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Abstract

Let S be a point set in the plane in general position, such that its elements are partitioned into k classes or *colors*. In this paper we study several variants on problems related to the Erdős-Szekeres theorem about subsets of S in convex position, when additional chromatic constraints are considered.

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1 Introduction

A set of n points in the plane is in general position if no three of the points are collinear. Such a set S will be referred to as an n-set. S is in convex position if every point of S appears on the boundary of conv(S), the convex hull of S.

The following result is commonly called the *Erdős–Szekeres theorem*:

Theorem 1.1 [13] *For every positive integer m there exists a smallest integer $f(m)$ such that any n-set, $n \geq f(m)$, contains an m-subset of points in convex position.*

This result has been attracting the attention of many researchers, both because of its beauty and elementary statement, and because finding the exact value of $f(m)$ turns out to be a very challenging problem. The reader is referred to the survey papers [5] and [25] for a history of the problem, a description of many variants, and a wide list of references. The best currently known bounds are

$$2^{m-2} + 1 \leq f(m) \leq \binom{2m-5}{m-2} + 2,$$

where the lower bound was obtained by Erdős and Szekeres [14] and the upper bound is due to Tóth and Valtr [33]. The lower bound is supposed to be sharp, according to a conjecture of Erdős and Szekeres.

Let A be a point set in the plane in general position. An m-point subset $B \subset A$ in convex position is called an *m-hole* in A if conv(B) is a polygon whose interior does not contain any point of A.

In 1978 Erdős [12] raised the following problem: is there a number $h(m)$, for every integer $m \geq 3$, such that every n-set with $n \geq h(m)$ contains an m-hole?

Obviously $h(3) = 3$, and it is easy to see that $h(4) = 5$. The 9-point configuration depicted on Figure 1 verifies that $h(5) \geq 10$. Harborth [15] proved in 1978 that $h(5) = 10$, and in 1983 Horton [16] showed that $h(m)$ does not exist for $m \geq 7$ by constructing arbitrarily large sets without a 7-hole. The existence of $h(6)$ is a problem that still remains open.

Although Erdős and Szekeres already mentioned the generalization of their original problem to higher dimensions, it is still far from being solved,

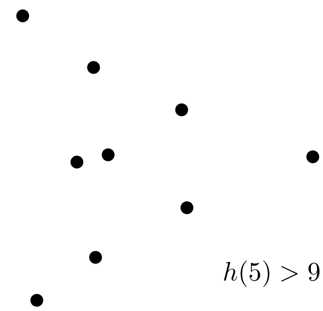


Figure 1: A set of 9 points with no 5-hole.

see [18] and [22] for the current bounds. But even in the plane many variants have been considered. Let us only mention a few examples. Bisztriczky and Fejes Tóth [8] proved a generalization replacing points with convex bodies. Bialostocki *et al.* [7], Caro [9] and Károlyi *et al.* [20] gave results on a conjecture (see [7]), according to which there is a number $h(m,q)$, for arbitrary positive integers $m \geq 3$ and q, such that any n-set, $n \geq h(m,q)$, contains an m-set for which the number of interior points is divisible by q. Several authors have studied the number of subsets in convex position of a given size that a sufficiently large point set can have [4, 6, 10, 23, 29, 30, 32, 35]. Let us finally mention the papers by Ambarcumjan [2], Hosono and Urabe [17], Károlyi [18] and Urabe [34], where several issues on partitioning a point set into subsets in convex position are considered.

Let $S = S_1 \dot{\cup} \cdots \dot{\cup} S_k$ be a partition of a planar point set S, in general position in the plane. We will assume that each set S_i is non-empty, and will refer to it as the set of points of *color* i. A subset $T \subset S$ is called *monochromatic* if all its points have the same color, and *polychromatic* otherwise. The term *heterochromatic* is used for sets in which every element has a different color.

In this paper we consider the following collection of problems. Given an integer m and a set S as above, possibly with additional requirements for $n = |S|$ to be large enough, can we find an m-hole of S falling into one of the three described chromatic classes? Or an m-subset in convex position?

Although colored versions of several results concerning finite point configurations have already been considered by many authors (see e.g. [1, 3, 11, 36]), the original motivation for us to study these problems came from a different area. A finite set Γ of curves in the plane is a *separator* for the sets $S_1, \ldots, S_k$ if every connected component in $\mathbb{R}^2 - \Gamma$ contains objects only from some S_i. We also say that each connected component is *monochromatic*. A thorough study of the subject is developed in [31].

When we have two sets, say the *red* points and the *blue* points, another way to approach their separability is to look for triangulations in which the number of monochromatic edges (or triangles) is as large as possible, which somehow helps in isolating the two populations.

As a consequence of the above motivation, it is the aim of this paper to study the conditions for the existence of certain configurations, and also consider how many *compatible* such configurations we can guarantee, where compatibility stands for having disjoint relative interiors. For example, if the configuration is a monochromatic edge, we also try to find how many monochromatic edges we can find without producing any crossing; the compatibility allows the edges to be completed to a triangulation.

Define $n_M(m,k)$ as the smallest integer with the property that any set of at least this many points, in general position in the plane, colored with k colors contains a monochromatic m-subset in convex position. For $n \geq k$, let $MC(n,m,k)$ be the largest number of compatible monochromatic m-holes that can be found in every k-colored n-set. In the present paper we determine, or give estimates for these numbers along with the numbers $n_H(m,k)$, $n_P(m,k)$, $HC(n,m,k)$ and $PC(n,m,k)$ which are defined in an analogous way for the heterochromatic and polychromatic cases, respectively. A related (yet quite different) problem is considered in [27]. More in the spirit of our problems, several results are described in [19, 21, 26, 28], but looking for configurations like cycles or paths when *edges* are colored.

We organize this paper as follows. In Section 2 we collect some simple observations regarding the functions n_M, n_H and n_P. Sections 3,4 and 5 are devoted to a detailed study of the functions MC, HC and PC, respectively. Due to Horton's result, discussed in Section 3, we only consider the values of these functions when $2 \leq m \leq 6$, since they are 0 otherwise. We summarize our results in tabular form in a concluding section.

2 Subsets in convex position

The exact values of the functions n_M and n_H can be easily determined.

Theorem 2.1
$$n_M(m,k) = k \cdot (f(m) - 1) + 1.$$

Proof: Given at least $k\cdot(f(m)-1)+1$ points, in general position in the plane, one of the color classes must contain at least $f(m)$ points. Then the existence of a monochromatic m-subset in convex position follows from Theorem 1.1. On the other hand, k disjoint copies of any set of size $f(m) - 1$ without a convex m-gon, each colored with a different color gives an example of a set without a monochromatic convex m-gon. □

Theorem 2.2 *(i) If $k \geq f(m)$, then $n_H(m,k) = f(m)$.*

(ii) If $k < f(m)$, then $n_H(m,k) = \infty$.

Proof: If $k, n \geq f(m)$, then we extract from any n-set a heterochromatic subset of size $f(m)$ and apply the Erdős–Szekeres theorem to find a heterochromatic convex m-gon. This proves the first part of the theorem.

To see the second part, assume that $k < f(m)$. Take a set of k points without a convex m-gon, and color the points with different colors. Replacing the point of the first color by $n - k + 1$ points of the same color, very close to each other, we obtain, for every $n \geq k$, a k-colored n-set without a heterochromatic convex m-gon. □

The situation is more subtle in the polychromatic case.

Theorem 2.3 *(i) If $k \geq f(m) - m + 2$, then $n_P(m,k) = f(m)$.*

(ii) If $k < f(m-1)$, then $n_P(m,k) = \infty$.

Proof: To prove the first part, assume that $n \geq f(m)$, and extract from any n-set a subset of size $f(m)$, which contains representatives of at least $f(m) - m + 2$ different colors. Applying Theorem 1.1 to this subset we obtain a convex m-gon whose vertices still represent at least two different colors.

As for the second part, assume that $2 \leq k < f(m-1)$. Take any k-set S without a convex $(m-1)$-gon, and color the points with different colors. Let $p_1, p_2, \ldots, p_t$ be the vertices of $\mathrm{conv}(S)$, in clockwise order. If $n > k$, then replace p_1 by points $b_0, \ldots, b_{n-k}$ very close to p_1 (so that the replacement of p_1 by any b_i does not change the order type of S), all colored by the

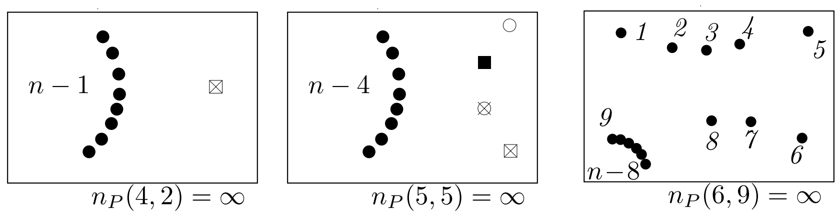


Figure 2: Polychromatic convex subsets.

same color as p_1, such that for $S' = S \cup \{b_0, \ldots, b_{n-k}\} \setminus \{p_1\}$, the vertices of $\mathrm{conv}(S')$ are $b_0, b_{n-k}, p_2, \ldots, p_t$, and moreover, $\mathrm{conv}\{b_0, \ldots b_{n-k}\}$ is a convex $(n-k+1)$-gon contained in the intersection of all the triangles $b_0 b_{n-k} p$, where $p \in S \setminus \{p_1\}$. A polychromatic subset in convex position may then contain at most two of the b_i's, hence finding a polychromatic convex m-gon would contradict the hypothesis that the original set had no convex $(m-1)$-gon. See Figure 2 for the case $m = 4$, $k = 2$. □

There is a gap remaining for $f(m-1) \leq k < f(m) - m + 2$. The examples of Figure 2 show that $n_P(5, f(4)) = n_P(5,5) = \infty$ and $n_P(6, f(5)) = n_P(6,9) = \infty$.

In fact, these examples can be viewed as a consequence of the fact that the Erdős–Szekeres conjecture is valid for $m = 4$ and $m = 5$:

Theorem 2.4 *If $m \geq 5$, then $n_P(m, 2^{m-3}+1) = \infty$.*

Proof: Since the Erdős–Szekeres conjecture is not verified for $m \geq 6$, and a complete precise proof would be rather tedious, we only sketch the main idea here. Construct first a set T of 2^{m-3} points without any convex $(m-1)$-gon, as described in [24], Exercise 14.31(b). This example is composed of $m-2$ clusters of points $T_0, \ldots, T_{m-3}$, where $|T_i| = \binom{m-3}{i}$. In particular, T_0 contains only one point q. Choose a point p very close to q to the left of q such that the line pq is horizontal. Now it can be proved that if $T \cup \{p\}$ contains a convex $(m-1)$-gon C, then C must contain p and q, and also there is a point r in $C \cap T_1$ such that p, q and r are consecutive vertices of C. Color each point of $T \cup \{p\}$ using a different color, and let $n \geq 2^{m-3}+1$ be an arbitrarily large integer. One can replace then p by a set P of $n - 2^{m-3}$ points, colored by the same color as p, very close to p along a carefully chosen convex arc with the following properties. First, if a convex polygon has at least 3 vertices from P, then it cannot have any vertex in T. Next, no convex polygon that has q and r as vertices, can have two vertices from P, no matter which point $r \in T_1$ is selected. Then it follows, along similar lines than in the proof of Theorem 2.3, that the n-set $T \cup P$ cannot contain a polychromatic convex m-gon. □

3 Monochromatic holes

3.1 Edges and triangles

Theorem 3.1 *If $n > k$, then $MC(n, 2, k) = 2\lceil n/k \rceil - 3$.*

Proof: The most frequent color has at least $\lceil \frac{n}{k} \rceil$ points, and any triangulation of these points gives at least $2\lceil \frac{n}{k} \rceil - 3$ compatible monochromatic edges. This is tight as shown by the following construction. Take n points in convex position and color them cyclically by the colors *1,2,. . . , k*. This way we obtain several groups of consecutive points colored *1,2,. . . , k* and a final group colored *1,2,. . . , t*. As in a set of compatible edges there can be at most one monochromatic edge between any two of these groups, the number of monochromatic edges cannot exceed the number of edges of a triangulation of $\lceil \frac{n}{k} \rceil$ points in convex position, which has $2\lceil \frac{n}{k} \rceil - 3$ edges. □

Theorem 3.2 *If $n \geq 5$, then $MC(n, 3, 2) = \lceil n/4 \rceil - 2$.*

Proof: First we verify the lower bound. In fact, we prove that there exist $\lceil n/4 \rceil - 2$ compatible empty monochromatic triangles of the same color. Assume without loss of generality that the number of red points is not less than the number of blue points. Let r' be the number of vertices of the convex hull of the red points, let r'' be the number of other red points, and let β be the number of blue points inside the red convex hull. On one hand, we consider any triangulation of the red points, such a triangulation has exactly $r' + 2r'' - 2$ triangles and trivially at most β of them contain some blue point, thus there are at least $r' + 2r'' - 2 - \beta$ empty compatible red triangles (notice that this number might be negative). On the other hand, we consider a triangulation of the β blue points inside the red convex hull, such a triangulation has at least $\beta - 2$ triangles (even more if the blue points are not in convex position) and at most r'' of them may contain a

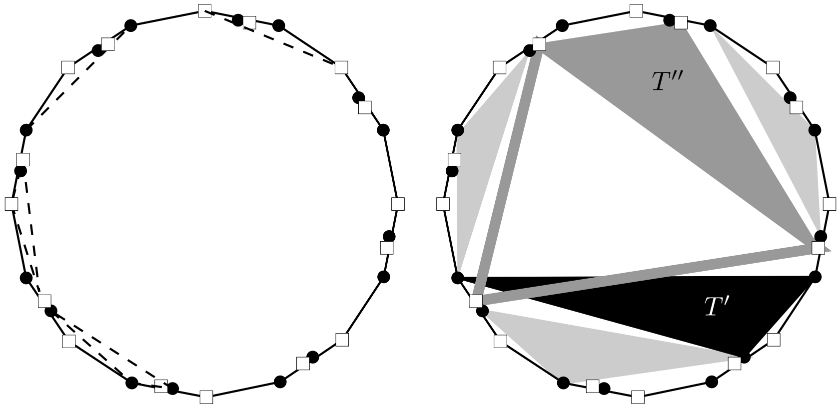


Figure 3: Tightness of $MC(n, 3, 2)$.

red point, thus there are at least $\beta - 2 - r''$ empty compatible blue triangles. The maximum of $r' + 2r'' - 2 - \beta$ and $\beta - 2 - r''$ is not smaller than $\frac{1}{2}(r' + 2r'' - 2 - \beta \ + \ \beta - 2 - r'') = \frac{r'+r''-4}{2} \geq \frac{n-8}{4}$.

To see that this bound is tight, assume first that n is a multiple of 4. Consider the example shown on the left side of Figure 3. It is constructed the following way. We rotate a regular $n/4$-gon about its center by a small angle and then shrink it by a ratio somewhat smaller than 1 to obtain a convex $n/2$-gon, all whose vertices are colored red. Another convex $n/2$-gon, all whose vertices are blue is then obtained as an appropriate mirror image of the red polygon such that only every second vertex of each polygon appears on the boundary of convex hull of the whole set. Take a maximal compatible collection of empty monochromatic triangles, and consider a blue triangle T' and a red triangle T'' that see each other. It is always possible to remove the blue triangle T' and to create a new empty red triangle which replaces the blue one, by linking the visible edge of the red triangle T'' to a red vertex neighboring one of the vertices of the removed blue triangle (see Figure 3-right). In this way we get a set of monochromatic empty triangles with maximal size, where all monochromatic empty triangles are red. The proof is easily finished now, because these triangles can be completed to a triangulation of the red points. Such a triangulation has $\frac{n}{2} - 2$ triangles. Now $\frac{n}{4}$ edges of the red convex hull have inside the hull a blue point very close them, respectively, and no two such edges can belong to the same triangle. We have to remove this number of triangles since they are non empty and

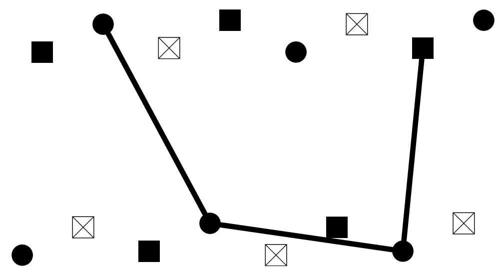


Figure 4: A Horton set of 16 points and a 4-cup.

thus we cannot get more than $\frac{n}{2}-2-\frac{n}{4}=\frac{n-8}{4}$ empty triangles. In the general case write $n=4j-r$, where $0\leq r<4$, and omit, from the construction with $4j$ points, r vertices of the convex hull. We still cannot have more than $j-2=\lceil\frac{n}{4}\rceil-2$ compatible empty monochromatic triangles. $\square$

3.2 Colored Horton sets

Results claiming the non-existence of monochromatic m-holes can be obtained by appropriate colorings of the so called Horton sets, which are examples of point sets without 7-holes.

A Horton set [16, 25] is a set H of n points sorted by x coordinates: $p_1<_x p_2<_x p_3<_x\ldots<_x p_n$, such that the odd points $p_1, p_3,\ldots$ and the even points $p_2, p_4,\ldots$ are Horton sets and such that any line through two even points (the *upper set*) leaves all odd points below and any line through two odd points (the *lower set*) leaves all even points above. A Horton set of size n is recursively obtained by adding a large vertical separation after intertwining in the x direction an upper Horton set H^+ of size $\lfloor\frac{n}{2}\rfloor$ and a lower set H^- of size $\lceil\frac{n}{2}\rceil$. Such a set is shown in Figure 4.

Given a Horton set S, we define an r-cup (resp. r-cap) as a subset of r points $p_{i_1},\ldots,p_{i_r}$ in convex position such that under the assumption $p_{i_1}<_x p_{i_2}<_x\ldots<_x p_{i_r}$, the upper (resp. lower) envelope of the convex hull is the segment $p_{i_1}p_{i_r}$. We say that such a cup is empty if no point of S lies above the lower envelope of its convex hull. An empty cap is defined similarly.

It is not difficult to check (see [16, 25]) that no Horton set contains an empty 4-cap or an empty 4-cup (Figure 4). Now it is easy to prove that no Horton set contains a 7-hole. A heptagon having only odd or only even numbered points is not empty by an induction hypothesis. Any heptagon with some odd points and some even points must have at least four points of the same kind. Assume, without loss of generality, that there are four odd points. They define a 4-cup in the odd Horton subset, which itself is a Horton set. Therefore there must be an odd point above this cup which makes it non-empty. This point is necessarily inside the heptagon, otherwise the property that a line through odd points leaves all even points above would be violated.

Theorem 3.3 *$MC(n, 3, 3) = 0$ for every positive integer n.*

Proof: Let us recall first that in a Horton set H, the indices of the vertices of an empty 2-cup differ by a power of two, as it can be easily shown by induction on the size of H. Indeed, the two vertices of the cup cannot be both odd, otherwise an even point would prevent the emptiness. Similarly, if one vertex is odd and the other one is even, their indices can only differ by one. Finally, if both vertices are even, their indices in H^+ differ by a power of two due to the inductional hypothesis, and one only has to multiply this number by 2 to get the difference of their indices in H. A similar argument shows that the difference between the indices of the vertices of an empty 2-cap also must be a power of two.

To prove the theorem, consider a Horton set H of size n and color the points with three colors R, G and B cyclically, so that the points are colored $RGBRGBRGB\ldots$ in the x order. This coloration splits well recursively, indeed the upper Horton set is colored $GRBGRB\ldots$ and the lower Horton set is colored $RBGRBG\ldots$, as one can see it on Figure 4. We will show that H does not contain an empty monochromatic triangle.

Consider a monochromatic triangle with two vertices in H^- and one in H^+. The difference between the indices of the two vertices in H^- is divisible by 3, thus these two vertices cannot form an empty 2-cup in H^-. Consequently, the triangle is not empty. Similarly, there is no empty monochromatic triangle with two vertices in H^+ and one in H^-.

The proof can be completed now by showing, using induction on n, that there cannot be empty monochromatic triangles all whose vertices belong to the same class H^+ or H^-. $\square$

Theorem 3.4 *$MC(n, 5, 2) = 0$ for every positive integer n.*

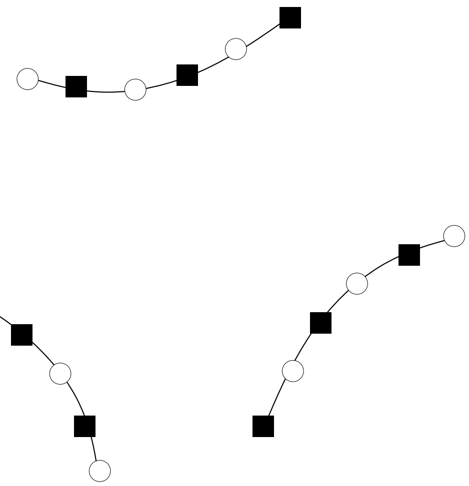


Figure 5: A set of 18 points with no monochromatic 4-hole.

Proof: Take the 3-colored Horton set H we considered during the proof of Theorem 3.3. Construct also a 2-coloring of H by identifying colors G and B as a single color denoted GB. Consider a monochromatic pentagon P in the 2-colored set, it can be either of color R or of color GB. In the second case at least three vertices of the pentagon have the same color with respect to the original 3-coloring. In all cases, P contains a monochromatic triangle in the 3-colored set, which cannot be empty, as it was shown in the previous proof. Thus, P cannot be empty either, the 2-colored set does not contain an empty monochromatic pentagon. $\square$

Finally we turn our attention to the case $k = 2$, $m = 4$. A construction with 18 points which contain no monochromatic 4-hole is shown on Figure 5 proving that $MC(n, 4, 2) = 0$ for $n \leq 18$.

Note that a bichromatic set with no monochromatic 4-hole cannot contain any 7-hole, otherwise such heptagon would have at least 4 points of the same color which would then form a monochromatic 4-hole. Thus to find examples, which would, in general, prove that $MC(n, 4, 2) = 0$, it is natural to look at Horton sets. However, this approach fails, as it is shown by the following result.

Theorem 3.5 *Every 2-colored Horton set of size* ≥ 64 *contains an empty monochromatic 4-hole.*

Proof: It is enough to prove the theorem for a Horton set H of size 64. Note that if the size of a Horton set is a power of 2 and we rotate it through π, we get another Horton set. In view of this observation, in such Horton sets we may change the roles of the lower and upper parts. Note also that in a Horton set, if two consecutive points of the lower class and two consecutive points of the upper class are all colored with the same color, then those 4 points form a monochromatic 4-hole. Moreover, if the points of a Horton set of size ≥ 7 are colored with two colors alternately in the x order, then the first 4 points of the lower class form a monochromatic 4-hole.

Thus, if H^- is colored alternately, then we are done. Otherwise in H^- there are two consecutive points of the same color, say red. If H^+ also contains two consecutive red points, then there is a red 4-hole. Consider the first 8 points of H^+. If they are colored alternately or there are two consecutive red points among them, then we are done. Hence we may assume that either there are at least 5 blue points among them or there are exactly 4 blue points, in which case both the first and the last point of that block of 8 points are red. Moreover, the same assumption can be made about the next block of 8 consecutive points of H^+. If we put together these assumptions we get that there are at least 9 blue points among the first 16 points of H^+. Similarly, we may assume that out of the last 16 points of H^+, also at least 9 are blue.

Then H^+ contains at least 18 blue points, and without loss of generality we may assume that at least 9 points of $(H^+)^-$ are blue. In particular, there are two consecutive blue points there. Mimicking the above argument we arrive at the conclusion that either there is a monochromatic 4-hole in H^+ or otherwise the 16-point Horton set $(H^+)^+$ contains at least 9 red points, and thus at most 7 blue points.

In this case, out of the 18 blue points in H^+, the 16-point set $(H^+)^-$ must contain at least 11. This implies that without loss of generality we may assume that the 8-point Horton set $((H^+)^-)^+$ contains at least 6 blue points. If either its lower or its upper class is completely blue, then it forms a blue 4-hole. Otherwise both the lower class and the upper class consist of 3 blue points and 1 red point. In particular, each class contains two consecutive blue points, and we are done. Since we have reviewed all the possible cases, and in each case we found a monochromatic 4-hole, the proof is complete. □

All these considerations give a strong support to the following conjecture.

Conjecture 3.1 *For n large enough, $MC(n, 4, 2) > 0$. In other words, every* *large bichromatic point set contains some monochromatic 4-hole.*

4 Heterochromatic holes

Theorem 4.1 *Let $n \geq k \geq 2$. Then*

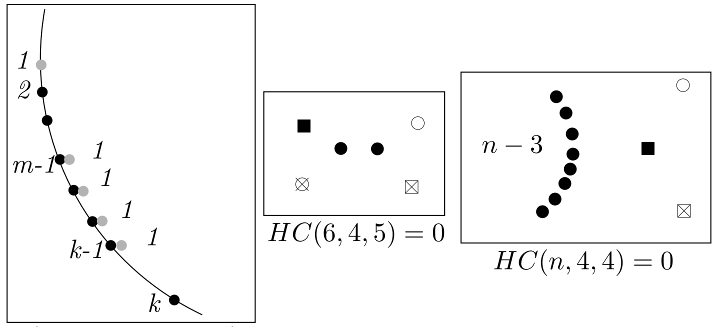
(i) $HC(n, 2, k) = n + k - 3$, *and*

(ii) $HC(n, 3, k) = k - 2$.

Proof: To prove the first part of the theorem, consider any k-colored n-set. Take a vertex p of its convex hull and denote by n_1 the cardinality of the color class of p. When we link p to the other points, we obtain $n - n_1$ heterochromatic edges. Enumerate these edges in polar order around p. Doing this, the color of the endpoint other than p will change at least $k - 2$ times. For each such change, link the pair of consecutive endpoints to create $k - 2$ new heterochromatic edges that still form a compatible set together with the previous $n - n_1$ edges. Each of the remaining $n_1 - 1$ points that have the same color as p can still be connected to a point of a different color without producing any crossings. This way we get a total of $n - n_1 + k - 2 + n_1 - 1 = n + k - 3$ compatible heterochromatic edges, proving $HC(n, 2, k) \geq n + k - 3$.

To see that this bound is tight, consider a set of n points in convex position such that the points with the same color appear consecutively on the convex hull. Any compatible collection of heterochromatic edges can be completed to a triangulation. Since out of the $2n - 3$ edges belonging to that triangulation $n - k$ edges that bound the convex hull are monochromatic, the number of heterochromatic edges cannot exceed $n + k - 3$.

To prove the second part of the theorem, choose k points of different colors from a given k-colored n-set, and construct first any triangulation of these k points. This way we obtain a compatible collection of at least $k - 2$ heterochromatic triangles, which are 3-holes within the subset of these k points. When we restore the remaining $n - k$ one by one, whenever an existing heterochromatic 3-hole disappears, a new one is created inside the 3-hole we just lost. Thus, after each step of this algorithm, there will still be a compatible collection of $k - 2$ heterochromatic 3-holes. This proves the inequality $HC(n, 3, k) \geq k - 2$.



$HC(2k-m+1,m,k)=0$ $HC(6,4,5)=0$ $HC(n,4,4)=0$

Figure 6: Examples without heterochromatic 4-holes.

We finally prove that this bound cannot be improved upon by considering the same construction as in the first part of the theorem. In this example, every compatible collection of heterochromatic triangles can be mapped to a compatible collection of the same number of triangles in a convex k-gon, hence the lower bound. $\square$

In the remaining part of this section we study the case $4 \leq m \leq 6$. Remember that $HC(n,m,k)=0$ if $m \geq 7$.

Theorem 4.2 *If* $m \geq 4$ *and* $n \geq 2k-m+1$*, then* $HC(n,m,k)=0$*.*

Proof: If $n=2k-m+1$, the construction of Figure 6 describes a point set without heterochromatic m-holes. First we place points $p_1, p_2, \ldots, p_k$, one from each color class, in this order along the graph of a strictly decreasing convex function. Then we place a point q_i of the first color (the *obstacle point*), very close to the point p_i, to the right of p_i, for $m-1 \leq i \leq k-1$. Any heterochromatic m-set C must contain three points $p_\alpha, p_\beta, p_\gamma$ with $\alpha < \beta < \gamma$ and $m-1 \leq \beta \leq k-1$. But then q_β is inside $\mathrm{conv}(C)$, and C cannot be an m-hole. For larger values of n, replace q_{m-1} by $n+m-2k$ points, all of the first color, very close to q_{m-1}. $\square$

On the other hand, for k large and $n-k$ small enough, there are always some heterochromatic m-holes, provided that $h(m)$ exists.

Theorem 4.3 *Let* $4 \leq m \leq 6$ *and* $n \geq k$*. If* $h(m) < \infty$*, then*

$$HC(n,m,k) \geq \lfloor \frac{k-2}{h(m)-2} \rfloor + k - n.$$

In particular, $HC(n,4,k) \geq \frac{4}{3}k - n - \frac{4}{3}$ *and* $HC(n,5,k) \geq \frac{9}{8}k - n - \frac{9}{8}$.

Proof: Given an n-set S, choose a vertex p of $\operatorname{conv}(S)$ and select $k-1$ other points of S to obtain a heterochromatic subset S' of size k. Denote these points by $p_0, p_1 \ldots p_{k-2}$, in clockwise order of visibility around p. Consider the $t = \lfloor (k-2)/(h(m)-2) \rfloor$ subsets

$$\{p, p_{(h(m)-2)i}, p_{(h(m)-2)i+1}, \ldots, p_{(h(m)-2)i+(h(m)-2)}\},$$

where $0 \leq i \leq t-1$, their convex hulls have mutually disjoint interiors. It follows then from the definition of $h(m)$ that S' contains a compatible collection of t heterochromatic m-holes. When we extend this configuration with the $n-k$ remaining points of S, still at least $t-(n-k)$ of these m-gons remain empty, hence the result. $\square$

Consequently, $HC(n,m,k)$ is positive for $h(m) \leq k \leq n \leq \frac{k-2}{h(m)-2} + k - 1$. However, for $\frac{k-2}{h(m)-2} + k - 1 < n < 2k - m + 1$ there is a gap in our results. In this interval we do not know, except of some particular cases, whether $HC(n,m,k)$ is zero or not. The particular values $HC(4,4,4)$ and $HC(6,4,5)$ are zero, as one can see from the examples of Figure 6.

5 Polychromatic holes

If $m = 2$, then polychromatic segments and heterochromatic segments are exactly the same concept. Thus it follows from Theorem 4.1 that $PC(n,2,k) = n + k - 3$ for $n \geq k \geq 2$. Therefore we first deal with the case $m = 3$.

Theorem 5.1 *If* $n \geq k$, *then* $PC(n,3,k) = n - 2$.

Proof: If n points are in convex position, then we cannot have more than $n-2$ compatible triangles at all.

To prove that this bound can be achieved we follow the construction shown in the Figure 7. We choose a point p of color *1* and link it to all points with different colors, inducing a partition of the plane into sectors with apex p (Figure 7-left). If there are n_1 points of color *1*, then at least $n - n_1 - 1$ of these sectors are convex regions. In each sector qpr, link all the points with color *1* to q and also link them in their radial order around q (Figure 7-center) to create this way a compatible set of $n_1 - 1$ empty

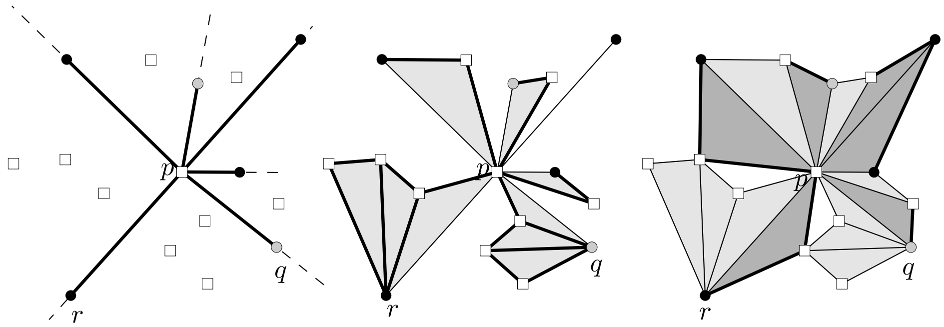


Figure 7: Triangulating with polychromatic triangles.

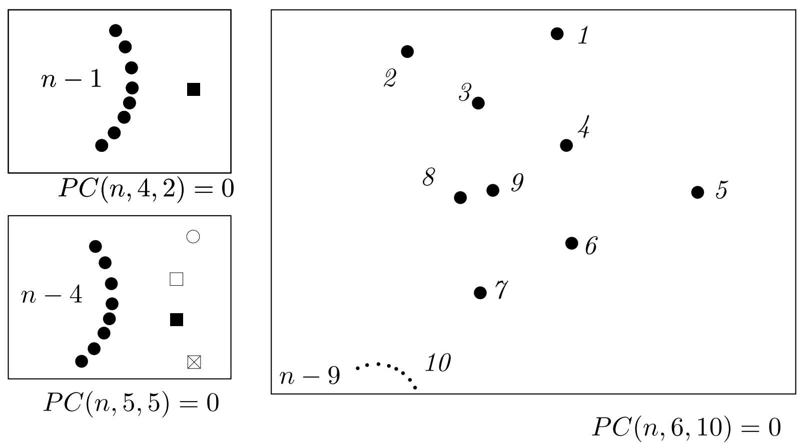


Figure 8: Sets of n points with no polychromatic m-hole.

polychromatic triangles. This set of triangles can be extended to a set of at least $(n_1 - 1) + (n - n_1 - 1) = n - 2$ compatible empty polychromatic triangles if in each convex sector qpr we link r and p to a point of color *1* which is closest to the ray pr (Figure 7-right). □

The major part of this section is devoted to the most interesting case of 4-holes. Note that $PC(n, 4, 2) = 0$, as it can be seen on Figure 8. This construction can be extended to obtain the following general upper bound.

Theorem 5.2 *If $n \geq k$, then $PC(n, 4, k) \leq k - 2$.*

Proof: If $n = k$, then $PC(n, 4, k) \leq PC(n, 3, k) = n - 2 = k - 2$ by Theorem 5.1. Thus we assume $n > k \geq 3$. Construct an n-set $S = \{p_1, p_2, \dots, p_n\}$ so that $p_1, p_2, \dots, p_{k+1}$ is a convex $k+1$-gon, and if $n > k + 1$, then points $p_{k+2}, \dots, p_n$ are inside the intersection of triangles $p_1p_2p_{k+1}$ and $p_1p_kp_{k+1}$ such that $p_1p_np_{n-1} \dots p_{k+1}$ is a convex $n - k + 1$-gon. Color points p_i by color i for $1 \leq i \leq k$ and assign the first color to all the remaining points. We claim that S does not contain more than $k - 2$ compatible polychromatic 4-holes.

To see this, let $\mathcal{P}$ denote a compatible collection of polychromatic 4-holes in S. Note that no element of $\mathcal{P}$ may have more than 2 vertices from the set $S' = \{p_1, p_{k+1}, p_{k+2}, \dots, p_n\}$. Consider an auxiliary convex polygon $Q = q_1q_2 \dots q_k$ and map p_i to q_i for $1 \leq i \leq k$, mapping all the remaining points of S to q_1. This way $\mathcal{P}$ is mapped to a compatible collection of polychromatic triangles and quadrilaterals in Q. This collection clearly can be extended to a triangulation of Q into $k - 2$ triangles, which proves our claim and completes the proof of the theorem. □

To prove some lower bounds as well we first study the particular case $k = 3$.

Theorem 5.3 *If $n \geq 5$, then $PC(n, 4, 3) = 1$.*

Proof: The upper bound $PC(n, 4, 3) \leq 1$ follows immediately from Theorem 5.2. To prove the lower bound, consider any 3-colored n-set S, $n \geq 5$. Let abc denote an empty heterochromatic triangle in S, such a triangle exists by Theorem 4.1. The lines ab, bc, ca divide the plane into one bounded and 6 unbounded regions which can be indexed by non-empty subsets $I \subseteq \{a, b, c\}$, denoting by T_I the region whose vertex set is I. In particular, no point of S lies inside $T_{\{a,b,c\}}$. If there is a point $p \in S \cap T_{\{a,b\}}$ that is closest to line ab, then $apbc$ is a polychromatic 4-hole. Thus we may assume that the regions $T_{\{a,b\}}$, $T_{\{b,c\}}$ and $T_{\{a,c\}}$ are empty. Hence we may assume without the loss of any generality that there is a point $p \in S$ inside $T_{\{a\}}$ that is closest to vertex a. Since $|S| \geq 5$, again by symmetry we may assume that there is a point $q \in S$ inside the convex region bounded by rays ap and ab. Triangle apb is empty, according to the choice of p. Consequently, if q has the additional property that it is closest to line pb, then the convex quadrilateral $apqb$ is a polychromatic 4-hole. □

It is not difficult to prove now the following general lower bound.

Theorem 5.4 *If $n \geq k$, then $PC(n,4,k) \geq \lfloor \frac{k-2}{3} \rfloor$.*

Proof: Given a k-colored n-set S, first take a heterochromatic subset S' of size k and sort the points of this subset according to the polar order around a vertex p of $\text{conv}(S')$. In this ordering of $S' \setminus \{p\}$, take points 4 by 4 with an overlap of 1 to obtain $\lfloor \frac{k-2}{3} \rfloor$ non-overlapping convex sectors with common apex p such that each sector contains a heterochromatic subset of size 5. The result follows immediately putting back the points of $S \setminus S'$ and applying Theorem 5.3 in each sector, respectively. □

A more subtle argument shows that the upper bound is tight, at least if n is large enough.

Theorem 5.5 *For every $k \geq 2$ there is an integer $n_0(k)$ such that if $n \geq n_0(k)$, then $PC(n,4,k) = k-2$.*

Proof: In view of Theorems 5.2 and 5.3, the statement is valid with $n_0(2) = 2$ and $n_0(3) = 5$. We will prove the theorem by induction, where the induction step depends on the following simple ham-sandwich argument we present for the sake of completeness.

Lemma 5.1 *Let A and B be disjoint finite sets of points in the plane such that $|A|$ is odd and no three points of $A \cup B$ are collinear. Then there is a line $\ell = ab$ with $a \in A$, $b \in B$ such that each open half-plane bounded by ℓ contains exactly $(|A|-1)/2$ points of A and at least $(|B|-2)/2$ point of B.*

Proof: Start with a directed line ℓ_0 through a point a of A that has exactly $(|A|-1)/2$ points of A on each side. Rotate ℓ_0 about a in clockwise direction. If it hits another point $a' \in A$, then continue rotating ℓ_0 about a'. After a while we arrive at a line ℓ_1 passing through a point $a_1 \in A$ and $b_1 \in B$ that has exactly $(|A|-1)/2$ points of A on each side. Let B_1^l and B_1^r denote the points of B lying on the left and the right hand side of ℓ_1, respectively, then $|B_1^l| + |B_1^r| = |B| - 1$. Repeating this process with ℓ_1 in place of ℓ_0, and then with ℓ_{i+1} in place of ℓ_i for $i = 1, 2, \ldots$ we arrive at a sequence of lines $\ell_i = a_i b_i$ where $a_i \in A$, $b_i \in B$, ℓ_i has exactly $(|A|-1)/2$ points of A on each side, and if B_i^l and B_i^r denote the points of B lying on the left and the right hand side of ℓ_i, respectively, then $|B_i^l| + |B_i^r| = |B| - 1$ and moreover $||B_{i+1}^l| - |B_i^l|| \leq 1$. After a complete half-turn ℓ_1 is transferred into ℓ_t with $a_t = a_1$, $b_t = b_1$, and consequently, $B_t^l = B_1^r$ and $B_t^r = B_1^l$. It follows that there must be an index $1 \leq i \leq t$ such that $|B_i^l| = \lfloor (|B|-1)/2 \rfloor$, and then $\ell = \ell_i$ gives the solution. □

To complete the proof of the theorem let $k \geq 4$ and assume that the theorem has already been proved for smaller values of k. Let P be a k-colored n-set where $n \geq k(2n_0(k^* + 2) - 2k^* - 4) + 1$ for

$$k^* \in \left\{ \left\lfloor \frac{k-2}{2} \right\rfloor, \left\lceil \frac{k-2}{2} \right\rceil \right\}.$$

Note that $k \geq 4$ implies $k^* + 2 < k$ and thus the existence of $n_0(k^* + 2)$ has already been established according to our assumption. Then one color class has at least $\max_i(2n_0(k_i + 2) - 2k_i - 3)$ points, where $k_1 = \lfloor \frac{k-2}{2} \rfloor$ and $k_2 = \lceil \frac{k-2}{2} \rceil$. Let A be a set of this many points of the same color, and let B be a set of $k - 1$ points such that $A \cup B$ represents all the k different colors. Applying the lemma we obtain two non-overlapping closed half-planes H_1 and H_2 such that H_i contains at least $n_0(k_i + 2)$ points of P among which there are at least $k_i + 2$ points of different colors. It follows that P contains $\lfloor \frac{k-2}{2} \rfloor$ compatible polychromatic 4-holes in H_1 and $\lceil \frac{k-2}{2} \rceil$ compatible polychromatic 4-holes in H_2 which together form a compatible collection of $k - 2$ polychromatic 4-holes. This implies that the result holds for k as well with $n_0(k) = \max_i(k(2n_0(k_i + 2) - 2k_i - 4) + 1)$, completing the proof of the theorem. □

We know very little about the cases $m = 5$ and $m = 6$. Note that $PC(n, m, k)$ is monotone increasing in the variable k. Consequently, the examples of Figure 8 also prove the additional results that $PC(n, 5, k) = 0$ for $k \leq 5$ and $PC(n, 6, k) = 0$ for $k \leq 10$.

6 Conclusion

Several results on a generalization of the Erdős–Szekeres theorem to colored sets of points have been presented in this paper.

One difference from the non-chromatic version is that if k is small enough with respect to m, then it is possible to construct arbitrarily large k-colored point sets with no heterochromatic or polychromatic subset of size m in convex position.

Results are more interesting for the problem of the existence of m-holes for $3 \leq m \leq 6$. We have succeeded in proving some results on the existence or non-existence of m-holes, but some intriguing problems remain open. Among them, our conjecture on the existence of a monochromatic 4-hole in any large enough bichromatic point set may be the most challenging one.

Note that if $k_1 \leq k_2$ and $m_1 \leq m_2$, then $MC(n, m_1, k_1) \geq MC(n, m_2, k_2)$. Thus we can summarize the results on the values of the function MC as follows.

$m \setminus k$	2	3	
2	$\longleftarrow$	$2\lceil \frac{n}{k} \rceil - 3$	$\longrightarrow$
3	$\lceil \frac{n-8}{4} \rceil$	$0 \longrightarrow$	
4	?	$0 \longrightarrow$	
5	0	$0 \longrightarrow$	
6	0	$0 \longrightarrow$	

It follows from our proof that in fact there are at least this many compatible empty monochromatic m-holes *of the same color* in every k-colored n-set.

The values of the function HC are contained in the following table.

$m \setminus k$	2	3	4	5	6	7	
2	$\longleftarrow$			$n+k-3$			$\longrightarrow$
3	$\longleftarrow$			$k-2$			$\longrightarrow$
4				0 if $n \geq 2k-3$			
5				0 if $n \geq 2k-4$			
6				0 if $n \geq 2k-5$			

Our last table contains the values of the function PC.

$m \setminus k$	2	3	4	5	6	7	8	9	10	11	
2	$\longleftarrow$				$n+k-3$					$\longrightarrow$	
3	$\longleftarrow$				$n-2$					$\longrightarrow$	
4	$\longleftarrow$				$k-2$ if $n \geq n_0(k)$					$\longrightarrow$	
5	0	0	0	0	?	?	?	?	?	?	
6							$\longleftarrow$		0	?	

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