An overview of Non Smooth Dynamical Systems. Moreau's Sweeping Process, Higher order systems and links with optimization

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1. A very short introduction to the Moreau's sweeping process

The Non Smooth Dynamical Systems (NSDS) are a very special kind of dynamical systems characterized by their nonsmoothness in the evolution with respect to time and by a set of non smooth generalized equations. The so-called Moreau's Sweeping Process[9, 10] is a special kind of differential inclusion with a maximal monotone operator[5] which appears to be a very nice formulation for the unilateral dynamics :

$$\begin{cases} \dot{x} + f(x,t) = \lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ -\lambda \in \mathcal{N}_{\Phi(t)}(x(t)) \end{cases} \tag{1}$$

where f is a smooth vector field and $\mathcal{N}_{\phi(t)}$ is the normal cone to a admissible set $\Phi(t)$ for the state x. On the equivalence between with other types of order one NSDS, we refer to [2] and to [6] for a review of various extension of Moreau's Sweeping Process and associated mathematical results with weaker assumptions on the regularity of the solution.

There are a lot of examples of NSDS which come from the engineering sciences. In electrical engineering, networks with idealized components (diodes, saturation, relay, ...) are easily formulated as in (1). If the dynamics is a linear one, the "Linear Complementarity Systems (LCS)" are often considered :

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ w = Cx + D\lambda \\ 0 \leq w \perp \lambda \geq 0 \end{cases} \tag{2}$$

For passive systems, this formulation is completely equivalent to (1), but we will see that this formulation is not well defined in all cases. In Mechanical engineering, Schatzman[15] has given a correct meaning to motion with measure acceleration for the Lagrangian systems with unilateral contact and Moreau have extended the sweeping process to the second order system[11] :

$$-du + f(t, q(t)) \in \mathcal{N}_{V(q)}(u(t)) \tag{3}$$

where $u = \dot{q}$ is the velocity assumed to a right continuous functions of bounded variations, $V(q)$ is the tangent cone to the admissible set ϕ at the position $q(t)$. Finally, the acceleration is replaced by a differential Measure du may be viewed as the derivation in the sense of distributions of the velocity u (for more details see [12]). This compact formulation is very powerful from the computational point of view but also from the pure mathematical point of view[8, 16, 4]. In control engineering, the standard problem of the optimal control a dynamical system with state constraints leads also to a dual problem which is also a NSDS. Finally, we can also mention that there is also a lot of applications in biology, in economics and everywhere a constraint is applied to the state variable.

The numerical methods inherit from the approach you have chosen to grasp for the NSDS. Two major approaches are widespread: the hybrid approach and the non smooth approach.

The hybrid approach is to consider the NSDS as a hybrid multi-modal dynamical system. In each mode separated by two events, the system is assumed to have a sufficient regularity to consider standard analysis and classical numerical methods. It leads the family of computational schemes, named the *Event–Driven schemes*. In this framework, there is no hope to establish a general convergence proof and the accumulation of events in finite time can not be circumvented.

The non smooth approach is based on the Moreau's Sweeping process and its variants. The key idea is to write a right approximation of measures on a finite interval, and this leads efficient and robust numerical schemes, called *Time–stepping schemes*. The first algorithm was the "Catching up algorithm". In the framework of multi-body dynamics, the derived algorithm is the "Non Smooth Contact Dynamics" method[13, 14, 7] which is able to treat several thousands of 3D frictional contact conditions. The time step is no longer of events but only fixed by a *a priori* error criterion. Therefore, accumulations of events or a large number of events in finite time are handled without difficulties. Furthermore, the convergence analysis of this family of schemes leads to existence of solutions for rather complicate systems [8, 16].

2. Higher relative degree Moreau's Sweeping Process

In a joint work with Bernard Brogliato[1] et Daniel Goeleven[3], higher relative degree systems are studied. In systems of the form (2), the relative degree r between the output w and the multiplier λ may be defined as the rank of the first non zero element in the sequence of Markov parameters $(D, CB, CAB, CA^2B, \ldots)$. There is a clear analogy between this relative degree and the differential index in differential algebraic equations. If the relative degree is greater or equal to 3, a generalized solution is a distribution of order more than 2. Therefore, there is no meaning to state a constraint of positivity to λ. In order to circumvent this problem, we propose a new formulation of such systems as a measure differential inclusion of higher order. This derivation is rather technical and to be concise as possible, we refer the reader to [3] for more details on it.

With this formulation, a precise meaning to solutions as distributions generated by a finite set of differential measures is given. Global existence and uniqueness results are also given in a certain class of regular function (analytical in every right neighborhood) and under some monotonicity assumptions. Finally, a efficient time-stepping scheme is designed. The convergence proof is on right way but still to be terminated. The applications for such type of systems are the electrical and mechanical systems with feedback control and the design of an indirect framework for solving the problem of optimal control with state constraints, based on necessary conditions.

3. Open problems for the Optimization community

As for the case of Lagrangian systems($r = 2$), the one-step problem leads to a set nested complementarity conditions, coupled with some logical conditions. For the case of initial value problem, a prediction of the conditional switch is done. If we have a boundary value problem, it is no longer reasonable. One question remains open: What is the good optimization strategy to use ?

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