Tensor approach to mixed high-order moments of absorbing Markov chains

Danil Nemirovsky

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Abstract: Moments of absorbing Markov chain are considered. First moments and non-mixed second moments are determined in classical textbooks such as the book of J. Kemeny and J. Snell "Finite Markov Chains". The reason is that the first moments and the non-mixed second moments can be easily expressed in a matrix form. Since the representation of mixed moments of higher orders in a matrix form is not straightforward, if ever possible, they were not calculated. The gap is filled by this paper. Tensor approach to the mixed high-order moments is proposed and compact closed-form expressions for the moments are discovered.

Key-words: Absorbing Markov Chain, high-order moment, tensor

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Une approche tensorielle pour le calcul des moments mixtes d'ordre supérieur des chaînes de Markov avec absorption

Résumé : Cet article s'intŕesse au calcul des moments d'une chaîne de Markov absorbante. Les premiers et les seconds moments non-mixtes sont déterminés dans les livres classiques, tel que le livre de J. Kemeny et J. Snell "Finite Markov Chains". La raison en est que les premiers et les seconds moments non-mixtes s'expriment facilement sous forme matricielle, ce qui n'est pas le cas des moment mixtes d'ordre arbitraire. Le fossé est comblé dans cet article oú grâce á une approche tensorielle des formules explicites pour les moments supérieurs mixtes d'ordre arbitraire sont obtenues.

Mots-clés : Chaînes de Markov absorbantes, moments d'ordre supérieur, tenseur.

Contents

1 Introduction

Let us consider an absorbing Markov chain and let matrix P be its transition matrix. By renumbering the states we can decompose matrix P in the following way

$$P = \begin{pmatrix} I & 0 \\ S & Q \end{pmatrix},$$

where submatrix Q is a substochastic matrix corresponding to transient states. Let T be the set of transient states and $\bar{T}$ be the set of absorbing states. We can define a fundamental matrix Z of the absorbing Markov chain

$$Z = (I - Q)^{-1} = I + Q + Q^2 + \dots .$$

Fundamental matrix $Z = \{z_{ij}\}_{i,j \in T}$ has the following probabilistic interpretation.

Definition 1.1 *Define N_j to be a function giving the total number of times before absorption that the absorbing Markov chain visits a transient state j.*

Let us denote by $E_i[N_j]$ the first moment of function N_j assuming that the Markov chain starts at state i, where $i, j \in T$. Then

$$Z = \{E_i[N_j]\}_{i,j \in T}$$

as it is noted in [3]. Non-mixed second moments $E_i[N_j^2]$ can also be found [3] with the help of matrix Z as

$$\{E_i[N_j^2]\}_{i,j \in T} = Z(2Z_{dg} - I),$$

where Z_{dg} is the same matrix as Z, but all the off-diagonal elements are set to zero.

However, the mixed second moments $E_i[N_j N_k]$ and the mixed higher-order moments $E_i\left[\prod_{j=0}^{m-1} N_{k_j}\right]$ are not so easy to calculate. Here we address this problem by tensor approach. Possible applications of our general result are Personalized PageRank [2] and polling system [1] which will be the topics of the subsequent publications.

2 Mixed second moments in matrix form

First we consider mixed second moments $E_i[N_j N_k]$ to show that calculation of them is not straightforward in matrix form.

Let us denote by u_j^k the indicator function $1_{\{X_k=j\}}$. We note that $N_j = \sum_{\varphi=0}^{\infty} u_j^{\varphi}$.

Theorem 2.1 *$E_i[N_j N_k]$ is finite.*

Proof. When we have proven the statement, it justifies our algebra with series below.

$$E_i[N_j N_k] = E_i\left[\left(\sum_{\varphi=0}^{\infty} u_j^{\varphi}\right)\left(\sum_{\psi=0}^{\infty} u_k^{\psi}\right)\right] = E_i\left[\sum_{\varphi=0}^{\infty}\sum_{\psi=0}^{\infty} u_j^{\varphi} u_k^{\psi}\right] = \sum_{\varphi=0}^{\infty}\sum_{\psi=0}^{\infty} E_i\left[u_j^{\varphi} u_k^{\psi}\right].$$

$E_i\left[u_j^{\varphi} u_k^{\psi}\right]$ is the probability that the process is in state j at step φ and in state k at step ψ, starting in state i.

We need to consider three cases

- Let $\varphi = \psi$. If states j and k are equal then we have that $E_i\left[u_j^{\varphi} u_k^{\varphi}\right] = p_{ij}^{(\varphi)}$; if states j and k are not equal then we have that $E_i\left[u_j^{\varphi} u_k^{\varphi}\right] = 0$ since the process cannot be in two different states at the same step $\varphi = \psi$. Hence, we write $E_i\left[u_j^{\varphi} u_k^{\varphi}\right] = p_{ij}^{(\varphi)}\delta_{jk}$, where δ_{jk} here and below is the Kronecker symbol.
- Let $\varphi < \psi$, and let $d_1 = \psi - \varphi$. Then, $E_i\left[u_j^{\varphi} u_k^{\psi}\right]$ is the probability that the process is in state j at step φ, and in state k at step $\varphi + d_1$. Hence, $E_i\left[u_j^{\varphi} u_k^{\psi}\right] = p_{ij}^{(\varphi)} p_{jk}^{(d_1)}$.
- Let $\varphi > \psi$, and let $d_2 = \varphi - \psi$. Then, $E_i\left[u_j^{\varphi} u_k^{\psi}\right]$ is the probability that the process is in state k at step ψ, and in state j at step $\psi + d_2$. Hence, $E_i\left[u_j^{\varphi} u_k^{\psi}\right] = p_{ik}^{(\psi)} p_{kj}^{(d_2)}$.

We proceed as following.

$$E_i\left[N_jN_k\right]=\sum_{\varphi=0}^{\infty}\sum_{\psi=0}^{\infty}E_i\left[u_j^{\varphi}u_k^{\psi}\right]=$$

$$=\sum_{\varphi=0}^{\infty}\left(\sum_{\psi=0}^{\varphi-1}E_i\left[u_j^{\varphi}u_k^{\psi}\right]+E_i\left[u_j^{\varphi}u_k^{\varphi}\right]+\sum_{\psi=\varphi+1}^{\infty}E_i\left[u_j^{\varphi}u_k^{\psi}\right]\right)=$$

$$=\sum_{\varphi=0}^{\infty}\left(\sum_{\psi=0}^{\varphi-1}p_{ik}^{(\psi)}p_{kj}^{(\varphi-\psi)}+p_{ij}^{(\varphi)}\delta_{jk}+\sum_{\psi=\varphi+1}^{\infty}p_{ij}^{(\varphi)}p_{jk}^{(\psi-\varphi)}\right).$$

According to [3, Corollary 3.1.2], there are numbers $b>0$, $0<d<1$ such that $p_{ij}^{\varphi}\leqslant bd^{\varphi}$, and we can give the following estimate.

$$E_i\left[N_jN_k\right]\leqslant\sum_{\varphi=0}^{\infty}\left(\sum_{\psi=0}^{\varphi-1}\left(bd^{\psi}\right)\left(bd^{\varphi-\psi}\right)+bd^{\varphi}\delta_{jk}+\sum_{\psi=\varphi+1}^{\infty}\left(bd^{\varphi}\right)\left(bd^{\psi-\varphi}\right)\right)=$$

$$=\sum_{\varphi=0}^{\infty}\left(b^2\sum_{\psi=0}^{\varphi-1}d^{\varphi}+bd^{\varphi}\delta_{jk}+b^2\sum_{\psi=\varphi+1}^{\infty}d^{\psi}\right)=$$

$$=\sum_{\varphi=0}^{\infty}\left(b^2\varphi d^{\varphi}+bd^{\varphi}\delta_{jk}+b^2d^{\varphi+1}\sum_{\psi=0}^{\infty}d^{\psi}\right)=\sum_{\varphi=0}^{\infty}\left(b^2\varphi d^{\varphi}+bd^{\varphi}\delta_{jk}+b^2\frac{d^{\varphi+1}}{1-d}\right)=$$

$$=\sum_{\varphi=0}^{\infty}b^2\varphi d^{\varphi}+\sum_{\varphi=0}^{\infty}bd^{\varphi}\delta_{jk}+b^2\frac{1}{1-d}\sum_{\varphi=0}^{\infty}d^{\varphi+1}=b^2\sum_{\varphi=0}^{\infty}\varphi d^{\varphi}+b\delta_{jk}\frac{1}{1-d}+b^2\frac{d}{(1-d)^2}.$$

Since $\frac{(\varphi+1)d^{\varphi+1}}{\varphi d^{\varphi}}=\frac{\varphi+1}{\varphi}d\to d<1$, when $\varphi\to\infty$, the series $\sum_{\varphi=0}^{\infty}\varphi d^{\varphi}$ converges. This completes the proof.

∎

Now that we have proven that $E_i[N_jN_k]$ is finite, let us calculate its value. We define matrix $\Lambda(i)$ as

$$\Lambda_{jk}(i)=E_i[N_jN_k].$$

Theorem 2.2 *The matrix of the mixed second order moment is given by*

$$\Lambda(i)=\sum_{\nu\in T}z_{i\nu}\left(D(\nu)Z+ZD(\nu)-D(\nu)\right),$$

where matrix $D(\nu)$ is defined by

$$D_{jk}(\nu) = \begin{cases} 1, & \text{if } \nu = j = k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let us calculate $E_\nu\left[N_j N_k\right]$. Following the principal idea of [3, Theorem 3.3.3] we ask where the process can go in one step, from its starting position ν. It can go to state φ with probability $p_{\nu\varphi}$. If the new state is absorbing, then we can never reach states j or k again, and the only possible contribution is from the initial state, which is $\delta_{\nu j}\delta_{\nu k}$. If the new state is transient then we will be in state j $\delta_{\nu j}$ times from the initial state, and N_j times from the later steps, and we will be in state k $\delta_{\nu k}$ times from the initial state, and N_k times from the later steps. Let us denote by T the set of transient states, and by $\bar{T}$ the set of absorbing states. We have

$$\begin{aligned}
E_\nu\left[N_j N_k\right] &= \sum_{\varphi \in \bar{T}} p_{\nu\varphi}\delta_{\nu j}\delta_{\nu k} + \sum_{\varphi \in T} p_{\nu\varphi} E_\varphi\left[\left(N_j + \delta_{\nu j}\right)\left(N_l + \delta_{\nu k}\right)\right] = \\
&= \sum_{\varphi \in \bar{T}} p_{\nu\varphi}\delta_{\nu j}\delta_{\nu k} + \sum_{\varphi \in T} p_{\nu\varphi}\left(E_\varphi\left[N_j N_k\right] + \delta_{\nu j} E_\varphi\left[N_k\right] + E_\varphi\left[N_j\right]\delta_{\nu k} + \delta_{\nu j}\delta_{\nu k}\right) = \\
&= \sum_{\varphi \in T} p_{\nu\varphi}\left(E_\varphi\left[N_j N_k\right] + \delta_{\nu j} E_\varphi\left[N_k\right] + E_\varphi\left[N_j\right]\delta_{\nu k}\right) + \delta_{\nu j}\delta_{\nu k} = \\
&= \sum_{\varphi \in T} p_{\nu\varphi} E_\varphi\left[N_j N_k\right] + \sum_{\varphi \in T} p_{\nu\varphi}\left(\delta_{\nu j} E_\varphi\left[N_k\right] + E_\varphi\left[N_j\right]\delta_{\nu k}\right) + \delta_{\nu j}\delta_{\nu k}. \qquad (1)
\end{aligned}$$

We recall that $z_{\varphi j} = E_\varphi\left[N_j\right]$. Let us denote $\varepsilon(\varphi, j, k) = E_\varphi\left[N_j N_k\right]$. Let us continue as follows

$$E_\nu\left[N_j N_k\right] = \sum_{\varphi \in T} p_{\nu\varphi} E_\varphi\left[N_j N_k\right] + \sum_{\varphi \in T} p_{\nu\varphi}\left(\delta_{\nu j} E_\varphi\left[N_k\right] + E_\varphi\left[N_j\right]\delta_{\nu k}\right) + \delta_{\nu j}\delta_{\nu k}.$$

$$\varepsilon(\nu, j, k) - \sum_{\varphi \in T} p_{\nu\varphi}\varepsilon(\varphi, j, k) = \sum_{\varphi \in T} p_{\nu\varphi}\left(\delta_{\nu j} z_{\varphi k} + z_{\varphi j}\delta_{\nu k}\right) + \delta_{\nu j}\delta_{\nu k}.$$

$$\sum_{\varphi \in T}\left(\delta_{\nu\varphi} - p_{\nu\varphi}\right)\varepsilon(\varphi, j, k) = \sum_{\varphi \in T} p_{\nu\varphi}\left(\delta_{\nu j} z_{\varphi k} + z_{\varphi j}\delta_{\nu k}\right) + \delta_{\nu j}\delta_{\nu k}.$$

Let us multiply the last expression by $z_{i\nu}$ and sum over ν.

$$\sum_{\nu \in T} z_{i\nu} \sum_{\varphi \in T}\left(\delta_{\nu\varphi} - p_{\nu\varphi}\right)\varepsilon(\varphi, j, k) = \sum_{\nu \in T} z_{i\nu} \sum_{\varphi \in T} p_{\nu\varphi}\left(\delta_{\nu j} z_{\varphi k} + z_{\varphi j}\delta_{\nu k}\right) + \delta_{\nu j}\delta_{\nu k}.$$

Next let us consider the lefthand side of the expression. Let us fix j, k for the moment. We can reformulate the lefthand side in matrix terms. We can consider $\varepsilon(\varphi, j, k)$ as a vector indexed by φ, let say $\varepsilon(\varphi, j, k) = \lambda_\varphi(j, k)$. Let us form matrices $Q = \{p_{\nu\varphi}\}_{\nu,\varphi \in T}$, $Z = \{z_{\varphi j}\}_{\varphi, j \in T}$, and $I = \{\delta_{\nu\varphi}\}_{\nu,\varphi \in T}$ is the identity matrix. One can see that the lefthand side can be formulated as

$$Z(I - Q)\lambda(j, k),$$

and, since $Z = (I - Q)^{-1}$, we have

$$Z(I - Q)\lambda(j,k) = \lambda(j,k),$$

or, written in a component form,

$$\sum_{\nu \in T} z_{i\nu} \sum_{\varphi \in T} (\delta_{\nu\varphi} - p_{\nu\varphi})\, \varepsilon(\varphi, j, k) = \varepsilon(i, j, k).$$

Now we consider the righthand side.

$$\begin{aligned} &\sum_{\varphi \in T} p_{\nu\varphi} (\delta_{ij} z_{\varphi k} + z_{\varphi j}\delta_{\nu k}) + \delta_{\nu j}\delta_{\nu k} = \\ &\delta_{\nu j} \sum_{\varphi \in T} p_{\nu\varphi} z_{\varphi k} + \delta_{\nu k} \sum_{\varphi \in T} p_{\nu\varphi} z_{\varphi j} + \delta_{\nu j}\delta_{\nu k}. \end{aligned} \tag{2}$$

One can see that $\delta_{\nu j}\delta_{\nu k} = D_{jk}(\nu)$ and $\delta_{\nu j} \sum_{\varphi \in T} p_{\nu\varphi} z_{\varphi k} = \{D(\nu)QZ\}_{jk}$, and $\delta_{ik} \sum_{\varphi \in T} p_{\nu\varphi} z_{\varphi j} = \{QZD(\nu)\}_{jk}$. Hence, we can write (2) in a matrix form.

$$D(\nu)QZ + QZD(\nu) + D(\nu).$$

Let us analyse the last expression.

$$\begin{aligned} & D(\nu)QZ + QZD(\nu) + D(\nu) = D(\nu)(Z - I) + (Z - I)D(\nu) + D(\nu) = \\ = \quad & D(\nu)Z - D(\nu) + ZD(\nu) - D(\nu) + D(\nu) = D(\nu)Z + ZD(\nu) - D(\nu). \end{aligned}$$

Thus, we can complete the proof by concluding that

$$\Lambda(i) = \sum_{\nu \in T} z_{i\nu} \left(D(\nu)Z + ZD(\nu) - D(\nu) \right).$$

∎

One can see that we have to consider the mixed second moments either as a vector $\lambda(j,k)$ depending on two indeces or as a matrix $\Lambda(i)$ depending on one indeces in the above proof. We need this trick because of poverty of matrix operations. In the contrast to the matrix approach, calculation of the mixed second moments and the mixed high-order moments is natural in a tensor form, as we shall show below.

3 Introduction to tensors

We give a brief introduction to basic facts from the tensor theory which we shall use in the further sections. We do not present the tensor theory in its completeness, we just define what we need for our application to the mixed high-order moments. Interested reader is referred to [6, 5] for more details.

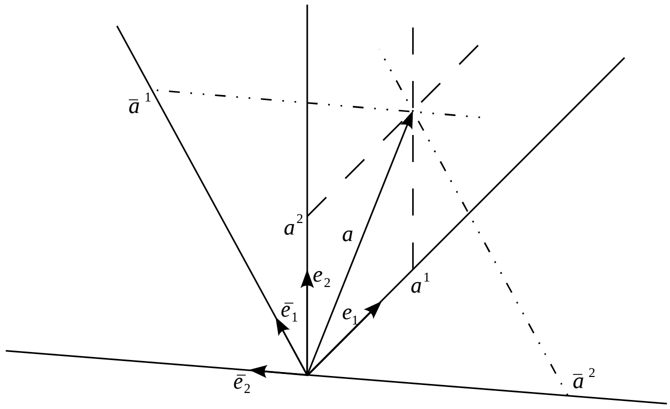

Figure 1: Change of a basis of a coordinate system. At the figure we have vector $\mathbf{a}$, basis (e_1, e_2), coordinates of vector $\mathbf{a}$ in the basis $\left(a^1, a^2\right)$, new basis $(\bar{e}_1, \bar{e}_2)$, coordinates of vector $\mathbf{a}$ in the new basis $\left(\bar{a}^1, \bar{a}^2\right)$.

Tensors are a generalization of such notions as vector and linear operator. Firstly, let us remind the notion of vector.

We consider a vector as an objective quantity having a magnitude and a direction. The vector does not depend on the way we describe the world. We denote the vector under consideration by $\mathbf{a}$. If we fix a coordinate system with its basis, $(e_1, e_2, \ldots, e_n)$, we can represent the vector as an array of real numbers, coordinates of the vector, $\left(a^1, a^2, \ldots, a^n\right)$,

$$\mathbf{a} = \sum_{i=1}^{n} a^i e_i.$$

When we change the basis or the coordinate system, we recalculate the coordinates by certain rules, but the vector itself does not change, see Figure 1. Let us assume now that we know only the vector and we do not know the coordinates of the vector. How can we determine them? It turns out that we can find a vector for each coordinate multiplying which by vector $\mathbf{a}$ by inner product we determine certain coordinate. Let e^i is such a vector for coordinate a^i.

$$a^i = \mathbf{a} \cdot e^i, \qquad i = 1, \ldots, n.$$

Vectors e^i are linear independent and, hence, form other basis which is called dual basis. Dual basis $\left(e^1, e^2, \ldots, e^n\right)$ relates to basis $(e_1, e_2, \ldots, e_n)$ as

$$e^i \cdot e_j = \delta^i_j,$$

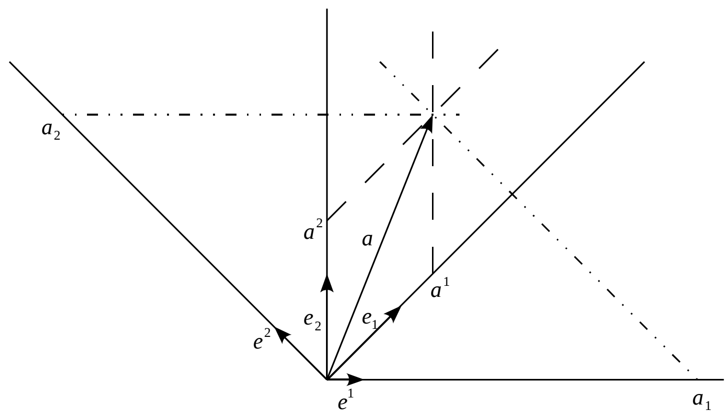

Figure 2: Dual basis. At the figure we have vector $\mathbf{a}$, basis (e_1, e_2), coordinates of vector $\mathbf{a}$ in the basis $\left(a^1, a^2\right)$, dual basis $\left(e^1, e^2\right)$, coordinates of vector $\mathbf{a}$ in the dual basis (a_1, a_2).

where δ^i_j is the Kronecker symbol. As in any other basis, we can find coordinates of vector $\mathbf{a}$ in the dual basis, $(a_1, a_2, \ldots, a_n)$, see Figure 2.

$$\mathbf{a} = \sum_{i=1}^{n} a_i e^i.$$

The coordinates of the vector in the main basis are called contravariant components of the vector. The coordinates of the vector in the dual basis are called covariant components of the vector. The notions “contravariant” and “covariant” are justified by the fact that when we change basis we use different rules to recalculate covariant and contravariant coordinates of the vector. Further, we shall always write covariant components with subscripts and contravariant components with superscripts. One can see that one-dimensional array of real numbers is enough to determine a vector.

But there are other entities for which one-dimensional array of real numbers is not enough. They are linear operators, linear mappings of a vector space to another vector space. If we fix coordinate systems in the vector spaces we can express a linear operator $\mathcal{A}$ by a matrix, say matrix A. If we change the basis, we recalculate entries of matrix A obtaining another matrix $\bar{A}$, but the both matrices correspond to same linear operator $\mathcal{A}$.

$$A = \{a^i_j\} \quad \bar{A} = \{\bar{a}^i_j\}.$$

Matrices corresponding to linear operator $\mathcal{A}$ can be written in the main basis and/or the dual basis. Let matrices A, B, C correspond to linear operator $\mathcal{A}$, but they are expressed in different bases.

$$A = \{a^i_j\} \quad B = \{b^{ij}\} \quad C = \{c_{ij}\}.$$

Components of matrix A are one-time contravariant and one-time covariant, components of matrix B are twice contravariant, components of matrix C are twice covariant, but all the matrices corresponds to same linear operator $\mathcal{A}$. One can see that a linear operator can be expressed by a two-dimensional array of real numbers.

But there are entities which cannot be represented by a two-dimensional array of real numbers. They are multilinear operators which are also called tensors. Let us express a tensor $\mathcal{A}$ by components which are n-times contravariant and m-times covariant. The order of tensor is $n+m$ and its component form is given by

$$a^{i_1 i_2 \ldots i_n}_{h_1 h_2 \ldots h_m}.$$

Let us introduce tensor operations which we need for further development. Tensor product $\otimes$ of a tensor $\mathcal{A}$ which is n-times contravariant and m-times covariant and a tensor $\mathcal{B}$ which is s-times contravariant and t-times covariant is a tensor $\mathcal{C}$ which is $n+s$-times contravariant and $m+t$-times covariant (3).

$$\mathcal{A} \otimes \mathcal{B} = \mathcal{C}, \tag{3}$$

where components of tensor $\mathcal{C}$ in some basis can be found by formula

$$a^{i_1 i_2 \ldots i_n}_{k_1 k_2 \ldots k_m} b^{p_1 p_2 \ldots p_s}_{h_1 h_2 \ldots h_t} = c^{i_1 i_2 \ldots i_n p_1 p_2 \ldots p_s}_{k_1 k_2 \ldots k_m h_1 h_2 \ldots h_t},$$

where indeces $i_1, i_2, \ldots, i_n$,$k_1, k_2, \ldots, k_m$,$p_1, p_2, \ldots, p_s$ and $h_1, h_2, \ldots, h_t$ take all possible values. Further we shall write tensor product $\otimes$ as

$$a^{i_1 i_2 \ldots i_n}_{k_1 k_2 \ldots k_m} \otimes b^{p_1 p_2 \ldots p_s}_{h_1 h_2 \ldots h_t} = c^{i_1 i_2 \ldots i_n p_1 p_2 \ldots p_s}_{k_1 k_2 \ldots k_m h_1 h_2 \ldots h_t},$$

assuming that indeces $i_1, i_2, \ldots, i_n$,$k_1, k_2, \ldots, k_m$,$p_1, p_2, \ldots, p_s$ and $h_1, h_2, \ldots, h_t$ take all possible values.

In some cases we need to consider only components of tensors having same indeces in tensor product (4).

$$a^{i_1 i_2 \ldots i_n}_{k_1 k_2 \ldots k_m} \otimes b^{i_1 i_2 \ldots i_n}_{h_1 h_2 \ldots h_t} = a^{i_1 i_2 \ldots i_n}_{k_1 k_2 \ldots k_m} b^{i_1 i_2 \ldots i_n}_{h_1 h_2 \ldots h_t} = c^{i_1 i_2 \ldots i_n}_{k_1 k_2 \ldots k_m h_1 h_2 \ldots h_t}. \tag{4}$$

Also let us define tensor contraction $\odot$ by formula in (5).

$$a^{i_1 i_2 \ldots i_n}_{k_1 k_2 \ldots k_m} \odot b^{k_1 k_2 \ldots k_m}_{h_1 h_2 \ldots h_t} = \sum_{k_1} \sum_{k_2} \cdots \sum_{k_m} a^{i_1 i_2 \ldots i_n}_{k_1 k_2 \ldots k_m} b^{k_1 k_2 \ldots k_m}_{h_1 h_2 \ldots h_t} = c^{i_1 i_2 \ldots i_n}_{h_1 h_2 \ldots h_t}. \tag{5}$$

We note that tensor contraction is equivalent to matrix product if the matrices are written in one-time contravariant and one-time covariant components.

We shall use tensors and the tensor operations in application to the mixed second moments and the mixed high-order moments.

4 Mixed second moments in tensor form

Having introduced tensor operation in the previous section, we shall show that the mixed second moments can be calculated from the tensor point of view without tricks which we used in the matrix form.

We denote $\varepsilon^i_j = E_i\left[N_j\right]$ and $\varepsilon^i_{jk} = E_i\left[N_j N_k\right]$, where ε^i_j and ε^i_{jk} we consider as tensors.

Theorem 4.1 *The mixed second moments are given by*

$$\varepsilon^i_{jk} \quad = \quad \varepsilon^i_\nu \odot \left(\varepsilon^\nu_j \otimes \delta^\nu_k + \varepsilon^\nu_k \otimes \delta^\nu_j - \delta^\nu_k \otimes \delta^\nu_j\right).$$

Proof. We begin the proof as in Theorem 2.2 arriving to expression (1).

$$E_\nu\left[N_j N_k\right] = \sum_{\varphi\in T} p_{\nu\varphi} E_\varphi\left[N_j N_k\right] + \sum_{\varphi\in T} p_{\nu\varphi}\left(\delta_{\nu j} E_\varphi\left[N_k\right] + E_\varphi\left[N_j\right]\delta_{\nu k}\right) + \delta_{\nu j}\delta_{\nu k}.$$

Now we rewrite the above expression in the tensor form. Hence, $E_\varphi\left[N_j\right] = \varepsilon^\varphi_j$, $E_i\left[N_j N_k\right] = \varepsilon^i_{jk}$. We consider matrix $Q = \{p_{\nu\varphi}\}_{\nu,\varphi\in T}$ as tensor q^ν_φ. Kroneker symbol $\delta_{\nu k}$ we treat as δ^ν_k tensor.

$$\varepsilon^\nu_{jk} = q^\nu_\varphi \odot \varepsilon^\varphi_{jk} + q^\nu_\varphi \odot \left(\varepsilon^\varphi_j \otimes \delta^\nu_k + \varepsilon^\varphi_k \otimes \delta^\nu_j\right) + \delta^\nu_j \otimes \delta^\nu_k,$$
$$\varepsilon^\nu_{jk} - q^\nu_\varphi \odot \varepsilon^\varphi_{jk} = q^\nu_\varphi \odot \left(\varepsilon^\varphi_j \otimes \delta^\nu_k + \varepsilon^\varphi_k \otimes \delta^\nu_j\right) + \delta^\nu_j \otimes \delta^\nu_k.$$

Since $\varepsilon^\nu_{jk} = \delta^\nu_\varphi \odot \varepsilon^\varphi_{jk}$, we write

$$\left(\delta^\nu_\varphi - q^\nu_\varphi\right) \odot \varepsilon^\varphi_{jk} = q^\nu_\varphi \odot \left(\varepsilon^\varphi_j \otimes \delta^\nu_k + \varepsilon^\varphi_k \otimes \delta^\nu_j\right) + \delta^\nu_j \otimes \delta^\nu_k.$$

Let us multiply the above expression from left by ε^i_ν by tensor product with contraction.

$$\varepsilon^i_\nu \odot \left(\delta^\nu_\varphi - q^\nu_\varphi\right) \odot \varepsilon^\varphi_{jk} = \varepsilon^i_\nu \odot \left(q^\nu_\varphi \odot \left(\varepsilon^\varphi_j \otimes \delta^\nu_k + \varepsilon^\varphi_k \otimes \delta^\nu_j\right) + \delta^\nu_j \otimes \delta^\nu_k\right).$$

Since tensor ε^i_ν corresponds to matrix Z, and tensor $\left(\delta^\nu_\varphi - q^\nu_\varphi\right)$ corresponds to matrix $I - Q$, and $Z = (I - Q)^{-1}$, we obtain that

$$\varepsilon^i_\nu \odot \left(\delta^\nu_\varphi - q^\nu_\varphi\right) \odot \varepsilon^\varphi_{jk} = \delta^i_\varphi \odot \varepsilon^\varphi_{jk} = \varepsilon^i_{jk}.$$

Now we can continue using the following observation. Since tensor q^ν_φ corresponds to matrix Q, and tensor ε^φ_j corresponds to matrix Z, and $QZ = Z - I$, then $q^\nu_\varphi \odot \varepsilon^\varphi_j = \varepsilon^\nu_j - \delta^\nu_j$.

$$\begin{aligned}
\varepsilon^i_{jk} &= \varepsilon^i_\nu \odot \left(q^\nu_\varphi \odot \left(\varepsilon^\varphi_j \otimes \delta^\nu_k + \varepsilon^\varphi_k \otimes \delta^\nu_j\right) + \delta^\nu_j \otimes \delta^\nu_k\right) = \\
&= \varepsilon^i_\nu \odot \left(q^\nu_\varphi \odot \left(\varepsilon^\varphi_j \otimes \delta^\nu_k\right) + q^\nu_\varphi \odot \left(\varepsilon^\varphi_k \otimes \delta^\nu_j\right) + \delta^\nu_j \otimes \delta^\nu_k\right) = \\
&= \varepsilon^i_\nu \odot \left(\left(q^\nu_\varphi \odot \varepsilon^\varphi_j\right) \otimes \delta^\nu_k + \left(q^\nu_\varphi \odot \varepsilon^\varphi_k\right) \otimes \delta^\nu_j + \delta^\nu_j \otimes \delta^\nu_k\right) = \\
&= \varepsilon^i_\nu \odot \left(\left(\varepsilon^\nu_j - \delta^\nu_j\right) \otimes \delta^\nu_k + \left(\varepsilon^\nu_k - \delta^\nu_k\right) \otimes \delta^\nu_j + \delta^\nu_j \otimes \delta^\nu_k\right) = \\
&= \varepsilon^i_\nu \odot \left(\varepsilon^\nu_j \otimes \delta^\nu_k - \delta^\nu_j \otimes \delta^\nu_k + \varepsilon^\nu_k \otimes \delta^\nu_j - \delta^\nu_k \otimes \delta^\nu_j + \delta^\nu_j \otimes \delta^\nu_k\right) = \\
&= \varepsilon^i_\nu \odot \left(\varepsilon^\nu_j \otimes \delta^\nu_k + \varepsilon^\nu_k \otimes \delta^\nu_j - \delta^\nu_k \otimes \delta^\nu_j\right).
\end{aligned}$$

Concluding that

$$\varepsilon^i_{jk} = \varepsilon^i_\nu \odot \left(\varepsilon^\nu_j \otimes \delta^\nu_k + \varepsilon^\nu_k \otimes \delta^\nu_j - \delta^\nu_k \otimes \delta^\nu_j\right),$$

we complete the proof. ∎

One can see that we used the natural tensor operations to calculate the mixed second moments in the above proof and we need not the trick with representation of the moments as in the matrix form.

5 Auxiliary combinatorial result

Before we deal with the mixed high-order moments, we need an auxiliary combinatorial result.

Let M be a finite set of elements of any nature with cardinality m. Let $M = \{k_0, k_1, \ldots, k_{m-1}\}$.

Let us enumerate all the combinations of the elements of set M having length j and let us index them by ψ, where $j = 0, \ldots, m$ and $\psi = 0, \ldots, \binom{m}{j} - 1$. Let us define a function $f(M, j, \psi)$. Value $f(M, j, \psi)$ is the combination of the elements of set M having length j and index ψ. Let us denote $\bar{f}(M, j, \psi) = M \setminus f(M, j, \psi)$.

Let us consider $f(M, j, \psi)$, where $\psi = 0, \ldots, \binom{m}{j} - 1$. Since the order of the elements in combination $f(M, j, \psi)$ does not matter, we can assume any order. Let $f(M, j, \psi) = \left\{k_{\omega_0}, k_{\omega_1}, \ldots, k_{\omega_{j-1}}\right\}$, where $\omega_x = 0, \ldots, m-1$, $x = 0, \ldots, j-1$. We shall assume that $\omega_0 \leqslant \omega_1 \leqslant \cdots \leqslant \omega_{j-1}$. According to [4] we can calculate ψ as

$$\psi = \sum_{x=0}^{j-1} \binom{\omega_x}{x+1},$$

where $\binom{a}{b} = 0$, if $a < b$. Such indexing provide lexico-graphic ordering to combinations $f(M, j, \psi)$. It means that, for example, when $m = 3$ and $j = 2$, combinations will be ordered like this: k_0k_1, k_0k_2, k_1k_3. We need the lexico-graphic ordering only to prove Proposition 5.1 below, although the proposition holds for any ordering. In any other discussion we assume any, but fixed, ordering.

Let us denote by A the set of all combinations of elements of set M with length $\varkappa$.

$$A = \left\{ f\left(M, \varkappa, \rho\right) \middle| \rho = 0, \ldots, \binom{m}{\varkappa} - 1 \right\}.$$

Let us denote by B the following multiset.

$$B = \left\{ f\left(f\left(M, j, \psi\right), \varkappa, \chi\right) \middle| \psi = 0, \ldots, \binom{m}{j} - 1, \chi = 0, \ldots, \binom{j}{\varkappa} - 1 \right\}.$$

One can see that multiset B consists of the same elements as set A. Let us establish a precise relation between set A and multiset B.

Proposition 5.1 *B is a multiset of the elements of set A and each element of set A is taken $\binom{m-\varkappa}{m-j}$ times.*

Proof. See Appendix. ∎

The auxiliary combinatorial result plays an important role in our treatment of the mixed high-order moment which we consider in the next section.

6 Mixed high-order moments

Let us now consider the mixed moments of higher order. Before we formulate the mixed high-order moments in tensor formalism, let us prove that the moments are finite.

Let us consider the conditional moment generating function of the absorbing Markov chain, $M_i(y) = E_i\left[e^{\sum y_j N_j}\right]$, where summation is performed over all states of the Markov chain and the process starts at a transient state i. We need to prove that the moment generating function is analytical in the origin.

Let us define vector ζ and matrix $\Theta = \{\vartheta_{ik}\}_{i,k\in T}$

$$
\begin{aligned}
\zeta_i &= e^{y_i}\left(1 - \sum_{k\in T} p_{ik}\right),\\
\vartheta_{ik} &= \delta_{ik} - e^{y_i} p_{ik}.
\end{aligned}
$$

Proposition 6.1 *If all y_i are small enough, moment generating function $M(y)$ is give by*

$$M(y) = \Theta^{-1}\zeta.$$

Proof. We ask where the process can go in one step, from its starting position i.

$$
\begin{aligned}
E_i\left[e^{\sum y_j N_j}\right] &= \sum_{k\in\bar{T}} p_{ik} e^{y_i} + \sum_{k\in T} p_{ik} E_k\left[e^{\sum_{j\neq i} y_j N_j + y_i(N_i+1)}\right] = \\
&= \sum_{k\in\bar{T}} p_{ik} e^{y_i} + \sum_{k\in T} p_{ik} E_k\left[e^{\sum y_j N_j + y_i}\right] = \\
&= \sum_{k\in\bar{T}} p_{ik} e^{y_i} + \sum_{k\in T} p_{ik} E_k\left[e^{\sum y_j N_j} e^{y_i}\right] = \\
&= e^{y_i}\left(1 - \sum_{k\in T} p_{ik}\right) + e^{y_i}\sum_{k\in T} p_{ik} E_k\left[e^{\sum y_j N_j}\right].
\end{aligned}
$$

And we solve the above equation in the following way.

$$
\begin{aligned}
E_i\left[e^{\sum y_j N_j}\right] &= e^{y_i}\left(1 - \sum_{k\in T} p_{ik}\right) + e^{y_i}\sum_{k\in T} p_{ik} E_k\left[e^{\sum y_j N_j}\right],\\
E_i\left[e^{\sum y_j N_j}\right] - e^{y_i}\sum_{k\in T} p_{ik} E_k\left[e^{\sum y_j N_j}\right] &= e^{y_i}\left(1 - \sum_{k\in T} p_{ik}\right),\\
\sum_{k\in T}\left(\delta_{ik} - e^{y_i} p_{ik}\right) E_k\left[e^{\sum y_j N_j}\right] &= e^{y_i}\left(1 - \sum_{k\in T} p_{ik}\right). \qquad (6)
\end{aligned}
$$

Then, we can rewrite (6) in matrix form

$$\Theta M(y) = \zeta.$$

Let us show that matrix Θ is invertible. Let us denote by $t = |T|$ and by Ξ the matrix

$$\Xi = \begin{pmatrix} e^{y_1} & 0 & \dots & 0 \\ 0 & e^{y_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{y_t} \end{pmatrix}.$$

We express matrix Θ as

$$\Theta = I - \Xi Q,$$

where Q corresponds to the transient states of the absorbing Markov chain. Since matrix Ξ is diagonal, then matrix ΞQ is matrix Q whose rows are multiplied by diagonal elements of matrix Ξ. Matrix Q is substochastic, hence,

$$Q\mathbf{1} = q,$$

where $q = (q_1, q_2, \dots, q_t)^T$, and $0 \leqslant q_i \leqslant 1$, $\forall i = 1, \dots, t$, and $\exists i : q_i < 1$. Then,

$$\Xi Q\mathbf{1} = q = (e^{y_1} q_1, e^{y_2} q_2, \dots, e^{y_t} q_t)^T.$$

Matrix ΞQ is substochastic, if $0 \leqslant e^{y_i} q_i \leqslant 1$, $\forall i = 1, \dots, t$, and $\exists i : e^{y_i} q_i < 1$. Therefore, if $y_i \leqslant -\ln q_i$, $\forall i = 1, \dots, t$, and $\exists i : y_i < -\ln q_i$, matrix Θ is invertible.

And we can determine the conditional moment generating function by

$$M(y) = \Theta^{-1}\zeta.$$

■

One can see that the conditional moment generating function $M(y)$ is analytical at the origin, and, hence, there exist all the mixed high-order moments and they are finite.

We denote

$$\begin{aligned} \varepsilon_j^i &= E_i\left[N_j\right], \\ \varepsilon_{jk}^i &= E_i\left[N_j N_k\right], \dots, \\ \varepsilon_{k_0 k_1 \dots k_{m-1}}^i &= E_i\left[\prod_{j=0}^{m-1} N_{k_j}\right], \end{aligned}$$

where m is a natural number. Let us denote $M = \{k_0, k_1, \dots, k_{m-1}\}$. The cardinality of set M is m. We call set M the basis set.

We have got the mixed moments of higher order in tensor representation. Since the product is a commutative operation, the order of indeces $k_0 k_1 \dots k_{m-1}$ in $\varepsilon_{k_0 k_1 \dots k_{m-1}}^i$ does not matter, and we can write $\varepsilon_{k_0 k_1 \dots k_{m-1}}^i = \varepsilon_M^i$.

Let us denote ${}_0a^{\nu}_{k_0k_1\ldots k_{m-1}} = \bigotimes_{\iota=0}^{m-1} \delta^{\nu}_{k_\iota}$. Let us define tensor ${}_ja^{\nu}_{k_0k_1\ldots k_{m-1}}$ as follows:

$$ {}_ja^{\nu}_{k_0k_1\ldots k_{m-1}} = \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon^{\nu}_{f(M,j,\psi)} \otimes {}_0a^{\nu}_{\bar{f}(M,j,\psi)}. $$

Since index ψ passes all possible values, the order of indeces $k_0k_1\ldots k_{m-1}$ in ${}_ja^{\nu}_{k_0k_1\ldots k_{m-1}}$ does not matter, and we can write ${}_ja^{\nu}_{k_0k_1\ldots k_{m-1}} = {}_ja^{\nu}_M$. We note that ${}_ma^{\nu}_M = \varepsilon^{\nu}_M$.

Let us define tensor ${}_jb^{\varphi\nu}_{k_0k_1\ldots k_{m-1}}$ as follows:

$$ {}_jb^{\varphi\nu}_{k_0k_1\ldots k_{m-1}} = \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon^{\varphi}_{f(M,j,\psi)} \otimes {}_0a^{\nu}_{\bar{f}(M,j,\psi)}. $$

Since index ψ passes all possible values, the order of indeces $k_0k_1\ldots k_{m-1}$ in ${}_jb^{\varphi\nu}_{k_0k_1\ldots k_{m-1}}$ does not matter, and we can write ${}_jb^{\varphi\nu}_{k_0k_1\ldots k_{m-1}} = {}_jb^{\varphi\nu}_M$. We note that ${}_mb^{\varphi\nu}_M = \varepsilon^{\varphi}_M, {}_0b^{\varphi\nu}_M = {}_0a^{\nu}_M$, and, in general, ${}_jb^{\nu\nu}_M = {}_ja^{\nu}_M$.

Let us define tensor ${}_{\varkappa j}c^{\nu}_{k_0k_1\ldots k_{m-1}}$, where $\varkappa \leqslant j$, as follows:

$$ {}_{\varkappa j}c^{\nu}_{k_0k_1\ldots k_{m-1}} = \sum_{\psi=0}^{\binom{m}{j}-1} {}_{\varkappa}a^{\nu}_{f(M,j,\psi)} \otimes {}_0a^{\nu}_{\bar{f}(M,j,\psi)}. $$

Since index ψ passes all possible values, the order of indeces $k_0k_1\ldots k_{m-1}$ in ${}_{\varkappa j}c^{\nu}_{k_0k_1\ldots k_{m-1}}$ does not matter, and we can write ${}_{\varkappa j}c^{\nu}_{k_0k_1\ldots k_{m-1}} = {}_{\varkappa j}c^{\nu}_M$.

Proposition 6.2 *The following formula takes place:*

$$ {}_{0j}c^{\nu}_M = \binom{m}{j} {}_0a^{\nu}_M. $$

Proof.

$$ {}_{0j}c^{\nu}_M = \sum_{\psi=0}^{\binom{m}{j}-1} {}_0a^{\nu}_{f(M,j,\psi)} \otimes {}_0a^{\nu}_{\bar{f}(M,j,\psi)} = \sum_{\psi=0}^{\binom{m}{j}-1} {}_0a^{\nu}_M = \binom{m}{j} {}_0a^{\nu}_M. $$

■

Proposition 6.3 *The following formula takes place:*

$$ {}_{jj}c^{\nu}_M = {}_ja^{\nu}_M. $$

Proof.

$$ {}_{jj}c_M^{\nu} = \sum_{\psi=0}^{\binom{m}{j}-1} {}_j a^{\nu}_{f(M,j,\psi)} \otimes_0 a^{\nu}_{\bar{f}(M,j,\psi)} = \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon^{\nu}_{f(M,j,\psi)} \otimes_0 a^{\nu}_{\bar{f}(M,j,\psi)} = {}_j a^{\nu}_M. $$

∎

Propositions 6.2 and 6.3 are the particular cases of the following proposition.

Proposition 6.4 *The following formula takes place:*

$$ {}_{\varkappa j}c_M^{\nu} = \binom{m-\varkappa}{m-j} {}_{\varkappa}a^{\nu}_M. $$

Proof. We can write ${}_{\varkappa j}c_M^{\nu}$ as follows

$$ {}_{\varkappa j}c_M^{\nu} = \sum_{\psi=0}^{\binom{m}{j}-1} {}_{\varkappa}a^{\nu}_{f(M,j,\psi)} \otimes_0 a^{\nu}_{\bar{f}(M,j,\psi)} = $$

$$ = \sum_{\psi=0}^{\binom{m}{j}-1} \sum_{\chi=0}^{\binom{j}{\varkappa}-1} \varepsilon^{\nu}_{f(f(M,j,\psi),\varkappa,\chi)} \otimes_0 a^{\nu}_{\bar{f}(f(M,j,\psi),\varkappa,\chi)} \otimes_0 a^{\nu}_{\bar{f}(M,j,\psi)}. \tag{7} $$

We can write ${}_{\varkappa}a^{\nu}_M$ as follows:

$$ {}_{\varkappa}a^{\nu}_M = \sum_{\psi=0}^{\binom{m}{\varkappa}-1} \varepsilon^{\nu}_{f(M,\varkappa,\psi)} \otimes_0 a^{\nu}_{\bar{f}(M,\varkappa,\psi)}. \tag{8} $$

Since operations "$+$" and "$\otimes$" are commutative, to prove the statement of the proposition it is enough to show that every term of summation (8) can be found in summation (7) $\binom{m-\varkappa}{m-j}$ times.

Indeces $f(f(M,j,\psi),\varkappa,\chi)$ in (7) make up multiset B from Section 5 and indeces $f(M,\varkappa,\psi)$ in (8) make up set A from Section 5. Therefore, we apply Proposition 5.1 and complete the proof. ∎

Theorem 6.1 *The mixed high-order moments of the absorbing Markov chain is given by*

$$ \varepsilon^i_M = \varepsilon^i_{\nu} \odot \sum_{\varkappa=0}^{m-1} (-1)^{m-\varkappa+1} {}_{\varkappa}a^{\nu}_M. $$

Proof. Let us assume that the theorem is proven for smaller values of m, particularly for $m = 2$ Theorem 4.1.

We start with the non-tensor representation of the high-order mixed moments, $E_i\left[\prod_{j=0}^{m-1} N_{k_j}\right]$.

Let us calculate $E_i\left[\prod_{j=0}^{m-1} N_{k_j}\right]$. Following the approach of [3, Theorem 3.3.3] we ask where the process can go in one step, from its starting position i. It can go to state φ with probability $p_{i\varphi}$. If the new state is absorbing, then we can never reach states k_j, where $j = 0, \dots, m-1$, again, and the only possible contribution is from the initial state, which is $\prod_{j=0}^{m-1} \delta_{ik_j}$. If the new state is transient then we will be in state k_j δ_{ik_j} times from the initial state, and N_{k_j}, where $j = 0, \dots, m-1$, times from the later steps. We have

$$E_i\left[\prod_{j=0}^{m-1} N_{k_j}\right] = \sum_{\varphi\in\bar{T}} p_{i\varphi} \prod_{j=0}^{m-1} \delta_{ik_j} + \sum_{\varphi\in T} p_{i\varphi} E_\varphi \left[\prod_{j=0}^{m-1} \left(N_{k_j} + \delta_{ik_j}\right)\right] =$$

$$\begin{aligned}
= & \sum_{\varphi\in\bar{T}} p_{i\varphi} \prod_{j=0}^{m-1} \delta_{ik_j} + \\
+ & \sum_{\varphi\in T} p_{i\varphi}\Bigg(E_\varphi\left[\prod_{\kappa\in M} N_\kappa\right] + \sum_{\psi=0}^{\binom{m}{m-1}-1} E_\varphi\left[\prod_{\kappa\in f(M,m-1,\psi)} N_\kappa\right] \prod_{\kappa\in\bar{f}(M,m-1,\psi)} \delta_{i\kappa} + \\
+ & \cdots + \sum_{\psi=0}^{\binom{m}{j}-1} E_\varphi\left[\prod_{\kappa\in f(M,j,\psi)} N_\kappa\right] \prod_{\kappa\in\bar{f}(M,j,\psi)} \delta_{i\kappa} + \cdots + \\
+ & \sum_{\psi=0}^{\binom{m}{1}-1} E_\varphi\left[\prod_{\kappa\in f(M,1,\psi)} N_\kappa\right] \prod_{\kappa\in\bar{f}(M,1,\psi)} \delta_{i\kappa} + \prod_{\kappa\in M} \delta_{i\kappa}\Bigg).
\end{aligned}$$

We note that

$$\sum_{\varphi\in\bar{T}} p_{i\varphi} \prod_{j=0}^{m-1} \delta_{ik_j} + \sum_{\varphi\in T} p_{i\varphi} \prod_{\kappa\in M} \delta_{i\kappa} = \prod_{\kappa\in M} \delta_{i\kappa} \left(\sum_{\varphi\in\bar{T}} p_{i\varphi} + \sum_{\varphi\in T} p_{i\varphi}\right) = \prod_{\kappa\in M} \delta_{i\kappa}.$$

And we continue

$$E_i\left[\prod_{j=0}^{m-1} N_{k_j}\right] = \prod_{\kappa\in M} \delta_{i\kappa} + \sum_{\varphi\in T} p_{i\varphi}\left(E_\varphi\left[\prod_{\kappa\in M} N_\kappa\right] + \sum_{j=1}^{m-1} \sum_{\psi=0}^{\binom{m}{j}-1} E_\varphi\left[\prod_{\kappa\in f(M,j,\psi)} N_\kappa\right] \prod_{\kappa\in\bar{f}(M,j,\psi)} \delta_{i\kappa}\right).$$

Let us rewrite the last expression in the tensor form. We will use index ν in place of i for further development. We note that $\prod_{\kappa\in\bar{f}(M,j,\psi)} \delta_{\nu\kappa}$ is represented in the tensor form as $\bigotimes_{\kappa\in\bar{f}(M,j,\psi)} \delta_\kappa^\nu = {}_0a^\nu_{\bar{f}(M,j,\psi)}$. Hence, we write

$$\begin{aligned}
\varepsilon_M^\nu &= {}_0a_M^\nu + q_\varphi^\nu \odot \left(\varepsilon_M^\varphi + \sum_{j=1}^{m-1} \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon^\varphi_{f(M,j,\psi)} \otimes {}_0a^\nu_{\bar{f}(M,j,\psi)}\right) = \\
&= q_\varphi^\nu \odot \varepsilon_M^\varphi + q_\varphi^\nu \odot \sum_{j=1}^{m-1} {}_jb_M^{\varphi\nu} + {}_0a_M^\nu.
\end{aligned}$$

Let us consider $q_{\varphi}^{\nu} \odot \sum_{j=1}^{m-1} {}_j b_M^{\varphi\nu} +_0 a_M^{\nu}$ and, in particular, $q_{\varphi}^{\nu} \odot_j b_M^{\varphi\nu}$ as one term of the summation.

$$q_{\varphi}^{\nu} \odot_j b_M^{\varphi\nu} = q_{\varphi}^{\nu} \odot \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon_{f(M,j,\psi)}^{\varphi} \otimes_0 a_{\bar{f}(M,j,\psi)}^{\nu}.$$

Next we consider $q_{\varphi}^{\nu} \odot \varepsilon_{f(M,j,\psi)}^{\varphi}$ and proceed further by induction.

$$q_{\varphi}^{\nu} \odot \varepsilon_{f(M,j,\psi)}^{\varphi} = q_{\varphi}^{\nu} \odot \varepsilon_{\mu}^{\varphi} \odot \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa+1} {}_{\varkappa} a_{f(M,j,\psi)}^{\mu}.$$

Since $q_{\varphi}^{\nu} \odot \varepsilon_{\mu}^{\varphi} = \varepsilon_{\mu}^{\nu} - \delta_{\mu}^{\nu}$.

$$\begin{aligned} q_{\varphi}^{\nu} \odot \varepsilon_{f(M,j,\psi)}^{\varphi} &= \left(\varepsilon_{\mu}^{\nu} - \delta_{\mu}^{\nu}\right) \odot \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa+1} {}_{\varkappa} a_{f(M,j,\psi)}^{\mu} = \\ &= \varepsilon_{f(M,j,\psi)}^{\nu} - \delta_{\mu}^{\nu} \odot \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa+1} {}_{\varkappa} a_{f(M,j,\psi)}^{\mu} = \\ &= \varepsilon_{f(M,j,\psi)}^{\nu} + \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa} {}_{\varkappa} a_{f(M,j,\psi)}^{\nu}. \end{aligned}$$

Now we come back to $q_{\varphi}^{\nu} \odot_j b_M^{\varphi\nu}$.

$$\begin{aligned} q_{\varphi}^{\nu} \odot_j b_M^{\varphi\nu} &= \sum_{\psi=0}^{\binom{m}{j}-1} \left(\varepsilon_{f(M,j,\psi)}^{\nu} + \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa} {}_{\varkappa} a_{f(M,j,\psi)}^{\nu} \right) \otimes_0 a_{\bar{f}(M,j,\psi)}^{\nu} = \\ &= {}_j a_M^{\nu} + \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa} \sum_{\psi=0}^{\binom{m}{j}-1} {}_{\varkappa} a_{f(M,j,\psi)}^{\nu} \otimes_0 a_{\bar{f}(M,j,\psi)}^{\nu} = \\ &= {}_j a_M^{\nu} + \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa} {}_{\varkappa j} c_M^{\nu} = \\ &= \sum_{\varkappa=0}^{j} (-1)^{j-\varkappa} {}_{\varkappa j} c_M^{\nu}. \end{aligned}$$

Now we come back to $q_\varphi^\nu \odot \sum_{j=1}^{m-1} {}_j b_M^{\varphi\nu} + {}_0 a_M^\nu$.

$$\begin{aligned} q_\varphi^\nu \odot \sum_{j=1}^{m-1} {}_j b_M^{\varphi\nu} + {}_0 a_M^\nu &= \sum_{j=1}^{m-1} \sum_{\varkappa=0}^{j} (-1)^{j-\varkappa} {}_{\varkappa j} c_M^\nu + {}_0 a_M^\nu = \\ &= \sum_{j=1}^{m-1} \sum_{\varkappa=1}^{j} (-1)^{j-\varkappa} {}_{\varkappa j} c_M^\nu + \sum_{j=1}^{m-1} (-1)^j {}_{0j} c_M^\nu + {}_0 a_M^\nu = \\ &= \sum_{\varkappa=1}^{m-1} \sum_{j=\varkappa}^{m-1} (-1)^{j-\varkappa} {}_{\varkappa j} c_M^\nu + \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} {}_0 a_M^\nu . \end{aligned}$$

We note that $\sum_{j=0}^{m} (-1)^j \binom{m}{j} = 0$, and, therefore,

$$\sum_{j=0}^{m-1} (-1)^{j-\varkappa} \binom{m}{j} = (-1)^{m+1} .$$

Let us consider $\sum_{j=\varkappa}^{m-1} (-1)^{j-\varkappa} {}_{\varkappa j} c_M^\nu$. We recall that ${}_{\varkappa j} c_M^\nu = \binom{m-\varkappa}{m-j} {}_\varkappa a_M^\nu$ according to Proposition 6.4, and we consider $\sum_{j=\varkappa}^{m-1} (-1)^{j-\varkappa} \binom{m-\varkappa}{m-j}$.

$$\begin{aligned} \sum_{j=\varkappa}^{m-1} (-1)^{j-\varkappa} \binom{m-\varkappa}{m-j} &= \sum_{j=0}^{m-\varkappa-1} (-1)^j \binom{m-\varkappa}{m-j-\varkappa} = \sum_{j=0}^{m-\varkappa-1} (-1)^j \binom{m-\varkappa}{j} = \\ &= \sum_{j=0}^{m-\varkappa} (-1)^j \binom{m-\varkappa}{j} - (-1)^{m-\varkappa} \binom{m-\varkappa}{m-\varkappa} = (-1)^{m-\varkappa+1} . \end{aligned}$$

Hence, for $q_\varphi^\nu \odot \sum_{j=1}^{m-1} {}_j b_M^{\varphi\nu} + {}_0 a_M^\nu$ we have

$$q_\varphi^\nu \odot \sum_{j=1}^{m-1} {}_j b_M^{\varphi\nu} + {}_0 a_M^\nu = \sum_{\varkappa=1}^{m-1} (-1)^{m-\varkappa+1} {}_\varkappa a_M^\nu + (-1)^{m+1} {}_0 a_M^\nu = \sum_{\varkappa=0}^{m-1} (-1)^{m-\varkappa+1} {}_\varkappa a_M^\nu .$$

And, finally, we obtain

$$\begin{aligned} \varepsilon_M^\nu &= q_\varphi^\nu \odot \varepsilon_M^\varphi + q_\varphi^\nu \odot \sum_{j=1}^{m-1} {}_j b_M^{\varphi\nu} + {}_0 a_M^\nu = \\ &= q_\varphi^\nu \odot \varepsilon_M^\varphi + \sum_{\varkappa=0}^{m-1} (-1)^{m-\varkappa+1} {}_\varkappa a_M^\nu . \end{aligned}$$

Next we consider $\varepsilon_M^\nu - q_\varphi^\nu \odot \varepsilon_M^\varphi$.

$$\varepsilon_M^\nu - q_\varphi^\nu \odot \varepsilon_M^\varphi = \left(\delta_\varphi^\nu - q_\varphi^\nu\right) \odot \varepsilon_M^\varphi,$$

and multiplying by ε_ν^i from left, and recalling that $\varepsilon_\nu^i \odot \left(\delta_\varphi^\nu - q_\varphi^\nu\right) = \delta_\varphi^i$ we have

$$\varepsilon_\nu^i \odot \left(\delta_\varphi^\nu - q_\varphi^\nu\right) \odot \varepsilon_M^\varphi = \delta_\varphi^i \odot \varepsilon_M^\varphi = \varepsilon_M^i.$$

We complete the proof with

$$\varepsilon_M^i = \varepsilon_\nu^i \odot \sum_{\varkappa=0}^{m-1} (-1)^{m-\varkappa+1} {}_\varkappa a_M^\nu.$$

■

One can see that tensor formalizm allows us to calculate the mixed high-order moments by compact formula. The mixed high-order moments are determined by the moments of lower orders.

7 Conclusion

We considered the mixed high-order moments of an absorbing Markov chain. While the first moments and the non-mixed seconds moment can be expressed in a matrix form, it can hardly be done for the mixed high-order moments. Using tensor formalizm, we developed a compact close-form expression for the mixed high-order moments.

8 Acknowledgements

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Appendix: Proof of Proposition 5.1

Let us consider the following set

$$D\left(M, j, \varkappa\right) = \left\{ \left(f\left(f\left(M, j, \psi\right), \varkappa, \chi\right), \left(\psi, \chi\right)\right) \middle| \psi = 0, \ldots, \binom{m}{j} - 1, \chi = 0, \ldots, \binom{j}{\varkappa} - 1 \right\}.$$

It is the set of elements of multiset B equipped by its indeces, therefore, we can distinguish equal elements of multiset B and compose the set. We shall write $D\left(M, j, \varkappa\right) = D$ if it does not produce any ambiguity.

Let us consider the following set

$$G = \left\{ (f(M, \varkappa, \rho), (\rho, \iota)) \middle| \rho = 0, \ldots, \binom{m}{\varkappa} - 1, \iota = 0, \ldots, \binom{m-\varkappa}{m-j} - 1 \right\}.$$

It is the set of the elements of set A equipped with its index ρ and auxiliary index ι. Due to index ι, each element of set A is repeated $\binom{m-\varkappa}{m-j}$ times in set G.

To prove the proposition we need to show that there is an one-to-one mapping from (ρ, ι) to (ψ, χ), which we denote by $(\psi, \chi)(\rho, \iota)$, such that

$$D = F,$$

where

$$F(M, j, \varkappa) = \left\{ (f(M, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \middle| \rho = 0, \ldots, \binom{m}{\varkappa} - 1, \iota = 0, \ldots, \binom{m-\varkappa}{m-j} - 1 \right\}.$$

We shall write $F(M, j, \varkappa) = F$ if it does not produce any ambiguity.

First of all, we prove that the cardinalities of sets D and F are equal. The cardinality of set D is equal to

$$|D| = \binom{m}{j}\binom{j}{\varkappa} = \frac{m!}{j!(m-j)!}\frac{j!}{\varkappa!(j-\varkappa)!} = \frac{m!}{(m-j)!\varkappa!(j-\varkappa)!}.$$

And the cardinality of set F is equal to

$$|F| = \binom{m}{\varkappa}\binom{m-\varkappa}{m-j} = \frac{m!}{\varkappa!(m-\varkappa)!}\frac{(m-\varkappa)!}{(m-j)!(m-j-m+\varkappa)!} = \frac{m!}{\varkappa!(m-j)!(\varkappa-j)!}.$$

Hence, one can see that $|D| = |F|$ and the one-to-one mapping can be potentially established. Therefore, the definition of set F is valid. We should prove that $D = F$.

We shall assume lexico-graphic ordering of combinations discussed above in the further development of the proof.

We shall continue the proof using the mathematical induction. We lead the induction by the cardinality of set M. Hence, let us prove the base of induction.

- Let $m = 1$ and $M = \{k_0\}$. We have following options for $(j, \varkappa)$: $(0, 0)$, $(1, 0)$, $(1, 1)$. Let us consider each option.
  - $(j, \varkappa) = (0, 0)$

$$\begin{aligned} & D(M, 0, 0) = \\ = \; & \left\{ (f(f(M, 0, \psi), 0, \chi), (\psi, \chi)) \middle| \psi = 0, \ldots, \binom{1}{0} - 1, \chi = 0, \ldots, \binom{0}{0} - 1 \right\} = \\ = \; & \{(\emptyset, (0, 0))\}. \end{aligned}$$

$$\begin{aligned}
& F\left(M,0,0\right) = \\
= \quad & \left\{ \left(f\left(M,0,\rho\right),\left(\psi,\chi\right)\left(\rho,\iota\right)\right) \middle| \rho = 0,\ldots,\binom{1}{0}-1, \iota = 0,\ldots,\binom{1-0}{1-0}-1 \right\} = \\
= \quad & \left\{\left(\emptyset,\left(\psi,\chi\right)\left(0,0\right)\right)\right\} = \\
= \quad & \left\{\left(\emptyset,\left(0,0\right)\right)\right\}.
\end{aligned}$$

- $(j,\varkappa) = (1,0)$

$$\begin{aligned}
& D\left(M,1,0\right) = \\
= \quad & \left\{ \left(f\left(f\left(M,1,\psi\right),0,\chi\right),\left(\psi,\chi\right)\right) \middle| \psi = 0,\ldots,\binom{1}{1}-1, \chi = 0,\ldots,\binom{1}{0}-1 \right\} = \\
= \quad & \left\{\left(\emptyset,\left(0,0\right)\right)\right\}.
\end{aligned}$$

$$\begin{aligned}
& F\left(M,1,0\right) = \\
= \quad & \left\{ \left(f\left(M,0,\rho\right),\left(\psi,\chi\right)\left(\rho,\iota\right)\right) \middle| \rho = 0,\ldots,\binom{1}{0}-1, \iota = 0,\ldots,\binom{1-0}{1-1}-1 \right\} = \\
= \quad & \left\{\left(\emptyset,\left(\psi,\chi\right)\left(0,0\right)\right)\right\} = \\
= \quad & \left\{\left(\emptyset,\left(0,0\right)\right)\right\}.
\end{aligned}$$

- $(j,\varkappa) = (1,1)$

$$\begin{aligned}
& D\left(M,1,1\right) = \\
= \quad & \left\{ \left(f\left(f\left(M,1,\psi\right),1,\chi\right),\left(\psi,\chi\right)\right) \middle| \psi = 0,\ldots,\binom{1}{1}-1, \chi = 0,\ldots,\binom{1}{1}-1 \right\} = \\
= \quad & \left\{\left(\left\{k_0\right\},\left(0,0\right)\right)\right\}.
\end{aligned}$$

$$\begin{aligned}
& F\left(M,1,1\right) = \\
= \quad & \left\{ \left(f\left(M,1,\rho\right),\left(\psi,\chi\right)\left(\rho,\iota\right)\right) \middle| \rho = 0,\ldots,\binom{1}{1}-1, \iota = 0,\ldots,\binom{1-1}{1-1}-1 \right\} = \\
= \quad & \left\{\left(\left\{k_0\right\},\left(\psi,\chi\right)\left(0,0\right)\right)\right\} = \\
= \quad & \left\{\left(\left\{k_0\right\},\left(0,0\right)\right)\right\}.
\end{aligned}$$

• Let $m = 2$ and $M = \{k_0, k_1\}$. We have following options for $(j,\varkappa)$: $(0,0)$, $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$, $(2,2)$. Let us consider each option.

  - $(j,\varkappa) = (0,0)$

$$\begin{aligned}
& D\left(M,0,0\right) = \\
= \quad & \left\{ \left(f\left(f\left(M,0,\psi\right),0,\chi\right),\left(\psi,\chi\right)\right) \middle| \psi = 0,\ldots,\binom{2}{0}-1, \chi = 0,\ldots,\binom{0}{0}-1 \right\} = \\
= \quad & \left\{\left(\emptyset,\left(0,0\right)\right)\right\}.
\end{aligned}$$

$$\begin{aligned}
& F(M,0,0) = \\
= \; & \left\{ \left(f(M,0,\rho), (\psi,\chi)(\rho,\iota) \right) \middle| \rho = 0,\ldots,\binom{2}{0} - 1, \iota = 0,\ldots,\binom{2-0}{2-0} - 1 \right\} = \\
= \; & \{ (\emptyset, (\psi,\chi)(0,0)) \} = \\
= \; & \{ (\emptyset, (0,0)) \} .
\end{aligned}$$

– $(j,\varkappa) = (1,0)$

$$\begin{aligned}
& D(M,1,0) = \\
= \; & \left\{ \left(f(f(M,1,\psi),0,\chi), (\psi,\chi) \right) \middle| \psi = 0,\ldots,\binom{2}{1} - 1, \chi = 0,\ldots,\binom{1}{0} - 1 \right\} = \\
= \; & \{ (\emptyset, (\psi,\chi)) \, | \psi = 0,\ldots,1, \chi = 0 \} = \\
= \; & \{ (\emptyset, (0,0)), (\emptyset, (1,0)) \} .
\end{aligned}$$

$$\begin{aligned}
& F(M,1,0) = \\
= \; & \left\{ \left(f(M,0,\rho), (\psi,\chi)(\rho,\iota) \right) \middle| \rho = 0,\ldots,\binom{2}{0} - 1, \iota = 0,\ldots,\binom{2-0}{2-1} - 1 \right\} = \\
= \; & \{ (\emptyset, (\psi,\chi)(\rho,\iota)) \, | \rho = 0, \iota = 0,\ldots,1 \} = \\
= \; & \{ (\emptyset, (\psi,\chi)(0,0)), (\emptyset, (\psi,\chi)(0,1)) \} = \\
= \; & \{ (\emptyset, (0,0)), (\emptyset, (1,0)) \} .
\end{aligned}$$

– $(j,\varkappa) = (1,1)$

$$\begin{aligned}
& D(M,1,1) = \\
= \; & \left\{ \left(f(f(M,1,\psi),1,\chi), (\psi,\chi) \right) \middle| \psi = 0,\ldots,\binom{2}{1} - 1, \chi = 0,\ldots,\binom{1}{1} - 1 \right\} = \\
= \; & \{ (f(f(M,1,\psi),1,\chi), (\psi,\chi)) \, | \psi = 0,\ldots,1, \chi = 0 \} = \\
= \; & \{ (\{k_0\}, (0,0)), (\{k_1\}, (1,0)) \} .
\end{aligned}$$

$$\begin{aligned}
& F(M,1,1) = \\
= \; & \left\{ \left(f(M,1,\rho), (\psi,\chi)(\rho,\iota) \right) \middle| \rho = 0,\ldots,\binom{2}{1} - 1, \iota = 0,\ldots,\binom{2-1}{2-1} - 1 \right\} = \\
= \; & \{ (f(M,1,\rho), (\psi,\chi)(\rho,\iota)) \, | \rho = 0,\ldots,1, \iota = 0 \} = \\
= \; & \{ (\{k_0\}, (\psi,\chi)(0,0)), (\{k_1\}, (\psi,\chi)(1,0)) \} = \\
= \; & \{ (\{k_0\}, (0,0)), (\{k_1\}, (1,0)) \} .
\end{aligned}$$

– $(j,\varkappa)=(2,0)$

$$
\begin{aligned}
&D\left(M,2,0\right)=\\
=\;&\left\{\left(f\left(f\left(M,2,\psi\right),0,\chi\right),\left(\psi,\chi\right)\right)\middle|\psi=0,\ldots,\binom{2}{2}-1,\chi=0,\ldots,\binom{2}{0}-1\right\}=\\
=\;&\left\{\left(f\left(f\left(M,2,\psi\right),0,\chi\right),\left(\psi,\chi\right)\right)|\psi=0,\chi=0\right\}=\\
=\;&\left\{\left(\emptyset,\left(0,0\right)\right)\right\}.
\end{aligned}
$$

$$
\begin{aligned}
&F\left(M,2,0\right)=\\
=\;&\left\{\left(f\left(M,0,\rho\right),\left(\psi,\chi\right)\left(\rho,\iota\right)\right)\middle|\rho=0,\ldots,\binom{2}{0}-1,\iota=0,\ldots,\binom{2-0}{2-2}-1\right\}=\\
=\;&\left\{\left(f\left(M,0,\rho\right),\left(\psi,\chi\right)\left(\rho,\iota\right)\right)|\rho=0,\iota=0\right\}=\\
=\;&\left\{\left(\emptyset,\left(\psi,\chi\right)\left(0,0\right)\right)\right\}=\\
=\;&\left\{\left(\emptyset,\left(0,0\right)\right)\right\}.
\end{aligned}
$$

– $(j,\varkappa)=(2,1)$

$$
\begin{aligned}
&D\left(M,2,1\right)=\\
=\;&\left\{\left(f\left(f\left(M,2,\psi\right),1,\chi\right),\left(\psi,\chi\right)\right)\middle|\psi=0,\ldots,\binom{2}{2}-1,\chi=0,\ldots,\binom{2}{1}-1\right\}=\\
=\;&\left\{\left(f\left(f\left(M,2,\psi\right),1,\chi\right),\left(\psi,\chi\right)\right)|\psi=0,\chi=0,\ldots,1\right\}=\\
=\;&\left\{\left(\{k_0\},\left(0,0\right)\right),\left(\{k_1\},\left(0,1\right)\right)\right\}.
\end{aligned}
$$

$$
\begin{aligned}
&F\left(M,2,1\right)=\\
=\;&\left\{\left(f\left(M,1,\rho\right),\left(\psi,\chi\right)\left(\rho,\iota\right)\right)\middle|\rho=0,\ldots,\binom{2}{1}-1,\iota=0,\ldots,\binom{2-1}{2-2}-1\right\}=\\
=\;&\left\{\left(f\left(M,1,\rho\right),\left(\psi,\chi\right)\left(\rho,\iota\right)\right)|\rho=0,\ldots,1,\iota=0\right\}=\\
=\;&\left\{\left(\{k_0\},\left(\psi,\chi\right)\left(0,0\right)\right),\left(\{k_1\},\left(\psi,\chi\right)\left(1,0\right)\right)\right\}=\\
=\;&\left\{\left(\{k_0\},\left(0,0\right)\right),\left(\{k_1\},\left(0,1\right)\right)\right\}.
\end{aligned}
$$

– $(j,\varkappa)=(2,2)$

$$
\begin{aligned}
&D\left(M,2,2\right)=\\
=\;&\left\{\left(f\left(f\left(M,2,\psi\right),2,\chi\right),\left(\psi,\chi\right)\right)\middle|\psi=0,\ldots,\binom{2}{2}-1,\chi=0,\ldots,\binom{2}{2}-1\right\}=\\
=\;&\left\{\left(f\left(f\left(M,2,\psi\right),2,\chi\right),\left(\psi,\chi\right)\right)|\psi=0,\chi=0\right\}=\\
=\;&\left\{\left(\{k_0,k_1\},\left(0,0\right)\right)\right\}.
\end{aligned}
$$

$$
\begin{aligned}
& F\left(M,2,2\right) = \\
= \; & \left\{ \left(f\left(M,2,\rho\right), \left(\psi,\chi\right)\left(\rho,\iota\right)\right) \middle| \rho = 0,\dots,\binom{2}{2}-1, \iota = 0,\dots,\binom{2-2}{2-2}-1 \right\} = \\
= \; & \left\{ \left(f\left(M,2,\rho\right), \left(\psi,\chi\right)\left(\rho,\iota\right)\right) \middle| \rho = 0, \iota = 0 \right\} = \\
= \; & \left\{ \left(\{k_0,k_1\}, \left(\psi,\chi\right)\left(0,0\right)\right) \right\} = \\
= \; & \left\{ \left(\{k_0,k_1\}, \left(0,0\right)\right) \right\}.
\end{aligned}
$$

Having proved the induction base, we continue with the induction step.

Let us consider set F. Since combinations $f\left(M,\varkappa,\rho\right)$ are ordered in lexico-graphic order, we know that combinations $f\left(M,\varkappa,\rho\right)$ containing element k_0 have indeces $\rho = 0,\dots,\binom{m-1}{\varkappa-1}-1$, and we can write

$$
\begin{aligned}
F \; = \; & \left\{ \left(f\left(M,\varkappa,\rho\right), \left(\psi,\chi\right)\left(\rho,\iota\right)\right) \middle| \rho = 0,\dots,\binom{m-1}{\varkappa-1}-1, \iota = 0,\dots,\binom{m-\varkappa}{m-j}-1 \right\} \cup \\
\cup \; & \left\{ \left(f\left(M,\varkappa,\rho\right), \left(\psi,\chi\right)\left(\rho,\iota\right)\right) \middle| \rho = \binom{m-1}{\varkappa-1},\dots,\binom{m}{\varkappa}-1, \iota = 0, \binom{m-\varkappa}{m-j}-1,\dots, \right\} = \\
= \; & F_1 \cup F_2.
\end{aligned}
$$

Combinations $f\left(M,\varkappa,\rho\right)$ as elements of set F_1 contain element k_0 and the combinations of set F_2 do not contain element k_0.

Let us consider set D. We again exploit that the combinations are ordered in lexico-graphic order, then combinations $f\left(M,j,\psi\right)$ containing element k_0 have indeces $\psi = 0,\dots,\binom{m-1}{j-1}-1$. Thus, we can write

$$
\begin{aligned}
D \; = \; & \left\{ \left(f\left(f\left(M,j,\psi\right),\varkappa,\chi\right), \left(\psi,\chi\right)\right) \middle| \psi = 0,\dots,\binom{m-1}{j-1}-1, \chi = 0,\dots,\binom{j}{\varkappa}-1 \right\} \cup \\
\cup \; & \left\{ \left(f\left(f\left(M,j,\psi\right),\varkappa,\chi\right), \left(\psi,\chi\right)\right) \middle| \psi = \binom{m-1}{j-1},\dots,\binom{m}{j}-1, \chi = 0,\dots,\binom{j}{\varkappa}-1 \right\} = \\
= \; & D_1 \cup D_2.
\end{aligned}
$$

We do the same with set D_1. Combinations $f\left(f\left(M,j,\psi\right),\varkappa,\chi\right)$ containing element k_0 have indeces $\chi = 0,\dots,\binom{j-1}{\varkappa-1}-1$, and we express D_1 as follows

$$
\begin{aligned}
D_1 \; = \; & \left\{ \left(f\left(f\left(M,j,\psi\right),\varkappa,\chi\right), \left(\psi,\chi\right)\right) \middle| \psi = 0,\dots,\binom{m-1}{j-1}-1, \chi = 0,\dots,\binom{j-1}{\varkappa-1}-1 \right\} \cup \\
\cup \; & \left\{ \left(f\left(f\left(M,j,\psi\right),\varkappa,\chi\right), \left(\psi,\chi\right)\right) \middle| \psi = 0,\dots,\binom{m-1}{j-1}-1, \chi = \binom{j-1}{\varkappa-1},\dots,\binom{j}{\varkappa}-1 \right\} = \\
= \; & D_a \cup D_b.
\end{aligned}
$$

Hence, we partition set D as $D = D_a \cup D_b \cup D_2$.

Let us prove that $D_a = F_1$. We can rewrite F_1 as follows

$$F_1 = \Bigg\{ \{k_0, (f(M \setminus \{k_0\}, \varkappa - 1, \rho)\}, (\psi, \chi)(\rho, \iota)) \Bigg| \rho = 0, \ldots, \binom{m-1}{\varkappa - 1} - 1, \iota = 0, \ldots, \binom{m-\varkappa}{m-j} - 1 \Bigg\}.$$

Considering set D_a one can see that each element of the set contains k_0 and all the combinations of the elements of set M containing element k_0 are counted.

$$\begin{aligned} D_a &= \left\{ (f(f(M, j, \psi), \varkappa, \chi), (\psi, \chi)) \middle| \psi = 0, \ldots, \binom{m-1}{j-1} - 1, \chi = 0, \ldots, \binom{j-1}{\varkappa - 1} - 1 \right\} = \\ &= \Bigg\{ (\{k_0, f(f(M \setminus \{k_0\}, j-1, \psi), \varkappa - 1, \chi)\}, (\psi, \chi)) \Bigg| \\ &\quad \psi = 0, \ldots, \binom{m-1}{j-1} - 1, \chi = 0, \ldots, \binom{j-1}{\varkappa - 1} - 1 \Bigg\}. \end{aligned}$$

One can see that $D_a = D(M \setminus \{k_0\}, j-1, \varkappa - 1)$, therefore, by induction, we can conclude that

$$D_a = F_1.$$

Next we shall prove that $F_2 = D_b \cup D_2$. One can easily see that

$$F_2 = \left\{ (f(M \setminus \{k_0\}, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \middle| \rho = \binom{m-1}{\varkappa - 1}, \ldots, \binom{m}{\varkappa} - 1, \iota = 0, \ldots, \binom{m-\varkappa}{m-j} - 1 \right\},$$

and, renumbering elements,

$$F_2 = \left\{ (f(M \setminus \{k_0\}, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \middle| \rho = 0, \ldots, \binom{m-1}{\varkappa} - 1, \iota = 0, \ldots, \binom{m-\varkappa}{m-j} - 1 \right\}.$$

Let us consider D_2. Since combinations $f(M, j, \psi)$ of set D_2 do not contain k_0, we write, renumbering elements,

$$D_2 = \left\{ (f(f(M \setminus \{k_0\}, j, \psi), \varkappa, \chi), (\psi, \chi)) \middle| \psi = 0, \ldots, \binom{m-1}{j} - 1, \chi = 0, \ldots, \binom{j}{\varkappa} - 1 \right\}.$$

One can see that $D_2 = D(M \setminus \{k_0\}, j, \varkappa)$, therefore, we conclude by induction that

$$D_2 = \left\{ (f(M \setminus \{k_0\}, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \middle| \rho = 0, \ldots, \binom{m-1}{\varkappa} - 1, \iota = 0, \ldots, \binom{m-1-\varkappa}{m-1-j} - 1 \right\}.$$

Let us consider D_b. Renumbering elements of set D_b we have

$$D_b = \left\{ (f(f(M, j, \psi), \varkappa, \chi), (\psi, \chi)) \middle| \psi = 0, \ldots, \binom{m-1}{j-1} - 1, \chi = 0, \ldots, \binom{j-1}{\varkappa} - 1 \right\}.$$

Since combinations $f\left(f\left(M,j,\psi\right),\varkappa,\chi\right)$ do not contains element k_0, we do not need it in combinations $f\left(M,j,\psi\right)$, hence, we write

$$D_b = \left\{ \left(f\left(f\left(M,j,\psi\right)\setminus\{k_0\},\varkappa,\chi\right),(\psi,\chi)\right) \middle| \psi=0,\dots,\binom{m-1}{j-1}-1, \chi=0,\dots,\binom{j-1}{\varkappa}-1 \right\},$$

or it is the same as

$$D_b = \left\{ \left(f\left(f\left(M\setminus\{k_0\},j-1,\psi\right),\varkappa,\chi\right),(\psi,\chi)\right) \middle| \psi=0,\dots,\binom{m-1}{j-1}-1, \chi=0,\dots,\binom{j-1}{\varkappa}-1 \right\}.$$

Now one can see that $D_2 = D\left(M\setminus\{k_0\},j-1,\varkappa\right)$, therefore, we conclude by induction that

$$D_b = \left\{ \left(f\left(M\setminus\{k_0\},\varkappa,\rho\right),(\psi,\chi)(\rho,\iota)\right) \middle| \rho=0,\dots,\binom{m-1}{\varkappa}-1, \iota=0,\dots,\binom{m-1-\varkappa}{m-j}-1 \right\}.$$

We renumber elements of D_b as follows

$$\begin{aligned} D_b = \Bigg\{ & \left(f\left(M\setminus\{k_0\},\varkappa,\rho\right),(\psi,\chi)(\rho,\iota)\right) \Bigg| \\ & \rho=0,\dots,\binom{m-1}{\varkappa}-1, \iota=\binom{m-1-\varkappa}{m-1-j},\dots,\binom{m-\varkappa}{m-j}-1 \Bigg\}, \end{aligned}$$

and we obtain that

$$D_b \cup D_2 = F_2.$$

Thus, we have

$$D_a \cup D_b \cup D_2 = F_1 \cup F_2,$$

and, consequently,

$$D = F,$$

which completes the proof.

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Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
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