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# Stabilization of switched linear systems: a sliding mode approach

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**Abstract:** In this paper, the existence of stabilizing sliding motions in the context of switched linear systems is investigated. We consider the case of switched systems with two subsystems and we propose conditions for the existence of stable sliding hyper-planes. For particular cases of switched systems (planar systems, integrator chains), we provide constructive analytical methods for deriving a stabilizing switching sequence.

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## 1. INTRODUCTION

Nowadays, control of hybrid systems is extensively studied because of its relevance in numerous applications such as embedded systems. Switched systems can be viewed as higher-level abstractions of hybrid systems, obtained by neglecting the details of the discrete behavior.

Roughly speaking, a switched system is made of a family of dynamical subsystems (linear or nonlinear), and a rule, called the switching law, that orchestrates the switching between them (Liberzon [2003]). A taxonomy of switched systems can be defined based on the switching law. One can distinguish the *controlled* aspect (when the switching function represents a discrete input) from the *autonomous* aspect (when, for example, switchings due to state space transitions appear). A survey of different switched system classes and the associated stability problems can be found in (Liberzon [2003], Decarlo et al. [2000], Sun and Ge [2005], Shorten et al. [2007]). In this paper, we will focus on the controlled aspect of the switching law, that is on the design of switching law that leads to the stabilization of a switched linear system. In particular, we address the case of switched linear systems described by differential equations of the form:

$$\dot{x}(t) = A_{\sigma(t)}x(t). \quad (1)$$

where  $x \in \mathbb{R}^n$  represents the state vector and  $\sigma(t)$ ,  $\sigma : \mathbb{R}^+ \rightarrow \mathcal{I} = \{1, 2, \dots, N\}$  is a piecewise constant function that represents the *switching law*. The stabilization problem can be formalized as follows :

**Problem :** for any  $t > 0$  design the switching signal  $\sigma(t)$  which makes the system (1) asymptotically stable.

Since the problem is trivial when at least one of the  $A_i$  matrices is Hurwitz stable, it is considered here that none of the matrices  $A_i$ ,  $i = 1, \dots, N$  is Hurwitz.

Several authors addressed the switching law design problem in the literature. A necessary condition for the stabilization by switching law was given in (Lin and Antsaklis [2007]): for at least one index  $k \in \mathcal{I}$  the matrix  $A_k^T + A_k$  has a negative eigenvalue. Using quadratic stability arguments, the results of (Wicks et al. [1994]) provide a

state dependent switching law, by verifying the existence of a global stable convex combination. For a system with two modes described by the matrices  $A_1, A_2$ , this implies finding an  $\alpha \in [0, 1]$  such that the equivalent matrix

$$A_{eq}(\alpha) = \alpha A_1 + (1 - \alpha)A_2 \quad (2)$$

is Hurwitz stable. This implies the existence of a symmetric positive definite matrix  $P = P^T \succ 0$  such that

$$A_{eq}^T(\alpha)P + PA_{eq}(\alpha) \prec 0. \quad (3)$$

For switched systems with two subsystems, the existence of a stable convex combination is strictly equivalent to the quadratic stabilization of the system (Shorten et al. [2007]), i.e. there exists a stable convex combination if and only if there exists a switching sequence  $\sigma^*(t) \quad \forall t \in [0, \infty)$  such that the quadratic Lyapunov functions  $V(x) = x^T P x$  is strictly decreasing along the solutions of the system with  $\sigma(t) = \sigma^*(t), \forall t \in [0, \infty)$ . Constructive methods for obtaining a state dependent switching law can be obtained using the min-projection strategy (Liberzon [2003]). In this case, the system is switching according to conic partitions of the state space.

However, the research of a stable convex combination of matrices is an NP hard problem (Blondel and Tsitsiklis [1997]). Moreover, these methods are based on quadratic Lyapunov functions which are conservative: there are cases of asymptotically stable switched linear systems that can be stabilized by switching while no quadratic Lyapunov function exists. In fact, it has been recently shown in (Blanchini and Savorgnan [2008]) that in general the existence of any smooth Lyapunov function is only sufficient for stabilization, but not necessary.

To reduce the conservatism of quadratic Lyapunov functions, several authors (Peleties and DeCarlo [1991], Geromel and Colaneri [2006]) proposed stabilization conditions based on multiple Lyapunov functions (Branicky [1994]). Stabilization criteria can be obtained using the  $S$ -procedure (Rantzer et al. [2000]). The results are expressed in terms of bilinear matrix inequalities (BMIs) which makes the approach difficult to apply in general. Other interesting approaches have been proposed based on polyhedral Lyapunov Functions (Lin and Antsaklis [2007]) which are necessary and sufficient for stabilization of uncertain switched

linear systems if no sliding motion is allowed. Stabilization conditions based on the explicit analysis of vector fields have been proposed in (Xu and Antsaklis [2000]).

Sliding mode control has been extensively studied for several years (see ??? for tutorials). Usually, a discontinuous control action is used so that the system trajectories are constrained to lie, after a finite time, within a neighbourhood of a given manifold of the state space. The result is a variable structure system, which may be regarded as a combination of subsystems where each subsystem has a fixed control structure and is valid for specified regions of system behaviour. The resulting behavior, called sliding mode, has some prescribed dynamics totally imposed by the parameters and the equations that define the sliding manifold. Robustness properties against various kinds of uncertainties such as parameter perturbations, external disturbances and measurement errors can be guaranteed. Sliding mode is still the object of many studies for both observation and control problems (see e.g. ???), with a large number of application fields (???).

However, in the case of switched linear systems, most of the existing switching law design approaches try to avoid their existence since they are difficult to characterize.

In this paper, we intend to investigate the existence of stabilizing sliding motions in the context of switched linear systems. The case of systems that switch between two subsystems is considered conditions for the existence of stable sliding hyper-planes are studied. For special cases of switched systems (second-order systems, integrators), constructive analytical methods for deriving a stabilizing switching sequence are provided.

The paper is organized as follows : Section 2 presents general concepts encountered in the domain of sliding mode control and gives a motivating example. In Section 3, conditions for the existence of a stable sliding hyperplane for switched linear systems with two subsystems are given. Numerical examples are given in Section 4.

## 2. PRELIMINARIES

### 2.1 Background on sliding modes

Consider a nonlinear controlled system of the form:

$$\dot{x} = f(x, u), \quad (4)$$

where  $f : \mathcal{O} \times \mathcal{U} \rightarrow \mathbb{R}^n$ , with  $\mathcal{O}$  an open subset of  $\mathbb{R}^n$  and  $\mathcal{U}$  a subset of  $\mathbb{R}^m$ .

A basic problem in the field of *variable structure control* is to design a  $m$ -dimensional control variable  $u$  (which should belong to the given control set  $\mathcal{U}$ ) such that the state variable  $x$  has to satisfy some a given constraint

$$s(x(t)) = 0 \text{ for all } t \geq T, \quad T > 0 \quad (5)$$

where  $s : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a fixed mapping which defines the sliding manifold  $\mathcal{S} = \{x : s(x) = 0\}$ . Such a problem can be tackled by designing  $u^*(x)$  as a discontinuous control where, in general, the components of  $u$  are switching between two functions. As a result the closed-loop system is usually described using a differential inclusion setting,

$$\dot{x} \in F(x, u^*(x)), \quad (6)$$

where the vector field  $f(x, u)$  in (4) with the discontinuous control  $u^*(x)$  is replaced by the set valued map

$F(x, u^*(x))$ . The map  $F$  is assumed to have good “properties” such as being “nonempty compact convex valued and upper semi-continuous” (ensuring thus existence of classical (i.e., almost everywhere) solutions to the initial value problem). Several approaches can be used for the construction of the set valued map  $F$ . One of them is due to Filippov (Filippov [1988]):

$$F(x) = \bigcap_{\epsilon > 0} \overline{\text{conv}} \{f[B(x, \epsilon) \setminus L] : \text{meas } L = 0\} \quad (7)$$

where  $\overline{\text{conv}}\{A\}$  denotes the closed convex hull of  $A$ , i.e., the intersection of all closed convex sets containing the set  $A$ .

*Definition 1.* A Filippov solution  $y$  to (6) is a locally absolutely continuous function  $y : [0, T] \rightarrow \mathbb{R}^n$  such that

$$\dot{y}(t) \in F(y(t)) \quad (8)$$

for almost every  $t \in [0, T]$ .

Thus, Filippov definition replaces the discontinuous differential equation (4) by the differential inclusion (6). In this context, assume that  $m = 1$  and that  $f$  in (4) is switching according:

$$f(x, u^*(x)) = \begin{cases} f^+(x), & \text{if } s(x) > 0, \\ f^-(x), & \text{if } s(x) < 0. \end{cases} \quad (9)$$

Then,  $F$  is defined by

$$F(x) = \begin{cases} f(x, u^*(x)) & \text{if } s(x) \neq 0, \\ \lim_{x \rightarrow x^* \in \mathcal{S}} \overline{\text{conv}} \{f^+(x), f^-(x)\} & \text{if } s(x) = 0. \end{cases}$$

Using the projection operator  $\text{Pr}_{\text{normal}}$  onto the normal to the manifold  $\mathcal{S}$  (oriented from  $\{x : s(x) > 0\}$  to  $\{x : s(x) < 0\}$ ), define

$$f_n^+(x) = \text{Pr}_{\text{normal}} \lim_{s(x) \rightarrow 0^+} f^+(x), \quad (10)$$

$$f_n^-(x) = \text{Pr}_{\text{normal}} \lim_{s(x) \rightarrow 0^-} f^-(x). \quad (11)$$

Then, Filippov results on the existence of solutions (see Filippov [1988]) lead to the existence of a “sliding motion” along the manifold  $\mathcal{S}$  when  $f_n^+ < 0$  and  $f_n^- > 0$ <sup>1</sup>:

$$\begin{cases} x \in \mathcal{S}, \\ \dot{x} = f_0(x), \end{cases} \quad (12)$$

with  $f_0(x) \in \overline{\text{conv}}\{f^+(x), f^-(x)\} \cap T_x \mathcal{S}$ . The “equivalent dynamic” or “sliding dynamic” is given by:

$$\frac{dx}{dt} = \left[ \frac{\langle \text{grad } s, f^- \rangle}{\langle \text{grad } s, f^- - f^+ \rangle} \right] f^+ - \left[ \frac{\langle \text{grad } s, f^+ \rangle}{\langle \text{grad } s, f^- - f^+ \rangle} \right] f^-, \quad (13)$$

where  $\langle a, b \rangle$  denotes the scalar product of the two vectors  $a$  and  $b$ . Indeed, the convex hull is defined by  $f_0(x) = \alpha f^+(x) + (1 - \alpha) f^-(x)$ ,  $\alpha \in [0, 1]$  and  $f_0 \in T_x \mathcal{S} \iff \langle \text{grad } s, f_0 \rangle = 0$ , leads to  $\alpha = \left[ \frac{\langle \text{grad } s, f^-(t, x) \rangle}{\langle \text{grad } s, (f^-(t, x) - f^+(t, x)) \rangle} \right]$ .

### 2.2 Motivating example

Here we provide an example that motivates the study of the existence of sliding modes in switched linear systems. Consider a second order switched system with two subsystems:

$$\frac{dx}{dt} = A_i x, \quad i = 1, 2 \quad (14)$$

<sup>1</sup> The two conditions for the existence of a sliding motion along the manifold  $\mathcal{S} : f_n^+(t, x) < 0$  and  $f_n^-(t, x) > 0$  are equivalent to the well-known condition  $s\dot{s} < 0$  (condition for the local attractivity of  $\mathcal{S}$ ).

with

$$A_1 = \begin{pmatrix} 1 & 1/\mu \\ -\mu & 1 \end{pmatrix}, \quad A_2 = T^{-1} \begin{pmatrix} 1 & -1/\mu \\ \mu & 1 \end{pmatrix} T, \quad (15)$$

$$T = \begin{pmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \quad (16)$$

where  $\mu = 0.1$ . Both systems have unstable complex eigenvalues. The corresponding vector fields are represented in Figures 1.

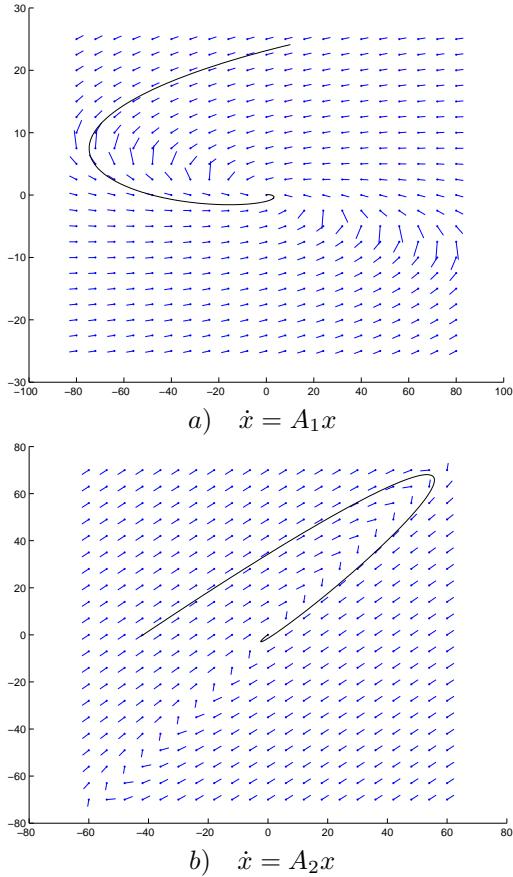


Fig. 1. Vector fields.

It can be shown that no stable combination exists for (14) and therefore there is no constant  $\alpha$  that solves the bilinear matrix inequality (3). Yet, it is rather intuitive that the system can be stabilized. In fact, through a qualitative analysis, one can define the switching lines  $\Gamma_1 x = 0$  and  $\Gamma_2 x = 0$  with  $\Gamma_1 = [1 \ 1]$  and  $\Gamma_2 = [0 \ 1]$ , such that

$$\sigma(t) = \begin{cases} 1, & \text{if } x^T \Gamma_1^T \Gamma_2 x < 0; \\ 2, & \text{otherwise.} \end{cases} \quad (17)$$

Then, the solution of the system exhibits a stable sliding motion along the surface  $\Gamma_1 x = 0$ , as it can be seen in Figure 2.

This implies, from the point of view of Filippov solutions, that, for all  $x$  such that  $\Gamma_1 x = 0$ , there exists locally an equivalent convex combination  $A_{eq}(\alpha(x))$  such that the solution of the system converges towards zero on the given surface in spite of the fact that no Hurwitz stable  $A_{eq}(\alpha)$  exists with a constant  $\alpha$  for all  $x$ . This example illustrates the usefulness of defining stable sliding

motions for switched systems when no global stable convex combination exists.

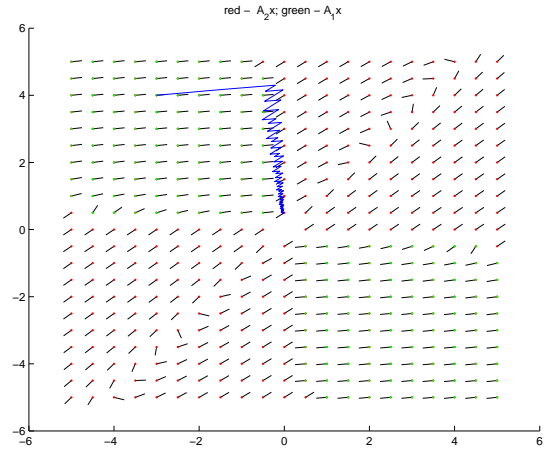


Fig. 2. Stabilization using (17)

### 3. A SLIDING MODE BASED STABILIZING SWITCHING LAW

#### 3.1 Bilinear formulation

Consider a switched system of the form (1) with  $N = 2$ . This system can be formulated as a bilinear system

$$\dot{x}(t) = Ax(t) + Bx(t)u(t) \quad (18)$$

with

$$A = \left( \frac{A_1 + A_2}{2} \right) \quad \text{and} \quad B = \left( \frac{A_1 - A_2}{2} \right),$$

where the control  $u(t) \in \{-1, 1\}$  is defined by

$$u(t) = \begin{cases} 1, & \text{if } \sigma(t) = 1; \\ -1, & \text{if } \sigma(t) = 2. \end{cases} \quad (19)$$

Using this notation, the control law design Problem 1 can be regarded as a particular sliding mode control synthesis problem for bilinear systems. The idea of this work is to investigate the existence of stable sliding modes on the manifold of the form:

$$s(t) = \Gamma x(t) = 0 \quad (20)$$

where  $\Gamma \in \mathbb{R}^{1 \times n}$  is a design parameter, for system (18). The problem is to design a switching control law  $u(t) \in \{-1, 1\}$  as well as the set of vectors  $\Gamma$  such that: (i)  $s(t) = 0$  is an *attractive* and *invariant* sliding hyperplane, (ii) the resulting sliding motion on  $s(t) = 0$  is asymptotically stable.

#### 3.2 Attractivity Conditions

In order to define the region of *attractivity* of the sliding manifold, we analyze the domain in  $R^n$  for which

$$s(t)\dot{s}(t) < 0,$$

that is

$$s(t)\dot{s}(t) = s(t) (\Gamma Ax(t) + \Gamma Bx(t)u(t)) < 0.$$

Consider the following switching rule:

$$u(t) = -\text{sign}(\Gamma Bx(t).s(t)). \quad (21)$$

Using the switching rule (21), one can notice that

$$s(t)\dot{s}(t) \leq |s(t)| (|\Gamma Ax(t)| - |\Gamma Bx(t)|).$$

This implies that the sliding manifold is attractive in the domain characterized by  $|\Gamma A x| < |\Gamma B x|$  i.e.

$$|\Gamma (A_1 + A_2) x| < |\Gamma (A_1 - A_2) x|. \quad (22)$$

This condition defines a conic partition of the state space:

$$x^T \left\{ (A_1 + A_2)^T \Gamma^T \Gamma (A_1 + A_2) - (A_1 - A_2)^T \Gamma^T \Gamma (A_1 - A_2) \right\} x < 0 \quad (23)$$

which gives the attractivity domain:

$$\{x : x \in \mathbb{R}^n, x^T A_1^T \Gamma^T \Gamma A_2 x < 0\}. \quad (24)$$

### 3.3 Equivalent sliding motion

Here,  $\Gamma$  has to be chosen such that the equivalent sliding motion on  $s = 0$  is asymptotically stable. Under the Filippov definition, the equivalent motion is given by the convex combination of vector fields. This implies that for all  $t$  such that  $\Gamma x(t) = 0$ , there exist a scalar  $\alpha(x(t)) \in [0, 1]$  such that the state of the system evolves according to the differential equation:

$$\dot{x}(t) = \alpha(x(t)) A_1 x(t) + (1 - \alpha(x(t))) A_2 x(t).$$

Using the definition of the sliding surface  $s = \Gamma x = 0$ , the function  $\alpha(x)$  satisfies the relation:

$$\alpha(x) = \frac{\langle \text{grad } s, A_2 x \rangle (x)}{\langle \text{grad } s, (A_2 - A_1) x \rangle} = \frac{\Gamma A_2 x}{\Gamma (A_2 - A_1) x}.$$

This implies that the equivalent dynamic on the sliding hyperplane is given by

$$\dot{x} = \frac{1}{\Gamma (A_2 - A_1) x} ((\Gamma A_2 x) A_1 x + (\Gamma A_1 x) A_2 x) \quad (25)$$

In the general case, the study of the stability of (25) is not straightforward. Hereafter two classes of systems are considered for which choices of a  $\Gamma$  leading to stable equivalent motions are fully characterized.

### 3.4 Stabilization of second order switched linear systems

For  $n = 2$ , one can choose without loss of generality:

$$\Gamma = [\gamma_1 \ 1], \quad s(x) = \gamma_1 x_1 + x_2. \quad (26)$$

Denoting by  $a_{kl}^i$  the entries of the matrices  $A_i$ , the switching law given by (21) gives rise to a ‘‘sliding motion’’ as soon as (22) holds. Moreover  $\alpha$  is independent of  $x$  and is given by:

$$\alpha = \frac{-\gamma_1^2 a_{12}^2 + \gamma_1 (a_{11}^2 - a_{22}^2) + a_{21}^2}{a \gamma_1^2 + b \gamma_1 + c} \quad (27)$$

$$a = (a_{12}^1 - a_{12}^2)$$

$$b = (a_{11}^2 - a_{11}^1 + a_{22}^1 - a_{22}^2)$$

$$c = (a_{21}^2 - a_{21}^1)$$

$$(28)$$

which gives the following equivalent dynamic

$$\dot{x}_1 = \beta x_1 \quad (29)$$

$$x_2 = -\gamma_1 x_1 \quad (30)$$

where  $\beta$  is given by

$$\beta = \frac{\mathfrak{d} \gamma_1^2 + \mathfrak{e} \gamma_1 + \mathfrak{f}}{a \gamma_1^2 + b \gamma_1 + c} \quad (31)$$

$$\mathfrak{d} = (a_{22}^1 a_{12}^2 - a_{22}^2 a_{12}^1)$$

$$\mathfrak{e} = (a_{11}^1 a_{22}^2 - a_{11}^2 a_{22}^1 + a_{21}^2 a_{12}^1 - a_{21}^1 a_{12}^2)$$

$$\mathfrak{f} = (a_{11}^2 a_{21}^1 - a_{11}^1 a_{21}^2).$$

$\gamma_1$  has to be chosen such that  $0 < \alpha < 1$  and  $\beta < 0$  (for a stable sliding motion). From equation (22), one can characterize the attractivity domain of the obtained sliding mode. For the case presented here, the attractivity domain is the conic region given by

$$\{x : x \in \mathbb{R}^n, x^T M(\gamma_1) x < 0\}. \quad (32)$$

with  $M(\gamma_1) =$

$$\begin{pmatrix} (\gamma_1 a_{11}^1 + a_{21}^1)(\gamma_1 a_{11}^2 + a_{21}^2) & (\gamma_1 a_{11}^1 + a_{21}^1)(\gamma_1 a_{12}^2 + a_{22}^2) \\ (\gamma_1 a_{12}^1 + a_{22}^1)(\gamma_1 a_{11}^2 + a_{21}^2) & (\gamma_1 a_{12}^1 + a_{22}^1)(\gamma_1 a_{12}^2 + a_{22}^2) \end{pmatrix}.$$

Notice that any  $x$  for which  $s(x) = 0$  can be expressed on the form  $x = c \times (1 \ -\gamma_1)^T$  with  $c \in \mathbb{R}$ . Therefore the sliding line is contained in the attractivity domain if the condition

$$(1 \ -\gamma_1) M(\gamma_1) (1 \ -\gamma_1)^T < 0 \quad (33)$$

is satisfied.

### 3.5 Stabilization of integrator cascade

Consider the case of two linear switching systems described by the following relation:

$$y^{(n)} + \sum_{i=1}^{n-1} a_i^j y^{(i)} = 0, \quad j = 1, 2 \quad (34)$$

In a state space representation, (34) is given by (1) where the two matrices  $A_j$  are

$$A_j = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -a_1^j & & & -a_{n-1}^j \end{pmatrix} \quad (35)$$

( $x$  being defined obviously). In that case, the resulting bilinear system (18) is defined by the matrices:

$$A = \left( \frac{A_1 + A_2}{2} \right) = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\frac{a_1^1 + a_1^2}{2} & & & -\frac{a_{n-1}^1 + a_{n-1}^2}{2} \end{pmatrix} \quad (36)$$

and

$$B = \left( \frac{A_1 - A_2}{2} \right) = \begin{pmatrix} 0 & 0 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ -\frac{a_1^1 - a_1^2}{2} & & & -\frac{a_{n-1}^1 - a_{n-1}^2}{2} \end{pmatrix}. \quad (37)$$

Define the sliding variable by:

$$s = \Gamma x, \quad (38)$$

where  $\Gamma = [\gamma_1, \dots, \gamma_n] \in \mathbb{R}^{(1 \times n)}$  (one can set  $\gamma_n = 1$  without loss of generality). According to section 2.1, the

equivalent dynamic is obtained using (13) where  $f_0$  is computed as

$$f_0(x) = (\alpha A_1 + (1 - \alpha)A_2)x$$

where  $\alpha A_1 + (1 - \alpha)A_2 =$

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -(\alpha a_1^1 + (1 - \alpha)a_1^2) & & & -(\alpha a_{n-1}^1 + (1 - \alpha)a_{n-1}^2) \end{pmatrix}. \quad (39)$$

From  $\Gamma x = 0$ , the resulting equivalent dynamics is described by :

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\dots = \dots \\ \dot{x}_{n-1} &= -\sum_{i=1}^{n-1} \gamma_i x_i \end{aligned} \quad (40)$$

which is stable if the vector  $\Gamma$  contains the coefficient of a Hurwitz polynomial. In this case, the domain of attraction obtained in (22) is described by the equation

$$\left| \sum_{i=1}^n \left( \gamma_{i-1} - \frac{a_i^1 + a_i^2}{2} \right) x_i \right| < \left| \sum_{i=1}^n \left( \frac{a_i^1 - a_i^2}{2} \right) x_i \right|. \quad (41)$$

#### 4. EXAMPLES

Two examples of planar systems are given here.

##### 4.1 Example 1

Consider again the example described by (14),(15), (16) with  $\mu = 0.1$ . Using the sliding line (26) and the obtained computation for  $\alpha$  and  $\beta$ , different sliding motions can be described. For this example, the parameter  $\alpha$  is constant and is given by:

$$\alpha = \frac{101\gamma_1^2 + 198\gamma_1 + 101}{301\gamma_1^2 + 198\gamma_1 + 103}.$$

The constraint  $\alpha \in [0, 1]$  is satisfied for any  $\gamma_1 \in \mathbb{R}$ . Stability of the sliding motion implies that

$$\beta = -\frac{1}{10} \frac{6890\gamma_1^2 + 8019\gamma_1 - 1129}{301\gamma_1^2 + 198\gamma_1 + 103} < 0,$$

which holds for any

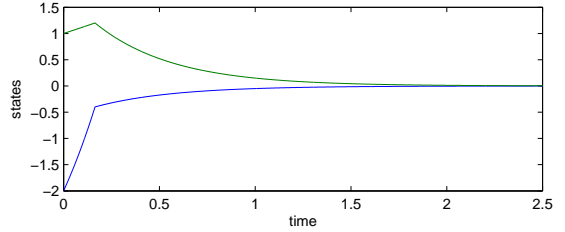
$$\gamma_1 \in \mathbb{R} \setminus \left[ \frac{-8019 - \sqrt{95419601}}{13780}, \frac{-8019 + \sqrt{95419601}}{13780} \right].$$

Simulations have been carried out with  $\gamma_1 = 3$  and with  $x_1(0) = -2$  and  $x_2(0) = 1$  as initial conditions (this implies that the conditions (32), (33) are satisfied). It can be seen in Figure 3 that the states tend to the origin once the sliding surface is reached.

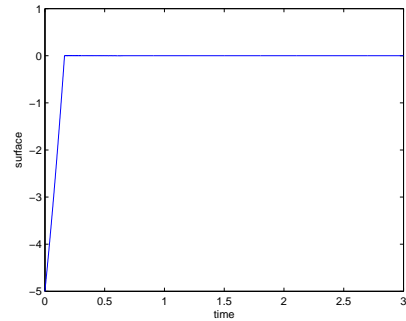
##### 4.2 Example 2

Consider the switched system given described by :

$$\frac{dx}{dt} = A_i x, i = 1, 2 \quad (42)$$



(a)



(b)

Fig. 3. Evolution of the states  $x(t)$  and of the sliding variable  $s(x)$  for the system in Example 1 with  $\gamma_1 = 3$  and  $x_1(0) = -2, x_2(0) = 1$

with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \quad (43)$$

where  $\mu = 1$ . In (Blanchini and Savorgnan [2008]), it was shown by means of this counterexample that a system which can be stabilized via switching control, does not necessarily admit a convex Lyapunov function, thus making difficult to apply usual methods given in the literature.

However, the constructive approach given here allows for the stabilization of this system. Indeed, the conditions

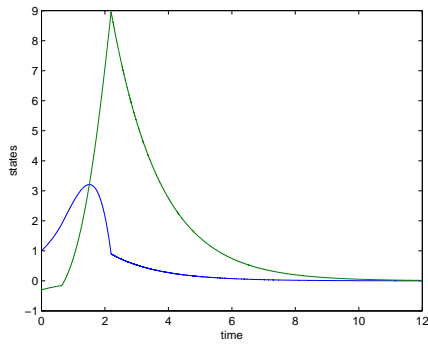
$$\alpha = \frac{\gamma_1^2 + 1}{(\gamma_1 - 1)^2} \in [0, 1], \beta = -\frac{\gamma_1^2 + 2\gamma_1 - 1}{(\gamma_1 - 1)^2} < 0,$$

can be satisfied by choosing  $\gamma_1$  such that  $\gamma_1 < 0$  and  $\gamma_1 \in \mathbb{R} \setminus [-1 - \sqrt{2}, -1 + \sqrt{2}]$ .

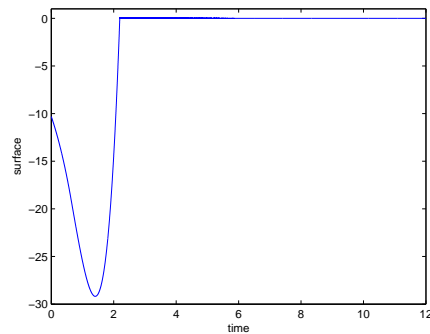
In the simulation reported in Figure 4,  $\gamma_1 = -10$  (satisfying (33)) and  $x_1(0) = 1$  and  $x_2(0) = -0.3$ . This particular set of initial conditions has been chosen because the attractivity condition (22) is not globally fulfilled in this case. Nevertheless, since the matrix  $A_2$  has complex eigenvalues, the trajectories of the system meet the attractivity domain after some finite time. Three specific behaviors can be seen in Figure 4(a):  $0 \leq t \leq 1.7$  unstability,  $1.7 \leq t \leq 2.2$  attractivity,  $t \geq 2.2$  sliding motion. This shows that global convergence can be obtained, even if the attractivity condition is not initially satisfied. This fact will be discussed in further work.

#### 5. CONCLUSION

In this paper, conditions for the existence of stabilizing sliding motions in the context of switched linear systems have been given. The case of switched systems with two



(a)



(b)

Fig. 4. Evolution of the states  $x(t)$  and of the sliding variable  $s(x)$  for the system in Example 2.

subsystems has been discussed and conditions for the existence of stable sliding hyper-planes have been provided. For particular cases of switched systems (planar systems, integrator chains), constructive analytical methods for deriving a stabilizing switching sequence have been proposed. In future work, we intend to investigate the use of more complex switching strategies, that ensure the global attractiveness of sliding hyper-planes.

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