A Locally Optimal Triangulation of the Hyperbolic Paraboloid

Pascal Desnogues, Olivier Devillers

A Locally Optimal Triangulation of the Hyperbolic Paraboloid

Pascal DESNOGUÈS * Olivier DEVILLERS †

Abstract: Given a set $\mathcal{S}$ of data points in $I\!R^2$ and corresponding data values for a specific non-convex surface, the unit hyperbolic paraboloid, we consider the problem of finding a locally optimal triangulation of $\mathcal{S}$ for the linear approximation of this surface. The chosen optimality criterion will be the L_2 norm: it means that we will try to find directly a triangulation that minimizes the L_2 error made when approximating locally the surface with triangles.
Keywords: data dependent triangulation, surface approximation.

Introduction

Given a set $\mathcal{S} = p_1, \ldots, p_n$ of n points, in the plane, for each $p_i = (x_i, y_i)$ corresponds a point P_i in the space belonging to the unit hyperbolic paraboloid, which will be henceworth referred as π; $P_i = (x_i, y_i, z_i)$ with $z_i = {x_i}^2 - {y_i}^2$. Let Ω be the convex hull of $\mathcal{S}$, and let $\mathcal{T}$ be a triangulation of Ω whose vertices are points of $\mathcal{S}$. To each triangle $t \in \mathcal{T}$ corresponds a plane containing the P_j's whose projection are the vertices of t: this plane defines a local linear approximation of the hyperbolic paraboloid (figure). Choosing a particular triangulation of $\mathcal{S}$ gives a particular piecewise linear interpolation of the surface.
The following definitions are the classic ones taken from the papers concerning data dependent triangulations (especially the papers from Dyn, Levin and Rippa [DLR90b], [DLR90a] or Brown [Bro91]). Given an interior edge ς of the triangulation $\mathcal{T}$, there are two triangles which have ς as common edge. These two triangles define a quadrilateral Q.

definition 1 *An interior edge ς of $\mathcal{T}$ is* locally optimal *with respect to a given cost function ν if one of the following condition holds:*

- *the quadrilateral Q is not convex.*
- *Q is strictly convex and $\nu(\mathcal{T}) \leq \nu(\mathcal{T}')$, where $\mathcal{T}'$ is obtained from $\mathcal{T}$ by replacing ς by the other diagonal of Q.*

definition 2 *A triangulation $\mathcal{T}$ of a set of points $\mathcal{S}$ is* locally optimal *with respect to a cost function ν if all its edges are locally optimal with respect to ν.*

definition 3 *Let $t = \Delta abc$ be a triangle whose vertices are belonging to $\mathcal{S}$. The* separation curve *of t with respect to a given cost function ν is the curve limiting the plane zone inside which no vertices of triangles neighboring t must lie if we want that t belongs to the locally optimal triangulation of $\mathcal{S}$ with respect to ν.*

For example, it is well-known that the separation curve of a triangle of the Delaunay triangulation is the circumcircle of the triangle: the chosen cost function can be any of the nice properties of this triangulation (maximization of the minimum angle, for example).
The main purpose of this work was suggested by a paper by Melissaratos [Mel93], where the Delaunay triangulation was proved to be optimal for triangulating the unit paraboloid $z = x^2 + y^2$ with respect to any L_p norm; but this property basically relies on the convexity of the paraboloid. Thus, we want in this paper to investigate the problem of triangulating a non convex surface. Namely we have chosen as a particular example the unit hyperbolic paraboloid whose equation is $z = x^2 - y^2$, and we are searching for a locally optimal linear approximation with respect to the L_2 norm.

definition 4 *Given a two variables function $f(x, y)$ defined over a plane domain $\mathcal{D}$, the L_2* norm *of f is given by:*

$$\|f\|_2 = \sqrt{\iint_{\mathcal{D}} f^2(x, y)\, dx\, dy}.$$

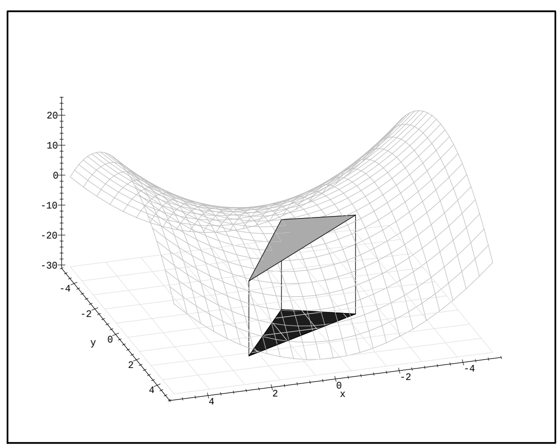

Figure 1: view of the hyperbolic paraboloid and of a lifted triangle.

*INRIA, B.P.93, 06902 Sophia-Antipolis cedex (France), Phone: +33 93 65 77 62, E-mail: `Pascal.Desnogues@sophia.inria.fr`, Fax: +33 93 65 76 43.
†INRIA, B.P.93, 06902 Sophia-Antipolis cedex (France), Phone: +33 93 65 77 63, E-mail: `Olivier.Devillers@sophia.inria.fr`, Fax: +33 93 65 76 43.

definition 5 *Given a two variables function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\mathcal{S}$ a set of data points, $\mathcal{T}$ a triangulation of $\mathcal{S}$, with each triangle t belonging to $\mathcal{T}$ corresponds a linear approximation p_t of the function f over t. Let $P_{\mathcal{T}}$ be the piecewise linear interpolation of f defined by the p_t's; $P_{\mathcal{T}}$ is defined all over the convex hull Ω of $\mathcal{T}$. Then, the L_2* error *made by the linear approximation of f over the triangulation $\mathcal{T}$ is:*

$$\begin{aligned} e(f,\mathcal{T}) &= \|f - P_{\mathcal{T}}\|_2 \\ &= \sqrt{\iint_{\Omega} \left(f(x,y) - P_{\mathcal{T}}(x,y)\right)^2 dx\, dy} \\ &= \sqrt{\sum_{t\in\mathcal{T}} \iint_t \left(f(x,y) - p_t(x,y)\right)^2 dx\, dy} \\ &= \sqrt{\sum_{t\in\mathcal{T}} \epsilon(f,t)}, \end{aligned}$$

where

$$\epsilon(f,t) = \iint_t \left(f(x,y) - p_t(x,y)\right)^2 dx\, dy.$$

Main results

As it has been stated, this paper will try to exhibit a local optimality criterion for the L_2 norm: the cost function to minimize here is the L_2 error made by a piecewise linear approximation of the hyperbolic paraboloid π. It will be shown how to find the separation curve of one triangle, and why the L_2 error is independant by translation of the triangle in the plane. In fact, we will first begin with an observation based on empirical researches, then give an interpretation of this observation and an idea of its proof, and finally we will show numerical results obtained with this triangulation.

Notations and conventions

Here are the main notations used in this paper:

- small letters are used for points in the plane, except t, which is used for a triangle of the plane, and f, which is the function to approximate (in this paper, $f(x,y) = x^2 - y^2$).
- capital letters are used for points in the 3 dimensional space: generally, they correspond to the lifted images of their small equivalent.
- As said before, $\mathcal{T}$ represents a triangulation, and $\mathcal{S}$ a set of points (in the plane).
- *e(f,$\mathcal{T}$)* is the error made over a triangulation (definition 5).
- $\epsilon(f,t)$ corresponds to the square value of the error made by the approximation of the surface with *one* triangle.
- $SC(t)$ is the *separation curve* of the triangle t.

The separation curves of a triangle

Calculus of the error

The main purpose here is to show how a choice was made for computing formally the error made when approximating linearly the hyperbolic paraboloid by one triangle.

Given three points in the plane a, b and c, we note A, B and C their respective lifted points on π. Let $\mathcal{P}_{\Delta ABC}$ be the plane that contains points A, B and C: its equation is given by $\mathcal{P}_{\Delta ABC} : z = p_{\Delta abc}(x,y)$. Then, the studied value is:

$$\epsilon(\Delta abc) = \iint_{\Delta abc} \left(p_{\Delta abc}(x,y) - (x^2 - y^2)\right)^2 dx\, dy.$$

This integral has been computed using the symbolic computation software *Maple* and can be expressed as a polynom of degree 5 in x and y, and of total degree 6 if we consider that the points a, b and c are also variables.

theorem 6 *the L_2 error made by the local linear approximation of the unit hyperbolic paraboloid with respect to a triangle is invariant by translation in the plane (xOy) of the vertices whose lifted images compose the chosen triangle.*

proof : the proof has been done using symbolic computation [DD]. □

This theorem is of big interest : henceforth, when searching for properties concerning one or two triangles, we will always have the right to choose one vertex as the origin by translating all the vertices. The calculus will be much simpler.

It means that the position of the hyperbolic paraboloid, considered as a lifting surface, is not important: the same property holds for the paraboloid when it is used for the Delaunay triangulation [Aur91].

The possible triangulations of four points

Now, we will take a look at how to use ϵ. Let's begin by a very simple remark.

Let $o = (0,0)$, $b = (x_b, y_b)$, $c = (x_c, y_c)$ be three points in the plane, and let $O = (0,0,0)$, $B = (x_b, y_b, z_B)$, $C = (x_c, y_c, z_C)$ be their respective lifted images on π.

Let $d = (x,y)$ be a new point in the plane. By looking at the position of d with regard to the other vertices, several cases may appear (figure 2):

- d is in a white zone: only one triangulation is possible (this is an uninteresting case).
- d is in a shaded zone: in such a zone, which will be called an *influence zone*, two triangulations of the four vertices are possible. For example, if d belongs to the influence zone of point o, we have the choice between Δobc and Δbcd or Δobd and Δocd.

The influence zones limit the plane domain where the separation curves of one triangle must be helpful for the

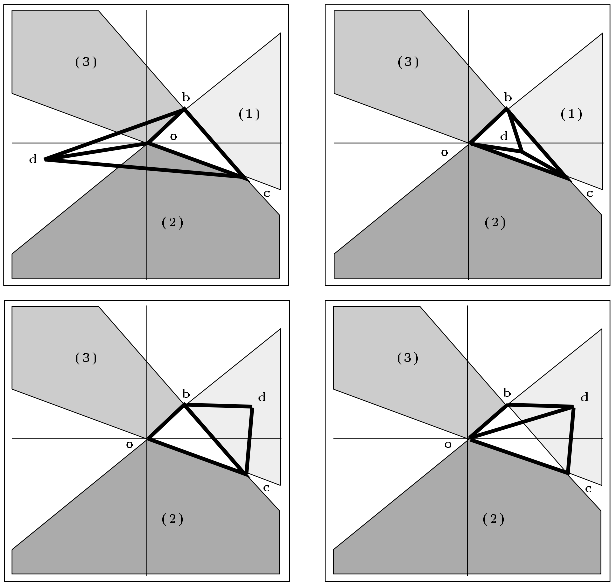

Figure 2: possible triangulations of four vertices.

final locally optimal triangulation. Examples of such separation curves can be found in figure 3.

Computation of the separation curve

There are many possibilities for the relative positions of the four points, but they all give the same result for the separation curve of triangle Δobc [DD].
We will only study one case here. So, let's suppose that d belongs to the influence zone of o. Then the quadrilateral $obcd$ is convex and two triangulations are possible. We have to compute:

$$\begin{aligned} e(\Delta obc + \Delta bcd) &= \sqrt{\epsilon(\Delta obc) + \epsilon(\Delta bcd)} \\ e(\Delta obd + \Delta ocd) &= \sqrt{\epsilon(\Delta obd) + \epsilon(\Delta ocd)} \end{aligned}$$

The value of interest is therefore:

$$\xi(\Delta obc, d) = [\epsilon(\Delta obc) + \epsilon(\Delta bcd)] - [\epsilon(\Delta obd) + \epsilon(\Delta ocd)]$$

and more presicely, ξ is interesting when it equals 0, because we have by definition:

$$d \in SC(\Delta obc) \quad \Leftrightarrow \quad \xi(\Delta obc, d) = 0.$$

The results given for $\xi(\Delta obc, d)$ by the formal language *Maple*, which is supposed to compute correctly polynomial integrals, are after simplifications:

$$\begin{aligned} &\xi(\Delta obc, d) = \\ &\lambda\,[\,3\,x^2 \;-\; 3\,y^2 \;-\; 2\,(x_c + x_b)\,x + 2\,(y_c + y_b)\,y \\ &\quad + 3\,({x_c}^2 - {y_c}^2) \;+\; 3\,({x_b}^2 - {y_b}^2) \;+\; 2\,(y_b\,y_c - x_b\,x_c)] \\ &\times [\,(x_b\,y_c - y_b\,x_c)\,(x^2 - y^2) \\ &\quad + (y_b\,{x_c}^2 - y_c\,{x_b}^2 + y_c\,{y_b}^2 - y_b\,{y_c}^2)\,x \\ &\quad + (x_b\,{y_c}^2 - x_c\,{y_b}^2 + x_c\,{x_b}^2 - x_b\,{x_c}^2)\,y\,], \end{aligned}$$

where λ is a positive constant, which is not important in this study.

When observing the expression of ξ and determining when it is null, the searched property becomes evident:

observation 7 *Let a, b and c be three points in the plane, and let A, B and C be their respective lifted images on the hyperbolic paraboloid. The* separation curve *of triangle Δabc consists in two hyperbolae:*

- *H_1 defined by $x^2 - y^2 = p_{\Delta abc}(x, y)$*

$$\begin{cases} z = x^2 - y^2 \\ z = \alpha\,x + \beta\,y + \gamma \qquad (\mathcal{P}_{\Delta ABC}) \\ A \in \mathcal{P}_{\Delta ABC},\ B \in \mathcal{P}_{\Delta ABC},\ C \in \mathcal{P}_{\Delta ABC} \end{cases}$$

- *H_2 whose equation is:*

$$\left(x - \frac{x_a + x_b + x_c}{3}\right)^2 - \left(y - \frac{y_a + y_b + y_c}{3}\right)^2 = 4\overline{HG}$$

with

$$G = \begin{pmatrix} \dfrac{x_a + x_b + x_c}{3} \\ \dfrac{y_a + y_b + y_c}{3} \\ {x_G}^2 - {y_G}^2 \end{pmatrix}, H = \begin{pmatrix} \dfrac{x_a + x_b + x_c}{3} \\ \dfrac{y_a + y_b + y_c}{3} \\ \dfrac{z_A + z_B + z_C}{3} \end{pmatrix}.$$

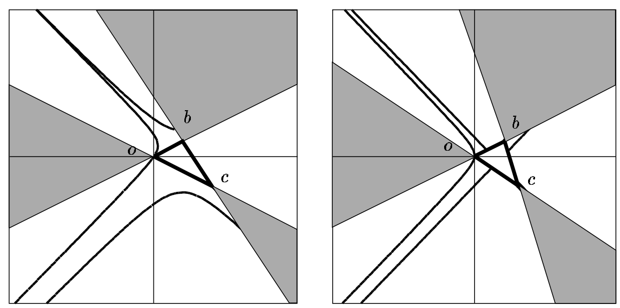

Figure 3: examples of separation curves in influence zones (white zones).

Interpretation of the curves

Hyperbola H_1 corresponds to the second term of ξ: it is in fact the projection in the plane of the intersection between π and the plane $\mathcal{P}_{\Delta ABC}$. This property can be related with the circumcircle of a triangle in the Delaunay triangulation: the circle is the projection of the intersection between the unit paraboloid and the space plane that contains the lifted images of the vertices of the triangle ([ES86] or [Aur91] for a survey).
The interpretation of hyperbola H_2 seems to be more difficult; indeed, its equation has been found thanks to empirical experiments and a few remaks. First of all, it must not be forgotten that the hyperbolic paraboloid is a doubly ruled surface.

proposition 8 *the lines belonging to the generating families of the unit hyperbolic paraboloid, when projected in the plane, are the lines parallel to the bisecting lines of the frame.*

proof : the equations of the generating lines of the surface, whose equation is $z = x^2 - y^2$, are:

$$\begin{cases} x - y &= \lambda z \\ x + y &= \dfrac{1}{\lambda} \end{cases} \text{ and } \begin{cases} x + y &= \mu z \\ x - y &= \dfrac{1}{\mu} \end{cases} \text{ with } \lambda, \mu \in I\!R - \{0\}$$

plus the two lines (corresponding to the plane $z = 0$):

$$\begin{cases} x + y &= 0 \\ z &= 0 \end{cases} \text{ and } \begin{cases} x - y &= 0 \\ z &= 0 \end{cases}.$$

Looking at the projection of these lines in the plane gives the correct answer. □

This proposition becomes very interesting when it is combined with:

proposition 9 *any hyperbola of the plane, whose asymptotes are parallel to the frame bissectrices, corresponds to the plane projection of the intersection between the unit hyperbolic paraboloid and a plane.*

proof : these hyperbolae have an equation of the following form

$$(x - x_0)^2 - (y - y_0)^2 = p,$$

where $x_0, y_0, p \in I\!R$.

Let $(\mathcal{P}) : z = \alpha\, x + \beta\, y + \gamma$ be the searched plane; then:

$$\begin{cases} \alpha &= 2\, x_0 \\ \beta &= -2\, y_0 \\ \gamma &= p - {x_0}^2 + {y_0}^2 \end{cases}$$

So, $\mathcal{P}$ always exists, and is unique. □

Now it can be said that the H_2, as H_1, corresponds to the intersection between the surface and a plane: moreover, it can be a little bit more precisely defined with the help of projective geometry and *polar conjugate* planes.

definition 10 *four points M, N, A, B lying on a common line are said to form an* harmonic division *if:*

$$\frac{MA}{MB} = \frac{NA}{NB}$$

and we can write in this case:

$$(M, N, A, B) = \frac{\overline{MA}}{\overline{MB}} \cdot \frac{\overline{NB}}{\overline{NA}} = -1.$$

definition 11 *two points M and N are said to be* conjugate *with respect to a quadric $\mathcal{Q}$ if $(M, N, A, B) = -1$ where A and B are the intersection points of line (MN) and $\mathcal{Q}$. If M does not belong to $\mathcal{Q}$, and if we choose for the line (MN) a tangent from M to $\mathcal{Q}$, then $A = B = N$ is the tangent point.*

The locus of the conjugate of M with respect to $\mathcal{Q}$ is a plane $\mathcal{P}$ passing through these tangent points, called the polar plane *of M with respect to $\mathcal{Q}$.*

Here, as the equation of the quadric π is $x^2 - y^2 - z = 0$, the equation of the polar plane $\mathcal{P}$ of M with respect to π can be obtained by polarizing in M the equation of the quadric:

$$2\, x\, x_M - 2\, y\, y_M - z_M = z.$$

Finally, when combining the equation of H_2 and the definition of a polar plane, we can state that this curve corresponds to the intersection between the hyperbolic paraboloid and the polar plane of the point $(x_G, y_G, 4z_H - 3z_G)$, with G and H defined in observation 7.

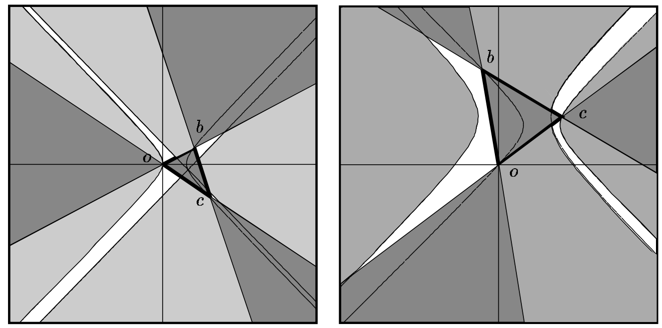

Figure 4: there must be no points of $\mathcal{S}$ in white zones if we want Δobc to be optimal.

Use of the separation curve

From the definition of ξ, it can be deduced that:

$$\xi(\Delta obc, d) \begin{cases} > 0 \Rightarrow \text{triangulation } \Delta obd, \Delta ocd \text{ is better} \\ = 0 \Rightarrow d \in SC(\Delta obc) \\ < 0 \Rightarrow \text{triangulation } \Delta obc, \Delta bcd \text{ is better} \end{cases}$$

And, finally, the desired criterion is obtained.

proposition 12 *Let $\mathcal{S}$ be a set of n points in the plane, a, b and c three distinct elements of $\mathcal{S}$, and let A, B, C be their respective lifted images on the unit hyperbolic paraboloid. Let*

$$\begin{cases} H_1 : h_1(x, y) = 0 \\ H_2 : h_2(x, y) = 0 \end{cases}$$

with

$$\begin{cases} h_1(x, y) &= (x - x_0)^2 - (y - y_0)^2 - p \\ h_2(x, y) &= (x - x_G)^2 - (y - y_G)^2 - 4\overline{HG} \end{cases}$$

be the two hyperbolae defined in proposition 7. Then, the triangle Δabc must be preserved in a final L_2 locally optimal triangulation of S for the linear approximation of the unit hyperbolic paraboloid iff there is no vertex (α, β) neighboring Δabc that verifies :

$$h_1(\alpha, \beta).h_2(\alpha, \beta) > 0.$$

proof : the proof has been done using symbolic computation [DD]. □

In fact, this proposition can be extended to a global optimality criterion: instead of only testing the adjacent vertices of a triangle, we must test all the vertices of $\mathcal{S}$. But this property has not been proved to be easily reached (it seems that, in general, a set of points $\mathcal{S}$ does not verify it).

Numerical results

In this section, we will summarize the main results of all the experiments made for testing our criterion. More details are available [DD].

As it has been stated, the task of this paper was to find a *locally* optimal triangulation: one of the algorithms commonly used for constructing this kind of triangulations is based on the local optimization procedure suggested by Lawson [Law77]:

- find an initial triangulation $\mathcal{T}$.
- while there exists an interior edge ς that is not locally optimal:

    swap the diagonals of the strictly convex quadrilateral whose diagonal is ς.
- the final triangulation is locally optimal.

It can be easily shown that such a procedure always finishes, mainly because there is a finite number of possible triangulations.

Two initial triangulations were tested: the Delaunay triangulation, which contributes to observe whether the results were good or not, and the triangulation obtained when inserting the points one after another without any swap.

Another approach has also been tested: instead of beginning with a special triangulation and all the points of the set, we introduce them one after one and construct an incremental triangulation using our criterion for swapping. In fact, it corresponds to apply the Lawson-like procedure every time a point is inserted, with the initial triangulation equal to the former one.

For both approaches, when making locally optimal triangulations, the order of the edges to be swapped may be of high importance, especially when it is almost sure that the flipping strategy will not lead to a globally optimal triangulation. Different selection strategies have been given by Dyn, Levin and Rippa [DLR90a]: swap at first edges that will give a maximum possible number of new convex quadrilaterals for the next iteration, swap at first the edge that minimizes the chosen criterion... The latter method is rather difficult to apply here because

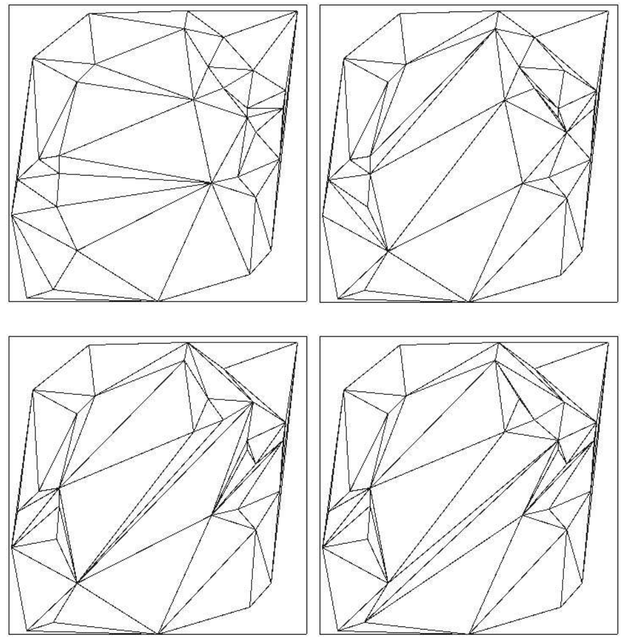

Figure 5: results for a set of 33 points: the Delaunay triangulation (first picture) gives a much more important error than the three others, obtained by different methods.

our criterion only gives a yes/no answer: but, it can be done by assimilating the criterion with the minimization of the L_2 error, which is a little annoying because it is the L_2 error that helps us to compare the results of the different triangulations.

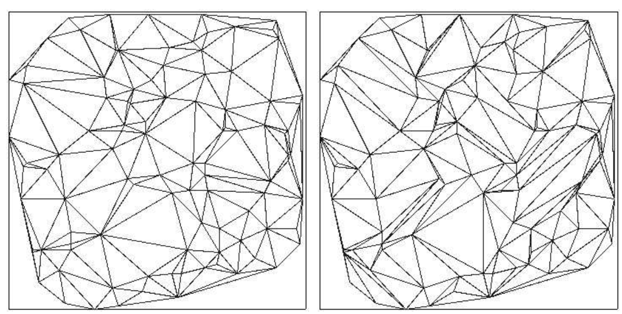

Figure 6: results for a set of 100 points: the first triangulation is Delaunay, the other one is obtained by swapping edges starting from the Delaunay triangulation.

Finally, we obtained various results: one method is not always the best one for every set of points (but this is a globally common result of the data dependent triangulations). If we want to compare the approximation error made with the Delaunay triangulation and with one of our triangulations, we can say that the improvement is on an average between 7 and 10% (it means that our triangulation gives a L_2 error 10% inferior to the one obtained with Delaunay). By now, only sets of points uniformly distributed in a square have been tested a lot:

sometimes, the improvement can be a lot better than 10%. For example, in figure 6, the final triangulation gives an error 20% inferior to Delaunay, for a set of 100 points uniformly distributed in the unit square. Moreover, for the set of 33 points (also in the unit square) of figure 5, we obtained 3 very different triangulations by use of our diverse methods, and the final error was always between 35% and 40% inferior to the error made by the Delaunay triangulation.

Conclusion

As it is shown in the numerical results, our criterion only remains local: it does not lead to a globally optimal triangulation, as it is the case for the use of the Delaunay triangulation for approximating the unit paraboloid.
Nevertheless, when applying the same argument as ours for the paraboloid, something very similar is obtained: the first term of the equation of the separation curve of a triangle t corresponds to the circumcircle of t (which is also the plane projection of the intersection between the paraboloid and the plane containing the lifted images of the three vertices of t). As for the hyperbolic paraboloid, the second term is the intersection between the polar conjugate plane of $(x_G, y_G, 4z_H - 3z_G)$ and the paraboloid,. Then, the equation of this curves gives something like $(x - x_G)^2 + (y - y_G)^2 = -\lambda^2$ which is clearly an imaginary curve: this explains why the separation curve of a Delaunay triangle is only a circle.
In fact, it is more than likely that the results obtained for the hyperbolic paraboloid can be generalized for all the quadrics.

Acknowledgments

We would like to thank Andreas Fabri for his helpful geometric programming tool for computing all our triangulations. We wish also to thank Monique Teillaud for all her interesting advice.

References

[Aur91] F. Aurenhammer. Voronoi diagrams: a survey of a fundamental geometric data structure. *ACM Comput. Surv.*, 23:345–405, 1991.

[Bro91] J. L. Brown. Vertex based data dependent triangulations. *Comput. Aided Geom. Design*, 8:239–251, 1991.

[DD] P. Desnoguès and O. Devillers. A locally optimal triangulation of the hyperbolic paraboloid. Technical Report, INRIA, Sophia Antipolis, France. to appear.

[DLR90a] N. Dyn, D. Levin, and S. Rippa. Algorithms for the construction of data dependent triangulations. In J. C. Mason and M. G. Cox, editors, *Algorithms for Approximation II*, pages 185–192. Chapman and Hall, London, 1990.

[DLR90b] N. Dyn, D. Levin, and S. Rippa. Data dependent triangulations for piecewise linear interpolation. *IMA Journal of Numerical Analysis*, 10:137–154, 1990.

[ES86] H. Edelsbrunner and R. Seidel. Voronoi diagrams and arrangements. *Discrete Comput. Geom.*, 1:25–44, 1986.

[Law77] C. L. Lawson. Software for C^1 surface interpolation. In J. R. Rice, editor, *Math. Software III*, pages 161–194, New York, NY, 1977. Academic Press.

[Mel93] E. A. Melissaratos. l_p optimal d dimensional triangulations for piecewise linear interpolation: A new result on data dependent triangulations. Technical Report RUU-CS-93-13, Dept. Comput. Science, Utrecht Univ., Utrecht, NL, 1993.