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# An Algorithm for Constructing the Convex Hull of a Set of Spheres in Dimension $d^*$

Jean-Daniel Boissonnat<sup>†</sup>    André Cérézo<sup>‡</sup>    Olivier Devillers<sup>†</sup>  
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## Abstract

We present an algorithm which computes the convex hull of a set of  $n$  spheres in dimension  $d$  in time  $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ . It is worst-case optimal in three dimensions and in even dimensions. The same method can also be used to compute the convex hull of a set of  $n$  homothetic convex objects of  $\mathbb{E}^d$ . If the complexity of each object is constant, the time needed in the worst case is  $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ .

**Keywords:** Computational Geometry, Convex Hull, Spheres

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# 1 Introduction

We present an algorithm which computes the convex hull of a set of  $n$  spheres in dimension  $d$  in time  $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ . It is worst-case optimal in three dimensions and in even dimensions. It can also be used to compute the convex hull of a set of  $n$  homothetic convex objects of  $\mathbb{E}^d$ .

Though the complexity and the computation of the convex hull of a set of points in any dimensions is a problem which has been studied extensively, only a few results about the convex hull of a set of spheres are known. The previous results, which are given below, are only for the case  $d = 2$  and  $3$ , and, as far as we know, there were no results about the computation of the convex hull of a set of homothetic objects.

The convex hull of a set of spheres is the smallest convex body that contains the spheres. In two dimensions, the boundary of such a convex hull consists of line segments and arcs of circles. In three dimensions, the convex hull boundary is composed of three different kinds of facets (see Figure 1).

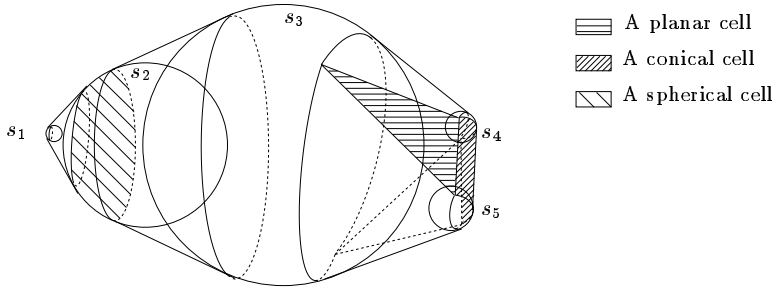


Figure 1: The convex hull of a set of spheres in 3 dimensions

- Planar facets, which are triangles included in planes tangent to three spheres.
- Conical facets, which are parts of cones tangent to two spheres.
- Spherical facets, which are parts of the spheres.

In the plane, the convex hull of a set of disks can be computed in  $O(n \log n)$  time (see [6]) which is optimal. In three-dimensional space, the complexity of the convex hull of a set of  $n$  spheres is  $\Theta(n^2)$  in the worst case, even for collections of pairwise disjoint spheres [7] (see Section 2 below). The convex hull of a set of  $n$  spheres in  $\mathbb{E}^3$  can be computed in  $O(nh)$  time [1], where  $h$  is the size of the output (i.e. of the convex hull). In the case where all spheres have the same radius, the convex hull of a set of spheres in  $\mathbb{E}^d$  can be easily deduced from the convex hull of the centers of the spheres (see Section 3.1), which can be computed in  $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ .

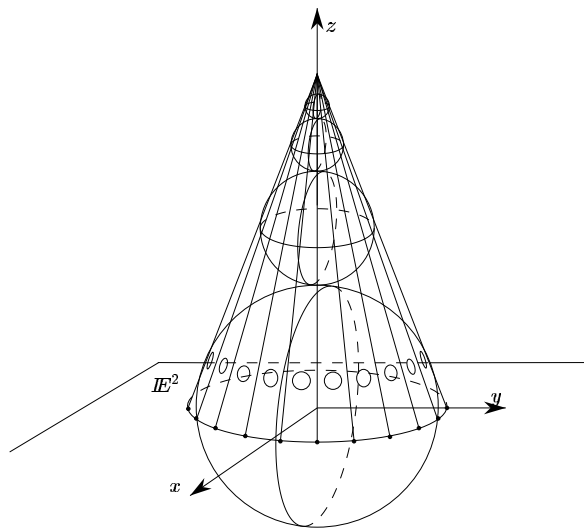
In dimension  $d$ , the boundary of the convex hull is composed of  $d$  different kinds of facets. Let a *supporting hyperplane of a set* be a hyperplane  $H$  which has a non empty

intersection with the set and such that the whole set is included in one of the closed halfspaces limited by  $H$ . Let a *supporting halfspace of a set* be a halfspace containing the set and limited by a supporting hyperplane of the set. The convex hull of a set of spheres in  $\mathbb{E}^d$  is the intersection of the supporting halfspaces of the set of spheres. A *facet of circularity  $i$*  ( $0 \leq i \leq d - 1$ ) is a maximal connected portion of the boundary of the convex hull consisting of points where the supporting hyperplanes are tangent to a given set of  $(d - i)$  spheres. For example, in dimension 3, the planar facets have circularity 0, the conical facets have circularity 1, the spherical facets have circularity 2.

The boundary of the convex hull of a set of spheres is the union of the closure of facets of circularity  $0, 1, 2, \dots, d - 1$ . The boundary of the convex hull is represented by the adjacency graph of these facets.

The paper is organized as follows: In the next Section we give a lower bound on the complexity of the convex hull of a set of  $n$  spheres. In Section 3 we present the algorithm to compute this convex hull and we show in Section 4 that it is optimal in three dimensions and in even dimensions. In Section 5 we extend our results to homothetic convex objects.

## 2 Lower Bounds



•  $n$  points on a circle

Figure 2: A set of spheres whose convex hull has size  $\Theta(n^2)$

In dimension 3, let us take  $n$  points, considered as spheres of radius 0, on a circle in the  $(x, y)$ -plane and take a point above this plane, on the  $z$ -axis. The convex hull of these  $n+1$  points is a pyramid. Now add  $n$  spheres having non-zero radii and centered on the  $z$ -axis, such that each sphere intersects each facet of this pyramid but none of its edges (see Figure 2). The complexity of the convex hull of this set of  $2n+1$  spheres is  $\Omega(n^2)$ .

By the upper bound theorem [5], the complexity of the convex hull of a set of  $n$  points in dimension  $d$  is  $O(n^{\lfloor \frac{d}{2} \rfloor})$  in the worst case. This bound is tight for cyclic polytopes. A point can be considered as a sphere of radius 0. Therefore, the complexity of the convex hull of a set of  $n$  spheres is at least equal to the complexity of the convex hull of a set of points, thus is  $\Omega(n^{\lfloor \frac{d}{2} \rfloor})$ .

We conjecture that the complexity of the convex hull of a set of  $n$  spheres is  $\Omega(n^{\lceil \frac{d}{2} \rceil})$ .

### 3 The Algorithm

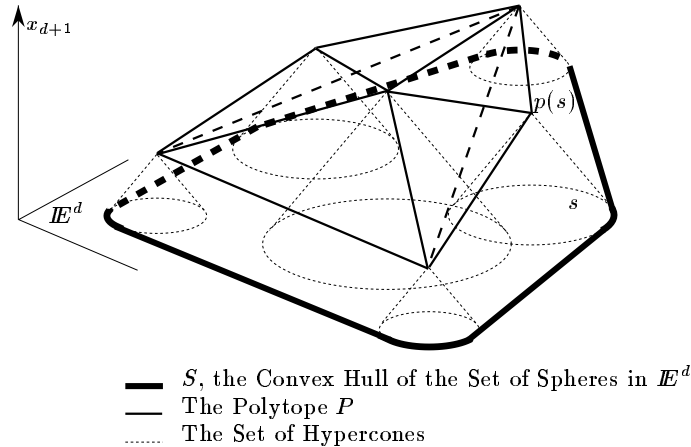


Figure 3: Embedding the spheres in  $\mathbb{E}^{d+1}$

We first introduce some notations, then recall some of the properties of duality, and finally give the algorithm that computes the convex hull of a set of spheres.

#### 3.1 Notations and Preliminaries

Let  $S$  be the *convex hull of a set of  $n$  spheres*  $\{s_1, \dots, s_n\}$  in  $\mathbb{E}^d$ . We embed  $\mathbb{E}^d$  in  $\mathbb{E}^{d+1}$  so that the hyperplane  $\{x_{d+1} = 0\}$  of  $\mathbb{E}^{d+1}$  contains all the spheres. The  $(d+1)$ -th axis will be called the vertical axis, and in the sequel, the expression *above* will refer to the  $(d+1)$ -th coordinate. Let  $s$  be a sphere in  $\mathbb{E}^d$  with center  $(x_1, \dots, x_d)$  and radius  $r$ . Let  $p$  be the mapping that associates to  $s$  the *point*  $p(s)$  in  $\mathbb{E}^{d+1}$  such that:

$$p : s \rightarrow p(s) = (x_1, \dots, x_d, r)$$

Let  $P$  be the *convex hull of the set of points*  $\{p(s_1), \dots, p(s_n)\}$  of  $\mathbb{E}^{d+1}$ . Let  $\lambda_0$  be a half lower hypercone with arbitrary apex, vertical axis and angle at the apex  $\pi/4$ .

For a sphere  $s$  in  $\mathbb{E}^d$ , let  $\lambda(s)$  be the translated copy of  $\lambda_0$ , with apex  $p(s)$ . Notice that the intersection between the hypercone  $\lambda(s)$  and the hyperplane  $\{x_{d+1} = 0\}$  is identical to the sphere  $s$ . Let  $\Lambda$  be the *convex hull of the set*  $\{\lambda(s_1), \dots, \lambda(s_n)\}$  of the  $n$  half lower hypercones of  $\mathbb{E}^{d+1}$  associated to the  $n$  spheres  $s_1, \dots, s_n$  (see Figure 3). The intersection between  $\Lambda$  and the hyperplane  $\{x_{d+1} = 0\}$  is equal to  $S$ .

Let  $O'$  be a point inside  $P$ .

**Theorem 1** *Any hyperplane of  $\mathbb{E}^d$  supporting  $S$  is the intersection with  $\{x_{d+1} = 0\}$  of a unique hyperplane  $H$  of  $\mathbb{E}^{d+1}$  satisfying the three properties:*

1.  $H$  supports  $P$ ,
2.  $H$  is the translated copy of a hyperplane tangent to  $\lambda_0$  along one of its generatrices,
3.  $H$  is above  $O'$ .

*Conversely, let  $H$  be a hyperplane of  $\mathbb{E}^{d+1}$  satisfying the above three properties. Its intersection with the hyperplane  $\{x_{d+1} = 0\}$  is a hyperplane of  $\mathbb{E}^d$  supporting  $S$ .*

*Proof:* Through any point of  $S$  of circularity  $i$  passes a hyperplane  $H$  which supports  $\Lambda$  along a generatrix of at least  $d - i$  of the hypercones  $\lambda(s_1), \dots, \lambda(s_n)$ .

This means that  $H$  supports  $P$  and is the translated copy of a hyperplane tangent to  $\lambda_0$ . As  $H$  supports  $\Lambda$ , it is above  $O'$ .

Conversely, if an hyperplane  $H$  supports  $P$  and is above  $O'$ , it is above  $P$ . As  $H$  is also the translated copy of a hyperplane tangent to  $\lambda_0$ , it supports  $\Lambda$ , along a generatrix of at least one of the hypercones  $\lambda(s_1), \dots, \lambda(s_n)$ . Its intersection with  $\{x_{d+1} = 0\}$  is a hyperplane of  $\mathbb{E}^d$  supporting  $S$ .  $\square$

In the case where all spheres have the same radius, it is easy to see that the convex hull of a set of spheres in  $\mathbb{E}^d$  can be obtained by *growing the faces* of the convex hull of the centers of the spheres, i.e. the convex hull of the spheres is exactly the Minkowski Sum of the convex hull of the centers and of a sphere of radius  $r$ . Notice that in this case, all the apices of the cones lie on the same horizontal plane  $\{x_{d+1} = r\}$ , and the growing mechanism can be interpreted as sweeping a plane  $\{x_{d+1} = t\}$ ,  $t$  varying from  $r$  to 0. Therefore, the complexity drops to  $O(n^{\lfloor \frac{d}{2} \rfloor})$  and the running time to  $O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n)$ .

### 3.2 Duality

We use *duality* to convert properties 1, 2 and 3 of the above theorem into simpler ones. Let us recall that  $O'$  is a point inside  $P$ . Let  $O'$  be the origin of a new coordinate system, whose axis are parallel to the axis of the previous coordinate system. New coordinates are denoted with a prime:  $X' = (x'_1, \dots, x'_{d+1})$ . Polarity with respect to  $O'$  is a one-to-one transformation which maps points of  $\mathbb{E}^{d+1}$  distinct from  $O'$  to hyperplanes of

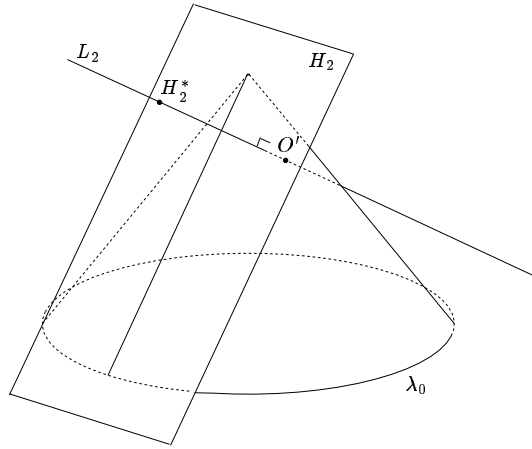


Figure 4: Pole of a hyperplane

$\mathbb{E}^{d+1}$  which do not contain  $O'$ . Let  $M$  be a point of  $\mathbb{E}^{d+1}$  distinct from  $O'$ .  $M^*$ , the polar hyperplane of  $M$ , is defined by the following relation:

$$M^* = \{X' \in \mathbb{E}^{d+1} \mid M.X' = 1\}$$

$H^*$ , the pole of a hyperplane  $H$  not containing  $O'$ , is defined by

$$H^*.X' = 1, \quad \forall X' \in H$$

We have  $(M^*)^* = M$  and  $(H^*)^* = H$ . Let  $H^{*-}$  be the halfspace bounded by  $H$  and containing  $O'$ .

Let the *polar set* of a set of hyperplanes be the set of poles of these hyperplanes.

**Proposition**

1. The polytope  $P^* = p(s_1)^{* -} \cap \dots \cap p(s_n)^{* -}$  of  $\mathbb{E}^{d+1}$  is dual to the polytope  $P$ , i.e. there is a bijection between the  $l$ -faces of  $P$  and the  $(d-l)$ -faces of  $P^*$  which reverses the relation of inclusion. Each hyperplane supporting  $P$  along a  $l$ -face  $F$  has its polar point on the corresponding  $(d-l)$ -face of  $F^*$  of  $P^*$ .
2. The polar set of the hyperplanes which are translated copies of the hyperplanes tangent to  $\lambda_0$  is a hypercone  $K$  with apex  $O'$ , a vertical axis, and an angle at the apex equal to  $\pi/4$ .
3. The polar set of the hyperplanes above  $O'$  is the half space  $x'_{d+1} > 0$ .

*Proof:*

The first assertion is well known.

Second assertion: Let  $H_2$  be a hyperplane tangent to  $\lambda_0$  (see Figure 4).  $H_2^*$ , the pole of  $H_2$ , belongs to the line  $L_2$  issued from  $O'$  and normal to  $H_2$ . The polar set of the hyperplanes parallel to  $H_2$  is  $L_2$ . The angle of  $H_2$  with the vertical axis is  $\pi/4$ . Therefore, the angle between  $L_2$  and the vertical axis is  $\pi/4$ . In order for  $H_2^*$  to be well defined,  $O'$  may be anywhere inside or outside the hypercone, but not on the hypercone.

As  $H_2$  moves around the hypercone  $\lambda_0$ , staying tangent to it,  $L_2$  moves on a hypercone  $K$  with apex  $O'$ , axis  $O'x'_{d+1}$ , angle at the apex  $\pi/4$ .  $K$  is a hypersurface of  $\mathbb{E}^{d+1}$ . The polar set of all the hyperplanes tangent to  $\lambda_0$  and of the translated copies of these hyperplanes is the hypercone  $K$ .

Third assertion: Let  $H_3$  be a hyperplane which lies above  $O'$ . Its equation is  $\sum_{i=1}^{d+1} h'_i x'_i = 1$ . Its intersection with the vertical axis,  $x'_{d+1}$ , is such that  $h'_{d+1} x'_{d+1} = 1$ . As this intersection is above  $O'$ , we have  $x'_{d+1} > 0$  and thus  $h'_{d+1} > 0$ . As the coordinates of  $H_3^*$  are  $(h'_1, \dots, h'_{d+1})$ ,  $H_3^*$  is in the halfspace  $\{x'_{d+1} > 0\}$ . Hence, the polar set of the hyperplanes lying above  $O'$  is the half space  $\{x'_{d+1} > 0\}$ .  $\square$

$P^*$	$S$
$\mathbb{E}^{d+1}$	$\mathbb{E}^d$
$i$ -face ( $1 \leq i \leq d$ )	facet of circularity $(d-i)$ ( $0 \leq d-i \leq d-1$ )
1-face	facet of circularity $d-1$ =part of a sphere
$d$ -face	facet of circularity 0=planar facet

Table 1: correspondence between faces of  $P$  and facets of  $S$

The polar set of the hyperplanes supporting the convex hull of the set of points  $P$ , tangent to at least one hypercone of  $\Lambda$  along a generatrix and above  $O'$  is

$$P^* \cap K \cap \{x'_{d+1} > 0\}$$



Let us consider the intersection of a  $i$ -face of  $P^*$  with  $K$ . The polar hyperplane of each point of this intersection supports  $P$  along a  $(d-i)$ -face and  $\Lambda$  along  $(d-i)$  generatrices of  $(d-i)$  hypercones. The polar set of this intersection is a family of hyperplanes whose intersection with  $\{x_{d+1} = 0\}$  is a facet of  $S$  of circularity  $(d-i)$  (see Table 1).

### 3.3 The Algorithm

1. Compute the convex hull  $P$  and choose a point  $O'$  inside  $P$
2. Compute the polytope  $P^*$  dual to  $P$  with respect to  $O'$ .
3. Compute the intersection between  $P^*$ , the hypercone  $K$  and the half space  $\{x'_{d+1} > 0\}$ .
4. Compute the incidence graph of the facets of  $S$  from the incidence graph of the faces of  $P^*$  intersecting  $K$  and the halfspace  $\{x_{d+1} > 0\}$ .

## 4 Complexity Analysis

Chazelle has shown that the convex hull of a set of  $n$  points in dimension  $d+1$  can be computed in optimal time  $\Theta(n^{\lfloor \frac{d+1}{2} \rfloor} + n \log n)$  (see [3]). Simpler randomized algorithms to compute the convex hull of a set of points can be found in [2, 4, 8]. The complexity of computing the polytope  $P^*$  dual to  $P$  is a linear function of the complexity of  $P$ . The complexity of computing the intersection of  $P^*$  with  $K$  and with  $\{x'_{d+1} > 0\}$  is a linear function of the complexity of  $P^*$  since  $K$  and  $\{x'_{d+1} > 0\}$  have constant complexity. The complexity of computing the incidence graph of the facets of  $S$  from the incidence graph of the faces of  $P^*$  intersecting  $K$  and the half space  $\{x_{d+1} > 0\}$  is a linear function of the complexity of  $P^*$ . Hence, the total amount of time needed to compute  $S$  is a linear function of the amount of time needed to compute  $P$ . Therefore, the time needed in the worst case to compute  $S$ , the convex hull of the set of spheres, is

$$O(n^{\lfloor \frac{d+1}{2} \rfloor} + n \log n) = O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$$

This is optimal in three and in even dimensions.

The given reduction from the problem of computing  $S$ , the convex hull of a set of spheres in  $\mathbb{E}^d$ , to that of computing  $P$ , the contour of the convex hull of a set of points in  $\mathbb{E}^{d+1}$ , does not preserve the output size. The complexity of  $P$  can exceed that of  $S$ . Therefore, using output-sensitive algorithms does not help here.

## 5 Extension to Homothetic Convex Objects

The algorithm for spheres generalizes to a set of homothetic convex objects having the same orientation. The case of non-convex homothetic objects can be reduced to the case of homothetic convex objects by taking the convex hull of each object. More precisely, let us take a convex object  $c$  of  $\mathbb{E}^d$  and let  $c_i$  ( $1 \leq i \leq n$ ) be a convex object obtained from  $c$  by some homothety and some translation. We compute *the convex hull*  $C$  of the set of convex objects  $\{c_1, \dots, c_n\}$ . The main point is that the hypercone  $K$  with angle at the apex  $\pi/4$  is now replaced by a more *general hypercone*  $G$ , which is no longer circular.

Let us associate a half lower hypercone  $\lambda(c)$  of  $\mathbb{E}^{d+1}$  to  $c$  by taking an apex  $p(c)$  above the object such that the vertical projection of the apex on  $\mathbb{E}^d$  is inside the convex object.  $\lambda(c)$  is the half hypercone consisting of the half lines joining  $p(c)$  and a point of  $c$ . Now we may associate to any object homothetic to  $c$  a hypercone  $\lambda(c_i)$  which is a translated copy of  $\lambda(c)$ , such that  $\{\lambda(c_i) \cap (x_{d+1} = 0)\} = c_i$ ,

As before  $P$  is the convex hull of  $\{p(c_1), \dots, p(c_n)\}$  and  $\Lambda$  the convex hull of  $\{\lambda(c_1), \dots, \lambda(c_n)\}$ .

Arguments similar to those of Section 3 can be used. If we replace the half lower hypercone  $\lambda_0$  by  $\lambda(c)$ , we have the following theorem:

**Theorem 2** *Any hyperplane of  $\mathbb{E}^d$  supporting  $C$  is the intersection with  $\{x_{d+1} = 0\}$  of a unique hyperplane  $H$  of  $\mathbb{E}^{d+1}$  satisfying the three properties:*

1.  $H$  supports  $P$
2.  $H$  is the translated copy of a hyperplane tangent to  $\lambda(c)$  along one of its generatrices.
3.  $H$  is above  $O'$ .

*Conversely, let  $H$  be a hyperplane of  $\mathbb{E}^{d+1}$  satisfying the above three properties. Its intersection with the hyperplane  $\{x_{d+1} = 0\}$  is a hyperplane of  $\mathbb{E}^d$  supporting  $C$ .*

The dual of the set of hyperplanes  $H$  satisfying Condition 2 is now a general hypercone  $G$  with apex  $O'$ , which is no longer circular.

The algorithm of Section 3 can be used if we replace the hypercone  $K$  by  $G$ .

We assume that the convex objects  $c$  have constant complexity. Hence the complexity of the hypercone  $G$  is constant. The complexity analysis remains the same as for spheres. Replacing  $K$  by  $G$  does not change the complexity of the algorithm since  $G$  has constant complexity. Therefore, the convex hull of  $n$  homothetic convex objects of constant complexity in dimension  $d$  can be computed in  $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$  time.

For example, the time needed to compute the convex hull of  $n$  homothetic ellipsoids in dimension  $d$  is  $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ .

Let us notice that, similarly to the case of spheres having the same radii, the convex hull of a set of translates of a given convex object can be easily computed in  $O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n)$ .

## 6 Conclusion

In this paper, we have reduced by a suitable geometric transform the problem of constructing the convex hull of  $n$  spheres in  $d$ -space to the problem of computing the intersection of a  $(d + 1)$ -polytope with  $n$  facets with a hypercone. We have shown that the convex hull of  $n$  spheres in dimension  $d$  can be computed in  $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$  time in the worst case, which is optimal in dimension 3 and in even dimensions. We conjecture that the algorithm is optimal in all dimensions.

We have extended these results to homothetic convex objects: If each object has constant complexity, the time needed in the worst case to compute their convex hull is  $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ . Computing the convex hull of general ellipsoids or convex objects in dimension  $d \geq 3$  remains an open problem.

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