A natural transfer function space for linear discrete time-invariant and scale-invariant systems

Daniel Alpay, Mamadou Mboup

A natural transfer function space for linear discrete time-invariant and scale-invariant systems

Daniel Alpay*
Department of mathematics
Ben Gurion University of the Negev, Israel
dany@cs.bgu.ac.il

Mamadou Mboup**
UFR mathématiques et informatique - CRIP5
Université Paris Descartes
45, rue des Saints-Pères - 75270 Paris cedex 06
Mamadou.Mboup@mi.parisdescartes.fr

***Abstract*— In a previous work, we have defined the scale shift for a discrete-time signal and introduced a family of linear scale-invariant systems in connection with character-automorphic Hardy spaces. In this paper, we prove a Beurling-Lax theorem for such Hardy spaces of order 2. We also study an interpolation problem in these spaces, as a first step towards a finite dimensional implementation of a scale invariant system. Our approach uses a characterization of character-automorphic Hardy spaces of order 2 in terms of classical de Branges Rovnyak spaces.**

I. INTRODUCTION

The self-similarity property is widely studied in the literature in the framework of stochastic process theory [1], [2], [3], and in the framework of systems theory [4], [5], [6]. In stochastic process theory the property is seen as a weighted form of stationarity in scale while in the systems theory approach, it is interpreted as a scale invariance. The scale shift, defined for a signal $x(t)$ by the operator $\alpha \mapsto x(\alpha t)$ thus plays a central rôle in the definition of self-similarity. Though simple and straightforward in the continuous-time domain, this operator is not well defined for discrete-time signal. In this paper, we use the definition given in [7]. Therein, a family of linear discrete both time- and scale-invariant systems is introduced in connection with character-automorphic Hardy spaces. In this paper, we prove a Beurling-Lax theorem for such Hardy spaces of order 2. We also study an interpolation problem in these spaces, as a first step towards a finite dimensional implementation of a scale invariant system. Our approach uses a characterization of character-automorphic Hardy spaces of order 2 in terms of classical de Branges Rovnyak spaces [8].

A. Scale shift for discrete-time signals

Let $f \in \mathbf{L}_1(\mathbb{R}_+)$ (that is, a continuous time signal), with Laplace transform $F(s)$, $\Re(s) \geqslant 0$. As it is well known, for every $\alpha = 1/\beta > 0$, the Laplace transform of $f(\beta t)$ is $\sqrt{\alpha}F(\alpha s)$. Therefore, time scaling has the same form both in the time and frequency domains. This remark is the starting point to define the scaling operator for discrete-time signals. Let θ be given such that $|\theta| < \frac{\pi}{2}$. Consider the Möbius transformation

$$G_\theta(s) = \frac{e^{i\theta} - s}{e^{-i\theta} + s},$$

which maps conformally the open right half-plane $\mathbb{C}_+$ onto the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Then, the scale shift

$$S_\alpha : s \mapsto S_\alpha(s) = \alpha s, \quad \alpha > 0$$

translates in the unit disc, into the hyperbolic transformation [7]

$$\gamma_{\{\alpha\}} = G_\theta \circ S_\alpha \circ G_\theta^{-1}. \tag{1}$$

Any transformation of this form maps the unit open disc (resp. the unit circle) into itself.

Conversely, the following lemma is true.

Lemma 1.1: For each hyperbolic transformation

$$\gamma(z) = \frac{\gamma_1 z + \gamma_2}{\overline{\gamma}_2 z + \overline{\gamma}_1}, \tag{2}$$

there exist $\alpha_\gamma > 0$, θ_γ with $|\theta_\gamma| < \frac{\pi}{2}$ and ξ_γ such that

$$e^{i\xi_\gamma}\gamma(z) = (G_{\theta_\gamma} \circ S_{\alpha_\gamma} \circ G_{\theta_\gamma}^{-1})(e^{i\xi_\gamma} z). \tag{3}$$

In particular, α_γ is given by the multiplier of the transformation

Proof: We assume that γ is normalized such that $|\gamma_1|^2 - |\gamma_2|^2 = 1$. Since γ is hyperbolic, we have the relation [9]

$$\frac{\gamma(z) - \xi_1}{\gamma(z) - \xi_2} = K\frac{z - \xi_1}{z - \xi_2} \tag{4}$$

where $\xi_1 = \frac{\sqrt{[\Re(\gamma_1)]^2 - 1} + i\Im(\gamma_1)}{\overline{\gamma_2}} = \frac{\lambda_\gamma}{\overline{\gamma_2}}$ and $\xi_2 = -\frac{\overline{\lambda_\gamma}}{\overline{\gamma_2}}$ are the two fixed points of γ. The positive constant K is called the multiplier of the transformation [9] and is given by $K + \frac{1}{K} = 4[\Re(\gamma_1)]^2 - 2$. Noting that $|\xi_1| = |\xi_2| = 1$, one may rearrange (4) to obtain

$$\frac{\lambda_\gamma - \overline{\lambda_\gamma}e^{i\xi_\gamma}\gamma(z)}{1 + e^{i\xi_\gamma}\gamma(z)} = K\frac{\lambda_\gamma - \overline{\lambda_\gamma}e^{i\xi_\gamma}z}{1 + e^{i\xi_\gamma}z}, \tag{5}$$

where $e^{i\xi_\gamma} = \frac{\lambda_\gamma}{\gamma_2}$. Dividing both sides of this equality by $|\lambda_\gamma|$, we get (3) by setting $e^{i\theta_\gamma} = \frac{\lambda_\gamma}{|\lambda_\gamma|}$ and $\alpha_\gamma = K$. ■

The set of all linear transformations as in (2) forms a group that we denote by Γ. The lemma then shows that the action of Γ on $\mathbb{D}$ is equivalent to the scale operator on $\mathbb{C}_+$. Therefore, we define below the discrete-time frequency domain scale shift by the action of the (hyperbolic) group of

*Earl Katz Chair in Algebraic System Theory
**EPI ALIEN, INRIA

automorphisms of $\mathbb{D}$. Given a discrete-time signal $\{x_n\}_{n\geq 0}$, consider its $\mathcal{Z}$ transform $X(z) = \sum_{n=0}^{\infty} x_n z^n$, which we assume convergent in a neighborhood of the origin. The scale $\alpha = \alpha_\gamma$ shift of the sequence $\{x_n\}_{n\geq 0}$ is the sequence $\{x_n(\gamma)\}_{n\geq 0}$ defined via the equation

$$X_\gamma(z) = \frac{1}{\overline{\gamma}_2 z + \overline{\gamma}_1} X(\gamma(z)) = \sum_{n\geq 0} x_n(\gamma) z^n. \qquad (6)$$

It is useful to note that this operator also makes sense for vector-valued functions.

B. *Scale-invariant systems*

A wide class of causal discrete time-invariant linear systems can be given in terms of convolution in the form

$$y_n = \sum_{m=0}^{n} h_{n-m} x_m, \quad n = 0, 1, \ldots, \qquad (7)$$

where $\{h_n\}$ is the impulse response and where the input sequence $\{x_m\}$ and output sequence $\{y_m\}$ are requested to belong to some pre-assigned sequences spaces. The $\mathcal{Z}$ transform of the sequence $\{h_n\}$, that is $H(z) = \sum_{n=0}^{\infty} z^n h_n$, is called the *transfer function* of the system, and there are deep relationships between properties of $H(z)$ and of the system; see [10] for a survey. In particular, it is well known that if the system is asymptotically stable, then $H(z)$ belongs to the classical Hardy space of order 2.

In this paper, we are interested in linear discrete-time systems which, in addition to the time invariance, are also invariant under a scale shift. Scale-invariance is defined [7] similarly to time-invariance: a scale shift in the input sequence induces the same scale shift on the corresponding output sequence. In terms of the $\mathcal{Z}$ transforms, this would directly mean that for all $\gamma \in \Gamma$,

$$Y_\gamma(z) = H(\gamma(z)) X_\gamma(z) = H(z) X_\gamma(z). \qquad (8)$$

Therefore, scale-invariance implies that the transfer function of the system be Γ-periodic. Now a function f satisfying $f \circ \gamma = f$, for all $\gamma \in \Gamma$ is said to be *automorphic* with respect to Γ. This makes sense only for discrete groups (see [9]).

In the following, we will be interested in the character-automorphic Hardy spaces of order 2. These are the natural transfer function spaces for the LTI and scale-invariant systems. In a previous study, see [8], we have given a characterization of these spaces in terms of associated classical de Branges Rovnyak spaces. In section II, we use this approach to prove a Beurling-Lax theorem and we study interpolation in these Hardy spaces in section III. Leech's theorem and the characterization of de Branges Rovnyak spaces given in [11, Theorem 3.1.2, p. 85] play an important role in the arguments.

In the remaining of the paper, the complex conjugation of is denoted par * and no longer by $^-$.

II. A NATURAL TRANSFER FUNCTION SPACE

A. *Definitions*

Now on, we discretise the scale axis and consider that Γ is a Fuchsian group of Widom type (with no elliptic element) [12]. We denote by $\mathfrak{z}$ its uniformizing map and by $\widehat{\Gamma}$ its dual group, *i.e.* the group of unimodular characters. Recall that a character α is a function defined on Γ and satisfying:

$$|\alpha(\gamma)| = 1 \quad \text{and} \quad \alpha(\gamma \circ \varphi) = \alpha(\gamma)\alpha(\varphi), \quad \forall \gamma, \varphi \in \Gamma.$$

A function f satisfying $f \circ \gamma = \alpha(\gamma) f$, $\forall \gamma \in \Gamma$ for $\alpha \in \widehat{\Gamma}$, is called *character-automorphic* with respect to Γ. Given a character α of $\widehat{\Gamma}$, the character-automorphic Hardy space $\mathcal{H}_2^\alpha(\mathbb{D})$ of order 2 is the space of character-automorphic functions which belong to the classical Hardy space $\mathcal{H}_2(\mathbb{D})$. Its reproducing kernel has been characterized in [13, Lemma 4.4.2 p. 387] as follows:

$$k^\alpha(z,\omega) = c(\alpha) \frac{\frac{k^{\alpha\mu}(z,0)}{b(z)} k^\alpha(\omega,0)^* - \left(\frac{k^{\alpha\mu}(\omega,0)}{b(\omega)}\right)^* k^\alpha(z,0)}{\mathfrak{z}(z) - \mathfrak{z}(\omega)^*} \qquad (9)$$

where $c(\alpha) = \frac{\mathfrak{z}(0)b(0)}{k^{\alpha\mu}(0,0)} > 0$. In (9), b is the Green's function of Γ, and the character associated to the Green's function is denoted by μ. Formula (9) expresses that the kernel is *structured*, and of the form of the kernels studied in the papers [14], [15]. It depends on a $\mathbb{C}^{1\times 2}$-valued function of one variable. Using (9) we proved in [8] that there exists a Schur function $\mathscr{S}_\alpha$, associated to the de Branges space $\mathcal{H}(\mathscr{S}_\alpha)$, such that

$$\mathcal{H}_2^\alpha(\mathbb{D}) = \left\{ F(z) = \frac{A^\alpha(z)}{1 - i\mathfrak{z}(z)} f(\sigma(z)) \; ; f \in \mathcal{H}(\mathscr{S}_\alpha) \right\} \qquad (10)$$

where $A^\alpha(z) = \sqrt{c(\alpha)} \left(\frac{k^{\alpha\mu}(z,0)}{b(z)} + i k^\alpha(z,0) \right)$ and $\sigma(z) = \frac{1 + i\mathfrak{z}(z)}{1 - i\mathfrak{z}(z)}$, and with the norm

$$\|F\|_{\mathcal{H}_2^\alpha(\mathbb{D})} = \|f\|_{\mathcal{H}(\mathscr{S}_\alpha)}.$$

We close this subsection with the

Definition 2.1: A causal linear time-invariant system is called scale-invariant with respect to Γ, if its transfer function is an element of $\mathcal{H}_2^\alpha(\mathbb{D})$ for some character α.
Note that such a system is not rational, unless Γ is a finite group.

B. *The shift operator in $\mathcal{H}_2^\alpha(\mathbb{D})$*

Set $\mathfrak{p} \in \mathbb{D}$ and given a function f analytic in $\mathbb{D}$, consider the operators

$$(R_\mathfrak{p} f)(\lambda) = \frac{f(\lambda) - f(\mathfrak{p})}{\lambda - \mathfrak{p}}.$$

These operators $R_\mathfrak{p}$ satisfy the resolvent equation

$$R_\mathfrak{p} - R_\mathfrak{q} = (\mathfrak{p} - \mathfrak{q}) R_\mathfrak{p} R_\mathfrak{q}.$$

Let $m(z) = \frac{A_\alpha(z)}{1 - i\mathfrak{z}(z)}$. The isomorphism

$$F(z) = m(z) f(\sigma(z)) \qquad (11)$$

between the de Branges space $\mathcal{H}(\mathcal{S}_\alpha)$ and the character-automorphic Hardy space allows one to define the following operators:

$$\mathcal{R}_{\mathfrak{p}}F = m(z)(R_{\mathfrak{p}}f)(\sigma(z)).$$

As we may directly check, these operators $\mathcal{R}_{\mathfrak{p}}$ also satisfy a resolvent equation. Therefore, they can be written in the form

$$\mathcal{R}_{\mathfrak{p}} = (\mathbf{T} - \mathfrak{p})^{-1}.$$

Here $\mathbf{T}$ is not an operator in general. It is a linear relation and is given by

$$(\mathbf{T}F)(z) = \sigma(z)F(z) + m(z)c_F,$$

where c_F is such that

$$\sigma(z)F(z) + m(z)c_F \in \mathcal{H}_2^\alpha(\mathbb{D}).$$

C. A Beurling theorem in $\mathcal{H}_2^\alpha(\mathbb{D})$

The classical Beurling theorem (see for instance [16, Théorème 17.21 p. 330]) gives a characterization of the closed subspaces $\mathcal{M}$ of the Hardy space $\mathcal{H}_2(\mathbb{D})$ of the unit disk $\mathbb{D}$: *Any such subspace is of the form $\mathcal{M} = j\mathcal{H}_2(\mathbb{D})$, where j is an inner function* (the case of vector-valued functions was first considered by Lax; see [17] for a discussion and references). The orthogonal complement of $\mathcal{M}$ is the reproducing kernel Hilbert space with reproducing kernel $K_j(z,w) = (1 - j(z)j(w)^*)/(1 - zw^*)$. If one replaces j by a Schur function s, that is, by a function analytic and contractive in $\mathbb{D}$, the kernel $k_s(z,w)$ is still positive in $\mathbb{D}$. Its associated reproducing kernel Hilbert space was denoted in the preceding subsection by $\mathcal{H}(\mathscr{S})$. These spaces are called de Branges Rovnyak spaces, and originate with the work [18]. When allowing $\mathscr{S}$ to be vector-valued, they have been fully characterized in [11, Theorem 3.1.2]. They are contractively included, but in general not isometrically included, in $\mathcal{H}_2(\mathbb{D})$.

Theorem 2.2: A Hilbert space $\mathcal{M}$ is contractively included in $\mathcal{H}_2^\alpha(\mathbb{D})$, and invariant under $\mathcal{R}_{\mathfrak{p}}$, and satisfies the inequality

$$\|\mathcal{R}_0F\|^2_{\mathcal{M}} \le \|F\|^2_{\mathcal{M}} - |F(0)|^2 \tag{12}$$

if, and only if, its reproducing kernel is of the form

$$\frac{A_\alpha(z)}{\sqrt{2}} \frac{1 - \mathscr{R}(\sigma(z))\mathscr{R}(\sigma(w))^*}{-i(\mathfrak{z}(z) - \mathfrak{z}(w)^*)} \frac{A_\alpha(w)^*}{\sqrt{2}},$$

where $\mathscr{R}$ is a vector-valued Schur function such that

$$\mathscr{S}_\alpha = \mathscr{R}\mathscr{R}_1, \tag{13}$$

with $\mathscr{R}_1$ also a vector-valued Schur function.

Remark 1: The inequality(12) is automatically satisfied if $\mathcal{M}$ is isometrically included in $\mathcal{H}_2^\alpha(\mathbb{D})$.

Proof: of 2.2 Associated to the space $\mathcal{M}$, there is, by the isomorphism (11), a Hilbert space $\mathcal{M}_\alpha$ which is $R_{\mathfrak{p}}$-invariant and satisfying

$$\|R_0f\|^2_{\mathcal{M}_\alpha} \le \|f\|^2_{\mathcal{M}_\alpha} - |f(0)|^2. \tag{14}$$

Using [11, Theorem 3.1.2], we see that the reproducing kernel of $\mathcal{M}_\alpha$ is of the form

$$\frac{1 - \mathscr{R}(\lambda)\mathscr{R}(\mathfrak{p})^*}{1 - \lambda\mathfrak{p}^*},$$

where $\mathscr{R}$ is a vector-valued Schur function. Since $\mathcal{M}_\alpha$ is contractively included in $\mathcal{H}(\mathscr{S}_\alpha)$, the kernel

$$\frac{\mathscr{R}(\lambda)\mathscr{R}(\mathfrak{p})^* - \mathscr{S}_\alpha(\lambda)\mathscr{S}_\alpha(\mathfrak{p})^*}{1 - \lambda\mathfrak{p}^*},$$

is positive. We therefore get the factorization (13) by using the Leech theorem. See [19, Theorem 2, p. 134] and [20, Example 1, p. 107]. We conclude by using the isomorphism (11) and the fact that (see equation (10)) the reproducing kernel of the space $\mathcal{H}_2^\alpha(\mathbb{D})$ is

$$\begin{aligned} k^\alpha(z,w) &= \frac{A_\alpha(z)}{1 - i\mathfrak{z}(z)} \frac{1 - \mathscr{S}_\alpha(\sigma(z))\mathscr{S}_\alpha(\sigma(w))^*}{1 - \sigma(z)\sigma(w)^*} \frac{A_\alpha(w)^*}{1 + i\mathfrak{z}(w)^*} \\ &= \frac{A_\alpha(z)}{\sqrt{2}} \frac{1 - \mathscr{S}_\alpha(\sigma(z))\mathscr{S}_\alpha(\sigma(w))^*}{-i(\mathfrak{z}(z) - \mathfrak{z}(w)^*)} \frac{A_\alpha(w)^*}{\sqrt{2}}. \end{aligned}$$

Reversing these different arguments allows one to establish the converse. ■

III. Interpolation

As we already mention, the elements of the character-automorphic Hardy space $\mathcal{H}_2^\alpha(\mathbb{D})$ are not rational functions unless we consider a finite number of possible scale shifts. Since the corresponding systems are of infinite dimension, a finite dimensional approximation step is necessary before their implementation. This is the motivation of the interpolation problem studied in this section.

To proceed, denote by

$$\mathcal{F} = \{z \in \mathbb{D} \,:\, |\gamma'(z)| < 1 \quad \text{for all } \gamma \in \Gamma, \gamma \ne id\} \tag{15}$$

the *normal fundamental domain* of Γ with respect to 0: there is no transformation in Γ, which sends one point of $\mathcal{F}$ into another point of $\mathcal{F}$.

So, we consider the following interpolation problem: *Given N complex numbers F_i and N points $z_i \in \mathcal{F} \cap \mathbb{D}$, describe the set of all functions $F \in \mathcal{H}_2^\alpha(\mathbb{D})$ with $\|F\|_{\mathcal{H}_2^\alpha(\mathbb{D})} \le 1$ satisfying*

$$F(z_i) = F_i,\ i = 1, \ldots N, \tag{16}$$

Note that this problem is different from the interpolation problems considered by Abrahamse in [21] and by Kupin and Yuditskii in [13]. These studies were interested in finding multipliers having given values at prescibed points while here we have the constraint that F must belong to $\mathcal{H}_2^\alpha(\mathbb{D})$.

The function

$$\mathscr{P}_\alpha = \frac{(1 - \mathscr{S}_\alpha)}{(1 + \mathscr{S}_\alpha)} \tag{17}$$

is analytic and has positive real part in $\mathbb{D}$. By the Herglotz representation theorem, we can write:

$$\mathscr{P}_\alpha = ic_\alpha + \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} d\sigma_\alpha(t),$$

with $c_\alpha \in \mathbb{R}$ and $d\sigma_\alpha$ is a postive measure $[0, 2\pi)$. Defining

$$\mathscr{Q}_\alpha(\lambda) \triangleq \frac{\sqrt{2}}{(1+\mathscr{S}_\alpha(\lambda))} = \frac{1+\mathscr{P}_\alpha(\lambda)}{\sqrt{2}}$$

where the second equality follows directly from (17), we see that can also write

$$\frac{\mathscr{P}_\alpha(\lambda)+\mathscr{P}_\alpha(\mathfrak{p})^*}{1-\lambda\mathfrak{p}^*} = \mathscr{Q}_\alpha(\lambda)\frac{1-\mathscr{S}_\alpha(\lambda)\mathscr{S}_\alpha(\mathfrak{p})^*}{1-\lambda\mathfrak{p}^*}\mathscr{Q}_\alpha(\lambda)^*. \tag{18}$$

Now this equation (18) implies that the operator of multiplication by $\mathscr{Q}_\alpha(\lambda)$ is a unitary transformation from the reproducing kernel space $\mathcal{L}(\mathscr{P}_\alpha)$, with kernel

$$\frac{\mathscr{P}_\alpha(\lambda)+\mathscr{P}_\alpha(\mathfrak{p})^*}{1-\lambda\mathfrak{p}^*}, \quad \lambda, \mathfrak{p} \in \mathbb{D}$$

to the space $\mathcal{H}(\mathscr{S}_\alpha)$. Moreover $\mathcal{L}(\mathscr{P}_\alpha)$ is the set of functions of the form

$$x(\lambda) = \int_0^{2\pi} \frac{e^{it}h(t)d\sigma_\alpha(t)}{e^{it}-\lambda}, \tag{19}$$

where h belongs to the closure (subsequently denoted $\mathcal{H}_2(\mathbb{D}, d\sigma_\alpha)$) of the set of functions $1/(1-e^{it}w^*)$ $(|w|<1)$ in $\mathbf{L}_2(d\sigma_\alpha)$. We therefore have the following:

Proposition 3.1: $F \in \mathcal{H}_2^\alpha(\mathbb{D})$ if, and only if,

$$F(z) = \frac{A^\alpha(z)}{1-i\mathfrak{z}(z)}\frac{1+\mathscr{S}_\alpha(\sigma(z))}{\sqrt{2}}\int_0^{2\pi}\frac{e^{it}h(t)d\sigma_\alpha(t)}{e^{it}-\sigma(z)} \tag{20}$$

with $h \in \mathbf{H}_2(d\sigma_\alpha)$.

The interpolation problem in the character-automorphic Hardy space then reduces to an interpolation problem in $\mathcal{L}(\mathscr{P}_\alpha)$, or more precisely, to an orthogonal projection in $\mathcal{H}_2(\mathbb{D}, d\sigma_\alpha)$:

Problem 1: Let $\lambda_\ell = \sigma(z_\ell)$ and

$$x_\ell = \frac{1-\mathfrak{z}(z_\ell)}{A^\alpha(z_\ell)}\frac{\sqrt{2}F_\ell}{1+\mathscr{S}_\alpha(\lambda_\ell))}, \quad \ell = 1, \dots N.$$

Find all $h \in \mathcal{H}_2(\mathbb{D}, d\sigma_\alpha)$ such that

$$\int_0^{2\pi}\frac{e^{it}h(t)d\sigma_\alpha(t)}{e^{it}-\lambda_\ell} = x_\ell, \quad \ell = 1, \dots N.$$

Now this is a classical Hilbert space problem: it admits a solution with minimum norm, corresponding to a function $h_{\min}$ of the form

$$h_{\min}(t) = \sum_{\ell=1}^{N}\frac{c_\ell}{e^{it}-\lambda_\ell},$$

and any other solution has the form

$$h_{\min} + h, \quad h \perp h_{\min}.$$

We thus deduce the description of all x of the form (19), and hence a description of the functions F by formula (20).

References

[1] R. Narasimha S. Lee, W. Zhao and R.M. Rao, "Discrete-time models for statistically self-similar signals," *IEEE Trans. Signal Processing*, vol. 51, no. 5, pp. 1221–1230, 2003.

[2] W. Leland, M. Taqqu, W. Willinger, and D. Wilson, "On the self-similar nature of Ethernet traffic (extended version)," *IEEE/ACM Trans. Networking*, pp. 1–15, Feb. 1994.

[3] B. B. Mandelbrot and J. W. Van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Review*, vol. 10, no. 4, pp. 422–437, 1968.

[4] M. Mboup, "On the structure of self-similar systems: A Hilbert space approach," in *Operator Theory: Advances and applications*, vol. OT-143, pp. 273–302. Birkhäuser-Verlag, 2003.

[5] B. Yazıcı and R. L. Kashyap, "A class of second-order stationary self-similar processes for $1/f$ phenomena," *IEEE Trans. Signal Processing*, vol. 45, no. 2, pp. 396–410, Feb 1997.

[6] C. J. Nuzman and H. Vincent Poor, "Reproducing kernel Hilbert space methods for wide-sense self-similar processes," *Ann. Appl. Probab.*, vol. 11, no. 4, pp. 1199–1219, 2001.

[7] M. Mboup, "A character-automorphic hardy spaces approach to discrete-time scale-invariant systems," in *MTNS'06*, Kyoto, Japan, 2006.

[8] D. Alpay and M. Mboup, "A characterization of Schur multipliers between character-automorphic hardy spaces," *Integral Equations and Operator Theory*, vol. 62, pp. 455–463, 2008.

[9] L. R. Ford, *Automorphic functions*, Chelsea, New-York, 2nd edition, 1951.

[10] D. Alpay, *Algorithme de Schur, espaces à noyau reproduisant et théorie des systèmes*, vol. 6 of *Panoramas et Synthèses*, Société mathématique de France, 1998.

[11] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, vol. 96 of *Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, 1997.

[12] M. V. Samokhin, "Some classical problems in the theory of analytic functions in domains of Parreau-Widom type," *Math USSR Sbornik*, vol. 73, pp. 273–288, 1992.

[13] S. Kupin and P. Yuditskii, "Analogs of the Nehari and Sarason theorems for character-automorphic functions and some related questions," in *Operator Theory: Advances and Applications*, vol. 95, pp. 373–390. Birkhäuser Verlag Basel, 1997.

[14] D. Alpay and H. Dym, "On reproducing kernel spaces, the Schur algorithm, and interpolation in a general class of domains," in *Operator theory and complex analysis (Sapporo, 1991)*, vol. 59 of *Oper. Theory Adv. Appl.*, pp. 30–77. Birkhäuser, Basel, 1992.

[15] D. Alpay and H. Dym, "On a new class of structured reproducing kernel spaces," *J. Funct. Anal.*, vol. 111, no. 1, pp. 1–28, 1993.

[16] Walter Rudin, *Analyse réelle et complexe*, Masson, Paris, 1980, Translated from the first English edition by N. Dhombres and F. Hoffman, Third printing.

[17] P. D. Lax and R. S. Phillips, *Scattering theory*, vol. 26 of *Pure and Applied Mathematics*, Academic Press Inc., Boston, MA, second edition, 1989, With appendices by Cathleen S. Morawetz and Georg Schmidt.

[18] L. de Branges and J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, New York, 1966.

[19] M. Rosenblum, "A corona theorem for countably many functions," *Integral Equations Operator Theory*, vol. 3, no. 1, pp. 125–137, 1980.

[20] M. Rosenblum and J. Rovnyak, *Hardy classes and operator theory*, Dover Publications Inc., Mineola, NY, 1997, Corrected reprint of the 1985 original.

[21] M. B. Abrahamse, "The Pick interpolation theorem for finitely connected domains," *Michigan Math. J.*, vol. 26, no. 2, pp. 195–203, 1979.