

A formal Γ -convergence approach for the detection of points in 2-D images

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*A formal Γ -convergence approach for the detection
of points in 2-D images.*

Daniele Graziani, Laure BLANC-FÉRAUD, Gilles Aubert

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Thème COG



*Rapport
de recherche*

3 **A formal Γ -convergence approach for the detection of**
4 **points in 2-D images.**

5 Daniele Graziani*, Laure BLANC-FÉRAUD*, Gilles Aubert†

6 Thème COG — Systèmes cognitifs
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9 **Abstract:** We propose a new variational model to isolate points in a 2-dimensional image.
10 To this purpose we introduce a suitable functional whose minimizers are given by the points
11 we want to detect. In order to provide numerical experiments we replace this energy with a
12 sequence of a more treatable functionals by means of the notion of Γ -convergence.

13 **Key-words:** points detection, curvature-depending functionals, divergence-measure fields,
14 Γ -convergence, biological 2-D images.

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15 A Γ -convergence approach for the detection of points in
16 2-D images.

17 **Résumé :**

18 **Mots-clés :**

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1 Introduction

Detecting fine structures, like points or curves in two or three dimensional images respectively, is an important issue in image analysis. In biological images a point may represent a viral particle whose visibility is compromised by the presence of other structures like cell membranes or some noise.

From a variational point of view the problem of isolating points is a difficult task, since it is not clear how these singularities must be classified in terms of some differential operator. Indeed, since these are usually defined as discontinuity without jump, we cannot use the gradient operator as in the classical problem of contours detection. As a consequence the functional framework we have to deal with may be not clear.

One possible strategy to overcome this obstacle is considering these kinds of pathology as a k -codimension object, meaning that they should be regarded as a singularity of a map $U : \mathbb{R}^{k+m} \rightarrow \mathbb{R}^k$, with $k \geq 2$ and $m \geq 0$ (see [7] for a complete survey on this subject). So that the detecting point case corresponds to the case $k = 2$ and $m = 0$.

In this direction, in [5], the authors have suggested a variational approach based on the theory of Ginzburg-Landau systems. In their work the isolated points in 2-D images are regarded as the topological singularities of a map $U : \mathbb{R}^2 \rightarrow \mathbb{S}^1$, where \mathbb{S}^1 is a unit sphere of \mathbb{R}^2 . So that it is crucial to construct, starting from the initial image $I : \mathbb{R}^2 \rightarrow \mathbb{R}$, an initial vector field $U_0 : \mathbb{R}^2 \rightarrow \mathbb{S}^1$ with a topological singularity of degree 1 at points in the initial image I where the intensity is high. How to do this in a rigorous way, it is a subject of a current investigation.

Here our purpose is to provide a lighter variational formulation, in which the singularity in the image is directly given in terms of a proper differential operator defined on vector fields. Another important difference is that in [5] points and curves are detected both as singularities, while in the present paper our aim is to isolate from the initial image points and at same time remove any other singularities like curves precisely. In our model $\Omega \subset \mathbb{R}^2$ is an open set and represents the image domain.

In order to detect the singularities of the image, we have to find a functional space whose elements generate, in a suitable sense, a measure concentrated on points. Such a space is $\mathcal{DM}^p(\Omega)$ introduced in [4] where $1 < p < 2$, the space of vector fields $U : \Omega \rightarrow \mathbb{R}^2$ whose distributional divergence is a Radon measure (see Subsection 2.2 for definitions and examples). The restriction $1 < p < 2$ is due to the initialization (see subsection 2.3).

Unfortunately even if we are capable of constructing an initial vector field U_0 (see below for such a construction) belonging to the space $\mathcal{DM}^p(\Omega)$, its singular set could contains several structures we want to remove from the original image like for instance curve or some noise. Hence, after the initialization we have to clear away all the structures we are not interested in by building up, starting from the initial data U_0 , a new vector field U whose singularities are given by the points of the image I we want to isolate.

Thus, from one hand we have to force the concentration set of the distributional divergence of U_0 to contain only the points we want to catch, and on the other hand we have to regularize the initial data U_0 outside the points of singularities. To this end we propose to minimize an energy involving a competition between a divergence term and the counting

80 Hausdorff measure \mathcal{H}^0 . More precisely the energy is the following

$$\int_{\Omega \setminus P} |\operatorname{div} U|^2 + \lambda \int_{\Omega} |U - U_0|^p + \mathcal{H}^0(P). \quad (1)$$

81 where $U \in L^{p,2}(\operatorname{div}; \Omega \setminus P)$ is the space of L^p -vector fields whose distributional divergence
 82 belongs to $L^2(\Omega \setminus P)$, P is the atomic set we want to target and λ is a positive weight. The
 83 first integral forces U to be regular outside P , while the term $\mathcal{H}^0(P)$ penalizes the presence
 84 of singular curves in the image.

85 From a practical point of view, this choice allows us to work with a first order differential
 86 operator and permits us to formulate the minimization problem in a common functional
 87 framework.

88 Clearly for initializing the minimization process we need to construct from the initial
 89 image, a vector field U_0 belonging to $\mathcal{DM}^p(\Omega)$. Such a vector field can be provided by the
 90 gradient of weak solution of the classical Dirichlet problem with measure data.

$$\begin{cases} \Delta f = I & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

91 In order to provide computer examples, we must approximate functional (1) by means of a
 92 sequence of more convenient functionals.

93 The approximation, we suggest in this paper, is based on the so called Γ -convergence,
 94 the notion of variational convergence introduced by De Giorgi (see [13, 14]). This theory
 95 is designed to approximate a variational problem by a sequence of different variational
 96 problems. The most important feature of the Γ -convergence relies on the fact that it implies
 97 the convergence of minimizers of the approximating functionals to those of the limiting
 98 functional. In this work we suggest a possible Γ -convergence approach for the detection of
 99 points. By the way we stress out that the Γ convergence result is only conjectured in this
 100 paper, whose purpose is to test a new variational method from an experimental point of
 101 view. For a rigorous variational approximation in a particular case, we refer the reader to
 102 [6].

103 Classically variational approximation techniques such as Γ -convergence or continuation
 104 method (see [2, 3] and [20] respectively) have been successfully employed in image and signal
 105 processing.

106 For instance in ([2, 3]) Ambrosio and Tortorelli have proven that the classical Mumford-
 107 Shah's functional for detecting 1-dimensional smooth boundaries, can be approximated by
 108 a sequence of elliptic functionals that are numerically more treatable.

The main difficulty here is related to the presence of a codimension 2 object, which is not
 a contour: the set P . In order to obtain a variational approximation close to the one provided
 in ([2, 3]), the crucial step is then to replace the term $\mathcal{H}^0(P)$ of functional (1) by a more
 handy, from a variational point of view, functional involving a smooth boundary and his
 perimeter given by the 1-dimensional Hausdorff measure \mathcal{H}^1 . Following some suggestion from
 the paper of Braides and Malchiodi and Braides and March (see [9, 10]) such a functional

is given by:

$$G_{\beta_\varepsilon}(D) = \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x),$$

109 where D is a proper regular set containing the atomic set P , κ is the curvature of its
110 boundary, the constant $\frac{1}{4\pi}$ is a normalization factor, and β_ε infinitesimal as $\varepsilon \rightarrow 0$. Roughly
111 speaking the minima of this functional are achieved on the union of balls of small radius, so
112 that when $\beta_\varepsilon \rightarrow 0$ the functional shrinks to the atomic measure $\mathcal{H}^0(P)$. On the other hand
113 the introduction of a curvature term requires a non trivial and convenient, for a numerical
114 point of view, approximation of the curvature-dependent functional. Such an approximation
115 is based on a celebrated conjecture due to De Giorgi (see [12]). By means of this argument
116 it is possible to substitute the curvature-depending functional with an integral functional
117 involving the Laplacian operator of smooth functions. Then it remains to approximate
118 the \mathcal{H}^1 -measure and this can be done by retrieving a classical gradient approach used in
119 [15, 16]. This strategy allows to deal with a functional whose Euler-Lagrange equations
120 can be discretized. A simple and intuitive explanation of the construction of the complete
121 approximating functionals will be given in section 3.

122 The paper is organized as follows: section 2 is intended to remind the reader of math-
123 ematical tools useful in the following. In section 3 we address the existence result for the
124 functional $F(U, P)$ defined in (1). In section 4 we state the two well-known Γ -convergence
125 results we need in the sequel. In section 5 we build in a formal way the approximating
126 sequence. In section 6 we present the discrete model and the detection strategy. Finally the
127 last section is devoted to some computer examples.

128 2 Preliminaries

129 2.1 Convergence for a set of points

130 For our purpose it will be crucial dealing with a notion of convergence for finite sets of points
131 introduced in [10].

132 **Definition 2.1.** We say that a sequence of a finite set of points $\{P_h\} \subset \overline{\Omega}$ converges to
133 a set $P \subset \overline{\Omega}$ if each of the sets P_h contains a number N of points $\{x_h^1, \dots, x_h^N\}$, with N
134 independent of h , such that $x_h^i \rightarrow x^i$ for any $i = 1, \dots, n$ and $\bigcup_{i=1}^N \{x_i\} = P$.

135 **Lemma 2.1.** Let $\{P_h\}$ be a sequence of a finite set of points such that $\mathcal{H}^0(P_h) \leq N_0$ for
136 every h with $N_0 \in \mathbb{N}$. Then there exists a subsequence $\{P_{h_k}\} \subset \{P_h\}$ and a set of points
137 $P \subset \overline{\Omega}$ such that P_{h_k} converges with respect to the convergence 2.1 to the set P .

138 **Proof.** Since $\mathcal{H}^0(P_h) \leq N_0$, we may find $N_1 \leq N_0$ such that every set P_h contains N_1 points.
139 For every $i = 1, \dots, N_1$ and there exists a subsequence $x_{h_k}^i \subset x_h^i$ converging to $x^i \in \overline{\Omega}$.

140 Then by setting $P_{h_k} = \bigcup_{i=1}^{N_1} x_{h_k}^i$ and $P = \bigcup_{i=1}^{N_1} x^i$, the thesis is achieved.

141 **Lemma 2.2.** Let $\{P_h\} \subset \overline{\Omega}$ be a sequence of finite set of points converging to a finite set of
142 points P . Then

$$\mathcal{H}^0(P) \leq \liminf_{h \rightarrow +\infty} \mathcal{H}^0(P_h) \quad (3)$$

Proof. From definition 2.1 it follows that

$$\liminf_{h \rightarrow +\infty} \mathcal{H}^0(P_h) \geq \liminf_{h \rightarrow +\infty} H^0(\{x_h^1, \dots, x_h^N\}) = N = \mathcal{H}^0(P). \quad \square$$

143 2.2 Distributional divergence

144 In this subsection we recall the definition of the space $L^{p,q}(\text{div}; \Omega)$ and $\mathcal{DM}^p(\Omega)$, introduced
145 in [4].

146 Let $\Omega \subset \mathbb{R}^2$ be an open set and let $U : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vector field.

147 **Definition 2.2.** We say that $U \in L^{p,q}(\text{div}; \Omega)$ if $U \in L^p(\Omega; \mathbb{R}^2)$ and if its distributional
148 divergence $\text{div}U \in L^q(\Omega)$. If $p = q$ the space $L^{p,q}(\text{div}; \Omega)$ will be denoted by $L^p(\text{div}; \Omega)$.

149 We say that $U \in L_{loc}^{p,q}(\text{div}; \Omega)$ if $U \in L^{p,q}(\text{div}; A)$ for every open set $A \subset\subset \Omega$.

Definition 2.3. For $U \in L^p(\Omega; \mathbb{R}^2)$, $1 \leq p \leq +\infty$, set

$$|\text{div}U|(\Omega) := \sup \left\{ \int_{\Omega} U \cdot \nabla \varphi dx dy : \varphi \in C_0^1(\Omega), |\varphi| \leq 1 \right\}.$$

We say that U is an L^p -divergence measure field, i.e. $U \in \mathcal{DM}^p(\Omega)$ if

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega; \mathbb{R}^2)} + |\text{div}U|(\Omega) < +\infty.$$

150 We say that $U \in \mathcal{DM}_{loc}^p(\Omega)$ if $U \in \mathcal{DM}^p(A)$ for every open set $A \subset\subset \Omega$.

151 **Remark 2.1.** If $U \in \mathcal{DM}^p(\Omega)$ then via Riesz Theorem it is possible to represent the distri-
 152 butional divergence of U by a Radon measure. More precisely there exists a Radon measure
 153 μ such that for every $\varphi \in C_0^1(\Omega)$ the following equality holds:

$$\int_{\Omega} U \cdot \nabla \varphi dx dy = - \int_{\Omega} \varphi d\mu.$$

154 For instance the field $U(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ belongs to $\mathcal{DM}_{loc}^1(\mathbb{R}^2)$ and its divergence
 155 measure is given by $-2\pi\delta_0$, where δ_0 is the Dirac mass.

156 Such a result can be proven by approximation. Let us define the following map:

$$U_{\varepsilon}(x, y) := \begin{cases} U(x, y) & \text{if } |x| \geq \varepsilon \\ (\frac{x}{\varepsilon^2}, \frac{y}{\varepsilon^2}) & \text{if } |x| < \varepsilon. \end{cases}$$

157 It is not difficult to check that u_{ε} is Lipschitz-map with divergence given by

158 $\frac{2}{\varepsilon^2} \chi_{B(0, \varepsilon)}$.

159 Then for every test function $\varphi \in C_0^1(\mathbb{R}^2)$ we have

$$\int U_{\varepsilon} \cdot \nabla \varphi dx dy = - \int \frac{2}{\varepsilon^2} \chi_{B(0, \varepsilon)} \varphi dx dy.$$

By applying the change of variables $x = \frac{x_1}{\varepsilon}$, $y = \frac{y_1}{\varepsilon}$ we obtain

$$\int U_{\varepsilon} \cdot \nabla \varphi dx dy = -2 \int \chi_{B(0, 1)} \varphi(\frac{x_1}{\varepsilon}, \frac{y_1}{\varepsilon}) dx_1 dy_1,$$

so that, letting $\varepsilon \rightarrow 0$, by the dominated convergence theorem we obtain

$$\int_{\Omega} U \cdot \nabla \varphi dx dy = -2\pi \varphi(0, 0) = -2\pi \int_{\Omega} \varphi d\delta_0$$

160 2.3 The Dirichlet problem with measure data

161 For the initialization of our algorithm we must build a vector field U_0 which should be such
 162 that its divergence is singular on points of the image I . Therefore we will use the gradient
 163 of the solution of the following Dirichlet problem (applied with $\mu = I$)

$$\begin{cases} \Delta f = \mu & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

164 where μ is a Radon measure.

165 Classical results (see [19]) guarantee the existence of a unique solution of problem (4).

166 Concerning the regularity it is known then $f \in W^{1,p}(\Omega)$ with $p < 2$.

167 3 Existence result

168 In this section we show the existence of a minimizing pair (U, P) for the functional F defined
169 in (1.)

170 Our argument needs two steps (see also [17] for a similar approach to minimize the
171 classical Mumford-Shah's functional). The first one consists in proving the existence a
172 minimizer of the functional (1) when the set P is fixed.

173 To this aim we adopte the following notation:

$$174 \quad F(U) = F(\cdot, P) = \int_{\Omega \setminus P} |\operatorname{div} U|^2 dx dy + \lambda \int_{\Omega} |U - U_0|^p dx dy + \mathcal{H}^0(P). \quad (5)$$

175 **Theorem 3.1.** *For every set P there exists a unique minimizer $U_P \in L^{p,2}(\Omega \setminus P; \operatorname{div})$ of*
176 *the functional (5).*

177 **Proof.** Let U_n be a minimizing sequence. Then we have the following bound

$$F(U_n) \leq M. \quad (6)$$

From the bound (6) and the classical inequality:

$$\|U_n\|_{L^p(\Omega \setminus P)}^p \leq 2^p \|U_n - U_0\|_{L^p(\Omega \setminus P)}^p + \|U_0\|_{L^p(\Omega \setminus P)}^p$$

it follows that

$$\|U_n\|_{L^p(\Omega \setminus P)}^p \leq M + \|U_0\|_{L^p(\Omega \setminus P)}^p := C.$$

Moreoveor we also have:

$$\|\operatorname{div} U_n\|_{L^2(\Omega \setminus P)}^2 \leq F(U_n) \leq M;$$

178 so that, up to subsequences, we obtain

$$\begin{cases} U_n \rightharpoonup U_P & \text{in } L^p(\Omega \setminus P) \\ \operatorname{div} U_n \rightharpoonup \operatorname{div} U_P & \text{in } L^2(\Omega \setminus P). \end{cases} \quad (7)$$

179 Therefore we can conclude that U_n weakly converges in $L^{p,2}(\Omega \setminus P; \operatorname{div})$ to a vector field
180 $U_P \in L^{p,2}(\Omega \setminus P; \operatorname{div})$.

Then we have thanks to semicontinuity properties of the L^p -norm with respect to the weak convergence:

$$\inf_U F(U) \leq F(U_P) \leq \liminf_{n \rightarrow +\infty} F(U_n) = \inf_U F(U).$$

181 Finally the strong convexity of functional (5) gives the uniqueness of the minimizer U \square

182 Once we obtained the existence of the minimizer U_P for every set P fixed, we focus on
183 the following functional.

$$184 \quad E(P) := F(U_P, P) = \int_{\Omega \setminus P} |\operatorname{div} U_P|^2 dx dy + \lambda \int_{\Omega} |U_P - U_0|^p dx dy + \mathcal{H}^0(P). \quad (8)$$

185 We extend U and $\operatorname{div}U$ by zero on P . However we keep the integration domain of $\operatorname{div}U$ to
 186 be $\Omega \setminus P$. We do that in order to make clear that $\operatorname{div}U$ is the distributional divergence of U
 187 on $\Omega \setminus P$ and not on Ω .

188 The following semicontinuity lemma plays a key role.

Lemma 3.1. *Assume that a sequence of finite sets of points $\{P_n\} \subset \overline{\Omega}$ converges to a finite set of point $P \subset \overline{\Omega}$. Then*

$$E(P) \leq \liminf_{h \rightarrow \infty} E(P_n)$$

Proof. Let us set $U_n = U_{P_n}$, we can assume that U_n and $\operatorname{div}U_n$ are both defined on all of Ω in the sense explained above. The sequence U_n is bounded in $L^p(\Omega)$. Indeed, by taking into account that U_n is a minimizer of the functional (5),

$$\|U_n\|_{L^p(\Omega)}^p \leq 2^p \|U_n - U_0\|_{L^p(\Omega)}^p + \|U_0\|_{L^p(\Omega)}^p \leq 2^p F(0) + \|U_0\|_{L^p(\Omega)}^p = \|U_0\|_{L^p(\Omega)}^p (2^p + 1).$$

189 In the same way one can show that the sequence $\operatorname{div}U_n$ is bounded in $L^2(\Omega)$. So that, up to
 190 subsequences, we may assume

$$\begin{cases} U_n \rightharpoonup U & \text{in } L^p(\Omega) \\ \operatorname{div}U_n \rightharpoonup V & \text{in } L^2(\Omega). \end{cases} \quad (9)$$

We claim that $\operatorname{div}U = V$ in $\Omega \setminus P$. In fact, take any test function φ with support in $\Omega \setminus P$, then since $P_n \rightarrow P$, we have for n large enough

$$\operatorname{supp}(\varphi) \subset \Omega \setminus P_n$$

and, consequently,

$$\int_{\operatorname{supp}(\varphi)} U_n \nabla \varphi dx = - \int_{\operatorname{supp}(\varphi)} \operatorname{div}U_n \varphi dx.$$

Therefore, by taking the weak limit by (9) we get

$$\int_{\operatorname{supp}(\varphi)} U \nabla \varphi dx = - \int_{\operatorname{supp}(\varphi)} V \varphi dx.$$

191 Then since the test function φ is arbitrary, we can conclude that $\operatorname{div}U = V$ on $\Omega \setminus P$.
 The thesis follows because, from the lower semicontinuity of the L^p -norm and Lemma
 2.2

$$E(P) \leq E(U, P) \leq \liminf_n E(U_n, P_n) \quad \square$$

192 We are now in position of proving the main result of this section.

193 **Theorem 3.2.** *There exist a minimizer (U, P) of the functional F , with $U \in L^{p,2}(\operatorname{div}; \Omega)$
 194 and $P \subset \Omega$ a finite set of points.*

195 **Proof.** For every P let U_P a minimizer for the functional F , whose existence is guaranteed
 196 by Theorem 3.1

Then we focus on the the functional $E(P) = F(U_P, P)$ and we take a minimizing sequence $\{P_n\}$. Then by Lemma 2.1 we have (up to a subsequences) that $P_n \rightarrow \bar{P} \subset \bar{\Omega}$ and $U_{P_n} \rightarrow U_{\bar{P}}$. By Lemma 3.1 we get

$$E(\bar{P}) \leq \liminf_{n \rightarrow +\infty} E(P_n).$$

197 Therefore

$$\inf_{(U,P)} F(U, P) \leq F(U_{\bar{P}}, \bar{P}) \leq \liminf_{n \rightarrow +\infty} E(P_n) \leq \liminf_{h \rightarrow +\infty} F(U_n, P_n) = \inf_P F(U_P, P) \leq F(U, P), \quad (10)$$

Now set $\tilde{P} := \bar{P} \setminus \partial\Omega$. Since for every P , U_P is a minimizer we get from (10)

$$F(U_{\bar{P}}, \tilde{P}) \leq F(U_{\bar{P}}, \bar{P}) \leq F(U_P, P) \leq F(U, P),$$

for every (U, P) . Hence we conclude that

$$F(U_{\bar{P}}, \tilde{P}) \leq \inf_{(U,P)} F(U, P). \quad \square$$

198 4 Γ -convergence

199 The key point of our strategy is to replace the functional (1) by means of more regular
200 functionals by following a formal Γ -convergence approach.

201 Therefore this section is devoted to a very simple presentation of the two results we need:
202 Modica-Mortola's theorem (see [15, 16]) concerning the approximation of the perimeter and
203 De Giorgi's conjecture (see [12]) about the approximation of curvature depending function-
204 als. For the definition of the Γ -convergence and its main properties we refer the reader to
205 [8, 11] and references therein.

206 4.1 Modica Mortola's approach

207 Modica-Mortola theorem states that it is possible to approximate, in the Γ -convergence
208 sense, a perimeter by means of the following sequence of functionals

$$F_\varepsilon^1(u) := \begin{cases} \int_\Omega (\varepsilon |\nabla u|^2 + \frac{V(u)}{\varepsilon}) dx & \text{if } u \in W^{1,2}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

209 where $V(u) = u^2(1-u)^2$ is a double well potential. Besides, since the minimizers of the
210 functional F_ε^1 may be trivial, some constraint on the functions u_ε must be added. Usually
211 a volume constraint of the type $\int_\Omega u dx dy = m$, is assumed.

212 Let us give an intuitive explanation of such a result. Since V has two absolute minimizers
213 at $u = 0, 1$, when ε is small, a local minimizer u_ε is close to 1 on a part of Ω and close to
214 0 on the other part, making a rapid transition of order ε between 0 and 1. When $\varepsilon \rightarrow 0$ the
215 transition set shrinks to a set of dimension 1, so that u_ε goes to a function taking values u
216 into $\{0, 1\}$ and the family of functionals Γ -converges to the measure of the perimeter of the
217 discontinuity set of u . Modica-Mortola's Theorem is the following.

218 **Theorem 4.1.** *The functionals $F_\varepsilon^1 : L^1(\Omega) \rightarrow [0, +\infty]$ Γ -converge with respect to the*
219 *L^1 -convergence to the following functional*

$$F^1(u) = \begin{cases} C_V \mathcal{H}^1(S_u) & \text{if } u \in \{0, 1\} \\ +\infty & \text{otherwise} \end{cases}$$

220 where, as usual, S_u denotes the set of discontinuities of u and C_V is a suitable constant
221 depending on the potential V .

222 4.2 De Giorgi's conjecture

The aim of De Giorgi was finding a variational approximation of a curvature depending
functional of the type:

$$F^2(D) = \int_{\partial D} (1 + \kappa^2) d\mathcal{H}^1;$$

223 where D is a regular set and k is a curvature of its boundary ∂D .

224 Since ∂D can be represented as the discontinuity set of the function $u_0 = 1 - \chi_D$,
 225 by Modica-Mortola's Theorem it follows that there is a sequence of non constant local
 226 minimizers such that $u_\varepsilon \rightarrow u_0$ with respect to the L^1 -convergence such that

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^1(u_\varepsilon) := C_V \mathcal{H}^1(\partial D).$$

227 Furthermore looking at the Euler-Lagrange equation associated to a contour length term,
 228 yields a contour curvature term κ , while the Euler-Lagrange equations for the functional
 229 $F_\varepsilon^1(u)$ contains a term $2\varepsilon\Delta u - \frac{V'(u)}{\varepsilon}$.

230 Then De Giorgi suggested to approximate the functional F^2 by adding to Modica-
 231 Mortola's approximating functionals the term

$$F_\varepsilon^2(u) = \int_{\Omega} (2\varepsilon\Delta u - \frac{V'(u)}{\varepsilon})^2 (\varepsilon|\nabla u|^2 + \frac{V(u)}{\varepsilon}) dx.$$

In [18] the authors have proven a simplified version of the De Giorgi's conjecture, where the integral above is replaced by the functional

$$F_\varepsilon^2(u) = \int_{\Omega} (2\varepsilon\Delta u - \frac{V'(u)}{\varepsilon})^2 dx.$$

232 5 The approximating functionals

233 In this section we present the energy we deal with and the construction of the approximating
234 sequence.

The energy we are interested in is given by

$$\int_{\Omega \setminus P} |\operatorname{div} U|^2 + \lambda \int_{\Omega} |U - U_0|^p + \mathcal{H}^0(P).$$

235 where $U \in L^{p,2}(\operatorname{div}; \Omega \setminus P)$, $U_0 \in \mathcal{DM}_{loc}^p(\mathbb{R}^2)$ and finally P is an atomic set consisting of a
236 finite number N of points, i.e. $P = \{x_1, \dots, x_N\}$.

As pointed out in the introduction, the first step is to substitute the counting measure $\mathcal{H}^0(P)$ with a more treatable term given by:

$$G_{\beta_\varepsilon}(D) = \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x);$$

237 where D is an union of regular simply connected sets $\{D_i\}$ with $i = 1, \dots, N$, such that
238 $x_i \in D_i$, $D_i \cap D_j = \emptyset$ for $i \neq j$. κ is the curvature of the boundary of the set D , the
239 constant $\frac{1}{4\pi}$ is a normalization factor and β_ε is infinitesimal as $\varepsilon \rightarrow 0$.

240 To understand why we can approximate $\mathcal{H}^0(P)$ with $G_{\beta_\varepsilon}(D)$ one should note that the
241 solution of the following minimum problem

$$\min_{D \supset P} G_{\beta_\varepsilon}(D) \tag{11}$$

242 is given by $D = \bigcup_i^N B(x_i, \beta_\varepsilon)$, where x_i are the points of P . We give an idea of a possible
243 proof in the case of a single point.

By the Young's inequality we have

$$G_{\beta_\varepsilon}(D) \geq \frac{1}{4\pi} \int_{\partial D} 2\kappa d\mathcal{H}^1(x)$$

244 and by applying the Gauss-Bonnet Theorem

$$G_{\beta_\varepsilon}(D) \geq \frac{1}{4\pi} (2)(2\pi) = 1 = \mathcal{H}^0(P).$$

245 Finally a simple calculation shows that, if we evaluate the functional G_{β_ε} on $B(x_1, \beta_\varepsilon)$,
246 we obtain the value 1, i.e. the number of points in P , i.e. $\mathcal{H}^0(P)$. The N points case can
247 be recovered with minor changes by the same argument.

For what follows it is convenient to split the functional G_{β_ε} in two terms:

$$G_{\beta_\varepsilon}(D) = G_{\beta_\varepsilon}^1(D) + G_{\beta_\varepsilon}^2(D)$$

248 where

$$G_{\beta_\varepsilon}^1(D) = \frac{1}{4\pi} \int_{\partial D} \frac{1}{\beta_\varepsilon} d\mathcal{H}^1(x);$$

249 and

$$G_{\beta_\varepsilon}^2(D) := \frac{1}{4\pi} \int_{\partial D} \beta_\varepsilon \kappa^2(x) d\mathcal{H}^1(x).$$

250 We can write an intermediate approximation of energy (1):

$$E_\varepsilon(U, D) = G_{\beta_\varepsilon}^1(D) + G_{\beta_\varepsilon}^2(D) + \int_{\Omega} (1 - \chi_D) |\operatorname{div}(U)|^2 + \lambda \int_{\Omega} |U - U_0|^p. \quad (12)$$

251 The advantage of such a formulation is that we know how to provide a variational ap-
 252 proximation of the perimeter measure $H^1 \llcorner \partial D$. Following Modica-Mortola's approach such
 253 an approximation can be obtained by using the following measure:

$$\mu_\varepsilon(w, \nabla w) dx = (\varepsilon |\nabla w|^2 + \frac{V(w)}{\varepsilon}) dx,$$

254 where $V(w) = w^2(1-w)^2$ is a double well functional.

255 Next step is expressing the curvature term by means of the function w . Thanks to
 256 the simplified version of the De Giorgi's conjecture we can replace the term κ by the term
 257 $2\varepsilon \Delta w - \frac{V'(w)}{\varepsilon}$.

So that we can formally write the complete approximating functional:

$$\begin{aligned} \Phi_\varepsilon(U, w) : &= \int_{\Omega} w^2 |\operatorname{div}(U)|^2 dx + \frac{1}{4\pi} \int_{\Omega} \beta_\varepsilon (2\varepsilon \Delta w - \frac{V'(w)}{\varepsilon})^2 dx + \frac{1}{\beta_\varepsilon} \int_{\Omega} \mu_\varepsilon(w, \nabla w) dx \\ &+ \lambda \int_{\Omega} |U - U_0|^p dx + \frac{2}{\mu_\varepsilon} \int_{\Omega} (1-w)^2 dx, \end{aligned} \quad (13)$$

258 where $U \in L^{p,2}(\operatorname{div}; \Omega)$ is equal to 0 on the $\partial\Omega$ and w is smooth function equal to 1
 259 on the boundary, i.e. $1-w \in C_0^\infty(\Omega)$, $\mu_\varepsilon \rightarrow 0$ when ε goes to 0. The last integral is a
 260 penalization term which prevents w_ε from converging to the function constantly equal to 0
 261 as $\varepsilon \rightarrow 0$.

262 Then if $(U_\varepsilon, w_\varepsilon)$ is a minimizing sequence of Φ_ε , then w_ε must be very close to the values
 263 1 when ε goes to 0, since the double well potential is positive except for $w_\varepsilon = 0, 1$ and w
 264 must be equal to 1 on $\partial\Omega$. On the other hand, near the points where the divergence is very
 265 big w_ε must be close to 0. Besides when $\varepsilon \rightarrow 0$, $\beta_\varepsilon \rightarrow 0$ goes to 0 as well, so that the singular
 266 set D is given by an union of balls of a small radius β_ε .

267 Therefore, while the functions U_ε approximate a minimizer U of the original functional,
 268 the level set $\{w_\varepsilon = 0\}$ approximate the original singular set P .

269 **Remark 5.1.** We point out that the Γ -convergence result is not proved in this paper, but
 270 only conjectured. A complete proof of the Γ -convergence result and the equicoerciveness of the
 271 sequence Φ_ε , in the particular case where the vector field U is a gradient, has been provided
 272 by the first and third author in [6].

The first variation of this functional leads to the following gradient flow system

$$\begin{aligned}\frac{\partial U}{\partial t} &= 2\nabla(w^2 \operatorname{div} U) + \lambda p |U - U_0|^{p-2} (U - U_0) \\ \frac{\partial w}{\partial t} &= -4 \frac{\Delta h}{\beta_\varepsilon} + \beta_\varepsilon h + \frac{2}{\varepsilon^2} \frac{1}{\beta_\varepsilon} V''(w) h - 2w |\operatorname{div} U|^2 + 2 \frac{1}{\mu_\varepsilon} (1 - w),\end{aligned}\quad (14)$$

273 where h is given by the equation

$$h = 2\varepsilon \Delta w - \frac{1}{\varepsilon} V'(w).$$

274 6 Detection

275 In our model the image contains an atomic Radon measure. Thus, in order to find an initial
 276 vector field which copies the singularities of the initial image, we use the gradient of the
 277 solution of the following Dirichlet problem:

$$\begin{cases} \Delta f = I & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \quad (15)$$

278 In this way we obtain a vector field whose divergence is singular on a proper set which
 279 contains the points we want to detect. In general this set could contain other structures.
 280 For instance if the initial image is a Radon measure concentrated both on points and on
 281 curves, the divergence of ∇f is singular on points and on curves at the same time. Besides
 282 if there is some noise in the image, it could be not clear how to differentiate the singular
 283 points due to the noise, from those we want to catch. As a consequence, by solving problem
 284 (15), we get just a predetection, which has to be refined.

285 To this purpose we search for a minimizer of the energy $\Phi_\varepsilon(U, w)$ via solving equations
 286 (14) with initial data U_0 given by ∇f . So that we obtain a vector field U whose divergence
 287 is relevant only on the set P and a function w whose zeros are given by the set P .

288 6.1 Discretization

The image is an array of size N^2 . We endowed the space $R^{N \times N}$ with the standard scalar product and standard norm. The gradient $\nabla I \in (R^{N \times N}) \times (R^{N \times N})$ is given by:

$$(\nabla I)_{i,j} = ((\nabla I)_{i,j}^1, (\nabla I)_{i,j}^2)$$

where

$$(\nabla I)_{i,j}^1 = \begin{cases} I_{i+1,j} - I_{i,j} & \text{if } i < N \\ 0 & \text{if } i = N, \end{cases}$$

$$(\nabla I)_{i,j}^2 = \begin{cases} I_{i,j+1} - I_{i,j} & \text{if } j < N \\ 0 & \text{if } j = 0. \end{cases}$$

289 We also introduce the discrete version of the divergence operator simply defined as the
 290 adjoint operator of the gradient: $\text{div} = -\nabla^*$. More in details if $v \in (R^{N \times N}) \times (R^{N \times N})$, we
 291 have

$$(\operatorname{div} v)_{i,j} = \begin{cases} v_{i,j}^1 + v_{i,j}^2 & \text{if } i, j = 1 \\ v_{i,j}^1 + v_{i,j}^2 - v_{i-1,j}^2 & \text{if } i = 1, 1 < j < N \\ v_{i,j}^1 - v_{i-1,j}^1 + v_{i,j}^2 - v_{i-1,j}^2 & \text{if } 1 < i < N, 1 < j < N \\ -v_{i-1,j}^1 + v_{i,j}^2 - v_{i-1,j}^2 & \text{if } i = N, 1 < j < N \\ v_{i,j}^1 - v_{i-1,j}^1 + v_{i,j}^2 & \text{if } 1 < i < N, j = 1 \\ v_{i,j}^1 - v_{i-1,j}^1 - v_{i-1,j}^2 & \text{if } 1 < i < N, j = N \\ -(v_{i-1,j}^1 + v_{i-1,j}^2) & \text{if } i, j = N. \end{cases}$$

292 Then we can define the discrete version of the Laplacian operator as $\Delta I = \operatorname{div}(\nabla I)$.

293 6.2 Discretization in time

We simply replace $\frac{\partial U}{\partial t}$ and $\frac{\partial w}{\partial t}$ by $\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\delta t}$ and $\frac{w_{i,j}^{n+1} - w_{i,j}^n}{\delta t}$ respectively. Then we write system (14) in the form (for simplicity we omit the dependence on ε)

$$\begin{cases} U_1^{n+1} = -\delta t \Phi_{U_1}(U_n, w_n) \\ U_2^{n+1} = -\delta t \Phi_{U_2}(U_n, w_n) \\ w^{n+1} = -\delta t \Phi_w(U_n, w_n). \end{cases}$$

294 We initialize our algorithm with $U(0) = \nabla f$, where f is the solution of problem (15). To do
295 this, we need to solve a Dirichlet problem with data measure I , therefore we regularize the
296 image by convolution with a Gaussian kernel G_σ with very small σ and then we solve, by
297 classical finite differences method, the problem:

$$\begin{cases} \Delta f = I_\sigma & \text{on } \Omega \\ f = 0 & \partial\Omega, \end{cases} \quad (16)$$

298 where $I_\sigma = I * G_\sigma$.

299 To initialize our algorithm, we need of an initial guess on w . So we choose $w(0) = 1$.

300 7 Computer examples

301 7.1 Parameter settings

302 Before running our algorithm all the parameters have to be fixed. The most important
 303 are ε , β_ε and μ_ε , which govern the set D approximating points we want to detect. Those
 304 parameters are related by the conditions $\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon |\log(\varepsilon)|}{\beta_\varepsilon} = 0$, $\lim_{\varepsilon \rightarrow 0} \frac{\beta_\varepsilon}{\mu_\varepsilon} = 0$. Furthermore, since
 305 the mesh grid size is 1 and β_ε gives the radius of a ball centered in the singular point we
 306 want to detect, from a discrete point of view the smallest value we can take is $\frac{\sqrt{2}}{2}$. Then we
 307 use the values 0.1 for ε , 0.7 for β_ε , and 0.8 for μ_ε . As exponent p of the discrepancy term
 308 we always take $p = 1.5$

309 Concerning the parameter λ we mainly used the value $\lambda = 0.1$, in order to force the
 310 algorithm to regularize as much as possible the initial data U_0 .

311 Since we deal with small values of ε , in order to have some stability, we must take a
 312 small discretization time step. Practically we mainly used the value $\delta t = 1 \times 10^{-6}$.

313 Concerning the stopping criterion we iterate the algorithm until $\max \left\{ \frac{\|U_1^{n+1} - U_1^n\|_1}{\|U_1^n\|_1}, \frac{\|U_2^{n+1} - U_2^n\|_1}{\|U_2^n\|_1}, \frac{\|w^{n+1} - w^n\|_1}{\|w^n\|_1} \right\}$
 314 $\leq 1 \times 10^{-2}$.

315 In all the computer examples the points are detected by means of the function w_ε . We
 316 display the level-set $\{w_\varepsilon \simeq 0\}$.

317 7.2 Commentaries

318 The figure 1 shows how resistant to the noise our model is. When the noise is larger the
 319 parameter ε must be as close as possible to the ideal value 0. Besides it is possible to see
 320 that for small values of ε our detection is very fine according to the continuous setting. More
 321 in details in the first row we display the initial image obtained by adding a Gaussian noise
 322 to a binary image of five points. The second row shows the behavior of w_ε for small values
 323 of ε and β_ε .

324 Looking at the histograms of the gray level of I and w_ε , one can see that it is easier
 325 fixing a threshold value starting from the function w_ε than from the initial image I . In the
 326 last row we display the set $\{w_\varepsilon \simeq 0\}$ obtained by plotting the set $\{w_\varepsilon \leq \alpha\}$ with threshold
 327 value $\alpha = 0.5$.

328 In figure 2 we test our algorithm on curves and points at the same time. In the first row
 329 we have a sequence of points and a curve with boundary inside Ω . In the second row we
 330 display the function w_ε and the level set $\{w_\varepsilon \simeq 0\}$ once again obtained by fixing a threshold
 331 value $\alpha = 0.5$. The result is that, as desired, our algorithm is capable of eliminating the
 332 curve from the initial image. According to the continuous setting when ε takes values close
 333 to 0 the approximating energy (13) behaves similarly to the limit energy (1), so that the
 334 presence of the curve is penalized in the minimization process. Then the set $\{w_\varepsilon \simeq 0\}$
 335 contains nothing else but points.

336 Finally in figure (a),(b),(c) and (d) we deal with a biological image. Our task is catch-
 337 ing the finest structure present in the image. In figure (d) the isolated points are quite

338 well detected, while the branches the cellule are not. Nevertheless due to the small time
 339 discretization step the computation time is quite large. To test the image in (a) of size
 340 500×500 , our algorithm takes 100 iterations and about 7mm. Certainly the algorithm can
 341 be accelerated by using more sophisticated techniques such as multigrid methods. Such a
 faster algorithm is the subject of our current investigation.

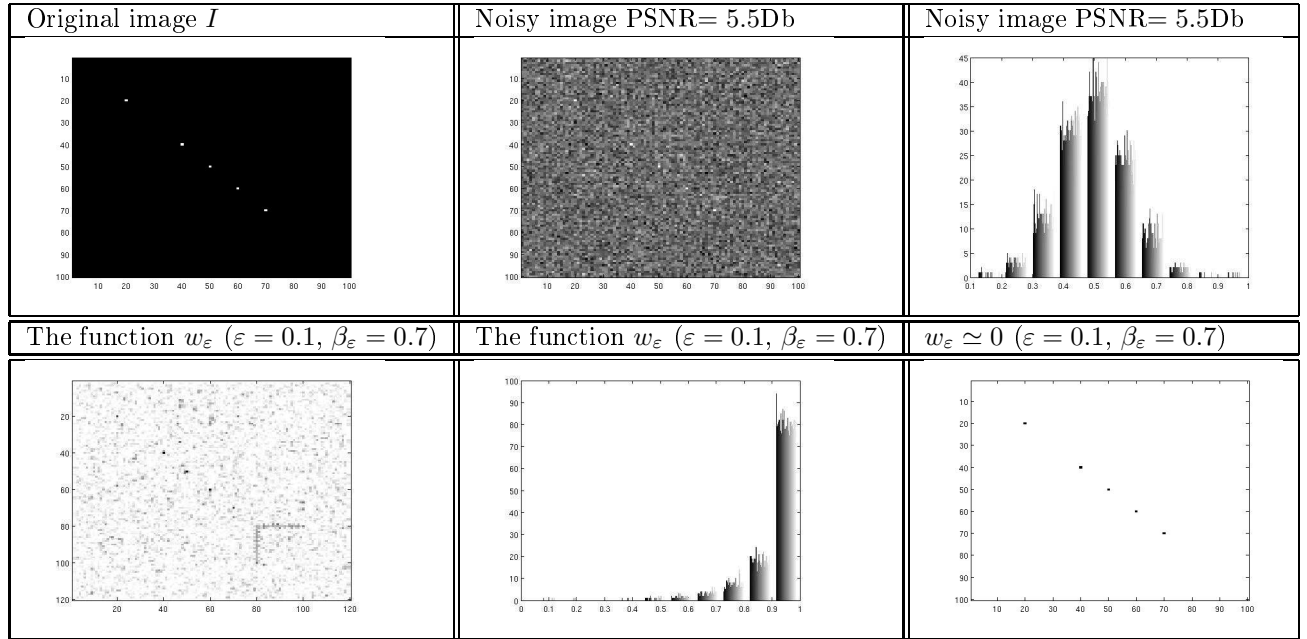


Figure 1: *Synthetic image: we test our algorithm on noisy images. When the parameters ε and β_ε are small as much as possible the detection is finer. The detection is refined by fixing a threshold value α for the function w_ε*

342

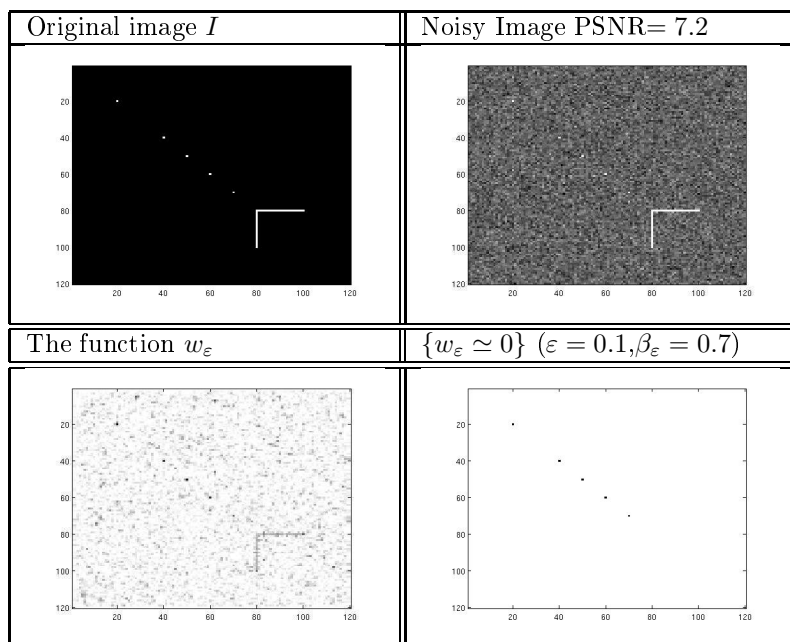
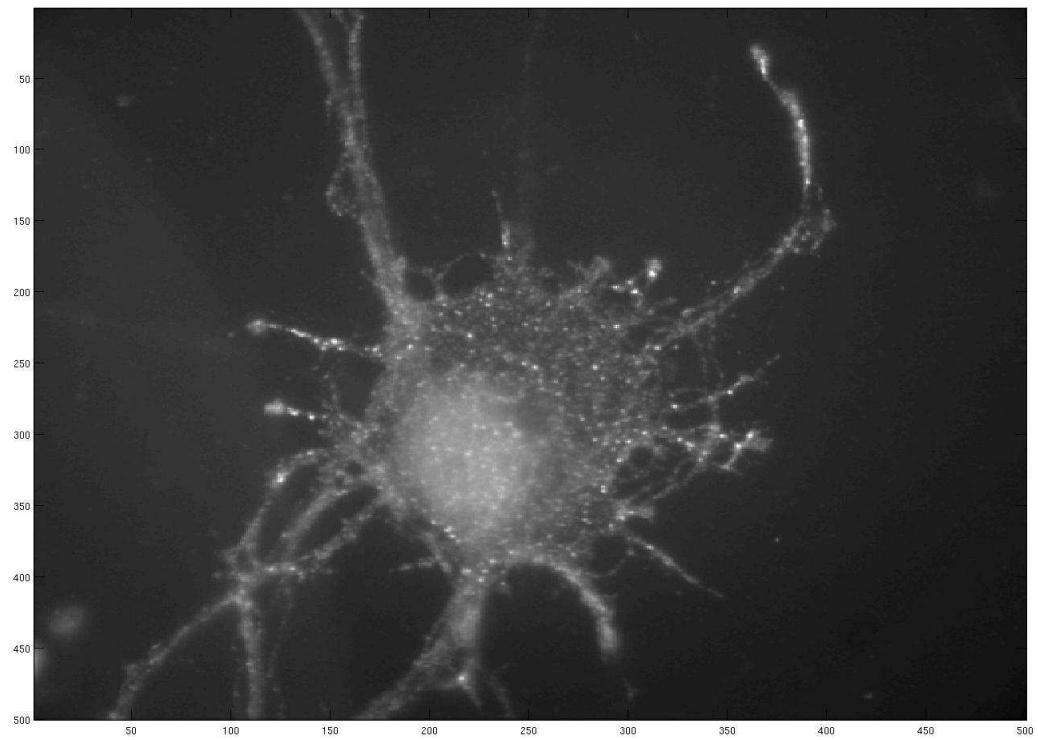
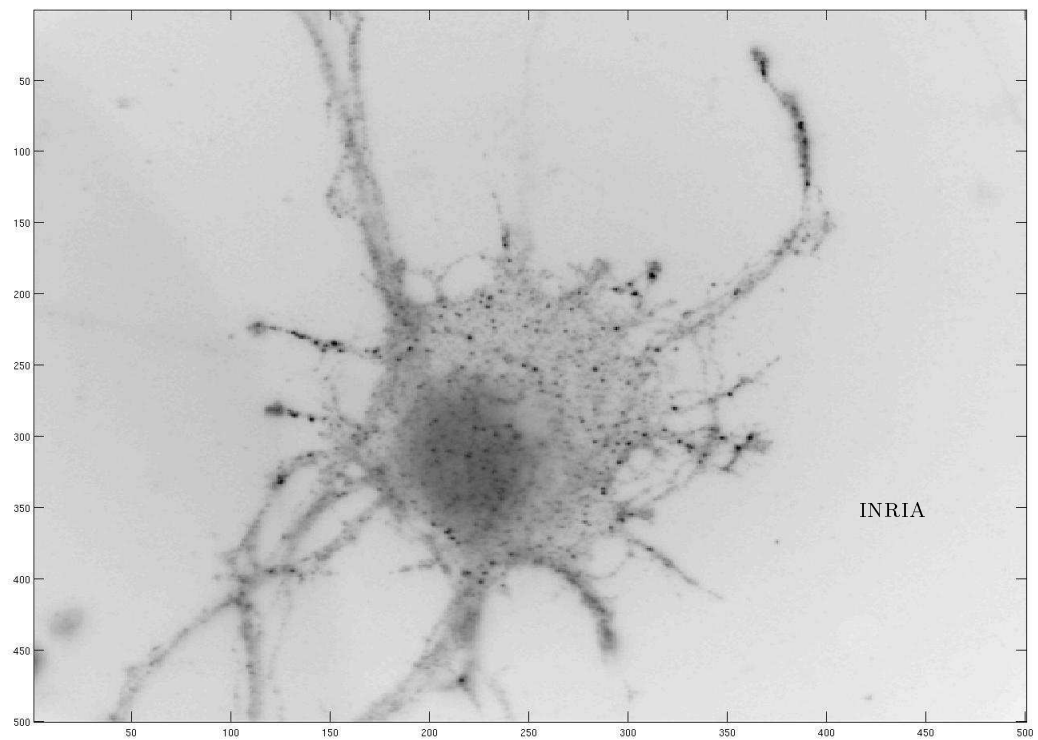
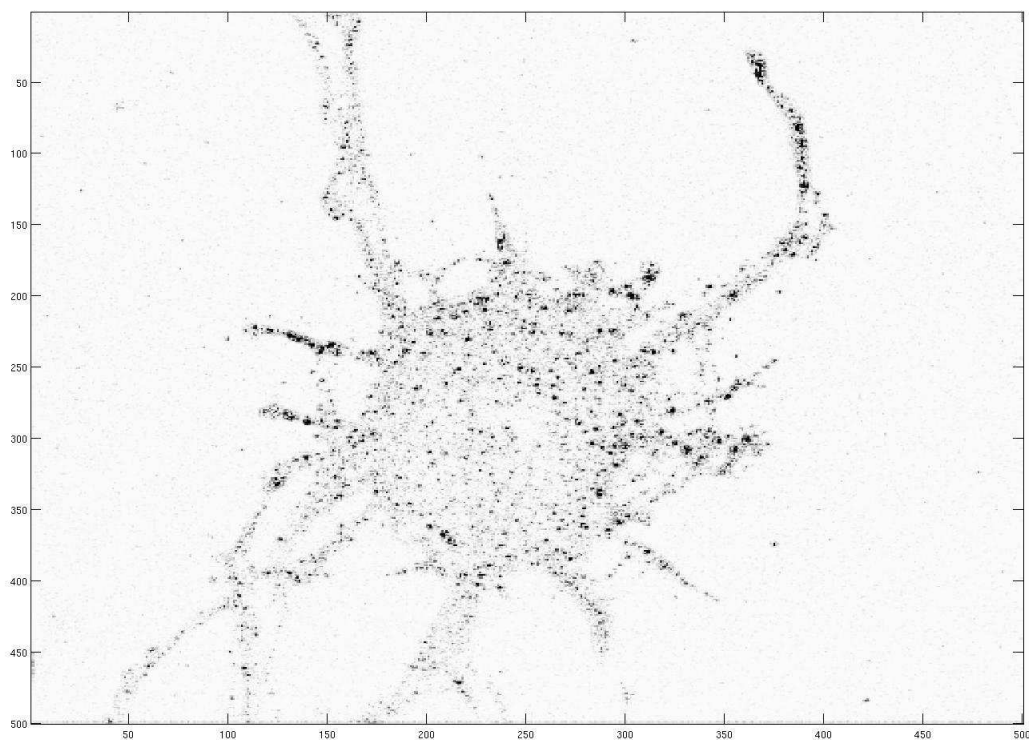


Figure 2: Synthetic image: curve points and noise are present in the initial image. As expected our method is capable of removing the curve from the image.

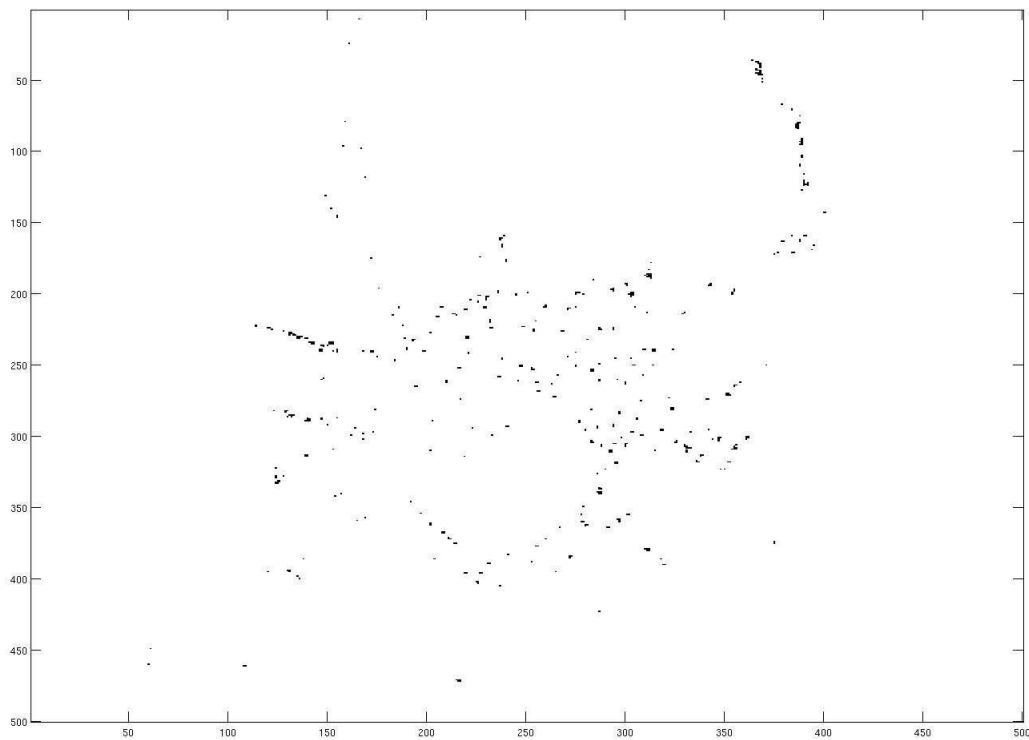


(a) Original Image

(b) Superposition of the original image with the set $\{w_\varepsilon \approx 0\}$ ($\varepsilon = 0.1, \beta_\varepsilon = 0.7$)



(c) The function w_ϵ ($\epsilon = 0.1$, $\beta_\epsilon = 0.7$)



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(d) the set $\{w_\epsilon \simeq 0\}$ ($\epsilon = 0.1$, $\beta_\epsilon = 0.7$)

343 8 Conclusion

344 In this work, a new variational method for spot detection in biological images has been
345 proposed and tested. We emphasize that, according to our knowledge, this is the first
346 method which makes possible isolating the spots from a filament in the observed image.
347 Moreover it also permits in a noisy image to fix a threshold value in a simple and direct way.
348 Moreover we believe that a suitable generalization of this method for the detection of spots
349 and even filaments in 3-D biological images can be provided. This is a subject of our current
350 investigation. Certainly there are many rooms for improvement both from a theoretical and
351 numerical point of view such as a deep investigation of the Γ -convergence approximation, as
352 a well as a significant acceleration of the algorithm.

References

- 353
- 354 [1] L.Ambrosio, N.Fusco, D.Pallara. *Functions of bounded variation and free discontinuity*
355 *problems*. Oxford University Press (2000).
- 356 [2] L.Ambrosio, V.M.Tortorelli. *Approximation of functionals depending on jumps by elliptic*
357 *functionals via Γ -convergence*. *Communic. Pure Appl. Math* **43** (1990), 999-1036.
- 358 [3] L.Ambrosio, V.M.Tortorelli. *Approximation of functionals depending on jumps by*
359 *quadratic, elliptic functionals via Γ -convergence*. *Boll.Un. Mat. Ital.* 6-B(1992), 105-123.
- 360 [4] G.Anzellotti *Pairings between measures and bounded functions and compensated com-*
361 *pactness*. *Ann. Mat. Pura Appl.* **135** (1983), 293-318.
- 362 [5] G.Aubert, J. Aujol, L.Blanc-Feraud *Detecting Codimension-Two Objects in an image*
363 *with Ginzburg-Landau Models*. *International Journal of Computer Vision* **65** (2005), 29-
364 42.
- 365 [6] G.Aubert, D.Graziani *Variational approximation for detecting points like-target problem*
366 *in 2-D biological images* submitted to ESAIM: COCV. Preprint available on [http://www-](http://www-sop.inria.fr/en/publications.php)
367 [sop.inria.fr/en/publications.php](http://www-sop.inria.fr/en/publications.php).
- 368 [7] F. Bethuel, H. Brezis and F. Hélein *Ginzburg-Landau Vortices*. Birkäuser, Boston (1994).
- 369 [8] A.Braides. *Γ -convergence for beginners*. Oxford University Press, New york (2000).
- 370 [9] A.Braides, A.Malchiodi *Curvature Theory of Boundary phases: the two dimensional*
371 *case*. *Interfaces Free. Bound.* **4** (2002), 345-370.
- 372 [10] A.Braides, R.March. *Approximation by Γ -convergence of a curvature-depending func-*
373 *tional in Visual Reconstruction* *Comm. Pure Appl. Math.* **59** (2006), 71-121.
- 374 [11] G.Dal Maso. *Introduction to Γ -convergence*. Birkhäuser, Boston(1993).
- 375 [12] E.De Giorgi *Some Remarks on Γ -convergence and least square methods, in* *Com-*
376 *posite Media and Homogenization Theory* (G. Dal Maso and G.F.Dell'Antonio eds.),
377 Birkhauser, Boston, 1991, 135-142.
- 378 [13] E.De Giorgi, T.Franzoni. *Su un tipo di convergenza variazionale*. *Atti Accad. Naz. Lincei*
379 *Rend. Cl. Sci. Mat. Natur.* **58** (1975), 842-850.
- 380 [14] E.De Giorgi, T.Franzoni. *Su un tipo di convergenza variazionale*. *Rend. Sem. Mat.*
381 *Brescia* **3** (1979), 63-101.
- 382 [15] L.Modica, S.Mortola. *Un esempio di Γ -convergenza*. *Boll.Un. Mat. Ital.* 14-B(1977),
383 285 – 299.

-
- 384 [16] L.Modica. *The gradient theory of phase transitions and the minimal interface criterion.*
385 Arch. Rational Mech. Anal. **98** (1987), 123-142.
- 386 [17] M.Morel, Solimini *Variational models in image segmentation*, Birkäuser, 1994.
- 387 [18] M.Röger, R.Shätzle. *On a modified conjecture of De Giorgi.* Math. Zeitschrift. **254**
388 (2006), 675-714
- 389 [19] G.Stampacchia *Le problème de Dirichlet pour les equations elliptiques du seconde ordre*
390 *à coefficients discontinuous.* Ann. Inst. Fourier (Grenoble), **15** (1965), 180-258.
- 391 [20] A.Witkin, D.Terzopoulos, M.Kass. *Signal mathcing trough scale space* International
392 Journal of Computer Vision **1 2** (1987), 133-144.



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