beta(P_{p,n}^N)\right)$$

In the above display, $P_{p,n}^N$ stands for the random Markov transitions defined as $P_{p,n}$ by replacing in (3.1) and (3.2) the transitions $\mathcal{M}_p$ by $\mathcal{M}_p^N$.

For linear functionals of the form (2.5), with $f_n \in \mathrm{Osc}_1(E_n)$, the $\mathbb{L}_r$-mean error estimate (3.4) is satisfied with a constant $c_{p,n}^N$ in (3.4) that can be chosen so that

$$c_{p,n}^N \leq \sum_{0 \leq q < p} \mathbb{E}\left(\beta\left(M_{p,\eta_{p-1}^N} \ldots M_{q+1,\eta_q^N}\right)\right) + \sum_{p \leq q \leq n} b_{q,n}^2\ \beta(S_{p,q}) \tag{3.5}$$

with the Markov transitions $S_{p,q}$ from E_p into E_q defined for any function $f \in \mathcal{B}(E_q)$ by the following formula $S_{p,q}(f) = Q_{p,q}(f)/Q_{p,q}(1)$.

We emphasize that the $\mathbb{L}_r$-mean error bounds described in the above theorem enter the stability properties of the semigroups $S_{p,q}$ and the one associated with the backward Markov transitions M_{n+1,η_n^N}. In several instances, the term in the r.h.s. of (3.5) can be uniformly bounded with respect to the time horizon. For instance, in the toy example we discussed in (2.6), we have the variance formula

$$\mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right) = (n+1)$$

and the non asymptotic $\mathbb{L}_r$-estimates

$$b_{p,n} = 1 \quad \text{and} \quad c^N_{p,n} \leq 1 \implies \sqrt{N}\, \mathbb{E}\left(\left|[\mathbb{Q}^N_n - \mathbb{Q}_n](F_n)\right|^r\right)^{\frac{1}{r}} \leq a_r\ (n+1)$$

In more general situations, these estimates are related to the stability properties of the Feynman-Kac semigroup. To simplify the presentation, let us suppose that the state space E_n, and the pair of potential-transitions (G_n, M_n) are time homogeneous $(E_n, G_n, H_n, M_n) = (E, G, H, M)$, and chosen so that the following regularity condition is satisfied

$$(M)_m \qquad \forall (x,x') \quad G(x) \leq \delta\ G(x') \quad \text{and} \quad M^m(x,dy) \leq \rho\ M^m(x',dy) \tag{3.6}$$

for some $m \geq 1$ and some parameters $(\delta, \rho) \in [1,\infty)^2$. Under this rather strong condition, we have

$$b_{p,n} \leq \rho\delta^m \quad \text{and} \quad \beta(S_{p,q}) \leq \left(1 - \rho^{-2}\delta^{-m}\right)^{\lfloor (q-p)/m \rfloor}$$

See for instance corollary 4.3.3. in [4] and the more recent article [2]. On the other hand, let us suppose that

$$\inf_{x,y,y'} \left(H(x,y)/H(x,y')\right) = \alpha(h) > 0$$

In this case, we have

$$M_{n,\eta}(x,dy) \leq \alpha(h)^{-2}\ M_{n,\eta}(x',dy) \Longrightarrow \beta\left(M_{p,\eta^N_{p-1}} \ldots M_{q+1,\eta^N_q}\right) \leq \left(1 - \alpha(h)^2\right)^{p-q}$$

For linear functional models of the form (2.5) associated with functions $f_n \in \mathrm{Osc}_1(E_n)$, it is now readily checked that

$$\sqrt{N}\, \mathbb{E}\left(\left|[\mathbb{Q}^N_n - \mathbb{Q}_n](F_n)\right|^r\right)^{\frac{1}{r}} \leq a_r\ b\ (n+1) \tag{3.7}$$

for some finite constant $b < \infty$ whose values do not depend on the time parameter n. The above non asymptotic estimates are not sharp for $r = 2$. To obtain better bounds, we need to refine the analysis of the variance using first order decompositions to analyze separately the bias of the particle model. In this context, we also prove in section 7.2 that

$$N\ \left|\mathbb{E}\left(\mathbb{Q}^N_n(F_n)\right) - \mathbb{Q}_n(F)\right| \leq c\ (n+1) \quad \text{and} \quad \mathbb{E}\left(W^{\mathbb{Q},N}_n(F_n)^2\right) \leq c\ (n+1)\left(1 + \frac{n+1}{N}\right) \tag{3.8}$$

for some finite constant $c < \infty$, whose values do not depend on the time parameter.

With some information on the constants a_r, the above $\mathbb{L}_r$-mean error bounds can turned to uniform exponential estimates w.r.t. the time parameter for normalized additive functionals of the following form

$$\overline{F}_n(x_0, \ldots, x_n) := \frac{1}{n+1} \sum_{0 \leq p \leq n} f_p(x_p)$$

To be more precise, by lemma 7.3.3 in [4], the collection of constants a_r in (3.7) can be chosen so that

$$a^{2r}_{2r} \leq (2r)!\ 2^{-r}/r! \quad \text{and} \quad a^{2r+1}_{2r+1} \leq (2r+1)!\ 2^{-r}/r! \tag{3.9}$$

In this situation, it is easily checked that for any $\epsilon > 0$, and $N \geq 1$, we have the following uniform Gaussian concentration estimates:

$$\frac{1}{N} \log \sup_{n \geq 0} \mathbb{P}\left(\left|[\mathbb{Q}^N_n - \mathbb{Q}_n](\overline{F}_n)\right| \geq \frac{b}{\sqrt{N}} + \epsilon\right) \leq -\epsilon^2/(2b^2)$$

This result is a direct consequence of the fact that for any non negative random variable U

$$\left(\forall r \geq 1 \quad \mathbb{E}\left(U^r\right)^{\frac{1}{r}} \leq a_r\ b\right) \Rightarrow \log \mathbb{P}\left(U \geq b + \epsilon\right) \leq -\epsilon^2/(2b^2)$$

To check this claim, we develop the exponential to prove that

$$\log \mathbb{E}\left(e^{tU}\right) \overset{\forall t \geq 0}{\leq} bt + \frac{(bt)^2}{2} \Rightarrow \log \mathbb{P}\left(U \geq b + \epsilon\right) \leq -\sup_{t \geq 0}\left(\epsilon t - \frac{(bt)^2}{2}\right)$$

4 A backward Markov chain formulation

This section is mainly concerned with the proof of the backward decomposition formula (2.7). Before proceeding, we recall that the measures (γ_n, η_n) satisfy the non linear equations

$$\gamma_n = \gamma_{n-1} Q_n \quad \text{and} \quad \eta_{n+1} := \Phi_{n+1}(\eta_n) := \Psi_{G_n}(\eta_n) M_{n+1}$$

and their semigroups are given by

$$\gamma_n = \gamma_p Q_{p,n} \quad \text{and} \quad \eta_n(f_n) := \eta_p Q_{p,n}(f_n)/\eta_p Q_{p,n}(1)$$

for any function $f_n \in \mathcal{B}(E_n)$. In this connection, we also mention that the semigroup of the pair of measures $(\Gamma_n, \mathbb{Q}_n)$ defined in (1.1) for any $0 \leq p \leq n$ and any $F_n \in \mathcal{B}(E_{[0,n]})$, we have

$$\Gamma_n(F_n) = \gamma_p D_{p,n}(F_n) \quad \text{and} \quad \mathbb{Q}_n(F_n) = \eta_p D_{p,n}(F_n)/\eta_p D_{p,n}(1) \tag{4.1}$$

These formulae are a direct consequence of the following observation

$$\eta_p D_{p,n}(F_n) = \int \mathbb{Q}_p(d(x_0, \ldots, x_p))\ \mathcal{Q}_{p,n}(x_p, d(x_{p+1}, \ldots, x_n)) F_n(x_0, \ldots, x_n)$$

Lemma 4.1 *For any $0 \leq p < n$, we have*

$$\gamma_p(dx_p)\ \mathcal{Q}_{p,n}(x_p, d(x_{p+1}, \ldots, x_n)) = \gamma_n(dx_n)\ \mathcal{M}_{n,p}(x_n, d(x_p, \ldots, x_{n-1})) \tag{4.2}$$

with

$$\mathcal{M}_{n,p}(x_n, d(x_p, \ldots, x_{n-1})) := \prod_{p \leq q < n} M_{q+1,\eta_q}(x_{q+1}, dx_q)$$

In particular, for any time $n \geq 0$, the Feynman-Kac path measures $\mathbb{Q}_n$ defined in (1.1) can be expressed in terms of the sequence of marginal measures $(\eta_p)_{0 \leq p \leq n}$, with the following backward Markov chain formulation

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) = \eta_n(dx_n)\ \mathcal{M}_{n,0}(x_n, d(x_0, \ldots, x_{n-1})) : \tag{4.3}$$

Before entering into the details of the proof of this lemma, we mention that (4.3) holds true for any well defined Markov transition $M_{n+1,\eta_n}(y, dx)$ from E_n into E_{n+1} satisfying the local backward equation

$$\Psi_{G_n}(\eta_n)(dx)\ M_{n+1}(x, dy) = \Phi_{n+1}(\eta_n)(dy)\ M_{n+1,\eta_n}(y, dx)$$

or equivalently

$$\eta_n(dx)\ Q_{n+1}(x, dy) = (\eta_n Q_{n+1})(dy)\ M_{n+1,\eta_n}(y, dx) \tag{4.4}$$

In other words, we have the duality formula

$$\Psi_{G_n}(\eta_n)\left(f\ M_{n+1}(g)\right) = \Phi_{n+1}(\eta_n)\left(g\ M_{n+1,\eta_n}(f)\right) \tag{4.5}$$

Also notice that for any pair of measures μ, ν on E_n s.t. $\mu \ll \nu$, we have $\mu M_{n+1} \ll \nu M_{n+1}$. Indeed, if we have $\nu M_{n+1}(A) = 0$, the function $M_{n+1}(1_A)$ is null ν-almost everywhere, and therefore μ-almost everywhere from which we conclude that $\mu M_{n+1}(A) = 0$. For any bounded measurable function g on E_n we set

$$\Psi^g_{G_n}(\eta_n)(dx) = \Psi_{G_n}(\eta_n)(dx)\ g(x) \ll \Psi_{G_n}(\eta_n)(dx)$$

From the previous discussion, we have $\Psi^g_{G_n}(\eta_n) M_{n+1} \ll \Psi_{G_n}(\eta_n) M_{n+1}$ and it is easily checked that

$$M_{n+1,\eta_n}(g)(y) = \frac{d\Psi^g_{G_n}(\eta_n) M_{n+1}}{d\Psi_{G_n}(\eta_n) M_{n+1}}(y)$$

is a well defined Markov transition from E_{n+1} into E_n satisfying the desired backward equation. These manipulations are rather classical in the literature on Markov chains (see for instance [18], and references therein). Under the regularity condition (H) the above transition is explicitly given by the formula (2.8).

Now, we come to the proof of lemma 4.1.

Proof of lemma 4.1:

We prove (4.2) using a backward induction on the parameter p. By (4.4), the formula is clearly true for $p = (n-1)$. Suppose the result has been proved at rank p. Since we have

$$\begin{aligned}
&\gamma_{p-1}(dx_{p-1})\ \mathcal{Q}_{p-1,n}(x_{p-1}, d(x_p, \ldots, x_n)) \\
&= \gamma_{p-1}(dx_{p-1})\ Q_p(x_{p-1}, dx_p)\ \mathcal{Q}_{p,n}(x_p, d(x_{p+1}, \ldots, x_n))
\end{aligned}$$

and

$$\gamma_{p-1}(dx_{p-1})\ Q_p(x_{p-1}, dx_p) = \gamma_p(dx_p)\ M_{p,\eta_{p-1}}(x_p, dx_{p-1})$$

Using the backward induction we conclude that the desired formula is also met at rank $(p-1)$. The second assertion is a direct consequence of (4.2). The end of the proof of the lemma is now completed. ■

We end this section with some properties of backward Markov transitions associated with a given initial probability measure that may differ from the one associated with the Feynman-Kac measures. These mathematical objects appear in a natural way in the analysis of the N-particle approximation transitions $\mathcal{M}_n^N$ introduced in (2.10).

Definition 4.2 *For any $0 \leq p \leq n$ and any probability measure $\eta \in \mathcal{P}(E_p)$, we denote by $\mathcal{M}_{n+1,p,\eta}$ the Markov transition from E_{n+1} into $E_{[p,n]} = (E_p \times \ldots \times E_n)$ defined by*

$$\mathcal{M}_{n+1,p,\eta}\left(x_{n+1}, d(x_p, \ldots, x_n)\right) = \prod_{p \leq q \leq n} M_{q+1,\Phi_{p,q}(\eta)}(x_{q+1}, dx_q)$$

Notice that this definition is consistent with the definition of the Markov transitions $\mathcal{M}_{p,n}$ introduced in lemma 4.1:

$$\mathcal{M}_{n+1,p,\eta_p}\left(x_{n+1}, d(x_p, \ldots, x_n)\right) = \mathcal{M}_{n+1,p}\left(x_{n+1}, d(x_p, \ldots, x_n)\right)$$

Also observe that $\mathcal{M}_{n+1,p,\eta}$ can alternatively be defined by the pair of recursions

$$\begin{aligned}
&\mathcal{M}_{n+1,p,\eta}\left(x_{n+1}, d(x_p, \ldots, x_n)\right) \\
&= \mathcal{M}_{n+1,p+1,\Phi_{p+1}(\eta)}\left(x_{n+1}, d(x_{p+1}, \ldots, x_n)\right) \times M_{p+1,\eta}(x_{p+1}, dx_p) \qquad (4.6) \\
&= M_{n+1,\Phi_{p,n}(\eta)}(x_{n+1}, dx_n)\ \mathcal{M}_{n,p,\eta}\left(x_n, d(x_p, \ldots, x_{n-1})\right)
\end{aligned}$$

The proof of the following lemma follows the same lines of arguments as the ones used in the proof of lemma 4.1. For the convenience of the reader, the details of this proof are postponed to the appendix.

Lemma 4.3 *For any $0 \leq p < n$ and any probability measure $\eta \in \mathcal{P}(E_p)$, we have*

$$\eta Q_{p,n}(dx_n)\ \mathcal{M}_{n,p,\eta}(x_n, d(x_p, \ldots, x_{n-1})) = \eta(dx_p)\ \mathcal{Q}_{p,n}(x_p, d(x_{p+1}, \ldots, x_n))$$

In other words, we have

$$\begin{aligned}
&\mathcal{M}_{n,p,\eta}(x_n, d(x_p, \ldots, x_{n-1})) \\
&= \frac{(\eta \times \mathcal{Q}_{p,n-1})(d(x_p, \ldots, x_{n-1}))G_{n-1}(x_{n-1})\ H_n(x_{n-1}, x_n)}{(\eta Q_{p,n-1})\left(G_{n-1}\ H_n(\,\cdot\,, x_n)\right)} \qquad (4.7)
\end{aligned}$$

with the measure $(\eta \times \mathcal{Q}_{p,n-1})$ defined below

$$(\eta \times \mathcal{Q}_{p,n-1})(d(x_p, \ldots, x_{n-1})) := \eta(dx_p)\ \mathcal{Q}_{p,n-1}(x_p, d(x_{p+1}, \ldots, x_{n-1}))$$

5 Particle approximation models

We provide in this section some preliminary results on the convergence of the N-particle measures $(\Gamma_n^N, \mathbb{Q}_n^N)$ to their limiting values $(\Gamma_n, \mathbb{Q}_n)$, as $N \to \infty$. Most of the forthcoming analysis is developed in terms of the following integral operators.

Definition 5.1 *For any $0 \leq p \leq n$, we let $D_{p,n}^N$ be the $\mathcal{F}_{p-1}^N$-measurable integral operators from $\mathcal{B}(E_{[0,n]})$ into $\mathcal{B}(E_p)$ defined below*

$$D_{p,n}^N(F_n)(x_p) := \int \mathcal{M}_p^N(x_p, d(x_0, \ldots, x_{p-1}))\mathcal{Q}_{p,n}(x_p, d(x_{p+1}, \ldots, x_n))F_n(x_0, \ldots, x_n)$$

with the conventions $D_{0,n}^N = \mathcal{Q}_{0,n}$, and resp. $D_{n,n}^N = \mathcal{M}_n^N$, for $p = 0$, and resp. $p = n$

The main result of this section is the following theorem.

Theorem 5.2 *For any $0 \leq p \leq n$, and any function F_n on the path space $E_{[0,n]}$, we have*

$$\mathbb{E}\left(\Gamma_n^N(F_n) \mid \mathcal{F}_p^N\right) = \gamma_p^N\left(D_{p,n}^N(F_n)\right) \quad \textit{and} \quad W_n^{\Gamma,N}(F_n) = \sum_{p=0}^n \gamma_p^N(1)\ V_p^N\left(D_{p,n}^N(F_n)\right)$$

Proof of theorem 5.2:

To prove the first assertion, we use a backward induction on the parameter p. For $p = n$, the result is immediate since we have

$$\Gamma_n^N(F_n) = \gamma_n^N(1)\ \eta_n^N\left(D_{n,n}^N(F_n)\right)$$

We suppose that the formula is valid at a given rank $p \leq n$. In this situation, we have

$$\begin{aligned} \mathbb{E}\left(\Gamma_n^N(F_n) \mid \mathcal{F}_{p-1}^N\right) &= \gamma_p^N(1)\ \mathbb{E}\left(\eta_p^N\left(D_{p,n}^N(F_n)\right) \mid \mathcal{F}_{p-1}^N\right) \\ &= \gamma_{p-1}^N(1) \int \eta_{p-1}^N(G_{p-1}H_p(.,x_p))\ \lambda_p(dx_p)\ D_{p,n}^N(F_n)(x_p) \end{aligned} \tag{5.1}$$

Using the fact that

$$\gamma_{p-1}^N(1)\ \eta_{p-1}^N(G_{p-1}H_p(.,x_p))\ \lambda_p(dx_p)\ M_{p,\eta_{p-1}^N}(x_p, dx_{p-1}) = \gamma_{p-1}^N(dx_{p-1})Q_p(x_{p-1}, dx_p)$$

we conclude that the r.h.s. term in (5.1) takes the form

$$\begin{aligned} &\int \gamma_{p-1}^N(dx_{p-1})\mathcal{M}_{p-1}^N(x_{p-1}, d(x_0, \ldots, x_{p-2}))\mathcal{Q}_{p-1,n}(x_{p-1}, d(x_p, \ldots, x_n))\ F_n(x_0, \ldots, x_n) \\ &= \gamma_{p-1}^N\left(D_{p-1,n}^N(F_n)\right) \end{aligned}$$

This ends the proof of the first assertion. The proof of the second assertion is based on the following decomposition

$$\begin{aligned} \left(\Gamma_n^N - \Gamma_n\right)(F_n) &= \sum_{p=0}^n \left[\mathbb{E}\left(\Gamma_n^N(F_n) \mid \mathcal{F}_p^N\right) - \mathbb{E}\left(\Gamma_n^N(F_n) \mid \mathcal{F}_{p-1}^N\right)\right] \\ &= \sum_{p=0}^n \gamma_p^N(1)\left(\eta_p^N\left(D_{p,n}^N(F_n)\right) - \frac{1}{\eta_{p-1}^N(G_{p-1})}\ \eta_{p-1}^N\left(D_{p-1,n}^N(F_n)\right)\right) \end{aligned}$$

where $\mathcal{F}_{-1}^N$ is the trivial sigma field. By definition of the random fields V_p^N, it remains to prove that

$$\eta_{p-1}^N\left(D_{p-1,n}^N(F_n)\right) = (\eta_{p-1}^N Q_p)\left(D_{p,n}^N(F_n)\right)$$

To check this formula, we use the decomposition

$$\begin{aligned}&\eta_{p-1}^N(dx_{p-1})\ \mathcal{M}_{p-1}^N(x_{p-1},d(x_0,\ldots,x_{p-2}))\ \mathcal{Q}_{p-1,n}(x_{p-1},d(x_p,\ldots,x_n))\\&=\eta_{p-1}^N(dx_{p-1})Q_p(x_{p-1},dx_p)\mathcal{M}_{p-1}^N(x_{p-1},d(x_0,\ldots,x_{p-2}))\ \mathcal{Q}_{p,n}(x_p,d(x_{p+1},\ldots,x_n))\end{aligned} \tag{5.2}$$

Using the fact that

$$\eta_{p-1}^N(dx_{p-1})Q_p(x_{p-1},dx_p)\quad=\quad(\eta_{p-1}^N Q_p)(dx_p)\ M_{p,\eta_{p-1}^N}(x_p,dx_{p-1})$$

we conclude that the term in the r.h.s. of (5.2) is equal to

$$(\eta_{p-1}^N Q_p)(dx_p)\ \mathcal{M}_p^N(x_p,d(x_0,\ldots,x_{p-1}))\ \mathcal{Q}_{p,n}(x_p,d(x_{p+1},\ldots,x_n))$$

This ends the proof of the theorem. ∎

Several consequences of theorem 5.2 are now emphasized. On the one hand, using the fact that the random fields V_n^N are centered given $\mathcal{F}_{n-1}^N$, we find that

$$\mathbb{E}\left(\Gamma_n^N(F_n)\right)=\Gamma_n(F_n)$$

On the other hand, using the fact that

$$\frac{\gamma_p(1)}{\gamma_n(1)}=\frac{\gamma_p(1)}{\gamma_p Q_{p,n}(1)}=\frac{1}{\eta_p Q_{p,n}(1)}$$

we prove the following decomposition

$$\overline{W}_n^{\Gamma,N}(F_n)=\sqrt{N}\left(\overline{\gamma}_n^N(1)\ \mathbb{Q}_n^N-\mathbb{Q}_n\right)(F_n)=\sum_{p=0}^n\overline{\gamma}_p^N(1)\ V_p^N\left(\overline{D}_{p,n}^N(F_n)\right) \tag{5.3}$$

with the pair of parameters $\left(\overline{\gamma}_n^N(1),\overline{D}_{p,n}^N\right)$ defined below

$$\overline{\gamma}_n^N(1):=\frac{\gamma_n^N(1)}{\gamma_n(1)}\quad\text{and}\quad\overline{D}_{p,n}^N(F_n)=\frac{D_{p,n}^N(F_n)}{\eta_p Q_{p,n}(1)} \tag{5.4}$$

Using again the fact that the random fields V_n^N are centered given $\mathcal{F}_{n-1}^N$, we have

$$\mathbb{E}\left(\overline{W}_n^{\Gamma,N}(F_n)^2\right)=\sum_{p=0}^n\mathbb{E}\left(\overline{\gamma}_p^N(1)^2\ \mathbb{E}\left[V_p^N\left(\overline{D}_{p,n}^N(F_n)\right)^2\ \mid\mathcal{F}_{p-1}^N\right]\right)$$

Using the estimates

$$\begin{aligned}\|D_{p,n}^N(F_n)\|&\leq\|Q_{p,n}(1)\|\ \|F_n\|\\\|\overline{D}_{p,n}^N(F_n)\|&\leq\|\overline{Q}_{p,n}(1)\|\ \|F_n\|\quad\text{with}\quad\overline{Q}_{p,n}(1)=\frac{Q_{p,n}(1)}{\eta_p Q_{p,n}(1)}\end{aligned} \tag{5.5}$$

we prove the non asymptotic variance estimate

$$\mathbb{E}\left(\overline{W}_n^{\Gamma,N}(F_n)^2\right)\leq\sum_{p=0}^n\mathbb{E}\left(\overline{\gamma}_p^N(1)^2\right)\ \|\overline{Q}_{p,n}(1)\|^2=\sum_{p=0}^n\left[1+\mathbb{E}\left([\overline{\gamma}_p^N(1)-1]^2\right)\right]\ \|\overline{Q}_{p,n}(1)\|^2$$

for any function F_n such that $\|F_n\|\leq 1$. On the other hand, using the decomposition

$$\left(\overline{\gamma}_n^N(1)\ \mathbb{Q}_n^N-\mathbb{Q}_n\right)=\left[\overline{\gamma}_n^N(1)-1\right]\ \mathbb{Q}_n^N+\left(\mathbb{Q}_n^N-\mathbb{Q}_n\right)$$

we prove that

$$\mathbb{E}\left(\left[\mathbb{Q}_n^N(F_n)-\mathbb{Q}_n(F_n)\right]^2\right)^{1/2} \leq \frac{1}{\sqrt{N}}\ \mathbb{E}\left(W_n^{\Gamma}(F_n)^2\right)^{1/2}+\mathbb{E}\left(\left[\overline{\gamma}_n^N(1)-1\right]^2\right)^{1/2}$$

Some interesting bias estimates can also be obtained using the fact that

$$\mathbb{E}\left(\mathbb{Q}_n^N(F_n)\right)-\mathbb{Q}_n(F_n)=\mathbb{E}\left(\left[1-\overline{\gamma}_n^N(1)\right]\ \ \left[\mathbb{Q}_n^N(F_n)-\mathbb{Q}_n(F_n)\right]\right)$$

and the following easily proved upper bound

$$\left|\mathbb{E}\left(\mathbb{Q}_n^N(F_n)\right)-\mathbb{Q}_n(F_n)\right| \leq \mathbb{E}\left(\left[1-\overline{\gamma}_n^N(1)\right]^2\right)^{1/2} \mathbb{E}\left(\left[\mathbb{Q}_n^N(F_n)-\mathbb{Q}_n(F_n)\right]^2\right)^{1/2}$$

Under the regularity condition $(M)_m$ stated in (3.6), we proved in a recent article [2], that for any $n \geq p \geq 0$, and any $N > (n+1)\rho\delta^m$ we have

$$\|\overline{Q}_{p,n}(1)\| \leq \delta^m \rho \quad \text{and} \quad N\ \mathbb{E}\left[\left(\overline{\gamma}_n^N(1)-1\right)^2\right] \leq 4\ (n+1)\ \rho\ \delta^m$$

From these estimates, we readily prove the following corollary.

Corollary 5.3 *Assume that condition $(M)_m$ is satisfied for some parameters (m,δ,ρ). In this situation, for any $n \geq p \geq 0$, any F_n such that $\|F_n\| \leq 1$, and any $N > (n+1)\rho\delta^m$ we have*

$$\mathbb{E}\left(\overline{W}_n^{\Gamma,N}(F_n)\right)=0 \quad \textit{and} \quad \mathbb{E}\left(\overline{W}_n^{\Gamma,N}(F_n)^2\right) \leq (\delta^m\rho)^2(n+1)\left(1+\frac{2}{N}\ \rho\delta^m(n+2)\right)$$

In addition, we have

$$N\ \mathbb{E}\left(\left[\mathbb{Q}_n^N(F_n)-\mathbb{Q}_n(F_n)\right]^2\right) \quad \leq \quad 2(n+1)\rho\delta^m\left(4+\rho\delta^m\left[1+\frac{2}{N}(n+2)\right]\right)$$

and the bias estimate

$$N\left|\mathbb{E}\left(\mathbb{Q}_n^N(F_n)\right)-\mathbb{Q}_n(F_n)\right| \leq 2\sqrt{2}\ (n+1)\rho\delta^m\left(4+\rho\delta^m\left[1+\frac{2}{N}(n+2)\right]\right)^{1/2}$$

6 Fluctuation properties

This section is mainly concerned with the proof of theorem 3.1. Unless otherwise is stated, in the further developments of this section, we assume that the regularity condition (H^+) presented in (3.3) is satisfied for some collection of functions (h_n^-, h_n^+). Our first step to establish theorem 3.1 is the fluctuation analysis of the N-particle measures $(\Gamma_n^N, \mathbb{Q}_n^N)$ given in proposition 6.2 whose proof relies on the following technical lemma.

Lemma 6.1

$$\begin{aligned}
&\mathcal{M}_n^N(x_n, d(x_0,\ldots,x_{n-1}))-\mathcal{M}_n(x_n, d(x_0,\ldots,x_{n-1}))\\
&=\sum_{0\leq p\leq n}\left[\mathcal{M}_{n,p,\eta_p^N}-\mathcal{M}_{n,p,\Phi_p\left(\eta_{p-1}^N\right)}\right](x_n, d(x_p,\ldots,x_{n-1}))\ \ \mathcal{M}_p^N(x_p, d(x_0,\ldots,x_{p-1}))
\end{aligned}$$

The proof of this lemma follows elementary but rather tedious calculations; thus it is postponed to the appendix. We now state proposition 6.2.

Proposition 6.2 *For any $N \geq 1$, $0 \leq p \leq n$, $x_p \in E_p$, $m \geq 1$, and $F_n \in \mathcal{B}(E_{[0,n]})$ such that $\|F_n\| \leq 1$, we have*

$$\sqrt{N}\ \mathbb{E}\left(\left|D^N_{p,n}(F_n) - D_{p,n}(F_n)(x_p)\right|^m\right)^{\frac{1}{m}} \leq a(m)\ b(n)\ \left(\frac{h^+_p}{h^-_p}(x_p)\right)^2 \tag{6.1}$$

for some finite constants $a(m) < \infty$, resp. $b(n) < \infty$, whose values only depend on the parameters m, resp. on the time horizon n.

Proof:

Using lemma 6.1, we find that

$$D^N_{p,n}(F_n) - D_{p,n}(F_n) = \sum_{0\leq q\leq p}\left[\mathcal{M}_{p,q,\eta^N_q} - \mathcal{M}_{p,q,\Phi_q\left(\eta^N_{q-1}\right)}\right]\left(T^N_{p,q,n}(F_n)\right)$$

with the random function $T^N_{p,q,n}(F_n)$ defined below

$$\begin{aligned} &T^N_{p,q,n}(F_n)(x_q,\ldots,x_p)\\ &:= \int \mathcal{Q}_{p,n}(x_p, d(x_{p+1},\ldots,x_n))\ \mathcal{M}^N_q(x_q, d(x_0,\ldots,x_{q-1}))\ F_n(x_0,\ldots,x_n)\end{aligned}$$

Using formula (4.7), we prove that for any $m \geq 1$ and any function F on $E_{[q,p]}$

$$\sqrt{N}\ \mathbb{E}\left(\left|\left[\mathcal{M}_{p,q,\eta^N_q} - \mathcal{M}_{p,q,\Phi_q\left(\eta^N_{q-1}\right)}\right](F)\,(x_p)\right|^m \ \mid \mathcal{F}^N_{q-1}\right)^{\frac{1}{m}} \leq a(m)\ b(n)\ \|F\|\ \left(\frac{h^+_p}{h^-_p}(x_p)\right)^2$$

for some finite constants $a(m) < \infty$ and $b(n) < \infty$ whose values only depend on the parameters m and n. Using these almost sure estimates, we easily prove (6.1). This ends the proof of the proposition. ■

Now, we come to the proof of theorem 3.1.
Proof of theorem 3.1:
Using theorem 5.2, we have the decomposition

$$W^{\Gamma,N}_n(F_n) = \sum_{p=0}^n \gamma^N_p(1)\ V^N_p\left(D_{p,n}(F_n)\right) + R^{\Gamma,N}_n(F_n)$$

with the second order remainder term

$$R^{\Gamma,N}_n(F_n) := \sum_{p=0}^n \gamma^N_p(1)\ V^N_p\left(F^N_{p,n}\right) \quad \text{and the function} \quad F^N_{p,n} := [D^N_{p,n} - D_{p,n}](F_n)$$

By Slutsky's lemma and by the continuous mapping theorem it clearly suffices to check that $R^{\Gamma,N}_n(F_n)$ converge to 0, in probability, as $N \to \infty$. To prove this claim, we notice that

$$\mathbb{E}\left(V^N_p\left(F^N_{p,n}\right)^2 \ \mid \mathcal{F}^N_{p-1}\right) \leq \Phi_p\left(\eta^N_{p-1}\right)\left(\left(F^N_{p,n}\right)^2\right)$$

On the other hand, we have

$$\begin{aligned} &\Phi_p\left(\eta^N_{p-1}\right)\left(\left(F^N_{p,n}\right)^2\right)\\ &= \int \lambda_p(dx_p)\ \Psi_{G_{p-1}}\left(\eta^N_{p-1}\right)(H_p(\,.\,,x_p))\ \ F^N_{p,n}(x_p)^2\\ &\leq \eta_p\left(\left(F^N_{p,n}\right)^2\right) + \int \lambda_p(dx_p)\ \left|\left[\Psi_{G_{p-1}}\left(\eta^N_{p-1}\right) - \Psi_{G_{p-1}}\left(\eta_{p-1}\right)\right](H_p(\,.\,,x_p))\right|\ F^N_{p,n}(x_p)^2\end{aligned}$$

This yields the rather crude estimate

$$\begin{aligned}
&\Phi_p\left(\eta_{p-1}^N\right)\left(\left(F_{p,n}^N\right)^2\right)\\
&=\int \lambda_p(dx_p)\ \Psi_{G_{p-1}}\left(\eta_{p-1}^N\right)(H_p(\,.\,,x_p))\ \ F_{p,n}^N(x_p)^2\\
&\leq \eta_p\left(\left(F_{p,n}^N\right)^2\right)+4\|Q_{p,n}(1)\|^2\int \lambda_p(dx_p)\ \ \left|\left[\Psi_{G_{p-1}}\left(\eta_{p-1}^N\right)-\Psi_{G_{p-1}}\left(\eta_{p-1}\right)\right](H_p(\,.\,,x_p))\right|
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
&\mathbb{E}\left(V_p^N\left(F_{p,n}^N\right)^2\right)\\
&\leq \int \eta_p(dx_p)\ \mathbb{E}\left[\left(F_{p,n}^N(x_p)\right)^2\right]\\
&\qquad +4\|Q_{p,n}(1)\|^2\int \lambda_p(dx_p)\ \mathbb{E}\left(\left|\left[\Psi_{G_{p-1}}\left(\eta_{p-1}^N\right)-\Psi_{G_{p-1}}\left(\eta_{p-1}\right)\right](H_p(\,.\,,x_p))\right|\right)
\end{aligned}$$

We can establish that

$$\sqrt{N}\ \mathbb{E}\left(\left|\left[\Psi_{G_{p-1}}\left(\eta_{p-1}^N\right)-\Psi_{G_{p-1}}\left(\eta_{p-1}\right)\right](H_p(\,.\,,x_p))\right|\right)\leq b(n)\ h_p^+(x_p)$$

See for instance section 7.4.3, theorem 7.4.4 in [4]. Using proposition 6.2,

$$\sqrt{N}\ \mathbb{E}\left(V_p^N\left(F_{p,n}^N\right)^2\right)\leq c(n)\left(\frac{1}{\sqrt{N}}\ \eta_p\left(\left(\frac{h_p^+}{h_p^-}\right)^4\right)+\lambda_p(h_p^+)\right)$$

for some finite constant $c(n)<\infty$. The end of the proof of the first assertion now follows standard computations. To prove the second assertion, we use the following decomposition

$$\sqrt{N}\ [\mathbb{Q}_n^N-\mathbb{Q}_n](F_n)=\frac{1}{\overline{\gamma}_n^N(1)}\ \overline{W}_n^{\Gamma,N}(F_n-\mathbb{Q}_n(F_n))$$

with the random fields $\overline{W}_n^{\Gamma,N}$ defined in (5.3). We complete the proof using the fact that $\overline{\gamma}_n^N(1)$ tends to 1, almost surely, as $N\to\infty$. This ends the proof of the theorem. ■

We end this section with some comments on the asymptotic variance associated to the Gaussian fields $W_n^{\mathbb{Q}}$. Using (4.1), we prove that

$$\mathbb{Q}_n=\Psi_{\overline{D}_{p,n}(1)}(\eta_p)P_{p,n}$$

with the pair of integral operators $(\overline{D}_{p,n},P_{p,n})$ from $\mathcal{B}(E_{[0,n]})$ into $\mathcal{B}(E_p)$

$$\overline{D}_{p,n}(F_n):=\frac{D_{p,n}(F_n)}{\eta_pQ_{p,n}(1)}=\frac{D_{p,n}(1)}{\eta_pQ_{p,n}(1)}\ P_{p,n}(F_n)\quad\text{and}\quad P_{p,n}(F_n):=\frac{D_{p,n}(F_n)}{D_{p,n}(1)}$$

from which we deduce the following formula

$$\begin{aligned}
&\overline{D}_{p,n}(F_n-\mathbb{Q}_n(F_n))(x_p)\\
&=\overline{D}_{p,n}(1)(x_p)\ \int[P_{p,n}(F_n)(x_p)-P_{p,n}(F_n)(y_p)]\ \ \Psi_{\overline{D}_{p,n}(1)}(\eta_p)(dy_p)
\end{aligned}\tag{6.2}$$

Under condition $(M)_m$, for any function F_n with oscillations $\mathrm{osc}(F_n)\leq 1$, we prove the following estimate

$$\|\overline{D}_{p,n}(1)\|\leq\delta^m\rho\Longrightarrow\mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right)\leq(\delta^m\rho)^2\sum_{p=0}^n\beta(P_{p,n})^2$$

7 Non asymptotic estimates

7.1 Non asymptotic $\mathbb{L}_r$-mean error estimates

This section is mainly concerned with the proof of theorem 3.2. We follow the same semigroup techniques as the ones we used in section 7.4.3 in [4] to derive uniform estimates w.r.t. the time parameter for the N-particle measures η_n^N. We use the decomposition

$$[\mathbb{Q}_n^N - \mathbb{Q}_n](F_n) = \sum_{0\leq p\leq n}\left(\frac{\eta_p^N D_{p,n}^N(F_n)}{\eta_p^N D_{p,n}^N(1)} - \frac{\eta_{p-1}^N D_{p-1,n}^N(F_n)}{\eta_{p-1}^N D_{p-1,n}^N(1)}\right)$$

with the conventions $\eta_{-1}^N D_{-1,n}^N = \eta_0 \mathcal{Q}_{0,n}$, for $p=0$. Next, we observe that

$$\begin{aligned}
&\eta_{p-1}^N D_{p-1,n}^N(F_n)\\
&= \int \eta_{p-1}^N(dx_{p-1})\mathcal{M}_{p-1}^N(x_{p-1}, d(x_0,\ldots,x_{p-2}))\mathcal{Q}_{p-1,n}(x_{p-1}, d(x_p,\ldots,x_n))F_n(x_0,\ldots,x_n)\\
&= \int \eta_{p-1}^N(dx_{p-1})Q_p(x_{p-1},dx_p)\\
&\qquad\qquad \times\mathcal{M}_{p-1}^N(x_{p-1}, d(x_0,\ldots,x_{p-2}))\mathcal{Q}_{p,n}(x_p, d(x_{p+1},\ldots,x_n))F_n(x_0,\ldots,x_n)
\end{aligned}$$

On the other hand, we have

$$\eta_{p-1}^N(dx_{p-1})Q_p(x_{p-1},dx_p) = \eta_{p-1}^N Q_p(dx_p)\ M_{p,\eta_{p-1}^N}(x_p, dx_{p-1})$$

from which we conclude that

$$\eta_{p-1}^N D_{p-1,n}^N(F_n) = (\eta_{p-1}^N Q_p)(D_{p,n}^N(F_n))$$

This yields the decomposition

$$[\mathbb{Q}_n^N - \mathbb{Q}_n](F_n) = \sum_{0\leq p\leq n}\left(\frac{\eta_p^N D_{p,n}^N(F_n)}{\eta_p^N D_{p,n}^N(1)} - \frac{\Phi_p(\eta_{p-1}^N)(D_{p,n}^N(F_n))}{\Phi_p(\eta_{p-1}^N)(D_{p,n}^N(1))}\right) \tag{7.1}$$

with the convention $\Phi_0(\eta_{-1}^N)=\eta_0$, for $p=0$. If we set

$$\widetilde{F}_{p,n}^N = F_n - \frac{\Phi_p(\eta_{p-1}^N)(D_{p,n}^N(F_n))}{\Phi_p(\eta_{p-1}^N)(D_{p,n}^N(1))}$$

then every term in the r.h.s. of (7.1) takes the following form

$$\frac{\eta_p^N D_{p,n}^N(\widetilde{F}_{p,n}^N)}{\eta_p^N D_{p,n}^N(1)} = \frac{\eta_p Q_{p,n}(1)}{\eta_p^N Q_{p,n}(1)} \times \left[\eta_p^N \overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N) - \Phi_p(\eta_{p-1}^N)\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N)\right]$$

with the integral operators $\overline{D}_{p,n}^N$ defined in (5.4). Next, we observe that $D_{p,n}^N(1) = Q_{p,n}(1)$, and $\overline{D}_{p,n}^N(1) = \overline{D}_{p,n}(1)$. Thus, in terms of the local sampling random fields V_p^N, we have proved that

$$\frac{\eta_p^N D_{p,n}^N(\widetilde{F}_{p,n}^N)}{\eta_p^N D_{p,n}^N(1)} = \frac{1}{\sqrt{N}} \times \frac{1}{\eta_p^N \overline{D}_{p,n}(1)} \times V_p^N \overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N) \tag{7.2}$$

and

$$\overline{D}_{p,n}^N(F_n) = \overline{D}_{p,n}(1) \times P_{p,n}^N(F_n) \quad \text{with} \quad P_{p,n}^N(F_n) := \frac{D_{p,n}^N(F_n)}{D_{p,n}^N(1)} \tag{7.3}$$

From these observations, we prove that

$$\frac{\Phi_p(\eta_{p-1}^N)(D_{p,n}^N(F_n))}{\Phi_p(\eta_{p-1}^N)(D_{p,n}^N(1))} = \frac{\Phi_p(\eta_{p-1}^N)(Q_{p,n}(1)\ P_{p,n}^N(F_n))}{\Phi_p(\eta_{p-1}^N)(Q_{p,n}(1))} = \Psi_{Q_{p,n}(1)}\left(\Phi_p(\eta_{p-1}^N)\right) P_{p,n}^N(F_n)$$

Arguing as in (6.2) we obtain the following decomposition

$$\begin{aligned}&\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N)(x_p)\\ &= \overline{D}_{p,n}(1)(x_p) \times \int \left[P_{p,n}^N(F_n)(x_p) - P_{p,n}^N(F_n)(y_p)\right]\ \Psi_{Q_{p,n}(1)}(\Phi_p(\eta_{p-1}^N))(dy_p)\end{aligned}$$

and therefore

$$\begin{aligned}\left\|\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N)\right\| &\leq b_{p,n}\ \mathrm{osc}(P_{p,n}^N(F_n))\\ &\leq b_{p,n}\ \ \beta(P_{p,n}^N)\ \ \mathrm{osc}(F_n) \quad \text{with} \quad b_{p,n} \leq \sup_{x_p,y_p} \frac{Q_{p,n}(1)(x_p)}{Q_{p,n}(1)(y_p)}\end{aligned}$$

We end the proof of (3.4) using the fact that for any $r \geq 1$, $p \geq 0$, $f \in \mathcal{B}(E_p)$ s.t. $\mathrm{osc}(f) \leq 1$ we have the almost sure Kintchine type inequality

$$\mathbb{E}\left(\left|V_p^N(f)\right|^r\ \mid \mathcal{F}_{p-1}^N\right)^{\frac{1}{r}} \leq a_r$$

for some finite (non random) constants $a_r < \infty$ whose values only depend on r. Indeed, using the fact that each term in the sum of (7.1) takes the form (7.2) we prove that

$$\sqrt{N}\ \mathbb{E}\left(\left|[\mathbb{Q}_n^N - \mathbb{Q}_n](F_n)\right|^r\right)^{\frac{1}{r}} \leq a_r \sum_{0\leq p\leq n} b_{p,n}^2\ \mathbb{E}\left(\mathrm{osc}(P_{p,n}^N(F_n))\right) \tag{7.4}$$

This ends the proof of the first assertion (3.4) of theorem 3.2. For linear functionals of the form (2.5), it is easily checked that

$$D_{p,n}^N(F_n) = Q_{p,n}(1) \sum_{0\leq q\leq p} \left[M_{p,\eta_{p-1}^N} \dots M_{q+1,\eta_q^N}\right](f_q) + \sum_{p<q\leq n} Q_{p,q}(f_q\ Q_{q,n}(1))$$

with the convention $M_{p,\eta_{p-1}^N} \dots M_{p+1,\eta_p^N} = Id$, the identity operator, for $q = p$. Recalling that $D_{p,n}^N(1) = Q_{p,n}(1)$, we conclude that

$$P_{p,n}^N(F_n) = f_p + \sum_{0\leq q<p} \left[M_{p,\eta_{p-1}^N} \dots M_{q+1,\eta_q^N}\right](f_q) + \sum_{p<q\leq n} \frac{Q_{p,q}(Q_{q,n}(1)\ f_q)}{Q_{p,q}(Q_{q,n}(1))}$$

and therefore

$$P_{p,n}^N(F_n) = \sum_{0\leq q<p} \left[M_{p,\eta_{p-1}^N} \dots M_{q+1,\eta_q^N}\right](f_q) + \sum_{p\leq q\leq n} \frac{Q_{p,q}(Q_{q,n}(1)\ f_q)}{Q_{p,q}(Q_{q,n}(1))}$$

$$\frac{Q_{p,q}(Q_{q,n}(1)\ f_q)}{Q_{p,q}(Q_{q,n}(1))} = \frac{S_{p,q}(\overline{Q}_{q,n}(1)\ f_q)}{S_{p,q}(\overline{Q}_{q,n}(1))} \quad \text{with} \quad S_{p,q}(g) = \frac{Q_{p,q}(g)}{Q_{p,q}(1)}$$

with the potential functions $\overline{Q}_{q,n}(1)$ defined in (5.5). After some elementary computations, we obtain the following estimates

$$\begin{aligned}&\mathrm{osc}(P_{p,n}^N(F_n))\\ &\leq \textstyle\sum_{0\leq q<p} \beta\left(M_{p,\eta_{p-1}^N} \dots M_{q+1,\eta_q^N}\right) \mathrm{osc}(f_q) + \sum_{p\leq q\leq n}\ b_{q,n}^2\ \beta(S_{p,q})\ \mathrm{osc}(f_q)\end{aligned}$$

This ends the proof of the second assertion (3.5) of theorem 3.2.

7.2 Non asymptotic variance estimates

This section is mainly concerned with the proof of the non asymptotic estimate stated in (3.8). Recalling that $\eta_p\overline{D}_{p,n}(1) = 1$, we readily check that

$$\frac{1}{\eta_p^N\overline{D}_{p,n}(1)} = 1 - \frac{1}{\eta_p^N\overline{D}_{p,n}(1)}\left(\eta_p^N\overline{D}_{p,n}(1) - \eta_p\overline{D}_{p,n}(1)\right) = 1 - \frac{1}{\sqrt{N}}\frac{1}{\eta_p^N\ \overline{D}_{p,n}(1)}\ \mathcal{W}_p^{N,\eta}(\overline{D}_{p,n}(1))$$

with the empirical random field $\mathcal{W}_p^{N,\eta}$ defined below

$$\mathcal{W}_p^{N,\eta} = \sqrt{N}\ [\eta_p^N - \eta_p]$$

We recall that

$$\forall f_p \in \mathrm{Osc}_1(E_p) \qquad \mathbb{E}\left(\left|\mathcal{W}_p^{N,\eta}(f_p)\right|^r\right)^{\frac{1}{r}} \leq a_r \sum_{0\leq q\leq p} b_{q,p}^2\ \beta(S_{q,p})$$

with the Markov transitions $S_{q,p}$ defined in theorem 3.2. See for instance the non asymptotic $\mathbb{L}_r$-estimates presented on page 36 in [3]. Using the above decomposition, the local terms (7.2) can be rewritten as follows

$$\frac{1}{\eta_p^N\overline{D}_{p,n}(1)}\ V_p^N(\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N)) = V_p^N(\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N)) - \frac{1}{\sqrt{N}}\frac{1}{\eta_p^N\overline{D}_{p,n}(1)}\ \mathcal{W}_p^{N,\eta}(\overline{D}_{p,n}(1)) \times V_p^N(\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N))$$

By (7.1), these local decompositions yield the following formula

$$W_n^{\mathbb{Q},N}(F_n) = I_n^N(F_n) + \frac{1}{\sqrt{N}}\ R_n^N(F_n)$$

with the first order term

$$I_n^N(F_n) := \sum_{0\leq p\leq n} V_p^N(\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N))$$

and the second order remainder term

$$R_n^N(F_n) := -\sum_{0\leq p\leq n}\frac{1}{\eta_p^N\overline{D}_{p,n}(1)}\ \mathcal{W}_p^{N,\eta}(\overline{D}_{p,n}(1)) \times V_p^N(\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N))$$

By construction, we have

$$\mathbb{E}\left(I_n^N(F_n)^2\right) = \sum_{0\leq p\leq n}\mathbb{E}\left(V_p^N(\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N))^2\right) \leq \sigma^2(n)$$

with some finite constant

$$\sigma^2(n) := \sum_{0\leq p\leq n} b_{p,n}^2\ \mathbb{E}\left(\mathrm{osc}(P_{p,n}^N(F_n)^2)\right)$$

Furthermore, using Cauchy-Schwartz inequality

$$\mathbb{E}\left(R_n^N(F_n)^2\right)^{\frac{1}{2}} \quad \leq \quad \sum_{0\leq p\leq n} b_{p,n}\ \mathbb{E}\left(\mathcal{W}_p^{N,\eta}(\overline{D}_{p,n}(1))^4\right)^{\frac{1}{4}}\ \mathbb{E}\left(V_p^N(\overline{D}_{p,n}^N(\widetilde{F}_{p,n}^N))^4\right)^{\frac{1}{4}} \leq r^N(n)$$

with some constant

$$r^N(n) := a_4^2 \sum_{0\leq p\leq n} b_{p,n}^2 \left(\sum_{0\leq q\leq p} b_{q,p}^2\ \beta(S_{q,p})\right)\ \mathbb{E}\left(\mathrm{osc}(P_{p,n}^N(F_n))\right)$$

We conclude that

$$\left|\mathbb{E}\left(W_n^{\mathbb{Q},N}(F_n)\right)\right| \leq \frac{1}{\sqrt{N}}\; r^N(n) \quad \text{and} \quad \mathbb{E}\left(W_n^{\mathbb{Q},N}(F_n)^2\right)^{\frac{1}{2}} \leq \sigma(n) + \frac{1}{\sqrt{N}}\; r^N(n)$$

Arguing as in section 3, under the regularity condition $(M)_m$ stated in (3.6), for linear functionals of the form (2.5), with $f_n \in \mathrm{Osc}_1(E_n)$, we readily check that

$$\sigma^2(n) \leq c\;(n+1) \quad \text{and} \quad r^N(n) \leq c^{1/2}\;(n+1)$$

for some finite constant $c < \infty$, whose values do not depend on the pair (n, N). In this case, we conclude that

$$\left|\mathbb{E}\left(W_n^{\mathbb{Q},N}(F_n)\right)\right| \leq \frac{1}{\sqrt{N}}\; c^{1/2}\;(n+1) \quad \text{and} \quad \mathbb{E}\left(W_n^{\mathbb{Q},N}(F_n)^2\right) \leq c\;(n+1)\left(1+\sqrt{\frac{n+1}{N}}\right)^2$$

This ends the proof of (3.8).

8 Comparisons with genealogical tree particle models

In this section, we provide with a brief comparison between these particle models and the genealogical tree particle interpretations of the measures $\mathbb{Q}_n$ discussed in (2.4).

8.1 Limiting variance interpretation models

Our first objective is to present a new interpretation of the pair of potential-transitions $(G_{p,n}, P_{p,n})$ defined in (3.2). We fix the time horizon n and we denote by $\mathbb{E}_{\mathbb{Q}_n}$ the expectation operator of a canonical random path $(X_0, \ldots, X_n)$ under the measure $\mathbb{Q}_n$. For any function $F \in \mathcal{B}(E_{[p,n]})$, $p \leq n$, using (2.7) we check that

$$\mathbb{E}_{\mathbb{Q}_n}\left(F(X_p, \ldots, X_n)\right) = \int \eta_n(dx_n) \prod_{p<q\leq n} M_{q,\eta_{q-1}}(x_q, dx_{q-1})\; F(x_p, \ldots, x_n)$$

This implies that for any $F \in \mathcal{B}(E_{[0,p]})$, we have the $\mathbb{Q}_n$-almost sure formula

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_n}\left(F(X_0, \ldots, X_p) \mid (X_p, \ldots, X_n)\;\right) &= \int \mathcal{M}_p(X_p, d(x_0, \ldots, x_{p-1}))F((x_0, \ldots, x_{p-1}), X_p) \\
&= \mathbb{E}_{\mathbb{Q}_n}\left(F(X_0, \ldots, X_p) \mid X_p\;\right)
\end{aligned}$$

Using elementary calculations, it is also easily checked that for any function $F \in \mathcal{B}(E_{[0,n]})$ we have the $\mathbb{Q}_n$-almost sure formula

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_n}\left(F(X_0, \ldots, X_n) \mid (X_0, \ldots, X_p)\;\right) \\
&= \tfrac{1}{\mathcal{Q}_{p,n}(1)(X_p)} \int \mathcal{Q}_{p,n}(X_p, d(x_{p+1}, \ldots, x_n))F((X_0, \ldots, X_p), (x_{p+1}, \ldots, x_n))
\end{aligned}$$

and therefore, for any function $F_n \in \mathcal{B}(E_{[0,n]})$, we prove that

$$\mathbb{E}_{\mathbb{Q}_n}\left(F_n(X_0, \ldots, X_n) \mid X_p\;\right) = P_{p,n}(F_n)(X_p)$$

In much the same way, if we denote by $\mathbb{Q}_n^{(p)}$ the time marginal of the measure $\mathbb{Q}_n$ with respect to the p-th coordinate, we have

$$\mathbb{Q}_n^{(p)} \ll \eta_p \quad \text{with} \quad \frac{d\mathbb{Q}_n^{(p)}}{d\eta_p} = G_{p,n}$$

For centered functions F_n s.t. $\mathbb{Q}_n(F_n) = 0$, by the functional central limit theorem 3.1, the limiting variance of the measures $\mathbb{Q}_n^N$ associated with the genetic model (2.2) with acceptance parameters $\epsilon_n = 0$ has the following interpretation:

$$\begin{aligned}\mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right) &= \sum_{p=0}^{n} \eta_p \left[G_{p,n}^2\ P_{p,n}(F_n)^2\right] \\ &= \sum_{p=0}^{n} \mathbb{E}_{\mathbb{Q}_n}\left(\frac{d\mathbb{Q}_n^{(p)}}{d\eta_p}(X_p)\ \mathbb{E}_{\mathbb{Q}_n}\left(F_n(X_0,\ldots,X_n)\,|X_p\right)^2\right)\end{aligned}$$

We end this section with some estimates of these limiting variances. Arguing as in (6.2), for any $F_n \in \mathcal{B}(E_{[0,n]})$, we readily prove the estimate

$$\mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right) \leq \sum_{0\leq p\leq n} b_{p,n}^2\ \mathrm{osc}(P_{p,n}(F_n))^2$$

For linear functionals of the form (2.5), with functions $f_n \in \mathrm{Osc}_1(E_n)$, using the same lines of arguments as those we used at the end of section 7, it is easily checked that

$$\mathrm{osc}(P_{p,n}(F_n)) \leq \sum_{0\leq q<p} \beta\left(M_{p,\eta_{p-1}}\ldots M_{q+1,\eta_q}\right) + \sum_{p\leq q\leq n} b_{q,n}^2\ \beta(S_{p,q})$$

Under the regularity condition $(M)_m$ stated in (3.6), the r.h.s. term in the above display is uniformly bounded with respect to the time parameters $0 \leq p \leq n$, from which we conclude that

$$\mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right) \leq c\ (n+1) \tag{8.1}$$

for some finite constant $c < \infty$, whose values do not depend on the time parameter.

8.2 Variance comparisons

We recall that the genealogical tree evolution models associated with the genetic type particle systems discussed in this article can be seen as the mean field particle interpretation of the Feynman-Kac mesures η_n defined as in (2.11), by replacing the pair (X_n, G_n) by the historical process $\mathcal{X}_n$ and the potential function $\mathcal{G}_n$ defined below:

$$\mathcal{X}_n := (X_0,\ldots,X_n) \quad \text{and} \quad \mathcal{G}_n(\mathcal{X}_n) := G_n(X_n)$$

We also have a non linear transport equation defined as in (2.1) by replacing $K_{n,\eta_{n-1}}$ by some Markov transition $\mathcal{K}_{n,\eta_{n-1}}$ from $E_{[0,n-1]}$ into $E_{[0,n]}$. In this notation, the genealogical tree model coincides with the mean field particle model defined as in (2.2) by replacing K_{n,η_{n-1}^N} by $\mathcal{K}_{n,\eta_{n-1}^N}$, where η_{n-1}^N stands for the occupation measure of the genealogical tree model at time $(n-1)$. The local sampling errors are described by a sequence of random field model $\mathcal{V}_n^N, \mathcal{V}_n$ on $\mathcal{B}(E_{[0,n]})$ defined as in (2.12) and (2.13), by replacing $K_{n,\eta}$ by $\mathcal{K}_{n,\eta}$. More details on the path space technique can be found in chapter 3 of the book [4].

The fluctuations of the genealogical tree occupation measures

$$\eta_n^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_{0,n}^i,\xi_{1,n}^i,\ldots,\xi_{n,n}^i)} \quad \text{and} \quad \gamma_n^N := \left(\prod_{0\leq p<n} \eta_p^N(\mathcal{G}_p)\right) \times \eta_n^N \tag{8.2}$$

around their limiting values η_n and γ_n are described by the pair of empirical random fields defined below

$$\mathcal{W}_n^{\gamma,N} := \sqrt{N}\left(\gamma_n^N - \gamma_n\right) \quad \text{and} \quad \mathcal{W}_n^{\eta,N} := \sqrt{N}\ [\eta_n^N - \eta_n]$$

To describe the limiting gaussian random fields $\mathcal{W}_n^\gamma$ and $\mathcal{W}_n^\eta$, we need another round of notation. Firstly, we observe that the pair of measures (γ_n, η_n) on the path space $E_{[0,n]}$ coincide with the measures $(\Gamma_n, \mathbb{Q}_n)$ we defined in the introduction of the present article. For these path space models, it is easily checked that

$$\gamma_n = \gamma_p \mathcal{D}_{p,n}$$

with the integral operator from $\mathcal{B}(E_{[0,n]})$ into $\mathcal{B}(E_{[0,p]})$ defined below

$$\mathcal{D}_{p,n}(F_n)(x_0, \ldots, x_p) := \int \mathcal{Q}_{p,n}(x_p, d(x_{p+1}, \ldots, x_n))\ F_n((x_0, \ldots, x_p), (x_{p+1}, \ldots, x_n))$$

In the above display $\mathcal{Q}_{p,n}$ is the integral operator defined in (3.1). Notice that

$$\mathcal{D}_{p,n}(1)(x_0, \ldots, x_p) = \mathcal{Q}_{p,n}(1)(x_p) = D_{p,n}(1)(x_p) = Q_{p,n}(1)(x_p)$$

As in (3.2), we consider be the pair of potential functions and Markov transitions $(\mathcal{G}_{p,n}, \mathcal{P}_{p,n})$ defined below

$$\mathcal{G}_{p,n}(x_0, \ldots, x_p) = G_{p,n}(x_p) \quad \text{and} \quad \mathcal{P}_{p,n}(F_n) = \mathcal{D}_{p,n}(F_n)/\mathcal{D}_{p,n}(1) \tag{8.3}$$

In terms of conditional expectations, we readily prove that

$$\mathbb{E}_{\mathbb{Q}_n}\left(F_n(X_0, \ldots, X_n) \mid (X_0, \ldots, X_p)\ \right) = \mathcal{P}_{p,n}(F_n)(X_0, \ldots, X_p) \tag{8.4}$$

for any function $F_n \in \mathcal{B}(E_{[0,n]})$.

It is more or less well known that the sequence of random fields $\mathcal{W}_n^{\gamma,N}$, resp. $\mathcal{W}_n^{\eta_n,N}$, converge in law, as $N \to \infty$, to the centered Gaussian fields $\mathcal{W}_n^\gamma$, resp. $\mathcal{W}_n^\eta$, defined as W_n^Γ, resp. $W_n^{\mathbb{Q}}$, by replacing the quantities $(V_p, G_{p,n}, D_{p,n}, P_{p,n}, \mathbb{Q}_n)$ by the path space models $(\mathcal{V}_p, \mathcal{G}_{p,n}, \mathcal{D}_{p,n}, \mathcal{P}_{p,n}, \eta_n)$; that is we have that

$$\begin{aligned}
\mathcal{W}_n^\gamma(F_n) &= \sum_{p=0}^n \gamma_p(1)\ \mathcal{V}_p\left(\mathcal{D}_{p,n}(F_n)\right) \\
\mathcal{W}_n^\eta(F_n) &= \sum_{p=0}^n \mathcal{V}_p\left(\mathcal{G}_{p,n}\ \mathcal{P}_{p,n}(F_n - \eta_n(F_n))\right)
\end{aligned}$$

A detailed discussion on these functional fluctuation theorems can be found in chapter 9 in [4]. Arguing as before, for centered functions F_n s.t. $\mathbb{Q}_n(F_n) = 0$, the limiting variance of the genealogical tree occupation measures η_n^N associated with the genetic model (2.2) with acceptance parameters $\epsilon_n = 0$ has the following interpretation:

$$\begin{aligned}
\mathbb{E}\left(\mathcal{W}_n^\eta(F_n)^2\right) &= \sum_{p=0}^n \mathbb{E}_{\mathbb{Q}_n}\left(\frac{d\mathbb{Q}_n^{(p)}}{d\eta_p}(X_0, \ldots, X_p)\ \mathbb{E}_{\mathbb{Q}_n}\left(F_n(X_0, \ldots, X_n)\,|(X_0, \ldots, X_p)\right)^2\right) \\
&= \mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right) + \sum_{p=0}^n\ \mathbb{E}_{\mathbb{Q}_n}\left(\frac{d\mathbb{Q}_n^{(p)}}{d\eta_p}(X_p)\ \mathrm{Var}_{\mathbb{Q}_n}\left(\mathcal{P}_{p,n}(F_n)\,|X_p\right)\right)
\end{aligned}$$

with the $\mathbb{Q}_n$-conditional variance of the conditional expectations (8.4) with respect to X_p given by

$$\begin{aligned}
&\mathrm{Var}_{\mathbb{Q}_n}\left(\mathcal{P}_{p,n}(F_n)\,|X_p\right) \\
&= \mathbb{E}_{\mathbb{Q}_n}\left(\left[\mathbb{E}_{\mathbb{Q}_n}\left(F_n(X_0, \ldots, X_n) \mid (X_0, \ldots, X_p)\right) - \mathbb{E}_{\mathbb{Q}_n}\left(F_n(X_0, \ldots, X_n)\,|X_p\right)\right]^2 |X_p\right)
\end{aligned}$$

For sufficiently regular models, and for linear functionals of the form (2.5), with local functions $f_n \in \mathrm{Osc}_1(E_n)$, we have proved in (8.1) that $\mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right) \leq c\ (n+1)$, for some finite constant

$c < \infty$, whose values do not depend on the time parameter. In this context, we also have that

$$\mathrm{Var}_{\mathbb{Q}_n}\left(\mathcal{P}_{p,n}(F_n)\,|X_p\right) = \mathbb{E}_{\mathbb{Q}_n}\left(\left[\sum_{0\leq q<p}\left(f_q(X_q) - \mathbb{E}_{\mathbb{Q}_n}\left(f_q(X_q)|X_p\right)\right)\right]^2 |X_p\right)$$

These local variance quantities may grow dramatically with the parameter p, so that the resulting variance $\mathbb{E}\left(\mathcal{W}_n^\eta(F_n)^2\right)$ will be much larger than $\mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right)$. For instance, in the toy model discussed in (2.6), we clearly have $\mathbb{Q}_n^{(p)} = \eta_p = \eta_0$ and

$$\mathbb{E}_{\mathbb{Q}_n}\left(F_n(X_0,\ldots,X_n)\,|(X_0,\ldots,X_p)\right) = \sum_{0\leq q\leq p} f(X_q)$$

from which we conclude that

$$\mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right) = (n+1) \quad \text{and} \quad \mathbb{E}\left(\mathcal{W}_n^\eta(F_n)^2\right) = \mathbb{E}\left(W_n^{\mathbb{Q}}(F_n)^2\right) + \frac{n(n+1)}{2}$$

8.3 Non asymptotic comparisons

For any function $F_n \in \mathcal{B}(E_{[0,n]})$, and any $r \geq 1$, it is known that

$$\sqrt{N}\ \mathbb{E}\left(\left|[\eta_n^N - \mathbb{Q}_n](F_n)\right|^r\right)^{\frac{1}{r}} \leq a_r \sum_{0\leq p\leq n} b_{p,n}^2\ \mathrm{osc}(\mathcal{P}_{p,n}(F_n))$$

with the occupation measure η_n^N of the genealogical tree model defined in (8.2). See for instance page 36 in [3]. Notice that the r.h.s. term is the same as in (7.4) by replacing $P_{p,n}^N$ by the integral operator $\mathcal{P}_{p,n}$ defined in (8.3). For linear functionals of the form (2.5), we have

$$\mathcal{P}_{p,n}(F_n)(x_0,\ldots,x_p) = \sum_{0\leq q<p} f_q(x_q) + \sum_{p\leq q\leq n} \frac{Q_{p,q}(Q_{q,n}(1)\ f_q)}{Q_{p,q}(Q_{q,n}(1))}(x_p)$$

Chosing local functions f_n s.t. $\mathrm{osc}(f_n) = 1$, we find that

$$\mathrm{osc}(\mathcal{P}_{p,n}(F_n)) \geq p \Longrightarrow \sum_{0\leq p\leq n} b_{p,n}^2\ \mathrm{osc}(\mathcal{P}_{p,n}(F_n)) \geq n(n+1)/2$$

In the reverse angle, under the regularity condition $(M)_m$, we prove in (7.4) and (3.7) that

$$\sum_{0\leq p\leq n} b_{p,n}^2\ \mathbb{E}\left(\mathrm{osc}(P_{p,n}^N(F_n)\right) \leq b\ (n+1)$$

for some finite constant $b < \infty$ whose values do not depend on the time parameter n.

Appendix

Proof of lemma 4.3

We prove the lemma by induction on the parameter $n(> p)$. For $n = p+1$, we have

$$\mathcal{M}_{p+1,p,\eta}(x_{p+1}, dx_p) = M_{p+1,\eta}(x_{p+1}, dx_p) \quad \text{and} \quad \mathcal{Q}_{p,p+1}(x_p, dx_{p+1}) = Q_{p+1}(x_p, dx_{p+1})$$

By definition of the transitions $M_{p+1,\eta}$, we have

$$\eta Q_{p+1}(dx_{p+1})\ \mathcal{M}_{p+1,p,\eta}(x_{p+1}, dx_p) = \eta(dx_p)\ \mathcal{Q}_{p,p+1}(x_p, dx_{p+1})$$

We suppose that the result has been proved at rank n. In this situation, we notice that

$$\begin{aligned}
&\eta(dx_p)\ \mathcal{Q}_{p,n+1}(x_p, d(x_{p+1}, \ldots, x_{n+1})) \\
&= \eta(dx_p)\ \mathcal{Q}_{p,n}(x_p, d(x_{p+1}, \ldots, x_n))Q_{n+1}(x_n, dx_{n+1}) \\
&= \eta Q_{p,n}(dx_n)\ Q_{n+1}(x_n, dx_{n+1})\ \mathcal{M}_{n,p,\eta}(x_n, d(x_p, \ldots, x_{n-1})) \\
&= \eta Q_{p,n}(1)\ \Phi_{p,n}(\eta)(dx_n)\ Q_{n+1}(x_n, dx_{n+1})\ \mathcal{M}_{n,p,\eta}(x_n, d(x_p, \ldots, x_{n-1}))
\end{aligned}$$

Using the fact that

$$\Phi_{p,n}(\eta)(dx_n)\ Q_{n+1}(x_n, dx_{n+1}) = \Phi_{p,n}(\eta)Q_{n+1}(dx_{n+1})\ M_{n+1,\Phi_{p,n}(\eta)}(x_{n+1}, dx_n)$$

and

$$\eta Q_{p,n}(1)\ \Phi_{p,n}(\eta)Q_{n+1}(dx_{n+1})\ = \eta Q_{p,n+1}(dx_{n+1})$$

we conclude that

$$\begin{aligned}
&\eta(dx_p)\ \mathcal{Q}_{p,n+1}(x_p, d(x_{p+1}, \ldots, x_{n+1})) \\
&= \eta Q_{p,n+1}(dx_{n+1})\ M_{n+1,\Phi_{p,n}(\eta)}(x_{n+1}, dx_n)\ \mathcal{M}_{n,p,\eta}(x_n, d(x_p, \ldots, x_{n-1})) \\
&= \eta Q_{p,n+1}(dx_{n+1})\ \mathcal{M}_{n+1,p,\eta}\ (x_{n+1}, d(x_p, \ldots, x_n))
\end{aligned}$$

This ends the proof of the lemma. ■

Proof of lemma 6.1:

Using the recursions (4.6), we prove that

$$\begin{aligned}
&\mathcal{M}_{n+1,p,\eta_p^N}\ (x_{n+1}, d(x_p, \ldots, x_n)) \\
&= \mathcal{M}_{n+1,p+1,\Phi_{p+1}(\eta_p^N)}\ (x_{n+1}, d(x_{p+1}, \ldots, x_n)) \times M_{p+1,\eta_p^N}(x_{p+1}, dx_p)
\end{aligned}$$

On the other hand, we also have

$$\mathcal{M}_{p+1}^N(x_{p+1}, d(x_0, \ldots, x_p)) = M_{p+1,\eta_p^N}(x_{p+1}, dx_p)\mathcal{M}_p^N(x_p, d(x_0, \ldots, x_{p-1}))$$

from which we conclude that

$$\begin{aligned}
&\mathcal{M}_{n+1,p+1,\Phi_{p+1}(\eta_p^N)}\ (x_{n+1}, d(x_{p+1}, \ldots, x_n))\, \mathcal{M}_{p+1}^N(x_{p+1}, d(x_0, \ldots, x_p)) \\
&= \mathcal{M}_{n+1,p,\eta_p^N}\ (x_{n+1}, d(x_p, \ldots, x_n))\, \mathcal{M}_p^N(x_p, d(x_0, \ldots, x_{p-1}))
\end{aligned}$$

The end of the proof is now a direct consequence of the following decomposition

$$\begin{aligned}
&\mathcal{M}_n^N(x_n, d(x_0, \ldots, x_{n-1})) - \mathcal{M}_n(x_n, d(x_0, \ldots, x_{n-1})) \\
&= \textstyle\sum_{1\leq p\leq n} \Big[\mathcal{M}_{n,p,\eta_p^N}\ (x_n, d(x_p, \ldots, x_{n-1}))\, \mathcal{M}_p^N(x_p, d(x_0, \ldots, x_{p-1})) \\
&\qquad\qquad -\mathcal{M}_{n,p-1,\eta_{p-1}^N}\ (x_n, d(x_{p-1}, \ldots, x_{n-1}))\, \mathcal{M}_{p-1}^N(x_{p-1}, d(x_0, \ldots, x_{p-2}))\Big] \\
&+\mathcal{M}_{n,0,\eta_0^N}\ (x_n, d(x_0, \ldots, x_{n-1})) - \mathcal{M}_{n,0,\eta_0}\ (x_n, d(x_0, \ldots, x_{n-1}))
\end{aligned}$$

with the conventions

$$\mathcal{M}_{n,0,\eta_0^N}\,(x_n, d(x_0,\ldots,x_{n-1}))\,\mathcal{M}_0^N(x_0, d(x_0,\ldots,x_1)) = \mathcal{M}_{n,0,\eta_0^N}\,(x_n, d(x_0,\ldots,x_{n-1}))$$

for $p=0$, and for $p=n$

$$\mathcal{M}_{n,n,\eta_n^N}\,(x_n, d(x_n,\ldots,x_{n-1}))\,\mathcal{M}_n^N(x_n, d(x_0,\ldots,x_{n-1}) = \mathcal{M}_n^N(x_n, d(x_0,\ldots,x_{n-1})$$

■

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