Culling a Set of Points for Roundness or Cylindricity Evaluations

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CULLING A SET OF POINTS FOR ROUNDNESS OR CYLINDRICITY EVALUATIONS*

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Abstract

Roundness and cylindricity evaluations are among the most important problems in computational metrology, and are based on sets of surface measurements (input data points). A recent approach to such evaluations is based on a linear-programming approach yielding a rapidly converging solution. Such a solution is determined by a fixed-size subset of a large input set. With the intent to simplify the main computational task, it appears desirable to cull from the input any point that cannot provably define the solution. In this note we present an analysis and an efficient solution to the problem of culling the input set. For input data points arranged in cross-sections under mild conditions of uniformity, this algorithm runs in linear time.

keywords: metrology, minimum zone circle, minimum zone cylinder, roundness, cylindricity, input culling

1 Introduction

Roundness and cylindricity evaluations are among the most important problems in computational metrology, since cylindrical surfaces are ubiquitous in industrial machining. It should be noted that roundness evaluation can be viewed as an approach to cylindricity evaluation when the latter is carried out measuring radial form (in sections normal to the nominal axis).

The standard characterization of roundness is the notion of *minimum zone circle*, i.e., the circular crown contained between two concentric circles with minimum radial separation and containing all the data points. Analogously,

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cylindricity is characterized by the notion of *minimum zone cylinder*, the cylindrical crown between two co-axial cylinders with minimum radial separation.

In a recent paper [2] Devillers and Ramos proved that, under certain conditions, the computation of the zone circle (the minimum width annulus) is an LP-type problem [6].

With regard to cylindricity, in an attempt to avoid the computationally onerous direct determination of the zone cylinder, Devillers and Preparata recently presented a linear-programming approach yielding an approximate solution rapidly converging to the zone cylinder [1]. This method is based on radial-form measurement, i.e., the data points are organized in planar ensembles.

The solution of the minimum-width annulus (and of the zone cylinder) is defined by a set of points belonging to its boundary. If we can show that a point cannot be a defining point of the solution, then we can remove it from the input. Culling such irrelevant points can be used as a preprocessing step to speed up the main algorithm designed to construct the solution. To analyze such a problem and to develop the corresponding algorithm is the objective of this note.

2 Definition of α-hulls

The α-hull of a set of points $\mathcal{P}$, as defined in [3], is a generalization of the convex hull.

For $\alpha > 0$, the α-hull of $\mathcal{P}$ is the intersection of all disks of radius $\frac{1}{\alpha}$ that contain $\mathcal{P}$. For $\alpha = 0$, the α-hull is just the convex hull and for $\alpha < 0$, the α-hull of $\mathcal{P}$ is the complement of the union of all disks of radius $-\frac{1}{\alpha}$ that do not contain any point of $\mathcal{P}$. Figure 1 illustrates α-hulls of $\mathcal{P}$ for three values of α, one positive and two negatives.

3 Minimum width annulus

Let $p_0, p_1, \ldots p_n$ be a set of points in counterclockwise order around the origin such that the origin belongs to their convex hull. We denote with $\mathcal{A}$ an annulus centered at the origin and containing all the points, C_i its internal circle and C_e the external one, where R and $R + \omega$ are the corresponding radii.

We are looking for the minimum width annulus containing the points which will be denoted $\mathcal{A}_0$; R_0 and $R_0 + \omega_0$ are the internal and external radii of $\mathcal{A}_0$, C_{0i} and C_{0e} the corresponding circles, respectively.

Lemma 1 *If a point p_j is on C_{0i} then p_j belongs to the α-hull of the points for $\frac{1}{\alpha} = -(R - \omega)$.*

Proof. It is easy to observe that a circle smaller than C_i cannot enclose a sector of C_i of angle greater than π. Thus, due to the assumption that the origin is inside the convex hull, any circle enclosing all the points must have a

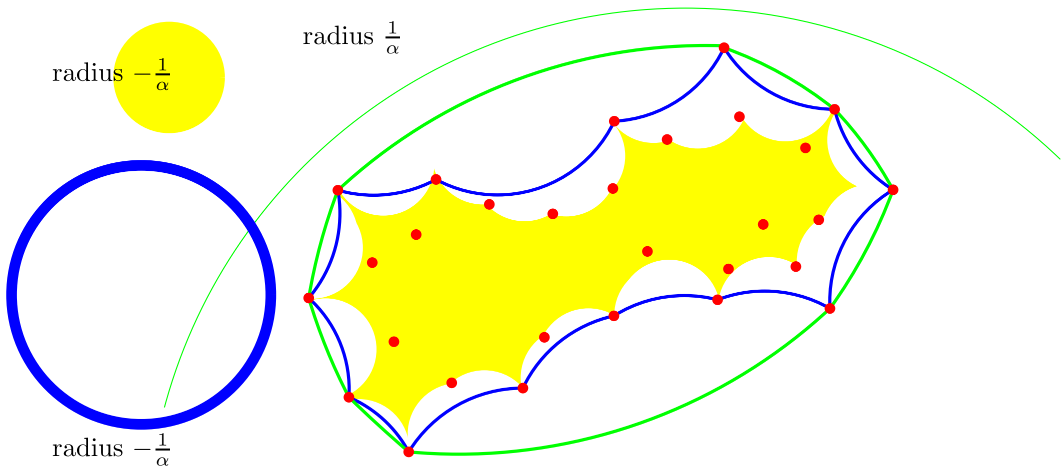

Figure 1: Two α-hulls of a set of points

radius not smaller than the radius R of C_i. This applies in particular to C_{0e}, of radius $R_0 + \omega_0$, i.e.:

$$R \leq R_0 + \omega_0$$

By definition of $\mathcal{A}_0$, its width is not larger than the width of $\mathcal{A}$, i.e.

$$\omega_0 \leq \omega$$

Combining these inequalities, we get

$$R_0 \geq R - \omega$$

From this it is clear that if a point p_j is on C_{0i}, then there exists an empty circle of radius $R - \omega$ passing through p_j as shown in Figure 2. In other words p_j is a vertex of the claimed α-hull of the points. $\diamondsuit$

Lemma 2 *If a point p_j is on C_{0e} then p_j belongs to the α-hull of the points for $\frac{1}{\alpha} = R + 2\omega$.*

Proof. A circle greater than C_e containing the origin must enclose a sector of $\mathcal{A}$ of angle greater than π, and thus contains at least one point. Therefore any empty circle containing the origin must have radius not larger that the radius $R + \omega$ of C_e. This condition applies to C_{0i}, of radius R_0, whence:

$$R + \omega \geq R_0$$

and, using the inequality $\omega_0 \leq \omega$,

$$R_0 + \omega_0 \leq R + 2\omega.$$

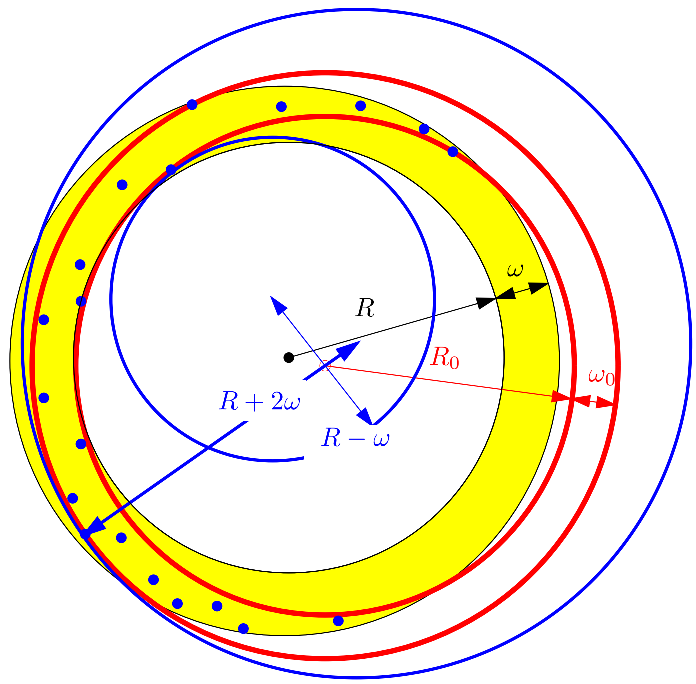

Figure 2: For the proof of Lemma 1 and 2

If p_j is on C_{0e}, then there is a circle of radius $R + 2\omega$ through p_j containing all the points (see Figure 2). The result follows. $\diamondsuit$

Theorem 3 *Given a set of points $\mathcal{P}$ such that*
— $\mathcal{P}$ is contained in an annulus $\mathcal{A}$
— the internal and external radii of $\mathcal{A}$ are R and $R + \omega$, respectively
— the center of $\mathcal{A}$ is inside the convex hull of $\mathcal{P}$,
the minimum width annulus $\mathcal{A}_0$ enclosing $\mathcal{P}$ verifies the conditions
— the internal circle of $\mathcal{A}_0$ passes through 2 points of the α-hull of $\mathcal{P}$ with $\frac{1}{\alpha} = -(R - \omega)$,
— the external circle of $\mathcal{A}_0$ passes through 2 points of the α-hull of $\mathcal{P}$ with $\frac{1}{\alpha} = R + 2\omega$.

Proof. The fact that there are two points on each circle is a classical result due to Rivlin [5] (who clearly specified the relative position of these points). The fact that the points are on the given α-hulls follows from Lemma 1 and 2. $\diamondsuit$

4 Minimum width zone cylinder

Culling the data points when searching for the minimum width zone cylinder can be done in an analogous way when the points belong to horizontal cross-sections.

Given a set of points $\mathcal{P}$ in three dimensions belonging to m horizontal planes $z = z_0 = -h/2$, $z = z_1 > z_0$, $z = z_2 > z_1$, $\ldots z = h/2 = z_{m-1} > z_{m-2}$, we make the following assumptions:

1. In each horizontal cross section the convex hull of the points contains the origin (z axis);
2. There exists a zone cylinder $\mathcal{C}$, with vertical axis (the z-axis) and radii R and $R + \omega$, containing all the points;
3. In the cross section at $z = z_0$ and $z = z_{m-1}$ each sector of angle ϕ (a parameter depending of the data sampling protocol) contains at least one data point.

We denote by $\mathcal{C}_0$ the minimum width zone-cylinder, with radii R_0 and $R_0+\omega_0$ and axis $\mathbf{v}_0$. The assumed existence of zone cylinder $\mathcal{C}$ allows us to establish that the axis of $\mathcal{C}_0$ cannot be too inclined with respect to that of $\mathcal{C}$.

Lemma 4 *With the above notation, the angle θ between $\mathbf{v}_0$ and the z-axis verifies the condition* $\sin\theta \leq \sigma_{\omega\phi} = \frac{8\omega + 4R\phi + 4\omega\phi + R\phi^2}{8h}$.

Proof. We consider the projections of the horizontal sections of $\mathcal{C}$ on a plane π orthogonal to $\mathbf{v}_0$. On π obviously zone cylinder $\mathcal{C}_0$ projects to a circular crown and any horizontal section of zone cylinder $\mathcal{C}$ projects to an elliptical crown,

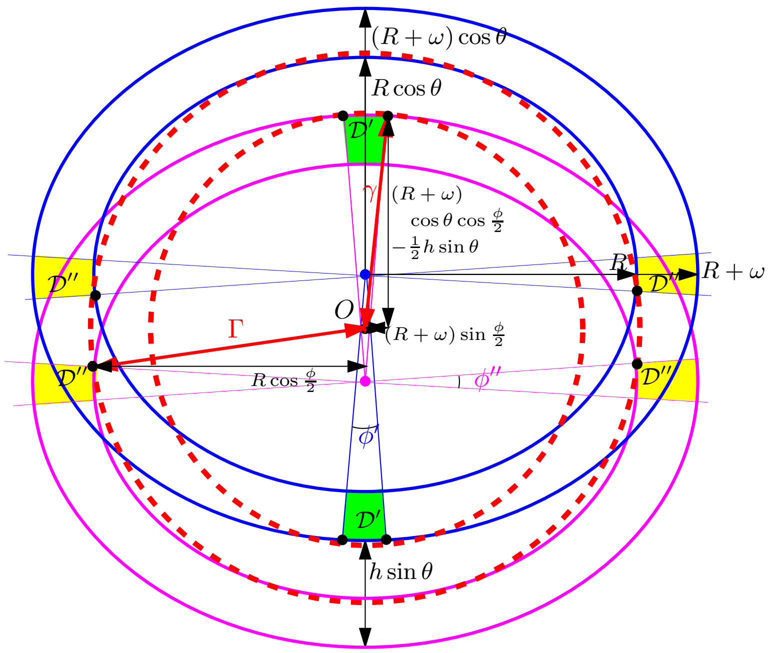

Figure 3: For the proof of Lemma 4

whose major semiaxis lengths are R and $R+\omega$ and whose minor semiaxis lengths are $R\cos\theta$ and $(R+\omega)\cos\theta$ (θ being the angle between the axes of $\mathcal{C}_0$ and $\mathcal{C}$). We consider the projection on π of the cross-sections at $z = z_0$ and $z = z_{m-1}$; these two elliptical crowns are offset by $h\sin\theta$, we denote by O the midpoint of the centers of these two crowns (see Figure 3). In this projection each sector of angle ϕ projects to a sector of the elliptical crown: specifically ϕ projects to ϕ' and ϕ'' when centered on the minor and major axis, respectively.

The intersection of the elliptical crowns with the sector of width ϕ' (shown as a lightly shaded domain $\mathcal{D}'$ in Figure 3) contains by hypothesis data points. Therefore, any circle that does not contain any point, and in particular the internal circle of $\mathcal{C}_0$, cannot strictly contain $\mathcal{D}'$. the largest circle which does not contain entirely one of the two green sectors has center O and radius γ (refer to Figure 3). Thus we have:

$$\begin{aligned} R_0 \le \gamma \quad &\le \quad \left[(R+\omega)\cos\theta\cos\frac{\phi}{2} - \frac{1}{2}h\sin\theta\right] + (R+\omega)\sin\frac{\phi}{2} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the triangle inequality)} \\ &\le \quad -\frac{1}{2}h\sin\theta + (R+\omega)(1+\sin\frac{\phi}{2}) \end{aligned}$$

Conversely, any circle, which contains all the points, and thus the external circle of $\mathcal{C}_0$, must intersect all the yellow sectors; the smallest such circle has center O and radius Γ (refer to Figure 3). Thus we have :

$$R_0 + \omega_0 \geq \Gamma \quad \geq \quad R\cos\frac{\phi}{2}$$

We now use the majorizations $1 \geq \cos\alpha \geq 1 - \alpha^2/2$ and $0 \leq \sin\alpha \leq \alpha$ for $0 \leq \alpha \leq 1$, and obtain:

$$\begin{aligned}
\omega &\geq \omega_0 = (R_0 + \omega_0) - R_0 \\
&\geq \Gamma - \gamma \\
&\geq R\cos\frac{\phi}{2} - (R+\omega)(1+\sin\frac{\phi}{2}) + \frac{1}{2}h\sin\theta \\
&\geq -R\frac{\phi^2}{8} - R\frac{\phi}{2} - \omega(1+\frac{\phi}{2}) + \frac{1}{2}h\sin\theta
\end{aligned}$$

whence:

$$\sin\theta \leq \frac{16\omega + 4R\phi + 4\omega\phi + R\phi^2}{4h}$$

$\diamondsuit$

Using the majorizations $\alpha \geq \sin\alpha \geq \sqrt{2/3}\alpha$ and $1 \geq \cos\alpha \geq 1 - \alpha^2/2 \geq 1-(3/4)\sin^2\alpha$ for $0 \leq \alpha \leq 1$, the preceding lemma has the following immediate corollary:

Corollary 5 *The ratio between the minor and the major axes of the horizontal section of the minimum zone cylinder is at least*

$$\lambda_{\omega\phi} \quad = \quad 1 - 3\left(\frac{16\omega + 4R\phi + 4\omega\phi + R\phi^2}{8h}\right)^2 \leq \cos\theta$$

Lemma 6 *If a point p_j is on $\mathcal{C}_{0i}$, then p_j belongs to the α-hull of the points in the plane $z = z_j$ for $\frac{1}{\alpha} = -(\lambda_{\omega\phi}R - \omega)(1 - \sigma_{\omega\phi})$.*

Proof.

The section of $\mathcal{C}_{0e}$ with the plane $z = z_j$ is an ellipse containing all the points on that section. Its minor axis has length $2(R_0 + \omega_0)$ and its major axis has length smaller than $2\frac{R_0+\omega_0}{\lambda_{\omega\phi}}$ by Corollary 5. It follows that the circle circumscribing the ellipse, of radius $\frac{R_0+\omega_0}{\lambda_{\omega\phi}}$, contains all the points in that section (see Figure 4). As in the proof of Lemma 1 we conclude that

$$R \quad \leq \quad \frac{R_0 + \omega_0}{\lambda_{\omega\phi}}$$

that is,

$$\begin{aligned}
R_0 &\geq \lambda_{\omega\phi}R - \omega_0 \\
&\geq \lambda_{\omega\phi}R - \omega
\end{aligned}$$

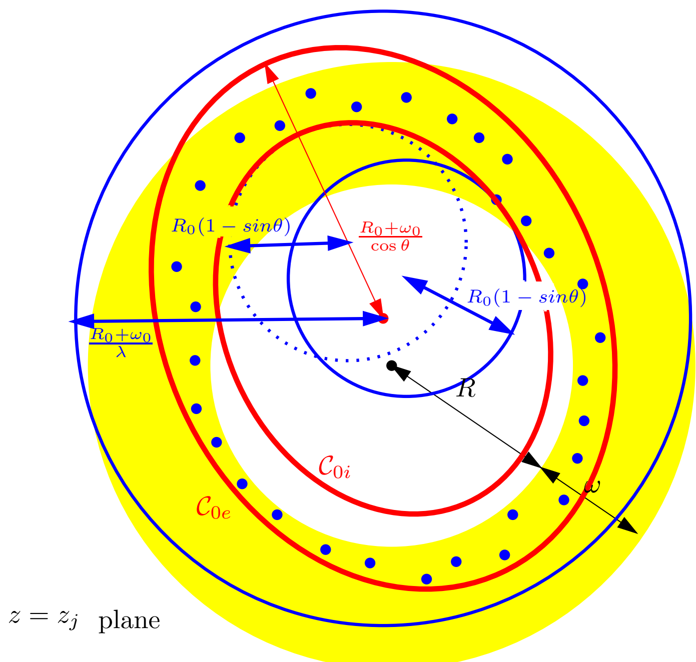

Figure 4: For the proof of Lemma 6

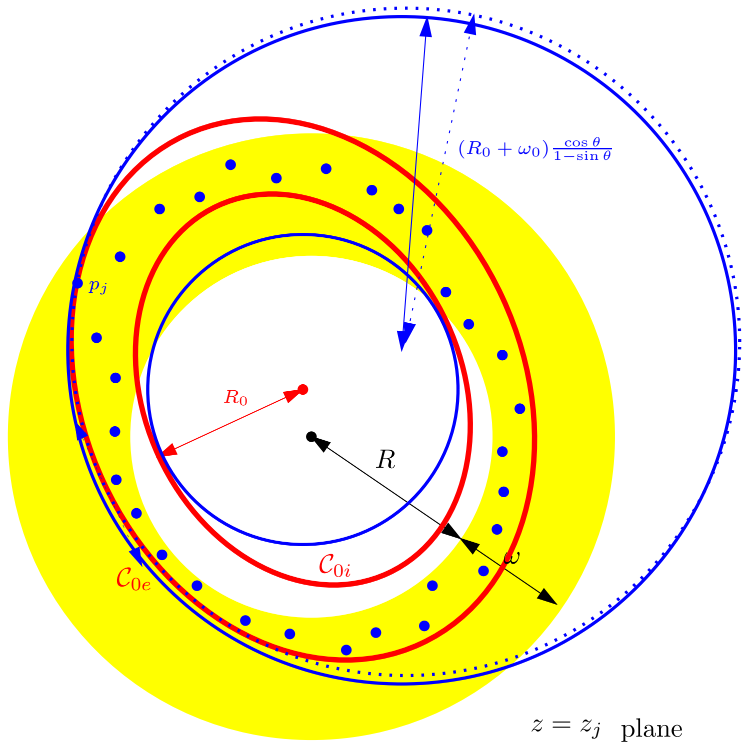

Figure 5: For the proof of Lemma 7

Conversely, the section of $\mathcal{C}_{0i}$ with the plane $z = z_j$ is an ellipse through p_j which does not contain any points. The radius of curvature of this ellipse is minimum in correspondence of the major axis and is $R_0(1-\sin\theta)$, so that the circle of radius $R_0(1-\sin\theta)$ tangent to the ellipse in p_j is internal to the ellipse and is therefore empty. $R_0(1-\sin\theta) \geq R_0(1-\sigma_{\omega\phi})$ by Lemma 4. Thus p_j is on the specified α-hull. $\diamondsuit$

Lemma 7 *If a point p_j is on $\mathcal{C}_{0e}$ then p_j belongs to the α-hull of the points in the plane $z = z_j$ for $\frac{1}{\alpha} = (R+2\omega)\frac{1}{1-\sigma_{\omega\phi}}$.*

Proof.

The section of $\mathcal{C}_{0i}$ with the plane $z = z_j$ is an ellipse containing none of the points on that section in its interior. Its minor axis has length $2R_0$. It follows that the circle inscribed in the ellipse, of radius R_0, has empty interior. As in the proof of Lemma 2 we conclude that

$$\begin{aligned} R+\omega &\geq R_0 \\ R_0+\omega_0 &\leq R+2\omega \end{aligned}$$

Conversely, the section of $\mathcal{C}_{0e}$ with the plane $z = z_j$ is an ellipse through p_j enclosing all of the points. Since the radius of curvature is maximum in correspondence of the minor axis and has value $(R_0 + \omega_0)\frac{\cos\theta}{1-\sin\theta}$, the circle with this radius tangent to the ellipse in p_j encloses all the points. Notice that $\frac{\cos\theta}{1-\sin\theta} \leq \frac{1}{1-\sigma_{\omega\phi}}$ by Lemma 4. Thus p_j is on the specified α-hull. $\diamondsuit$

Theorem 8 *Given a set of points $\mathcal{P}$ organized in horizontal cross sections such that*
— $\mathcal{P}$ is contained in a vertical zone cylinder $\mathcal{C}$
— the internal and external radii of $\mathcal{C}$ are R and $R+\omega$, respectively
— the axis of $\mathcal{C}$ is inside the convex hull of all cross section of $\mathcal{P}$,
— first and last cross sections are densely sampled, i.e. in each sector of angle ϕ there is at least one point of $\mathcal{P}$
the minimum width zone cylinder $\mathcal{C}_0$ enclosing $\mathcal{P}$ verifies the conditions
— the internal cylinder of $\mathcal{C}_0$ passes through points of the α-hull of cross sections of $\mathcal{P}$ with $\frac{1}{\alpha} = -(\lambda_{\omega\phi}R - \omega)(1 - \sigma_{\omega\phi})$,
— the external cylinder of $\mathcal{C}_0$ passes through points of the α-hull of cross sections of $\mathcal{P}$ with $\frac{1}{\alpha} = (R + 2\omega)\frac{1}{1-\sigma_{\omega\phi}}$,
where $\lambda_{\omega\phi} = 1 - 3\left(\frac{16\omega+4R\phi+4\omega\phi+R\phi^2}{8h}\right)^2$ and $\sigma_{\omega\phi} = \frac{8\omega+4R\phi+4\omega\phi+R\phi^2}{8h}$.

Proof. From Lemma 6 and 7. $\diamondsuit$

5 Graham scan for α-hull

If we enforce the very reasonable condition that the polar order of the points around the origin is the same as the order on the α-hull boundary (for the points belonging to the boundary), then the classical Graham scan [4] to compute the conventional two-dimensional convex hull easily generalizes to our problem by just replacing the orientation test of a triangle by a test comparing the radius of of the circle defined by three points with $\frac{1}{\alpha}$.

Specifically, by L we denote a doubly-linked list of points in the plane, assumed to be in angular order around the origin, with forward link L and reverse link L^{-1}. For each point v in L, $pos(v)$ is the index of v in L. Here the relation symbol γ is either $>$ or $<$. Recalling that given three points p_1, p_2, p_3 in the plane, the radius of their defined circle is given by the formula

$$\frac{\text{dist}(p_1p_2)\text{dist}(p_1p_3)\text{dist}(p_2p_3)}{4\text{area}(p_1p_2p_3)},$$

the adapted algorithm is as follows:

Algorithm $culling(L, \gamma, \alpha, START)$

```
ℓ ← length(L)
for  j ← 1  to  ℓ  do
    U(j) ← 1
v ← START
a ← dist(v, L(v))
b ← dist(v, LL(v))
c ← dist(L(v), LL(v))
while  (L(v) ≠ START)  do
    A ← area(v, L(v), LL(v))
    if  abc/4A γ 1/α
        then
            v ← L(v)
            a ← c
            b ← dist(v, LL(v))
            c ← dist(L(v), LL(v))
        else
            U(pos(L(v))) ← 0
            delete  L(v)  from  L
            v ← L^{-1}(v)
            c ← b
            a ← dist(v, L(v))
            b ← dist(v, LL(v))
return  U
```

As is well known, this algorithm runs in time $O(|L|)$.

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