Lower and upper bounds on the number of empty cylinders and ellipsoids

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Abstract

Given a set $\mathcal{S}$ of n points in three dimensions, we study the maximum numbers of quadrics spanned by subsets of points in $\mathcal{S}$ in several ways. Among various results we prove that the number of empty circular cylinders is between $\Omega(n^3)$ and $O(n^4)$ while we have a tight bound $\Theta(n^4)$ for empty ellipsoids. We also take interest in pairs of empty homothetic ellipsoids, with application to the number of combinatorially distinct Delaunay triangulations obtained by orthogonal projections of $\mathcal{S}$ on a two-dimensional plane, which is $\Omega(n^4)$ and $O(n^5)$.

A side result is that the convex hull in d dimensions of a set of n points, where one half lies in a subspace of odd dimension $\delta > \frac{d}{2}$, and the second half is the (multi-dimensional) projection of the first half on another subspace of dimension δ, has complexity only $O\left(n^{\frac{d}{2}-1}\right)$.

1 Introduction

Counting incidences between geometric objects, and analyzing the number of objects spanned by a finite set of points, are topics of frequent interest in computational and combinatorial geometry. Prominent examples are counting the number of faces of many cells in an arrangement of curves [13], determining the number of straight lines touching a given set of spheres [14], bounding the number of empty spheres defined by a finite set of points (i.e., the number of vertices of the Voronoi diagram), and counting the number of plane graphs spanned by a planar set of points [5].

In this paper, we provide combinatorial bounds on the maximum number of 'combinatorially different' quadrics of various types in $\mathbb{R}^3$. Given a set $\mathcal{S}$ of n points in $\mathbb{R}^3$, we consider quadrics having all points of $\mathcal{S}$ on the same side. For ellipsoids or cylinders, inside and outside are clearly defined, and such a quadric will be called *enclosing* or *empty*. We also prove bounds concerning pairs of *homothetic* quadrics. Our original motivation for studying such pairs was the case of circular cylinders, in order to count the number of combinatorially different 2D Delaunay triangulations obtainable by orthogonal projection of $\mathcal{S}$. Such projections are commonly used in 3D surface reconstruction, e.g., for recovering terrains.

Naturally, our bounds depend on the number of degrees of freedom of the considered object class, which is the dimension of a manifold describing the class in any representation, or, in other words, the minimum number of independent real parameters needed to (locally) specify an object of the class. Before describing our results in more detail, let us briefly recall some facts about quadrics.

Types of quadrics

A general quadric is defined by its equation that contains 9 coefficients plus a constant factor, that is, a quadric has 9 degrees of freedom. Given a subset of k points from $\mathcal{S}$ and a family of quadrics having d degrees of freedom, there exists a family of quadrics with $d-k$ degrees of freedom that are passing through these points. Consequently, a quadric (and also an ellipsoid) can be defined by constraining it to pass through 9 given points. We will call two quadrics *combinatorially different* if they have different subsets of $\mathcal{S}$ on their boundaries. Combinatorially different quadrics spanned by $\mathcal{S}$ are the objects of interest in our complexity bounds.

Circular cylinders, which have only 5 degrees of freedom, can be locally specified by 5 points. (In fact, up to six cylinders through these points may exist [8]). It turns out that finding bounds for empty circular cylinders is more involved than for general quadrics. We also study the somehow intermediate case of general (elliptic) cylinders, which have 7 degrees of freedom. A complete classification of quadrics can be found in Dupont et al. [10]. Observe that two quadrics are homothetic if their equations share the same quadratic part. Thus, a second quadric homothetic to a given one has 4 degrees of freedom (3 for translation and one for the homothety factor). For the particular case of cylinders, translations along the cylinder axis are irrelevant, and thus the second cylinder has 3 degrees of freedom. For circular cylinders, being homothetic is the same as being parallel. From this discussion we deduce that the
- number of circular cylinders (not necessarily empty) defined by a set of n points is $\Theta(n^5)$,
- number of pairs of parallel circ. cylinders is $\Theta(n^8)$,
- number of general cylinders is $\Theta(n^7)$,
- number of pairs of homothetic cylinders is $\Theta(n^{10})$,

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- number of quadrics is $\Theta(n^9)$, and
- number of pairs of homothetic quadrics is $\Theta(n^{13})$.

Results and related work

This paper provides improved asymptotic upper and lower bounds on the number of combinatorially different quadrics defined by a set of n points in $\mathbb{R}^3$. Upper and lower bounds match in the case of general quadrics, $\Theta(n^4)$, and pairs of them, $\Theta(n^6)$. Surprisingly, both bounds are also valid for general cylinders. Thus, despite the fact that general cylinders have 2 degrees of freedom less than quadrics, the intrinsic complexity does not decrease. Gaps remain in our results for circular cylinders, $\Omega(n^3)$ and $O(n^4)$, and for pairs of parallel circular cylinders, $\Omega(n^4)$ and $O(n^5)$.

The last mentioned result carries over to the number of combinatorially different 2D Delaunay triangulations obtainable by projecting a given point set in $\mathbb{R}^3$. To our knowledge, no nontrivial bounds have been known on this problem, which can be viewed as a three-dimensional generalization of the problem of finding the different orderings of the projection of a set of points. The latter problem is well known as finding circular sequences [11] and has a complexity of $\Theta(n^2)$.

Lower bounds for maximizing the number of certain types of quadrics are derived from a generic construction and its variants. Upper bounds are obtained by applying tailored linearization schemes, introducing a space of quadrics in 9 dimensions, and a space of homothetic quadrics in 13 dimensions (that can be reduced to dimension 12 for cylinders). These spaces are similar to well-known notions such as the spaces of circles, spheres, or conics [11].

Our second linearization scheme becomes powerful in combination with a new theorem (of separate interest) that reduces the complexity of the convex hull of a set of n points in d dimensions for certain special configurations. Namely, if $\frac{n}{2}$ of the points lie in a subspace of dimension $\delta > \frac{d}{2}$, and the second half is the (multi-dimensional) projection of the first half on another subspace of dimension δ, then the complexity is $O\left(n^{\lfloor \frac{\delta}{2} \rfloor + \lfloor \frac{d-\delta}{2} \rfloor}\right)$. This reduces the general upper bound of $O\left(n^{\lfloor \frac{d}{2} \rfloor}\right)$ to $O\left(n^{\frac{d}{2}-1}\right)$ if d is even and δ is odd. The particular case $\delta = d-1$ was already proven [7].

Related work mostly concerns circular cylinders, a kind of quadric useful, for example, in metrology [2, 9] for quality measure of mechanical devices, or in 3D modeling in order to fit special surfaces in a given data set (e.g., trying to model pipes in a factory). Agarwal et al. [3] compute the set of empty unit radius cylinders in $O(n^{3+\varepsilon})$ time and show an $\Omega(n^2)$ complexity bound. Devillers [7] gives $O(n^4)$ and $\Omega(n^3)$ bounds for the number of coaxial circular cylinders defined by 6 points with all points between the two cylinders. There is special interest in smallest enclosing cylinders. Agarwal et al. [3] propose a construction algorithm with near cubic time complexity. Chan [6] proposes a faster approximation algorithm, and Schömer et al. [12] give an algorithm with bit-complexity analysis.

2 Upper bounds

2.1 Empty and enclosing ellipsoids or cylinders

Lemma 1 *The number of combinatorially different* empty *(or* enclosing*) ellipsoids or cylinders defined by a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ is $O(n^4)$.*

Proof. We use a classical linearization scheme similar to the one used in the next section, □

2.2 Pairs of empty homothetic ellipsoids

Lemma 2 *The number of combinatorially different pairs of* empty homothetic *ellipsoids or cylinders defined by a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ is $O(n^6)$.*

Proof. The linearization scheme here is similar to the one used in [1, 7]. To a given point $p = (x_p, y_p, z_p) \in \mathbb{R}^3$ we associate not one but two points in $\mathbb{R}^{13}$:

$$p^\star = (x_p^2, y_p^2, z_p^2, x_p y_p, x_p z_p, y_p z_p, x_p, y_p, z_p, 0, 0, 0, 0) \in \mathbb{R}^{13}$$

$$p^\dagger = (x_p^2, y_p^2, z_p^2, x_p y_p, x_p z_p, y_p z_p, 0, 0, 0, x_p, y_p, z_p, 1) \in \mathbb{R}^{13}.$$

Given two homothetic quadrics Q_ϕ and Q'_ϕ with equations

$$\phi_1 x^2 + \phi_2 y^2 + \phi_3 z^2 + \phi_4 xy + \phi_5 xz + \phi_6 yz + \phi_7 x + \phi_8 y + \phi_9 z = \phi_0$$

$$\phi_1 x^2 + \phi_2 y^2 + \phi_3 z^2 + \phi_4 xy + \phi_5 xz + \phi_6 yz + \phi'_7 x + \phi'_8 y + \phi'_9 z = \phi'_0,$$

we define a hyperplane

$$H_{\phi,\phi'} : \phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \phi_4\chi_4 + \phi_5\chi_5 + \phi_6\chi_6 + \phi_7\chi_7 + \phi_8\chi_8 + \phi_9\chi_9 + \phi'_7\chi_{10} + \phi'_8\chi_{11} + \phi'_9\chi_{12} + (\phi_0 - \phi'_0)\chi_{13} = \phi_0$$

in dimension 13 (where χ_i are the coordinates in $\mathbb{R}^{13}$). Now we have

$$\begin{aligned} p \in Q_\phi &\iff p^\star \in H_{\phi,\phi'} \\ \text{and} \quad p \in Q_{\phi'} &\iff p^\dagger \in H_{\phi,\phi'}. \end{aligned}$$

The following two statements are equivalent: (1) An ellipsoid Q_ϕ passes through points $p_1, p_2, \ldots, p_j \in \mathcal{S}$, and a ellipsoid $Q_{\phi'}$ homothetic to Q_ϕ passes through $p_{j+1}, p_{j+2}, \ldots, p_k \in \mathcal{S}$, and both Q_ϕ and $Q_{\phi'}$ are empty of other points of $\mathcal{S}$. (2) The corresponding hyperplane $H_{\phi,\phi'}$ passes through $p_1^\star, p_2^\star, \ldots, p_j^\star, p_{j+1}^\dagger, \ldots, p_k^\dagger \in \mathcal{S}^{\star\dagger} = \{p^\star, p^\dagger \mid p \in \mathcal{S}\}$ and all other points of $\mathcal{S}^{\star\dagger}$ are above $H_{\phi,\phi'}$.

Thus pairs of empty ellipsoids correspond to supporting hyperplanes of the convex hull of $\mathcal{S}^{\star\dagger}$, which has a complexity of $O\left(n^{\lfloor \frac{d}{2} \rfloor}\right)$, where $d = 13$ is the dimension of the underlying space, that is, a complexity of $O(n^6)$. □

2.3 Pairs of empty homothetic cylinders

Lemma 3 *The number of combinatorially different pairs of* empty homothetic *cylinders defined by a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ is $O(n^5)$.*

Proof. Assume, without loss of generality, that there is no horizontal cylinder. The second cylinder can be defined from the first one by a homothety and a horizontal translation. It yields to two homothetic cylinders Q_ϕ and Q'_ϕ whose equations verify $\phi_9 = \phi'_9$ which allows to reduce the dimension of the linearization space by one, going down to $\mathbb{R}^{12}$.

Now we are dealing with the convex hulls of points $p^\star$ belonging to a 9 dimensional space and points $p^\dagger$ which are the projection of the points $p^\star$ on another 9 dimensional space parallel to a 3 dimensional vector space. Thus we can apply our general projection theorem (Theorem 4 in the following section) and get an upper bound of $O\left(n^{\lfloor\frac{9}{2}\rfloor+\lfloor\frac{3}{2}\rfloor}\right) = O(n^5)$. □

3 General projection theorem

For special configurations of points in d dimensions, the complexity of the convex hull cannot reach the worst case of $\Theta\left(n^{\lfloor\frac{d}{2}\rfloor}\right)$. For example, the projection theorem in [7] proves that for a point set $\mathcal{V}$ of size $2n$, constructed by taking n points in a hyperplane and their parallel projections on another hyperplane, the complexity of its convex hull $CH(\mathcal{V})$ reduces to $O\left(n^{\lfloor\frac{d-1}{2}\rfloor}\right)$. This is of interest if d is even.

A multi-dimensional projection is defined by a vector space $\vec{V}$ of dimension $d-\delta$. Given a point p, its projection p' on Π' is the intersection of Π' and the affine subspace parallel to $\vec{V}$ passing through p; this projection is a unique point if $\vec{V}$ is supplementary to the direction of Π' (no $\vec{v} \in \vec{V}$ is parallel to Π').

We generalize the projection theorem to the case where the points in $\mathbb{R}^d$ live in two δ-dimensional affine subspaces, with $\frac{d}{2} < \delta \leq d-1$. If d is even and δ is odd, we save a linear factor like in the original version of the projection theorem. More precisely, we have:

Theorem 4 (General projection theorem) *For any constant dimension d, let Π, Π' be two δ-dimensional affine subspaces in $\mathbb{R}^d$, with $\delta > \frac{d}{2}$, and let $\vec{V}$ be a $(d-\delta)$-dimensional vector space of $\mathbb{R}^d$ supplementary to the directions of both Π and Π'. Let $\mathcal{U} \subset \Pi$ be a set of n points, such that $CH(\mathcal{U}) \cap \Pi' = \emptyset$, and let $\mathcal{U}' \subset \Pi'$ be the projection parallel to $\vec{V}$ of $\mathcal{U}$ on Π'. Let $\mathcal{V} = \mathcal{U} \cup \mathcal{U}'$. The convex hull $CH(\mathcal{V})$ of $\mathcal{V}$ has asymptotic complexity $O\left(n^{\lfloor\frac{\delta}{2}\rfloor+\lfloor\frac{d-\delta}{2}\rfloor}\right)$. If d is even and δ is odd, the complexity becomes $O\left(n^{\frac{d}{2}-1}\right)$.*

Proof. (Sketch) (see [4] for details). A full dimensional face of $CH(\mathcal{V})$ combines a face of $CH(\mathcal{U})$ with a face of $CH(\mathcal{U}')$. The proof then goes in two steps: first a facet of $CH(\mathcal{V})$ contains a $(\delta-1)$-dimensional face that projects on a facet of $CH(\mathcal{U})$ (and there is a constant number of such faces for each facet of $CH(\mathcal{U})$); second the number of facets of $CH(\mathcal{V})$ containing a given $(\delta-1)$-dimensional face has the complexity of a convex hull in dimension $d-\delta$. Combining these two points gives the claimed complexity. □

4 Lower bounds

We now turn our attention to lower bound constructions. We exhibit families of points lying on a few straight lines that allow for a large number of empty (or enclosing) quadrics of several types. By slightly perturbing these sets we can achieve general position without altering the number of objects counted.

4.1 Empty circular cylinders

Lemma 5 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different empty circular cylinders defined by $\mathcal{S}$ is $\Omega(n^3)$.*

Proof. Consider the points $p_0 = (1,0,0)$ and $q_0 = (0,1,0)$. Then the ellipses in the plane $z=0$ and tangent to the x axis at p_0 and to the y axis at q_0 form a continuous family that can be parameterized by $\lambda \in [1,\infty[$, the ratio of the long axis to the small axis (Figure 1a). Denote with E_λ one of these ellipses and consider the ellipses $E_{1+\frac{k}{n}}$ for $0 \leq k < N$ for $N = \frac{n}{3}$.

Through each of these ellipses, we can fit one circular cylinder such that the vector of the direction of the cylinder axis is positive in all three coordinates (there is another one with only x and y values negative).

Now, since the slopes of those cylinders are different, we get disjoint ellipses if we consider a cross section of the cylinders by a plane $z = h$ for large enough h. Thus we can add points r_k on the line $x - y = z - h = 0$ that separate these ellipses as indicated in Figure 1b. By slightly tilting the cylinders until for every k the cylinder corresponding to $E_{1+\frac{k}{n}}$ touches r_k, we create a linear size family of cylinders through p_0, q_0 and r_k which are tangent to the x and y axes.

Considering points $p_i = (1+\frac{i\varepsilon}{n},0,0)$ and $q_j = (0,1+\frac{j\varepsilon}{n},0)$, $0 \leq i,j < N$, for small enough ε performs a small perturbation on the ellipses and so does neither change the existence of the cylinders nor the fact that they are empty, since all cylinders are tangent to the x and y axes which contain the p_i and q_j points (Figure 1cd). This construction produces empty circular cylinders through p_i, q_j, r_k for all i,j,k. □

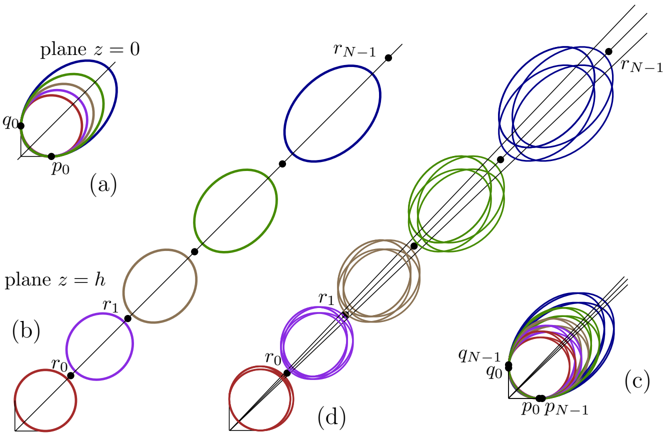

Figure 1: (ab) Cross sections of cylinders through p_0 and q_0 and tangent to x and y axes. (cd) Cross sections of cylinders through p_i and q_j and tangent to x and y axes.

4.2 Empty pairs of parallel circular cylinders

Lemma 6 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different pairs of empty parallel circular cylinders defined by $\mathcal{S}$ is $\Omega(n^4)$.*

Proof. Let now $N = \frac{n}{4}$. We add a family of points to the construction in the proof of Lemma 5, namely $s_l = (L + \frac{l\varepsilon}{n}, 0, 0)$, $0 \le l < N$, and a single point $u = (L-1, 1, 0)$, for L large enough. For each of the $\Omega(n^3)$ cylinders through p_i, q_j, r_k for all $0 \le i, j, k < N$ we can place a disjoint parallel circular cylinder through u and any of the points s_l and tangent to the x axis, avoiding all the points p_i, q_j, r_k. ☐

4.3 Empty general cylinders (and pairs of them)

Lemma 7 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different empty general cylinders defined by $\mathcal{S}$ is $\Omega(n^4)$. Moreover, the number of combinatorially different pairs of empty homothetic general cylinders defined by $\mathcal{S}$ is $\Omega(n^5)$.*

Proof. Omitted [4] ☐

4.4 Empty ellipsoids (and pairs of them)

Lemma 8 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different empty ellipsoids defined by $\mathcal{S}$ is $\Omega(n^4)$. Moreover, the number of combinatorially different pairs of empty homothetic ellipsoids defined by $\mathcal{S}$ is $\Omega(n^6)$.*

Proof. Omitted [4] ☐

4.5 Enclosing ellipsoids or cylinders

Lemma 9 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different enclosing ellipsoids defined by $\mathcal{S}$ is $\Omega(n^4)$, and a set such that the number of combinatorially different enclosing cylinders defined by $\mathcal{S}$ is $\Omega(n^3)$.*

Proof. Omitted [4] ☐

5 2D Delaunay triangulations from projection

Corollary 10 *Given a set of n points in $\mathbb{R}^3$, the number of combinatorially different 2D Delaunay triangulations obtainable by projecting these points on a plane is $O(n^5)$. Moreover, there exist sets of n points in $\mathbb{R}^3$ such that this number is $\Omega(n^4)$.*

Proof. (Sketch) If a direction of projection is represented by a point on $\mathbb{S}_2$, there exists a map on $\mathbb{S}_2$ whose cells contain directions, giving the same Delaunay triangulation. Vertices of this map correspond to pairs of parallel empty circular cylinders. ☐

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