Optimal Gathering Protocols on Paths under Interference Constraints

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Abstract

We study the problem of gathering information from the nodes of a multi-hop radio network into a predefined destination node under reachability and interference constraints. In such a network, a node is able to send messages to other nodes within reception distance, but doing so it might create interference with other communications. Thus, a message can only be properly received if the receiver is reachable from the sender and there is no interference from another message being simultaneously transmitted. The network is modeled as a graph, where the vertices represent the nodes of the network and the edges, the possible communications. The interference constraint is modeled by a fixed integer $d \geq 1$, which implies that nodes within distance d in the graph from one sender cannot receive messages from another node. In this paper, we suppose that each node has one unit-length message to transmit and, furthermore, we suppose that it takes one unit of time (slot) to transmit a unit-length message and during such a slot we can have only calls which do not interfere (called compatible calls). A set of compatible calls is referred to as a round. We give protocols and lower bounds on the minimum number of rounds for the gathering problem when the network is a path and the destination node is either at one end or at the center of the path. The protocols are shown to be optimal for any d in the first case, and for $1 \leq d \leq 4$, in the second case.

Key words: Gathering algorithms, interference, multi-hop radio network, path.

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1 Introduction

1.1 Problem statement

The problem we consider in this paper was motivated by a question asked by FRANCE TELECOM about “how to provide Internet connection to a village” (see [5]) and is related to the following scenario. Suppose we are given a set of communication devices placed in houses in a village (for instance, network interfaces that connect computers to the Internet). They require access to a gateway (for instance, a satellite antenna) to send and receive data through a multi-hop wireless network. In this network, the devices communicate exclusively by means of radio transmissions, referred to as *calls*. A call involves a message and two devices, the *sender* and the *receiver*. The communication is subject to the following technological constraints:

Reachability constraint: since every device has limited transmission power, the message transmitted in a call may not reach some other devices in the network. Thus, in order to be reached by a call, the receiver of this call must be within reachability distance of the sender.

Interference constraint: a call may interfere with calls that are in the neighborhood of the receiver, or a message can be properly received only if no other senders are in the neighborhood of the receiver. For this reason, a device that is within interference distance of the sender of one call cannot be the receiver of another call.

Considering these two constraints, a message transmitted in a call can only be properly received if the receiver is reachable from the sender and there is no interference by another message being simultaneously transmitted. An illustration is given in Figure 1, where the blue (light grey) region represents the transmission zone of the senders, and the red (dark) region (including the blue part) represents the interference zone. So node 1 can reach nodes 0 and 2, and node 4 can reach 3 and 5. But nodes 2 and 3 cannot receive messages from node 1 and node 4, respectively, in the same time slot because they are within interference distance of both nodes 1 and 4. In this context, our gathering problem can be formulated as the following:

t-gathering problem: suppose each device of the network has a piece of information. The t-gathering consists of collecting (gathering) all these pieces of information into a special device t, called the *gathering node*, by the means of calls subject to the two constraints described before. The t-gathering problem is to realize such a constrained gathering without concatenating messages and with the minimum delay.

A slight variation of this problem has received much attention in the context of sensor networks. In such networks, each device contains a sensor and the gathering problem corresponds to the situation where information collected at each sensor has to be gathered to a single central device (base station). However, most of the articles are concerned with minimizing the energy consumption and allow aggregation of data. The work which is

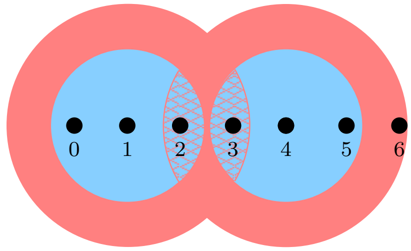


Fig. 1. Network with 7 nodes and two simultaneous transmissions.

most related to ours is [10], in which reachability and interference constraints are also assumed, but most of its results apply for the case of directional antennas.

A different formulation is the so-called *t-personalized broadcast* in which a single device (the gateway in the problem of France Telecom) has a different piece of information to broadcast to every other device in the network. It is not hard to show that there is a one-to-one correspondence between the solutions of this problem and the t-gathering problem. We will focus only on the gathering problem.

In this paper, we propose solutions to the gathering problem for the particular case of a path. Before going into details about our results and related work, let us introduce the mathematical formulation of the problem.

1.2 Model and assumptions

According to the model adopted in [2], the network described above is represented by an undirected graph $G = (V, E)$, where V is the set of nodes, each of them representing a communication device, and E is the set of edges, representing the pairs of nodes involved in possible calls. Denote by $X_{s,r}$ a call where a (sender) node $s \in V$ sends message X to another (receiver) node $r \in V$. Let $d_G(s, r)$ indicate the distance in G, defined as the length of a shortest path between s and r. We model the reachability and the interference constraints by two positive integers, respectively $d_T \geq 1$ and $d \geq d_T$. A node $r \in V$ is *reachable* from $s \in V$, $s \neq r$, if and only if $d_G(r, s) \leq d_T$. An important case is $d_T = 1$, which means that a node is able to communicate only with its neighbors in the graph (or G is the communication graph). The second parameter d models the interference constraint as follows: if a receiver is within distance d from a sender, then this node cannot receive any other message. If s sends a message X to r, then the call $X_{s,r}$ interferes with every node $w \in V$ such that $d_G(s, w) \leq d$.

We assume that every occurrence of a call takes one unit of time (or one slot) and involves a one unit-length message. Two calls are said to be *compatible* if they do not interfere with each other (otherwise, they are *incompatible*). More precisely, two calls X_{s_1,r_1} and Y_{s_2,r_2}, for $r_1, r_2, s_1, s_2 \in V$, are compatible if $d_G(s_1, r_2) > d$ and $d_G(s_2, r_1) > d$. Observe that one of the consequences of the interference constraint is that $s_1 \neq r_2$ and $s_2 \neq r_1$, which implies that a node is not able to send and receive messages simultaneously. A *round* is a

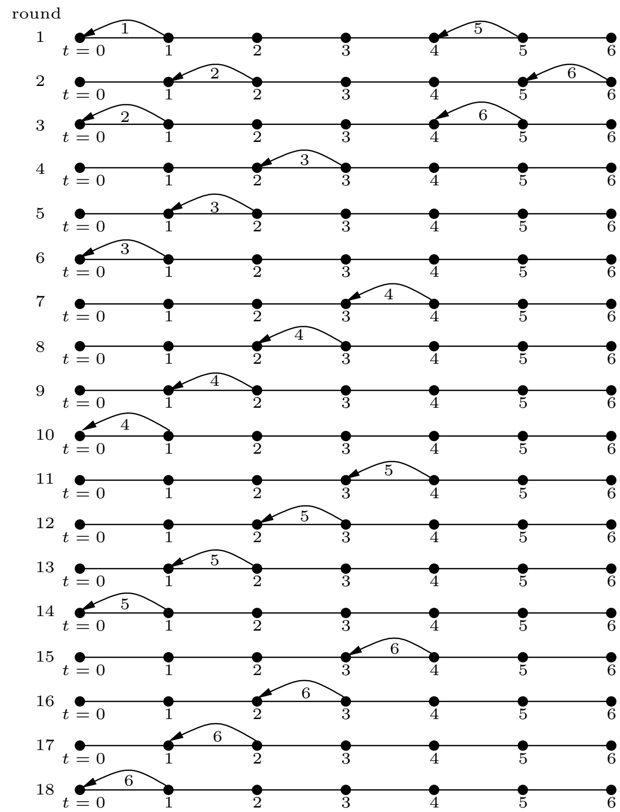


Fig. 2. Gathering protocol for a path with 7 nodes, $d = 2$, and $d_T = 1$.

set of compatible calls, whereas a *protocol* is a sequence of rounds (in which the calls of each round occur simultaneously).

Let us illustrate this model with the example of a path with 7 nodes, and with $d_T = 1$ and $d = 2$. In the protocol shown in Figure 2, all the rounds consist of a single call or two compatible calls. Notice, for example, that the call $1_{1,0}$ cannot appear in the same round as $4_{4,3}$ because $d_G(1, 3) \leq 2 = d$. On the other hand, the calls $1_{1,0}$ and $5_{5,4}$ are compatible, and are then allowed to appear in a single round (which is the case in round 1). It will be shown later that the protocol consisting of the sequence of 18 rounds is in fact optimal.

In this paper, our aim is to find a t-gathering protocol using a minimum number of rounds in the specific case where $d_T = 1$ and G is a path. In fact, this stems from the assumption that the village consists of one main street. To our great surprise, the gathering problem is not so simple in this case if one wants to obtain an exact optimal gathering protocol when the gathering node is at the center of a path.

A final remark with respect to the model just described is that another possibility would be to represent the radio devices as nodes in the plane, and to state the reachability and interference constraints according to the euclidean distances. However, since we only consider paths, this alternative model is equivalent to the one adopted in this paper.

1.3 Related works

The broadcasting and gossiping problems have been widely studied for wired networks (see the book [14]), including models that assume no concatenation of messages [3]. For radio networks, the case when $d_T = d = 1$ is studied only for broadcasting in [9,11] and gossiping in [7,8,13]. Note that broadcasting is different from our problem which is the reverse of personalized broadcasting; indeed, in a broadcast the same information has to be transmitted to all the other nodes and so flooding techniques can be used. With respect to the gathering problem, in companion papers to this one different cases have been studied. In [4], optimal solutions are provided for the two-dimensional square grid. In [2], the size of information in each node is assumed to be arbitrary. Then, a protocol for general graphs with an approximation factor of at most 4 is presented. It is also shown that the problem of finding an optimal gathering protocol (one that uses a minimum number of rounds) does not admit a Fully Polynomial Time Approximation Scheme if $d > d_T$, unless P=NP, and is NP-HARD if $d = d_T$. An extension of the problem where messages can be released over time is considered in [6] and a 4-approximation algorithm is presented for that case.

As mentioned before, sensor networks have been the subject of many papers. But, most of them deal with minimizing the energy consumption or maximizing the life time of the sensor network. A model that is closer to ours is considered in [10] where, different from our model, each node is equipped with directional antennas and no buffering capacity is available in the nodes. This corresponds to the case in our model when $d_T = 1$, interference distance is zero and each node is not allowed to receive more than one message at a time. Under their assumptions, they give optimal (polynomial) gathering protocols for paths and tree networks. Their work has been extended to general graphs in [12] for unitary messages. With the additional assumptions that multiple channels are allowed, gathering protocols minimizing the delay are presented in [16]. Another related model can be found in [15], where the authors study the case in which steady-state flow demands between each pair of nodes have to be satisfied.

1.4 Our results

In this paper, we consider the case when G is a path and $d_T = 1$. The results of this paper, summarized in Table 1, are presented in the remaining sections as follows. In Section 2, we deal with the case where the gathering node is at one end of the path. Let $g_d(p)$ be the minimum number of rounds of a gathering protocol for a path of length p (so with $p + 1$ nodes). We describe an optimal protocol which uses $g_d(p)$ rounds to complete the gathering process. This result is similar to Theorem 4.1 of [10] (where the proof was not given). Note that their theorem is only valid with the gathering node at the end of the path and under the assumption of no buffering (but in this case buffering is not needed for optimal protocols). In Section 3, we consider the case where the gathering node is at the center of the path. Let $g_d(p,p)$ be the minimum number of rounds of a gathering protocol for a path of length $2p$. We first give a lower bound for $g_d(p,p)$ by only considering the interference constraint (this lower bound is also valid for the flow model of [15]). Then, we design an algorithm which meets the lower bound for $p \leq p_1$, where $d = 2k+1$ or $d = 2k+2$, depending upon its parity, and $p_1 = d+1+k(k+1)/2$. Next we show how to strengthen the preceding lower bound. In fact, we show that, for $p \geq d+2$, any algorithm for the path of length $2p$ with the gathering node at the center needs $2k+1$ more rounds than that for the path of length p with the gathering node at one end. Our algorithm meets this strengthened lower bound for $d = 1,2,3,4$ (which correspond to the practical cases). We close the paper with some concluding remarks in Section 4. Note that some results of this paper have been presented at the conference CIAC06 [1].

The path of length p with a gathering node at one end	
$p \leq d+2$	$g_d(p) = p(p+1)/2$
$p \geq d+2$	$g_d(p) = p(d+2) - \frac{(d+1)(d+2)}{2}$

The path of length $2p$ with a gathering node at the center	
$p \leq k+1$	$g_d(p,p) = 2g_d(p) = p(p+1)$
$k+2 \leq p \leq p_1$	$g_d(p,p) = p(d+1) - \lfloor d/2 \rfloor (k+1)$
$p \geq p_1$	$g_d(p) + \max\{2k+1, \frac{(k+1)(k+2)}{2} - (p-p_1)\} \leq g_d(p,p) \leq g_d(p) + \frac{(k+1)(k+2)}{2}$

Table 1
Summary of results for $g_d(p)$ and $g_d(p,p)$: the minimum number of rounds of a gathering protocol for a path and $p_1 = d+1+k(k+1)/2$ ($d = 2k+1$ or $d = 2k+2$).

2 Paths with the gathering node at one end

Let Π_p be the path of length p (consisting of p edges and $p+1$ nodes). The nodes are denoted $0,1,2,\cdots,p$, and the edges are of the type $(i,i-1)$, for all $1 \leq i \leq p$. To simplify the notation, we write X_i the call $X_{i,i-1}$ and $g_d(p)$ for the minimum number of rounds. The recursive scheduler described in Algorithm 1 produces a gathering algorithm that is used to prove the result below when the gathering node is $t=0$ (see Figure 2 for an example with $p=6$ and $d=2$). It should be noted that the loops of Phases I and II are indexed by decreasing indices.

Theorem 1 *For the path Π_p and $d \geq 1$,*

$$g_d(p) = \begin{cases} p(p+1)/2, & \text{if } p \leq d+2 \\ (d+2)(2p-d-1)/2 = p(d+2) - \frac{(d+1)(d+2)}{2}, & \text{otherwise} \end{cases}$$

Algorithm 1 Gathering scheduler for Π_p

1: **if** $p > 0$ **then**
2: Call recursively the gathering scheduler on Π_{p-1}

`// Phase I: schedule calls involving the message` P
`// and nodes at distance at least` $d+3$ `from` t

3: Let $x = p - (d+2)$
4: **for** $j \leftarrow p, \ldots, d+4, d+3$ **do**
5: Let $i = j - (d+2)$
6: Schedule P_j in the same round as X_i

`// Phase II: schedule the remaining calls involving`
`// the message` P

7: **for** $j \leftarrow \min\{p, d+2\}, \ldots, 2, 1$ **do**
8: Schedule P_j in a new round

PROOF. The upper bound is given by the number of rounds produced by Algorithm 1. Suppose that all calls involving messages smaller than P are scheduled as indicated in line 2. The calls involving the message P leaving a node $j \geq d+3$ are scheduled in existing rounds during Phase I as indicated in lines 4–6. New rounds are then created for the remaining calls in Phase II. Hence, proceeding by recurrence,

$$
\begin{aligned}
g_d(p) &\leq g_d(p-1) + \min\{p, d+2\} \\
&= \sum_{i=1}^{p} \min\{i, d+2\},
\end{aligned}
$$

which gives the upper bound of the lemma.

To show the lower bound, note that the information X of a node x must be transmitted via the calls X_j, $1 \leq j \leq x$. Furthermore, the interference constraint implies that at most one call X_j, for $1 \leq j \leq d+2$, can occur in a round. So, to bring X, for each $1 \leq x \leq p$, from node x to the gathering node, we need at least $\min\{x, d+2\}$ rounds, all containing only one call in the interval $[0, d+2]$. Therefore,

$$
g_d(p) \geq \sum_{i=1}^{p} \min\{i, d+2\}.
$$

$\square$

3 Paths with the gathering node at the center

3.1 Preliminaries

Let $\Pi_{-p,p}$ denote the path of length $2p$ with the $2p{+}1$ vertices $-p, -(p{-}1), \cdots, -1, 0, 1, 2, \cdots, p$, and with the edges $(-i, -(i-1))$ and $(i, i-1)$, for all $1 \le i \le p$. In this section, we discuss bounds on the minimum number of rounds $g_d(p,p)$ performed by a protocol for $\Pi_{-p,p}$ when the gathering node is $t = 0$. For the ease of explanation, we write $d = 2k+1$ or $d = 2k+2$, depending whether d is odd or even, respectively. Note that this notation for d means that $k = \lceil d/2 \rceil - 1$.

Clearly, $g_d(p,p) \ge g_d(p)$ since $\Pi_{-p,p}$ is composed by two symmetric paths of length p. However, such lower bound would be tight only if the calls in one path are paired with calls in the other. We will show that such a complete pairing is not possible due to the interference constraints and the natural order induced by the calls involving a single message (for example, 5_2 must be before 5_1 in any protocol). If the number of nodes is small, then all the calls are incompatible and every protocol is optimal.

Proposition 2 *If $p \le k+1$, for $k = \lceil d/2 \rceil - 1$, then $g_d(p,p) = 2g_d(p) = p(p+1)$.*

For $p \ge k+2$, an optimal protocol requires some compatible calls to be appropriately paired. Like in the previous section, write X_i (referred to as a *positive call*) and $-X_i$ (a *negative call*) for the calls $X_{i,i-1}$ and $-X_{-i,-(i-1)}$, respectively. Special attention needs to be devoted to the *critical calls*, which are the calls in the *critical interval* $[-(d{+}2), d{+}2]$ of nodes. In the critical interval, two positive calls X_i and Y_j interfere, and so do two negative calls $-X_i$ and $-Y_j$. Moreover, two calls $-X_i$ and Y_j interfere if and only if $i + j \le d+1$ because the distance between nodes $-i$ and $j-1$ is $i+j-1$. For example, a call $-X_1$ can be paired only with the critical calls Y_{d+1} or Y_{d+2}. Consequently, every round contains at most two critical calls and, in addition, a round contains two critical calls $-X_i$ and Y_j only if $i + j \ge d+2$.

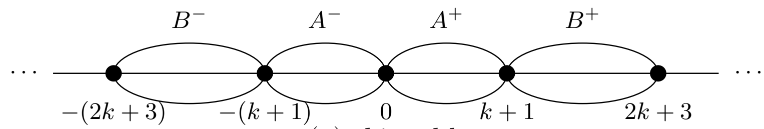
The set of critical calls can be decomposed as follows (see Figure 3). Let

$$A^+ = \bigcup_{i=1}^{k+1} \{X_i \mid i \le x \le p\} \text{ and } A^- = \{-X_i \mid X_i \in A^+\}. \tag{1}$$

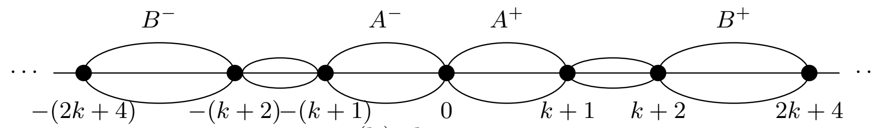
These two sets are defined such that a call in A^+ cannot be paired with any call in A^-, which means that two critical calls can be paired only if at least one of them does not belong to $A^+ \cup A^-$. We define the sets

$$B^+ = \bigcup_{i=d-k+1}^{p'} \{X_i \mid i \le x \le p\} \text{ and } B^- = \{-X_i \mid X_i \in B^+\}, \tag{2}$$

where $p' = \min\{p, d+2\}$. When d is odd, these sets partition the set of critical calls since



(a) d is odd.



(b) d is even.

Fig. 3. Decomposition of the critical calls of $\Pi_{-p,p}$ when $p \geq d+2$.

$d-k+1 = k+2$. But when d is even, there are also all the calls $-X_{k+2}$ and X_{k+2}.

3.2 A lower bound when $p \geq k+2$

When the number of nodes in each side of the path is at least $k+2$, then some positive calls can be paired with some negative ones without violating the interference constraint. Taking into account the compatible pairs that can be formed in this way, we obtain the following lower bound.

Theorem 3 *$g_d(p,p) \geq p(d+1) - \lfloor d/2 \rfloor (k+1)$, for $k = \lceil d/2 \rceil - 1$.*

PROOF. To obtain the lower bound, we count the maximum number M of compatible pairs $\{-X_i, Y_j\}$ and we have $g_d(p,p) \geq 2g_d(p) - M$. As we have seen that there cannot be pairs formed by two elements in $A^+ \cup A^-$, so each pair must contain at least one element of $B^+ \cup B^-$, and in the case of d even, we can also pair $-X_{k+2}$ with X_{k+2} for $k+2 \leq x \leq p$. Thus, $M = |B^+| + |B^-| + \epsilon(p-k-1)$, where $\epsilon = 1$ if d is even and 0 if d is odd.

First consider $p \leq d+2$, in which case Theorem 1 gives $g_d(p,p) \geq p(p+1) - M$. If d is odd, we get from (2): $|B^+| = |B^-| = \sum_{k+2 \leq i \leq p} p - i + 1 = \frac{(p-k-1)(p-k)}{2}$ and so $2g_d(p) - M = p(p+1) - (p-k-1)(p-k) = p(2k+2) - k(k+1)$. If d is even, then $|B^+| = |B^-| = \sum_{k+3 \leq i \leq p} p - i + 1 = \frac{(p-k-2)(p-k-1)}{2}$ and so $M = (p-k-1)^2$ and $2g_d(p) - M = p(p+1) - (p-k-1)^2 = p(2k+3) - (k+1)(k+1)$.

For $p \geq d+3$, when p is incremented by 1, $g_d(p)$ is incremented by $d+2$, both $|B^+|$ and $|B^-|$ are incremented by $k+2$, and M is incremented by $2(d+2) - 2(k+2) - \epsilon = 2(d-k) - \epsilon$. Recall that in the case of d even, we also pair $-X_{k+2}$ and X_{k+2} and $\epsilon = 1$ ($\epsilon = 0$ if d is odd). Therefore, it follows that the lower bound of the theorem is incremented by $M = d+1$ when p is incremented by 1, as claimed. $\square$

In this subsection, we present a gathering algorithm whose number of rounds meets the lower bound described in the previous subsection when $p \leq p_1 = d+1+k(k+1)/2$. This gathering algorithm corresponds to the sequence of rounds scheduled with Algorithm 2. A round is called an *obstruction* if it contains only one critical call. In the general case of $p \geq 1$ and $d \geq 1$, the rounds are scheduled recursively in the sense that the rounds involving the calls P_j and $-P_j$, for all $j \in \{1, 2, \ldots, p\}$, are scheduled after all the calls associated with the path consisting of $p-1$ positive and negative nodes are scheduled in line 2. This is done without modifying the order of rounds, but only by including the new calls in existing rounds, when possible, or creating new pairs and obstructions. When applied to the case $d = 3$, this algorithm produces the rounds shown in Figure 4 and summarized in Table 2. The proof that it is also optimal for larger values of p with $1 \leq d \leq 4$ will be given in the next subsection.

Before going into details, three observations can be made in connection with Algorithm 2. First, one should note that the index j of the loops in Phases I, II, and III (lines 4, 8 and 14) is decreasing. A consequence of this fact is that Phases I and II apply only for $p \geq d+3$ and $p \geq k+2$, respectively. The second observation is that Algorithm 2 schedules the calls in a sequence of pairs of symmetric rounds in such a way that, if a pair of compatible critical calls $\{X_i, -Y_j\}$, with $x < y$, is scheduled in a certain round, then the round immediately after includes the symmetric counterpart $\{-X_i, Y_j\}$. Similarly, if a round is constituted by a single positive call X_i, the next round consists of the single negative call $-X_i$. Finally, the third observation is with respect to line 13, where it is assumed that a round compatible with $-P_j$ is available. It stems from the second observation that this turns out to be always the case. The gathering algorithm obtained for $d = 3$ and $d = 4$ are illustrated in Tables 2 and 3, respectively.

First we illustrate how Algorithm 2 produces the rounds in Figure 4 for $d = 3$ ($k = 1$) and $p = 6$. After the recursive invocation in line 2, 18 rounds are scheduled by the algorithm, corresponding to the solution of the gathering problem for $p - 1 = 5$. At this point of the execution, there are 6 obstructions left, namely 4_1, -4_1, 5_2, -5_2, 5_1 and -5_1. Then, Phase I is applied with $j = 6$ for $p = 6$. By line 6, call 6_6 is scheduled in the same round as 1_1 and, symmetrically, call -6_6 is scheduled in the same round as -1_1 by line 7 (such a symmetric round is omitted in Table 2). In the sequel, Phase II is executed for $j = 5, 4, 3$, in this order. For $j = 5$, by lines 9–10, call 6_5 is scheduled with call -4_1 and -6_5 with 4_1 (in the symmetric rounds 14 and 13, respectively). Similarly, for $j = 4$, call 6_4 (resp. -6_4) is scheduled with -5_2 (resp. 5_2). For $j = 3$, call 6_3 is scheduled in a new round (round 19), by line 12, since there are no more obstructions compatible with it. Notice that this new round is itself compatible with -6_3, leading by line 13 to the pair $\{-6_3, 6_3\}$ in round 19. Finally, in Phase III, the rounds 20–23 are scheduled with the obstructions 6_2, -6_2, 6_1 and -6_1. Similarly to this example, the application of Algorithm 2 for $d = 4$ gives the rounds shown in Table 3.

Algorithm 2 Gathering scheduler for $\Pi_{-p,p}$

1: **if** $p > 0$ **then**
2: Call recursively the gathering scheduler for $\Pi_{-(p-1),(p-1)}$

`// Phase I: greedily schedule noncritical calls`
`// involving the messages` P `and` $-P$

3: Let $x = p - (d+2)$
4: **for** $j \leftarrow p, \dots, d+4, d+3$ **do**
5: Let $i = j - (d+2)$
6: Schedule P_j in the same round as X_i
7: Schedule $-P_j$ in the same round as $-X_i$

`// Phase II: greedily schedule critical calls`
`// involving the messages` P `and` $-P$
`// and nodes at distance at least` $k+2$ `from` t

8: **for** $j \leftarrow \min\{p, d+2\}, \dots, k+3, k+2$ **do**
9: **if** there is an obstruction compatible with P_j **then**
10: Schedule P_j in the smallest round that is compatible with P_j
11: **else**
12: Schedule P_j in a new round
13: Schedule $-P_j$ in the smallest round that is compatible with $-P_j$

`// Phase III: greedily schedule the remaining critical calls`
`// involving the messages` P `and` $-P$

14: **for** $j \leftarrow \min\{p, k+1\}, \dots, 2, 1$ **do**
15: Schedule P_j in a new round
16: Schedule $-P_j$ in a new round

A call is included in an existing round whenever line 6, 7 or 10 is executed. In particular, the new calls scheduled in Phase I are outside the critical interval and they can be included in existing rounds. Whereas a noncritical call is always included in an existing round, critical calls may also create new pairs (Phase II) or obstructions (Phase III). The execution of Phase II, line 13, corresponds to the inclusion of a call from B^- in an existing round (if line 10 is executed) or to the creation of a new pair (if line 12 is executed instead). If a new (critical) call is paired with an obstruction in line 10, then its symmetric counterpart is created in line 13 by including the new call in an existing round. Otherwise, an obstruction is created with the new call in line 12 and is then paired in line 13.

The main point in the analysis of the Algorithm 2 is to show that all calls in B^- are paired, as well as $-P_{k+2}$ and P_{k+2} are paired together if d is even, leading to the following upper bound for $g_d(p,p)$.

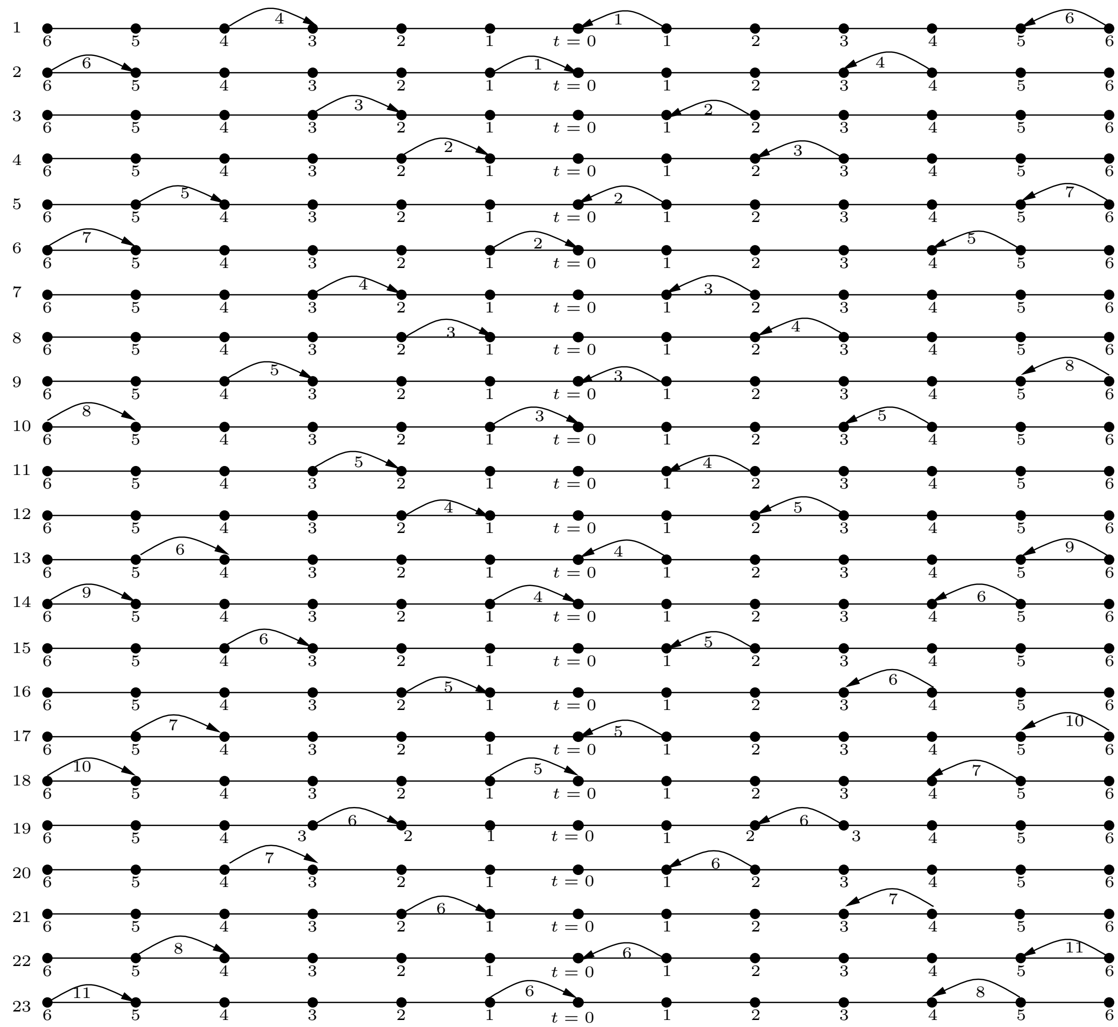


Fig. 4. Some of the rounds produced by Algorithm 2 for $d = 3$ and $p \geq 11$. Only the calls involving nodes in the subpath $\Pi_{-6,6}$ are shown.

Theorem 4 *Let $p \geq k+1$, and $p_1 = d+1+k(k+1)/2$, then* $g_d(p,p) \leq p(d+1) - \lfloor d/2 \rfloor (k+1) + \max\{0, p - p_1\}$.

Note that, for $p \geq k+1$, when p is incremented by 1, then $g_d(p,p)$ increases by $d+1$ till p_1 and then by $d+2$ (like $g_d(p)$). Combined with Theorem 3, we get the following exact result:

Theorem 5 *Let $k+1 \leq p \leq p_1$, where $p_1 = d+1+k(k+1)/2$, then*

$$g_d(p,p) = p(d+1) - \lfloor d/2 \rfloor (k+1)$$

.

Proof of Theorem 4 To prove the theorem, we count the number of rounds $r_d(p)$ created in Phases II and III, and we show that they are created as pairs in Phase II and obstructions in Phase III. For this purpose, let A_d^p denote the sequence of calls that define

Round	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$	$p=11$
1	1_1			-4_4		6_6			-9_9		11_{11}
3		2_2	-3_3				7_7	-8_8			
5		2_1			-5_5		7_6			-10_{10}	
7			3_2	-4_3				8_7	-9_8		
9			3_1		-5_4			8_6		-10_9	
11				4_2	-5_3				9_7	-10_8	
13				4_1		-6_5			9_6		-11_{10}
15					5_2	-6_4				10_7	-11_9
17					5_1		-7_5			10_6	
19						$\{-6_3, 6_3\}$					$\{-11_8, 11_8\}$
20						6_2	-7_4				11_7
22						6_1		-8_5			11_6
24							$\{-7_3, 7_3\}$				
25							7_2	-8_4			
27							7_1		-9_5		

Table 2
Pairs and obstructions in the rounds derived from Algorithm 2 for $d = 3$. For every round shown in the table but those between horizontal lines, the algorithm also includes its symmetric counterpart.

obstructions left after line 2 and that are paired in Phase II with the sequence

$$\langle -P_\ell, -P_{\ell-1}, \ldots, -P_{d-k+1}\rangle, \ell = min\{p, d+2\}. \tag{3}$$

The first element of A_d^p is paired with $-P_\ell$, the second with $-P_{\ell-1}$ and so on. For each value of $p \geq k+2$, we need to determine A_d^p and, in addition, we need to show that the new pairs created in Phase II are composed by the call $P_{k+2} \notin A_d^p$ paired with $-P_{k+2}$, when p is even, and the call $P_{d-k+1} \notin A_d^p$ paired with $-P_{d-k+1}$, if p is large enough.

d is odd: Let $d = 2k+1$. The proof is by induction on p. The basis of the induction follows from the observation that every call X_j or $-X_j$ such that $1 \leq j \leq x \leq k+1$ is an obstruction. So, the induction starts at $r_d(k+1) = (k+1)(k+2)$. For the induction step, three cases are distinguished, as follows:

(1) $k+2 \leq p \leq 2k+2 = d+1$. Note that $\min\{p, d+2\} = p$ in this case, which means that $-P_p$ is in the sequence (3). Write $i = p-k-1$. If $i = 1$ ($p = k+2$), the sequence (3) is $\langle -P_{k+2}\rangle$. Since the only call already scheduled and compatible with $-P_{k+2}$ is $(k+1)_{k+1}$, we have $A_d^{k+2} = \langle (k+1)_{k+1}\rangle$. It remains to show that

$$A_d^{k+1+i} = \langle (k-i+2)_{k-i+2}, (k-i+4)_{k-i+3}, \ldots, (k+i)_{k+1}\rangle, \tag{4}$$

for all other values of i. Assume inductively that it is indeed the case until $i-1$. Then, $-P_{p=k+1+i}$ is only compatible with an X_{k-i+2}, the first such a call available being $x = (k-i+2)$. Consider now $-P_{k+i}$. The first call already scheduled and compatible with it is $(k-i+3)_{k-i+3}$ which, by the induction hypothesis, belongs to A_d^{k+i}. Since it is not available, the next candidate is the call $(k-i+4)_{k-i+3}$. Indeed,

Round	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$
1	1_1				-5_5		7_7		
3		2_2		-4_4				8_8	
5		2_1				-6_6		8_7	
7			$\{-3_3, 3_3\}$						$\{-9_9, 9_9\}$
8			3_2		-5_4				9_8
10			3_1			-6_5			9_7
12				$\{-4_3, 4_3\}$					
13				4_2		-6_4			
15				4_1			-7_6		
17					$\{-5_3, 5_3\}$				
18					5_2		-7_5		
20					5_1			-8_6	
22						$\{-6_3, 6_3\}$			
23						6_2	-7_4		
25						6_1		-8_5	
27							$\{-7_3, 7_3\}$		
28							7_2	-8_4	
30							7_1		-9_6
32								$\{-8_3, 8_3\}$	
33								8_2	-9_5
35								8_1	
37									$\{-9_4, 9_4\}$
38									$\{-9_3, 9_3\}$
39									9_2
41									9_1

Table 3
Similar to Table 2, but for $d = 4$.

it is available because it could be paired only with $-(p-1)_{k+i}$ which, again by the induction hypothesis, is paired with $(k-i+3)_{k-i+3}$. Inductively, $-P_{p-h}$, $0 \le h \le i$, can only be paired with X_j such that $j \ge k+2-i+h$ and $x = k+2-i+2h$. Therefore, all the pairs $-P_j$, $k+2 \le j \le p$, are paired, giving $|A_d^p| = i$, for $p = k+1+i$. It follows that only $2(k+1) = d+1$ obstructions are created in Phase III and, so,

$$r_d(p) = r_d(p-1) + d + 1. \tag{5}$$

(2) $2k+3 \le p \le p_1 = d+1+k(k+1)/2$. Write $p' = k(k-1)/2$, which means that $p_1 = 3k+2+p'$. Let $\oplus$ denote a concatenation of sequences and $i = p-2k-2$, where $1 \le i \le k+p'$. There are two sub-cases for the sequence A_d^p:

(a) for $1 \le i \le p'$,

$$A_d^{2k+2+i} = A_{d-2}^{2k+i} \oplus \langle (2k+1+i)_{k+1} \rangle.$$

This follows directly from the fact that, by (4), the obstructions of the type X_i, for $x \le 2k$ and $k > 1$, left by the recursive call when $i = 1$ are exactly the same left by the gathering algorithm produced when $d = 2k-1$ and $p = 2k$. The calls in the recursive term are paired with the calls in the sequence (3) except $-P_{k+2}$, which is then paired with single call in the sequence of the additional term.

(b) for $p' + 1 \leq i \leq k + p'$,

$$A_d^{2k+2+i} = \langle (k+i)_1, (k+1+i)_2, \ldots, (p'+2k)_{p'+k-i+1} \rangle \oplus \langle (p'+2k+1)_{p'+k-i+1}, (p'+2k+2)_{p'+k-i+2}, \ldots, (2k+i+1)_{k+1} \rangle.$$

The main observation in this case is that the call $(p'+2k+1)_{p'+k-i+2}$ is paired in line 2. If $i = p'+1$, this fact is derived from the recursion of sub-case 2a, as follows. If $k = 1$, then 3_2 is paired with -4_3. Otherwise, $(p'+2k+1)_{p'+k-i+2}$ is paired with the additional term of the recursion. For $i > p'+1$, the call $(p'+2k+1)_{p'+k-i+2}$ is in A_d^{2k+1+i}.

The sequences of obstructions in this case are better understood with examples. As a first example, assume $d = 3$ (Figure 4 and Table 2 illustrate this example). Then, we have $k = 1$, $p' = 0$ and only sub-case 2b applies for $p = 5$, leading to

$$A_3^5 = \langle 2_1, 3_1, 4_2 \rangle.$$

In the second example, we take $d = 5$ ($k = 2$ and $p' = 1$). Sub-case 2a applies for $p = 7$ ($i = 1$) and gives

$$A_5^7 = A_3^5 \oplus \langle 6_3 \rangle = \langle 2_1, 3_1, 4_2, 6_3 \rangle.$$

Sub-case 2b applies for $p = 8, 9$ ($i = 2, 3$) and gives

$$A_5^8 = \langle 4_1, 5_2, 6_2, 7_3 \rangle, \quad A_5^9 = \langle 5_1, 6_1, 7_2, 8_3 \rangle.$$

Similarly, if $d = 7$ ($k = 3$), sub-case 2a applies for $9 \leq p \leq 11$ ($1 \leq i \leq 3$), giving the sequences

$$A_7^9 = A_5^7 \oplus \langle 8_4 \rangle, \quad A_7^{10} = A_5^8 \oplus \langle 9_4 \rangle, \quad A_7^{11} = A_5^9 \oplus \langle 10_4 \rangle.$$

The sequences for $12 \leq p \leq 14$ ($4 \leq i \leq 6$) correspond to sub-case 2b, and are given as

$$A_7^{12} = \langle 7_1, 8_2, 9_3, 10_3, 11_4 \rangle, \quad A_7^{13} = \langle 8_1, 9_2, 10_2, 11_3, 12_4 \rangle,$$
$$A_7^{14} = \langle 9_1, 10_1, 11_2, 12_3, 13_4 \rangle.$$

These three examples illustrate the sequences determined in sub-cases 2a and 2b, and show that the $k+2$ calls $-P_j$, $k+2 \leq j \leq d+2$, are paired. Thus, in general we still have

$$r_d(p) = r_d(p-1) + d + 1. \tag{6}$$

(3) $p \geq p_1 + 1$, in which case we write $p = p_1 + i$, $i \geq 1$. The $k+1$ pairs P_j such that $k+3 \leq j \leq d+2$ are paired with the obstructions

$$A_d^p = \langle (p_1 - j - 1 - i)_1, (p_1 - j - i)_2, \ldots, (p_1 + i)_{k+1} \rangle.$$

In this case, the call P_{k+2} is paired with $-P_{k+2}$, creating a new round. Again, $2(k+1)$ obstructions are created, so giving

$$r_d(p) = r_d(p-1) + d + 2. \tag{7}$$

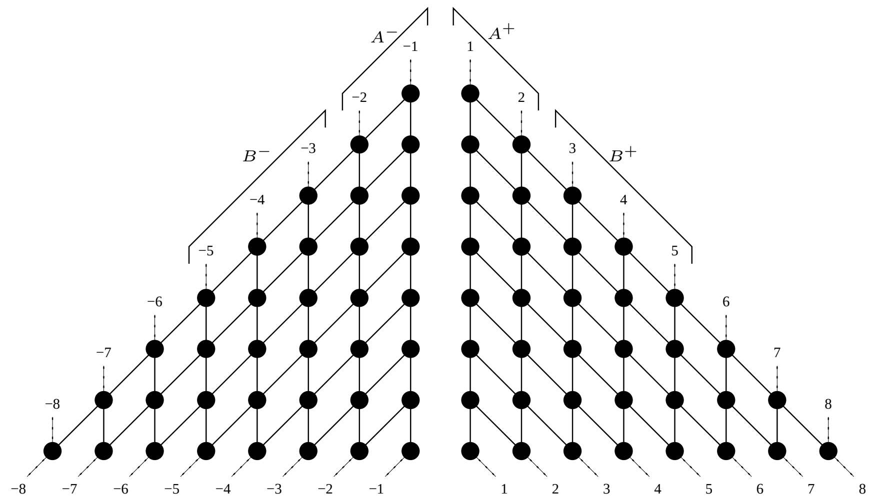


Fig. 5. Partial order $\preceq$ on the calls of a protocol for $p = 8$ and $d = 3$.

d is even. Let $d = 2k + 2$ and in this case, we obtain the result from the case $d - 1$ odd (see Table 3). First, observe that, if $p < 3k + 3 + p'$, then $A^p_{2k+2} = A^{p-1}_{2k+1} \oplus \langle P_{k+2} \rangle$. Otherwise, A^p_d includes P_{k+2} and pairs kept from A^{p-1}_{2k+1} and A^p_{2k+1} depending on the call X_{k+2}, where $x = 3k+3+p'$. A call Y_j is kept from A^{p-1}_{2k+1} if $y < x$ or ($y = x$ and $j > k+2$), and from A^p_{2k+1} otherwise. $\square$

3.4 A lower bound for $p \geq d + 2$

A natural property satisfied by every protocol is that a call X_i (resp. $-X_i$) appears in a round occurring before that of another call X_j (resp. $-X_j$) if $i > j$. Note that the gathering protocols produced with Algorithms 1 and 2 share a common additional property: X_i (resp. $-X_i$) appears after Y_i (resp. $-Y_i$) if $x > y$. However, one can easily modify any gathering protocol to enforce this latter property without increasing the number of rounds. For this purpose, it suffices to send the message with the smallest x in every call. Hereafter, we only consider protocols in which the calls satisfy the partial order $\preceq$ illustrated in Figure 5 and defined as follows.

Definition 6 *$X_i \preceq Y_j$ if $x \leq y$ and $i \geq j$ and $-X_i \preceq -Y_j$ if $x \leq y$ and $i \geq j$. We may use the notation $X_i \prec Y_j$ when $x \neq y$ or $i \neq j$.*

In the rest of this subsection, we present a lower bound for $p \geq d + 2$. This lower bound is based on the minimum number of obstructions that are induced by $\preceq$.

The following lemma will be used repeatedly in the rest of the paper.

Lemma 7 (Non-Crossing Lemma) *A gathering protocol cannot have two different pairs $\{-X_i, W_j\}$ and $\{-Y_k, Z_\ell\}$ with either $-Y_k \prec -X_i$ and $W_j \prec Z_\ell$, or $-X_i \prec -Y_k$ and $Z_\ell \prec W_j$.*

PROOF. By the sake of contradiction, suppose a gathering protocol that contains both $\{-X_i, W_j\}$ and $\{-Y_k, Z_\ell\}$ and, in addition, $-Y_k \prec -X_i$ and $W_j \prec Z_\ell$. Since $-Y_k \prec -X_i$, the round in which $\{-X_i, W_j\}$ appears must occur after that of $\{-Y_k, Z_\ell\}$, and this by the definition of the order $\preceq$. Similarly, $W_j \prec Z_\ell$ implies that $\{-Y_k, Z_\ell\}$ must appear after $\{-X_i, W_j\}$, leading to a contradiction. A similar argument applies when $-X_i \prec -Y_k$ and $Z_\ell \prec W_j$. □

We will make the following assumption to facilitate the proof later.

Assumption 8 *If X_i (resp. $-X_i$) is an obstruction, then the two following conditions hold:*

(1) either $i = 1$ or X_{i-1} (resp. $-X_{i-1}$) is an obstruction; and
(2) either $x = p$ or $(X+1)_i$ (resp. $-(X+1)_i$) is an obstruction.

Indeed, for any given protocol, it is possible to adjust some rounds to make the above assumption to be true, as follows.

Lemma 9 *There exists an optimal protocol that satisfies Assumption 8.*

PROOF. Consider an optimal protocol which does not satisfy Assumption 8. Then let X_i be the smallest (for the order defined above) obstruction contradicting the assumption. So one of X_{i-1} or $(X+1)_i$ must be paired. Let us consider the first pair appearing after X_i in the protocol. Suppose it is $(-Y_j, X_{i-1})$ (the proof is similar if it is a pair containing $(X+1)_i$). In the rounds between X_i and $(-Y_j, X_{i-1})$, there can only be Z_k (if we only look at the positive side) where $z > x$ and $k > i$, or $z < x$ and $k < i$ (by the minimal choice of X_i). So we can move X_i until the round before X_{i-1} without generating conflict. Then we pair X_i with $-Y_j$ (which is possible as $i - 1 + j \geq d + 2$) and let X_{i-1} be an obstruction. Repeat this process until Assumption 8 is satisfied. □

An obstruction is called *positive* if its unique critical call is positive, and is called *negative* otherwise. Let s_A^+ (resp. s_A^-) be the number of positive (resp. negative) obstructions involving calls of A^+ (resp. A^-). Similarly, let s_B^+ and s_B^- stand for the number of positive and negative obstructions in B^+ and B^-, respectively. The total number of positive (resp. negative) obstructions is given by $s^+ = s_A^+ + s_B^+$ (resp. $s^- = s_A^- + s_B^-$), and $s^+ = s^-$.

Proposition 10 *P_1 and $-P_1$ are obstructions.*

PROOF. We cannot have both $-P_1$ and P_1 paired because, otherwise, the pairs would be $\{-P_1, Y_j\}$, with $j \geq d+1$, and $\{-Z_i, P_1\}$, with $i \geq d+1$, contradicting the Non-Crossing Lemma as $-Z_i \prec -P_1$ and $Y_j \prec P_1$. So, without loss of generality, suppose P_1

is an obstruction. Due to $s^+ = s^-$, there is a negative obstruction and, by Assumption 8, $-P_1$ is an obstruction. $\square$

An immediate consequence is the optimality of Algorithm 2 for $d = 1, 2$.

Corollary 11 *For $d = 1, 2$ ($k = 0$) and $p \geq d + 2$,*

$$g_d(p, p) = g_d(p) + 1 = \begin{cases} 3p - 2, & \text{if } d = 1 \\ 4p - 5, & \text{if } d = 2 \end{cases}$$

However, if $d \geq 3$, the number of obstructions increases. In particular, the calls $-P_2$ and P_2 cannot be paired simultaneously in any gathering protocol because, otherwise, the pairs would be $\{-P_2, Y_j\}$, with $j \geq 3$, and $\{-Z_i, P_2\}$, with $i \geq 3$, contradicting the Non-Crossing Lemma as $-Z_i \prec -P_2$ and $Y_j \prec P_2$. Indeed, $-P_2$ and P_2 are two out of the $(k+1)(k+2)$ obstructions in the gathering protocol produced by Algorithm 2. In the sequel, it is proved that the gathering protocol produced by Algorithm 2 is also optimal for some values of d greater than 2 based on the fact that the number of calls in B^- (resp. B^+) that can be paired with A^+ (resp. A^-) is limited by the order $\preceq$.

Theorem 12 *If $p \geq d + 2$, then every gathering protocol has at least $2k + 1$ positive and $2k + 1$ negative obstructions.*

PROOF.

By contradiction; suppose that $d = 2k + 1$ and that there exists a gathering protocol with $2k$ positive and $2k$ negative obstructions. First, we show that a certain number of calls in B^- are paired with calls in B^+ in this protocol. The following lemma plays an essential role in the proof of this theorem.

Lemma 13 *Let $d = 2k+1$ and suppose a gathering protocol with at most $2k$ positive and $2k$ negative obstructions. Then, for $1 \leq h \leq p - (d + 1)$, there are at least h pairs with one call in B_h^- and the other in B_h^+, where*

$$B_h^+ = \{X_i \in B^+ \mid p - h + 1 \leq x \leq p\} \text{ and } B_h^- = \{-X_i \mid X_i \in B_h^+\}.$$

Now we use this lemma and give the proof afterward. For $d = 2k + 1$, the proof that there are at least $2k+1$ positive and negative obstructions is by contradiction with the situation corresponding to Lemma 13 for $h = p - (d+1)$. Assume that $s = s_A^+ + s_B^- = s_A^- + s_B^- \leq 2k$. As there are $|A^+| - s_A^+$ elements of A^+ paired with elements of B^- and s_B^- obstructions in B^-, there are at most $|B^-| - |A^+| + s_A^+ - s_B^-$ pairs with one call in B^- and the other in B^+. Applying Lemma 13 with $h = p - (d+1)$, we get $s_A^+ - s_B^- \geq p - (d+1) - |B^-| + |A^+|$. Using

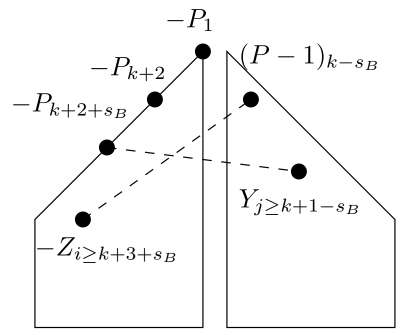


Fig. 6. Contradiction in the basis of the induction of the proof of Lemma 13. The dashed lines indicate the pairs analyzed.

$|A^+| = (k+1)p - \frac{k(k+1)}{2}$ and $|B^-| = (k+2)p - \frac{(k+1)(3k+6)}{2}$ we get $s_A^+ - s_B^- \geq (k+1)(k+1)$, a contradiction with the initial hypothesis $s_A^+ + s_B^+ = s_A^- + s_B^- \leq 2k$.

For $d = 2k+2$, every compatible gathering protocol remains compatible if the interference distance is reduced to $2k+1$. Then the arguments above also show the existence of at least $2k+1$ positive and negative obstructions when d is even. $\square$

Proof of Lemma 13. By induction on h, assuming $p \geq d+2$ (otherwise, there is nothing to prove). First we consider the case $h = 1$, that is we want to prove that there exists at least one pair of the form $\{-P_i, P_j\}$, $d-k+1 \leq i, j \leq d+2$.

A first observation is the following fact: at least one of P_{k+1} or $-P_{k+1}$ is an obstruction because the Non-Crossing Lemma prevents them from being both paired. Wlog let us suppose that P_{k+1} is an obstruction and so by Assumption 8 $P_1, P_2, \ldots, P_{k+1}$ are obstructions.

A second observation is that $(P-1)_{k-s_B^+}$ must be paired as otherwise there would be at least $2k+1$ positive obstructions, namely $P_1, \ldots, P_{k+1}, (P-1)_1, \ldots, (P-1)_{k-s_B^+}$ (by Assumption 8) and the s_B^+ positive obstructions.

Finally, observe that if $-P_{k+1}$ is paired, then $s_B^- = 0$ by Assumption 8. On the other hand, if $-P_{k+1}$ is an obstruction, then we can wlog assume that $s_B^+ \geq s_B^-$. It turns out that there are at most $k+1+s_B^+$ negative obstructions. Therefore, $-P_{k+2+s_B^+}$ is paired. Let $-Z_i$, with $i \geq k+3+s_B^+$, be the call paired with $(P-1)_{k-s_B^+}$, and Y_j, with $j \geq k+1-s_B^+$, be the call paired with $-P_{k+2+s_B^+}$. Since $y \leq p-1$ contradicts the Non-Crossing Lemma, we have $Y_j = P_j$. In addition, since P_{k+1} is an obstruction, we have $j \geq k+2$ by Assumption 8. Consequently, $-P_{k+2+s_B^+}$ of B^- is paired with P_j of B^+.

For the induction step, suppose the lemma is true until some $h < p-(d+1)$ and let $x = p-h$. Since there are at most $2k$ positive (and $2k$ negative) obstructions, both calls X_{k+2} and $-X_{k+2}$ are paired.

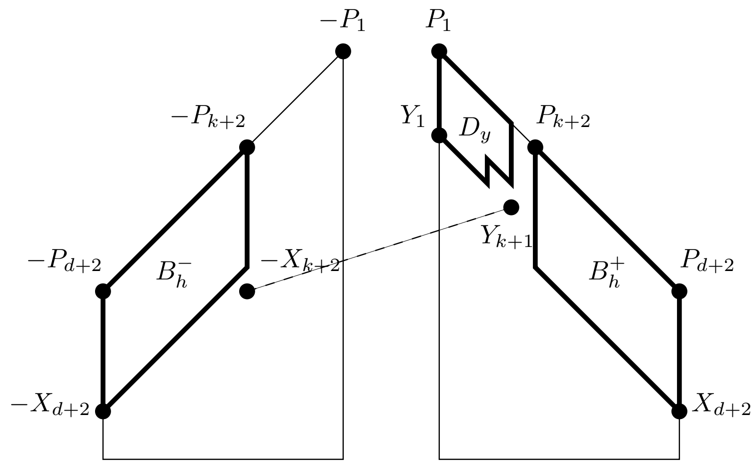


Fig. 7. Induction step in the proof of Claim 14, with $j = k+1$ and $x = p - h$.

Claim 14 *Let $x = p - h$, then the call Y_j (resp. $-Z_i$) paired with $-X_{k+2}$ (resp. X_{k+2}) is such that $y \geq x$ (resp. $z \geq x$).*

Proof of Claim 14. We analyze two cases separately for the pair $\{-X_{k+2}, Y_j\}$:

(1) $j = k+1$. Define the set

$$D_y = \{W_{k+1} \mid y+1 \leq w \leq p\} \cup \{W_\ell \mid y \leq w \leq p, 1 \leq \ell \leq k\},$$

as depicted in Figure 7. Observe that Y_j is a lower bound for $D_y \subseteq A^+$ and $|D_y| = (k+1)(p-y)+k$. Since D_y contains at most s_A^+ obstructions, at least $(k+1)(p-y)+k-s_A^+$ calls of D_y need to be paired with calls in B^- which, by the Non-Crossing Lemma, are not smaller than $-X_{k+2}$ in $\preceq$. This set of calls is exactly B_h^-, whose cardinality is $h(k+2)$. But, by the induction hypothesis, at least h of them are paired with B_h^+, and so cannot be paired with D_y. It remains at most $h(k+1)$ calls to be paired, which means that $h(k+1) \geq (k+1)(p-y)+k-s_A^+$. Since $s_A^+ \leq 2k$, we get $h(k+1) \geq (k+1)(p-y)-k$. Therefore, $h \geq (p-y) - \frac{k}{k+1}$ and we get $y \geq p-h$ as y is integer.

(2) $j \geq k+2$. It can be seen in Figure 8 that the call $-X_{k+2}$ is an upper bound for the set $E_x = \{-Z_i \in B^- \mid -Z_i \prec -X_{k+2}\}$, which contains $|E_x| = |B^-| - |B_h^-| - 1$ calls. Among them, at most $|B^-| - |A^+| + s_A^+ - s_B^- - (h+1)$ pair with a call in B^+. To see this, recall that there are at most $|B^-| - |A^+| + s_A^+ - s_B^-$ calls of B^- paired with a call of B^+, at least h calls paired with a call in B_h^- by the induction hypothesis, and the pair involving $-X_{k+2}$. So, by the Non-Crossing Lemma, at least $|A^+| - |B_h^-| - s_A^+ + s_B^- + h$ calls have to be paired with the calls of

$$F_y = \{W_\ell \mid 1 \leq \ell \leq k+1, \ell \leq w \leq y-1\},$$

which consists of the calls of A^+ that do not succeed Y_j in $\preceq$. Considering that $s_B^- \geq 0$, $s_A^+ \leq 2k$ and $|B_h^-| = h(k+2)$, we get $|F_y| \geq |A^+| - 2k - h(k+1)$. Then, it follows

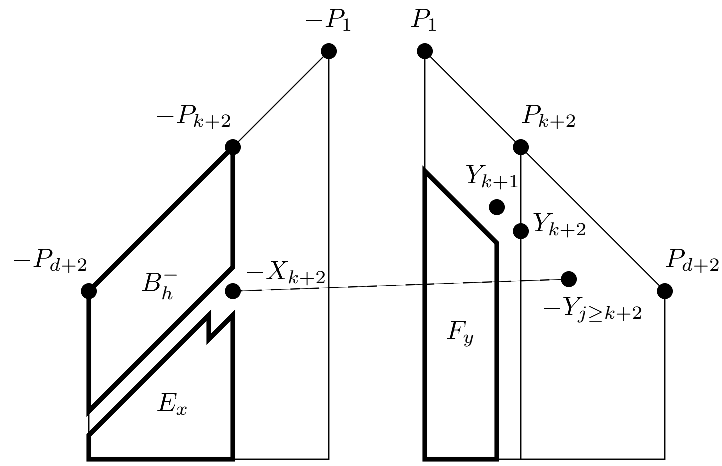


Fig. 8. Induction step in the proof of Claim 14, with $j \geq k+2$ and $x = p - h$.

from $|A^+| = (k+1)p - \frac{k(k+1)}{2}$ and $|F_y| = \sum_{\ell=1}^{k+1}(y-\ell) = y(k+1) - \frac{k(k+1)}{2} - (k+1)$ that $y \geq p - h - \frac{k-1}{k+1}$. This ensures that $y \geq p - h$ because $\frac{k-1}{k+1} < 1$ and y is integer.

The Non-Crossing Lemma yields that $i < k+2$ and $j < k+2$ cannot occur simultaneously in such a scenario. So, $j \geq k+2$ or $i \geq k+2$, which means that $\{-X_{k+2}, Y_j\}$ or $\{-Z_i, X_{k+2}\}$ is a new pair with one call in B_{h+1}^- and the other in B_{h+1}^+. Therefore, the lemma is true for $h+1$. This concludes the proof of the lemma. $\square$

As a consequence of Theorem 12, the gathering protocol derived from Algorithm 2 is optimal for $d = 3, 4$.

Theorem 15 *For $d = 3, 4$ $(k = 1)$ and $p \geq d+2$,*

$$g_d(p,p) = g_d(p) + 3 = \begin{cases} 5p - 7, & \text{if } d = 3 \\ 6p - 12, & \text{if } d = 4 \end{cases}$$

4 Concluding remarks

In this article we have given lower bounds and upper bounds for $g_d(p,p)$ the minimum number of rounds of a gathering protocol for a path of length $2p+1$, where the gathering node is at the center of the path. When, $p \leq p_1$, where $p_1 = d+1+k(k+1)/2$ the bounds coincide and therefore the problem is completely solved. The bounds also coincide for $d = 1, 2, 3, 4$ (which correspond to the practical cases). However the determination of the exact value for other d and any p seems a difficult task. We conjecture that the algorithm given in the article is optimal and so one should improved the lower bounds by proving that for $p > p_1$ there are $\frac{(k+1)(k+2)}{2}$ positive and negative obstructions.

Conjecture For $p \geq p_1 = d + 1 + k(k+1)/2$, $g_d(p,p) = g_d(p) + \frac{(k+1)(k+2)}{2}$

The results presented in the previous sections can also be extended to more general cases, for instance, when the gathering node is placed anywhere in the path. However, for a path of length $2p$, the choice of the center is the one that minimizes the number of rounds. Note that the results also apply to the converse problem called personalized broadcasting, where a node wants to send different messages to the other nodes of the network, as it suffices to reverse the calls and rounds in the protocols.

The gathering problem subject to the constraints in the general case is NP-complete. But if we restrict the structures of networks, then the solution is not always clear. For example, in the case of trees when $d = 1$, there exists polynomial solutions. It would be interesting to investigate the problems for different classes of networks as well as for the case when the size of messages are not the same. Another direction can be considered is when the buffering is not allowed in the the process of communications.

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