Graph Theory and Traffic Grooming in WDM Rings

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Abstract

This paper has a double purpose. In the first part of the paper we give an overview of different aspects of graph theory which can be applied in communication engineering, not trying to present immediate results to be applied neither a complete survey of results, but to give a flavor of how graph theory can help research in optical networks.

The second part of this paper is a detailed example of the usage of graph theory, but it is also a complete survey of recent results in minimization of the number of add–drop multiplexers (ADMs) required in a WDM ring with traffic grooming.

1 Introduction. Some words about maths and engineering.

It is not in the news that the development of sciences and technology makes it impossible to be aware of all fields of knowledge, andspecialization cannot be avoided in the scientific research where the rapid advances make it difficult even to follow the new results in the own specific discipline.This unavoidable fact turns out to be a strong limitation in thedevelopment of science, since many achieved results in somedisciplines that could be useful in other fields of knowledge are inmost cases unknown by scientists.It is desirable that researchers know at least the existence of otherfield's advances. Although not being specialist on these other areas,they should know where to look for information in benefit of their ownresearch.

It is specially interesting for engineers to be aware of mathematics since maths provide tools that can be immediately used in their own research. Probably no one doubts of the usefulness of mathematics in engineering, but it might also happen that when reading the previous sentence an engineer basically thinks on the ”classical” mathematics he learned and on the well stated mathematical models he was taught such as the Maxwell laws. One cannot expect to have a library of ”plug and play” mathematical models for new problems, but the art of creating a mathematical model is one of the examples in which cooperation of engineers and mathematicians iscrucial in order to get valuable results. But dialogue is not always easy and requires some effort from both, mathematicians and engineers. The translation of a problemarising from engineering to a mathematical language requires a deepknowledge of the discipline in which the problem is contextualized.Setting of assumptions and hypothesis of the model is always acompromise between realism and tractability. Finally, the resultsgiven by the mathematical model must be validated with someexperimental or simulation results.On the other hand it is important not to forget that mathematics is also a continuously developing science and that there are fields of mathematics that maybe are not taught at the university but still could be very

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useful in engineering if they were known. As an extreme example consider the case of an engineer that did never hear of discrete mathematics. Surely he will not think on mathematical tools when he needs to perform some discrete optimization.While discussing with some engineers we got the impression that they are somehow living on a "Best effort" world: They will try to do their best to improve existing systems but they usually deal with heuristics and simulation as their main tools.We are of course not opposite to heuristics, and we recognize that the "best effort" engineering lead in many cases to ingenious advances in technology. It is also true that heuristics and simulation are suitable in many cases specially when no other tools are available, or to start having some insight on a problem. But in some cases engineers could overuse those techniques just becausethey do not the existence of other mathematical tools or the either because they do not dare to use them or they are not confident in obtaining new results by using them: One can argue that mathematics will not help since real optimization problems are intractable. Even being that the case, and even not existing a mathematical solution, the mathematical formulation couldgive lots of insights providing new directions to look at in order to improve existing results. Precisely, translating a problem to mathematics have the following advantages:

- Translating a problem to maths gives some insight on it.
- A translated problem avoids getting confused by irrelevant data of the problem.
- A problem translated to maths is understandable by other scientific communities that can help solving it.
- If a problem is translated in terms of a mathematical structure, it is possible to use the tools an results provided by mathematics to solve it.
- Problems coming from other disciplines may have the sametranslation and might have been solved before.

Graph theory is a field in discrete mathematics that seems to have plenty of applications in communications engineering. In the next section we revise different aspects of graph theory which can be applied in communication engineering, not trying to present immediate results to be applied neither a complete survey ofresults, but we only want to give a flavor of how graph theory can help this field.The remaining of the paper will be devoted to show a more detailed example of the usage of graph theory in optical networks: By using graph theory it is optimized the number of ADMs in a WDM ring network when traffic grooming is used.

2 Graph theory and optimization in networks

We assume that the reader is familiar with the basic graph theory concepts. In any case the reader is referred to [6, 16] for general concepts and definitions.Let us just recall that a graph $G = (V, E)$ is an ordered pair consisting of a set $V = V(G)$ of elements called *vertices* and a set $E = E(G)$ ofunordered pairs of vertices called *edges*. In other words, a graph is a set of binary relationships within the elements of a genericset.With this so general definition, it is not surprising that graphs appear in many different contexts of the scientific knowledge such as sociology,economy, or engineering. Precisely a graph appears, at least implicitly, as soon as a binary relationship is considered.

2.1 Network topologies

Concerning to interconnection networks it is straightforward that anetwork topology can be modeled by a graph in which the vertices ofthe graph play the role of nodes in the network and the edges modelthe point-to-point links between different nodes. The graph can beundirected in the case of full-duplex communication links, or directedin case the links are unidirectional or in case of highly asymmetriccommunication protocols such as ADSL. Alternatively, if some nodesshare a common physical or logical bus, the bus can be modeled by ahyper-edge, and hence the whole network by a hyper-graph. This might bethe case of networks that use CSMA/CD-like protocols such as Ethernet.When a network is modeled by a graph structure, many of the problems arising in the context of networks, specially those which depend on the network topology can be also translated to graph theoryYet in the early 50's graph theory was used to model telephonynetworks, and from the very beginning interesting results came fromthese studies like, for instance, the studies for non-blockingmultistage switching networks (the Clos Network, [20]). Since then lots of studies have been done in graph theory when consideringgraphs as topological models for different kinds of networks: Design of dense networks [12, 16], traffic congestion (forwarding index) [19, 34], broadcasting algorithms and dissemination (gossiping) of information [35], fault tolerance (surviving route graph [26], connectivity [30]),...At a first sight one might thinkthat, from the graph theoretic point of view, there is not anydifference between optical networks and generic interconnectionnetworks, since graph theory does not care about what are the wires madeof. But besides the general results and models for interconnectionnetworks that can be also useful in our context, new specific problemsappeared in the study of all optical networks. These problems are notexclusive of this field but have now a particular interest.In particular, we are not concerned with classicalnetworks in which some electrical wires are replaced by optical fibersand at each endpoint there is a electric-to-optical shift, but onnetworks in which switching is also done using lightwave technology.Some of the properties of all-optical networks, that lead to newproblems in Graph theory are the following:

- While electrical buffering is technologically easy to implement, this is not the case of optical systems. Routing must be done without storing the information in the routers and packets cannot wait to be delivered through a link without optical to electrical conversion [29]. Fast routing algorithms must be realized (such as MPLS [2, 3]) and, in case of network congestion, deflection routing [17, 41] could be a good solution.

- New routing constraints appear when using WDM, different from those in TDM. An example of problems arising from WDM will be given in the next section.

- New models for networks are being developed due to the new devices in optical technology. In this context there have appeared many studies on bus networks [11, 25] in which the network topology is no more modeled by a graph but by a hyper-graph.

It may also happen that old solved problems in graph theory become solutions for new problems in the optical network context. For instance, it is shown in [24] that networks based on the Optical Transpose Interconnection System (OTIS) architecture [39] have a topology which is highly related with the very well known families of directed graphs called Kautz and De Bruijn graphs. Those families have been proposed many times as topologies for interconnection networks due to their good properties.

2.2 Competition for resources

Modeling of network topologies is not the only way graphs can be used in interconnection networks. Another typical way of using graphs is to model the competition for resources.Indeed graphs can easily model any situation in which binaryrelationships outcome like for instance the competition of agentstowards the same resources: vertices of the graph represent thedifferent agents, and two vertices are adjacent if they want to usethe same resource. In order to compute the minimum number of resourcesneeded one can use the resulting "incompatibility" graph. In fact,this problem turns to be a standard problem in graph theory calledvertex coloring: i.e. assign colors to the vertices of the graph suchthat not two adjacent vertices must have the same color [6]For the sake of illustration, let us consider WDM optical networks consisting ofrouting nodes interconnected by point-to-point fiber-optic links whichcan support a certain number of wavelengths. Switching will besupposed to be also optical, and packets are supposed not to change ofwavelength at the intermediate nodes. When a communication must beestablished between to nodes of the network a virtual path is reservedand a wavelength is assigned to that transmission.An important restriction in this model is that two different virtualpaths sharing at least one common link must have differentwavelengths, as if they did not, it would not be possible todistinguish both signals at the next intermediate node.To compute the minimum number of wavelengths required to establish aset of virtual paths we define the *Conflict Graph* the followingway: Vertices of the graph stand for the different virtual paths andtwo vertices will be adjacent if the corresponding virtual path shareat least one link.The wavelength assignment problem can besolved in the Conflict Graph by assigning colors (wavelengths) to itsvertices in such a way that adjacent vertices have different colors[4].One can object that this *a priori* planing is not realistic, andit is true. But this *static* and simple setting of the problemcan be improved by using more realistic dynamic approximations based on on-line algorithms [1].On the other hand, even being a simple model it gives some insight on the problem. For instance it is interesting to point out a question with a (maybe) surprising answer:The recent Dense WDM (DWDM) allows to use a considerable number ofwavelengths in the same fiber. Is it equivalent to use a single fiberwith DWDM that using multiple fibers with conventional WDM to join twonodes in the network (supposing that both solutions allow the sametotal number of wavelengths per link)?Many factors should be considered to decide which solution is better.But some Graph theory background can help to realize that bothsolutions are not topologically equivalent [15]. In fact, itis easy to see that the second alternative turns out to be moreefficient from the point of view of the usage of the resources(wavelengths), as shown in the following example:Suppose a simple network consisting on four nodes connected in a startopology as shown in Figure 1 and imagine we want toestablish three simultaneous communications between nodes 1–2, 2–3and 1–3. The goal will be assigning wavelengths to these transmissionrequirements using the minimum number of wavelengths. If links consist on two fibers each, the requirement can be satisfiedby using one wavelength per fiber, while two wavelengths per fiber arenot enough to solve the assignment with a single fiber per link. Thiscan be easily seen in terms of graph colorings, since the conflictgraph in this former case is a triangle, and vertices in a trianglecannot be colored with two colors.

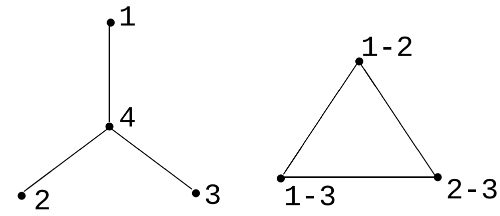

Figure 1: A network and the conflict graph for 3 requirements.

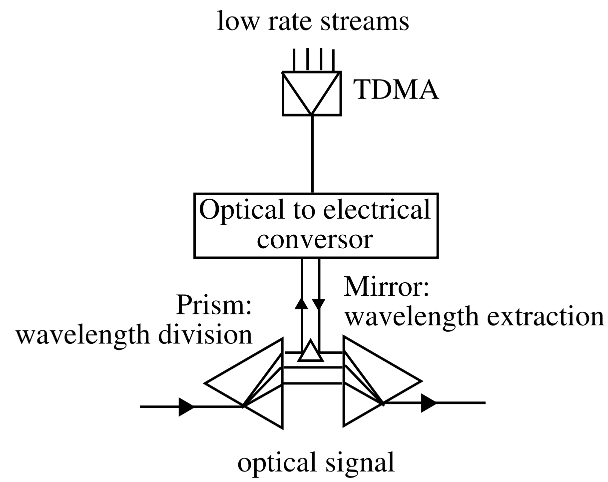

Figure 2: Block scheme of an Add Drop Multiplexer

3 Traffic Grooming in WDM rings

We show in this section a more detailed example of how graph theorycan help engineering in optical networks. Besides being an interesting example, the results shown in this section are a survey of recent results in optimization of traffic grooming.

3.1 Definition of the problem

Many of the nowadays network infrastructures are based on the synchronous optical network (SONET). A SONET ring typically consists on a set of nodes connected by an optical fiber in a unidirectional ring topology. Nodes of the network insert and/or extract the data streams on a wavelength by means of an add drop multiplexer (ADM).

A WDM or DWDM optical network can handle many wavelengths, each of them with a huge bandwidth available. On the other hand a single user seldom needs such large bandwidth. Therefore, by using multiplexed access such as TDMA or CDMA, different users can share the same wavelength optimizing the bandwidth usage of the network. Traffic grooming is the generic term for packing low rate signals intohigher speed streams (see the surveys [28, 40, 42]). By using traffic grooming, not only the bandwidth usage is optimized but also the cost of the network can be reduced by avoiding the total amount of ADMs used in the network:If traffic grooming is used, one node may use or may not use the same wavelength (and therefore the same ADM device) in the communication with several nodes. Depending on these choices the total amount of ADMs required in the network may be reduced. Let us recall that the problem of minimizing the number of ADMs is different from that of minimizing the number of wavelengths. Indeed, it is known that even for the simpler network which is the unidirectional-ring, the number of wavelengths and the number of ADMs cannot always besimultaneously minimized (see [31], or [18] for uniform traffic) though in many cases both parameters can be minimized simultaneously.Both minimization problems have been considered by many authors. See surveys [5, 27] for minimization of the number of wavelengths and [31, 32, 36, 43, 46] for minimization of ADMs. Also and numerical results, heuristics and tables have been given (see for example those in [44]). Let us consider the particular case of unidirectional rings (therouting is unique) with static uniform symmetric all-to-all traffic(there is exactly one request of a given size from i to j for eachcouple (i, j)) and with no possible wavelength conversion.Given a pair of nodes, $\{i, j\}$, let us associate a circle containing both the request from i to j and from $j$to i. We will assume that both requests use the same wavelength.In the conditions of uniform symmetric traffic in an unidirectional ring, this assumption is not an important restriction and it will allow us to focus on the grooming phase with independence of the routing. Notice that a circle is a reservation of a fraction of the Bandwidth in the whole ring network corresponding to a communication between two nodes. (It is also possible to consider more general classes other than circles containing two symmetric requests packed into the same wavelength. These components are called circles [18, 46], circuits [44] or primitive rings [22, 23].)If each circle requires only $\frac{1}{C}$ of the bandwidth ofa wavelength, we can "groom" C circles on the same wavelength. $C$is called the *grooming ratio* (or grooming factor). Forexample, if the request from i to j (and from j to i) is packed in an OC-12 and a wavelength can carry up to an OC-48, the grooming factor is 4. Given the grooming ratio C and the size N of the ring, theobjective is to minimize the total number of (SONET) ADMs used,denoted $A(C, N)$, which will imply reducing the network cost by eliminating asmany ADMs as possible compared to the "no grooming case". For example, let $N = 4$; we have 6 circles corresponding to the 6 pairs$\{0, 1\}$, $\{1, 2\}$, $\{2, 3\}$, $\{0, 2\}$, $\{0, 3\}$, $\{1, 3\}$.If we don't use grooming, that is if we assign one wavelength percircle, we need 2 ADMs per circle, and thus a total of 12. Supposenow that $C = 3$, that is we can groom 3 circles on one wavelength. A possible solutions is to groom the circles associated with $\{0, 1\}, \{1, 2\}, \{2, 3\}$ on wavelength 1 (requiring 4 ADMs) and the circlesassociated with $\{0, 2\}, \{0, 3\}, \{1, 3\}$ on wavelength 2 (requiring 4 ADMs).The total number of ADMs in the network given by this arrangement is 8. Notice that a better solution consists in grooming the circles associated with$\{0, 1\}, \{1, 2\}, \{0, 2\}$ on wavelength 1 (using 3 ADMs) and the circles associated with $\{0, 3\}, \{1, 3\}, \{2, 3\}$ on wavelength 2 for a total number of 7 ADMs.

3.2 Mathematical translation of the problem

The problem of minimizing the number of ADMs in a unidirectional ring can be modeled by graphs as it is shown in [9].In order to show this translation, let us first set up the notation that will be used in this paper:

- As mentioned above, we will restrict to the case of unidirectional rings with static uniform symmetric all-to-all traffic with no possible wavelength conversion (some of the ideas can be applied to other situations though).

- N will denote the number of nodes of the unidirectional ring $\overrightarrow{C}_N$

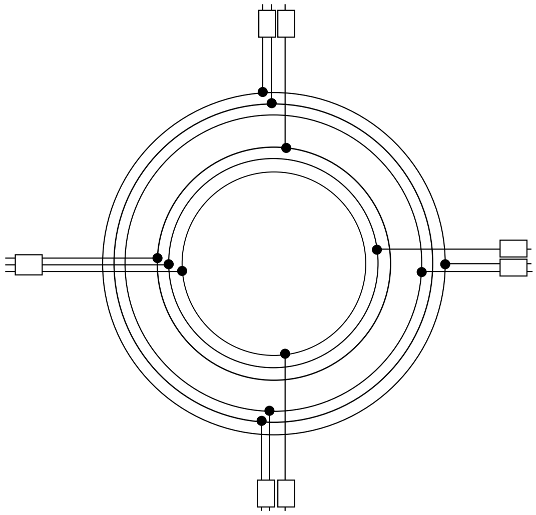

Figure 3: An optimized distibution of ADMs in a ring network with 4 nodes. Every wavelength is represented by a set of three circles corresponding to groomed signals

- $C_{\{i,j\}}$ will denote a *circle* associated to the pair $\{i,j\}$, i.e. containing both a unitary request from i to j and from j to i. Recall that $C_{\{i,j\}}$ uses all the arcs of $\overrightarrow{C}_N$.
- R is the total number of circles. In the case of unidirectional rings, with uniform unitary traffic, each pair $\{i,j\}$ is associated to a unique circle $C_{\{i,j\}}$ and thus $R = \frac{N(N-1)}{2}$.
- C is the grooming ratio (or grooming factor). Let us recall that C is also the number of circles a wavelength can contain [18]. Similarly, $\frac{1}{C}$ indicates the part of the bandwidth of a wavelength that can be used by a circle. For example, if a wavelength is running at the line rate of OC-N, it can carry $C = \frac{N}{M}$ low speed OC-M. Typical values of C are $3, 4, 8, 12, 16, 48$ and 64.
- Let $A(C,N)$ be the minimum number of ADMs required in a ring $\overrightarrow{C}_N$ with grooming ratio C.

Given a a ring $\overrightarrow{C}_N$ with grooming ratio C let us consider the complete graph K_N, i.e. a graph with N vertices in which there is an edge (i,j) for every pair of vertices i and j. Notice that the number of edges of K_N equals the number of circles $R = \frac{N(N-1)}{2}$. Moreover there is a one–to–one mapping between the circles of $\overrightarrow{C}_N$, $C_{\{i,j\}}$, and the edges of K_N, (i,j).Let $\mathcal{S}$ be a an assignment of wavelengths and time slots for all the requirements among all possible pairs of nodes requiring A ADMs. Let B_λ be a subgraph of K_N representing the usage of a given wavelength λ in the assignment $\mathcal{S}$. Precisely, let the edges in $E(B_\lambda)$ correspond to the circles $C_{\{i,j\}}$ groomed onto the wavelength λ, and let the vertices in $V(B_\lambda)$ correspond to the nodes of $\overrightarrow{C}_N$ using wavelength λ. Notice that the number of vertices of B_λ, $|V(B_\lambda)|$ is the number of nodes using wavelength λ or, alternatively, the number of ADMs required for wavelength λ. Notice also that the total number of edges of B_λ, $E(B_\lambda)$ is at most the grooming ratio C.Realize also of the existence of the following one–to–one correspondences in this interpretation:

- The edges of K_N correspond to the circles $C_{\{i,j\}}$.
- Each subgraph B_λ corresponds to a wavelengths.
- Each vertex of each B_λ corresponds to an ADM.

With these correspondences it is immediate to see that the original problem of finding the minimum number of ADMs, $A(C,N)$, required in a ring $\overrightarrow{C}_N$ with grooming ratio C is equivalent to the following problem in graphs:

Problem 3.1

Input: *A number of nodes N and a grooming ratio C*

Output: *A partition of the edges of K_N into subgraphs B_λ, $\lambda = 1,\dots,W$, such that $|E(B_\lambda)| \leq C$*

Objective: *To minimize $\sum_{1\leq\lambda\leq W} |V(B_\lambda)|$*

For the sake of illustration, Figure 4 depicts two possible grooming arrangements proposed for$N = 4$ and $C = 3$ described in Section 3.1. The second solution corresponds to Figure 3.

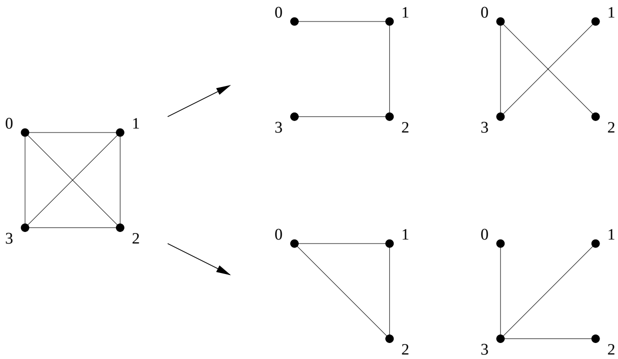

Figure 4: 2 partitions of K_4 with grooming ratio $C = 3$.

3.3 Solving the problem

Once the problem has been formulated in terms of graphs it is understandable for non experts in optical networks, and it can be explainedwith no need of introducing technical aspects (WDM, ADMs,...) that are not required to look for a solution.

Problem 3.1 can be easily solved for given values of C and N by some mathematical method like integer linear programming (see Section 3.3.1).

It is possible that an engineer feels satisfied at this point and he may think that the problem is solved as soon as the computer gives a table of results like Table 1.

Numerical results are important of course, and they were maybe the main reason to study this problem. But they are not the only useful results that can be expected from mathematics. Indeed, deeper mathematics will give some insight in the problem while numerical methods will seldom do.

Moreover like many combinatorial problems, Problem 3.1 has non polynomial computational complexity [18]. This implies that the numerical results that a computer may obtain are limited to small values of N even with a large computation power.

Heuristics and approximated methods should be used for larger values of N. But in order to evaluate the goodness of an heuristic method it should not only be compared with other existing methods ("best effort" engineering), but also with some upper and lower bounds or even with the real solution when possible.

Various heuristics have been developed mainly based on greedy algorithms, simulated annealing or genetic algorithms [18, 31, 44]. Most of them have obtained very good results even without good knowledge of the gap with optimality. For instance in [44] it is shown that $21 \leq A(48, 14) \leq 24$ while according to Theorem 3.8, we have $A(48, 14) = 24$.

At this point graph theory can help engineering and in fact, the problem we are dealing with is a particular instance of graph decomposition problems, which are classical difficult problems in graph theory (under different assumptions and restrictions).

In spite of the difficulty of the problem simple graph theoretic ideas may give bounds and solutions that wouldn't be easy to discover without translating the problem. See for instance, that the results in Section 3.3.2come from very easy concepts in graph theory.In addition, already known results in graph theory can directly be used in the problem [13, 14], and in any case, there exist lots of techniques that will help to solve it. See for instance the design theory tools used in Section 3.3.3.

3.3.1 ILP formulation

The problem described in the previous section can be easily formulated in terms of integer linearprogramming (ILP) which may be solved using CPLEX:Let $e^l_{i,j} = 1$ whenever the subgraph B_l contains the edge $\{i, j\}$, and0 otherwise, and let $n^l_i = 1$ if $i \in V(B_l)$. The objective is:

$$Minimize \sum_i \sum_l n^l_i$$

subject to the restrictions

$$\forall \{i, j\} \in V, \quad \sum_l e^l_{i,j} \quad \geq \quad 1$$

and

$$\begin{array}{lrcl} \forall l, & e^l_{i,j} & \leq & n^l_i \\ & e^l_{i,j} & \leq & n^l_j \\ & \sum_{\{i,j\}\in V} e^l_{i,j} & \leq & \min\left\{C, \frac{|V_l|(|V_l|-1)}{2}\right\} \end{array}$$

This simple formulation might be improved a lot by adding some other constraints and therefore reducing the search space. But even with a careful selection of the constrains it is difficult to get results for values of N larger than 10. An interested reader can find another recently proposed ILP formulation of this problem in [37].

Table 1 shows some values of $A(C,N)$ for $N \leq 16$ and small C, like a table in [44].

N/C	3	4	12	16	48	64
3	3	3	3	3	3	3
4	7	7	4	4	4	4
5	12	10	5	5	5	5
6	17	15	9	6	6	6
7	21	21	12	11	7	7
8	31	28	16	14	8	8
9	36	36	18	18	9	9
10	48	45	24	20	10	10
11	57	55	30	26	16	11
12	69	66	35	32	19	15
13	78	78	39	36	22	19
14	95	91	47	41	24	22
15	105	105	55-56	45	30	25
16	124	120	60	53-54	32	28

Table 1: Value of $A(C,N)$ for $N \leq 16$ and $C = 3, 4, 12, 16, 48, 64$

3.3.2 Lower bounds

In [9] it is shown the importance of choosing graphs B_λ in the partition of K_N with the highest ratio$\rho(B_\lambda) = \frac{|E(B_\lambda)|}{|V(B_\lambda)|}$ as possible.Indeed, if $\rho_{\max}(C)$ denotes the maximum ratio among all possible graphs with at most C edges, i.e., $\rho_{\max}(C) = \max\{\rho(B_\lambda) \mid |E(B_\lambda)| \leq C\}$, then the following theorem holds:

Theorem 3.2 (Lower Bound [9]) *The number of ADMs required in a unidirectional ring with N nodes and grooming ratio C is lower bounded by the expression*

$$A(C,N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}.$$

Proof: Let us consider a partition of K_N into subgraphs $B_1, \ldots, B_W$ that minimizes $\sum_{1\leq\lambda\leq W} |V(B_\lambda)|$.The following inequalities hold:

$$A(C,N) = \sum_{\lambda=1}^{W} |V(B_\lambda)| = \sum_{\lambda=1}^{W} \frac{|E(B_\lambda)|}{\rho(B_\lambda)} \geq$$

Table 2: Values of $\rho_{\max}(C)$ for small C

C	1	2	3	4	5	6	7	8	9	10
$\rho_{\max}$	$\frac{1}{2}$	$\frac{2}{3}$	1	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{9}{5}$	2
C	11	12	13	14	15	16	24	32	48	64
$\rho_{\max}$	2	2	$\frac{13}{6}$	$\frac{14}{6}$	$\frac{5}{2}$	$\frac{5}{2}$	3	$\frac{32}{9}$	$\frac{9}{2}$	$\frac{64}{11}$

$$\geq \frac{1}{\rho_{\max}(C)} \sum_{\lambda=1}^{W} |E(B_\lambda)| = \frac{R}{\rho_{\max}(C)}$$

□

The value for $\rho_{\max}(C)$ was also determined in [9], and it is given by the following proposition:

Proposition 3.3 ([9]) *Let $k \in \mathbb{N}$ be an integer such that $\frac{k(k-1)}{2} \leq C < \frac{(k+1)k}{2}$. Then, the maximum ratio $|E|/|V|$ among all possible graphs with at most C edges is given by the expression*

$$\rho_{\max}(C) = \begin{cases} \frac{k-1}{2} & \text{if} C \leq \frac{(k+1)(k-1)}{2} \\ \frac{C}{k+1} & \text{if} C > \frac{(k+1)(k-1)}{2} \end{cases}$$

Moreover, the value of $\rho_{\max}(C)$ is attained by the following graphs:

- *The complete graph on k vertices, K_k if $C \leq \frac{(k+1)(k-1)}{2}$.*
- *Any graph with C edges and $k+1$ vertices whenever $C > \frac{(k+1)(k-1)}{2}$.*

For the sake of illustration, Table 2 shows the valuesof $\rho_{\max}(C)$ for small values of C, also plotted in Figure 5.

Theorem 3.2 seems like the minimum number of ADMs will be achieved when choosing subgraphs such that the average ratio is maximized, or roughly speaking, when choosing subgraphs with a ratio equal to $\rho_{\max}(C)$ whenever it is possible (although this last sentence is not strictly true). Note that according to Proposition 3.3, these subgraphs need not to have exactly C edges.Therefore it might happen that in some cases the minimum could not be attainable for $W = W_{\min}$ (minimum number of subgraphs).Indeed, it was proved in [9] that the minimum number of ADMs, $A(C,N)$, for unidirectional

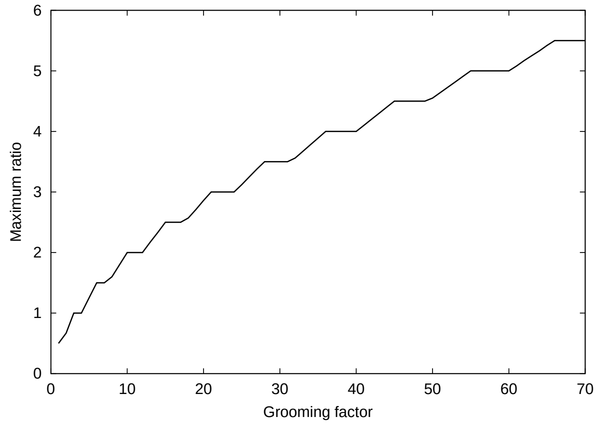

Figure 5: Values of $\rho_{\max}(C)$

rings with uniform unitary traffic is not necessarily obtained using the minimum number of wavelengths, disproving conjectures of[18] for many values of C (the first one being $C = 7$) and of [36] for $C = 16$.For example, let $C = 7$. If a subgraph has 7 edges, its ratio is atmost $\frac{7}{5} = 1.4$. But a subgraph with 6 edges can have a ratio$\frac{6}{4} = 1.5$ (and it is attained by the complete graph on 4 vertices, K_4). Any other subgraphhas a ratio at most $\frac{5}{4}$. Therefore, an optimal solution for thenumber of ADMs uses K_4 and not subgraphswith 7 edges and 5 vertices. A solution minimizing the numberof wavelengths, uses graphs with 7 edges instead.For a numerical example, let $N = 13$. In Section 3.3.3 it is explained that $A(6, 13) = 52$ (Table 3). Moreover K_{13} can be decomposed into13 complete graphs with 4 vertices each. Namely, if vertices are labeled with numbers $0, \ldots, 12$, the reader can check that the cliques induced by vertices $\{i, i+1, i+4, i+6\}$ perform a perfect decomposition(additions are taken modulo 13). On the other hand, a construction using as few subgraphs (wavelengths) as possible ($\lceil \frac{78}{7} \rceil = 12$) needs at least 54 ADMs.A more careful analysis shows that the lower bound given in Theorem 3.2 is not always attainable, and it is possible to improve it for given values of N and C.

3.3.3 Upper bounds and optimal results

An upper bound is usually obtained by giving a construction that needs a certain number of ADMs. That number will be, for sure, an upper bound for the value of $A(C, N)$.

Results from Design theory A first approach to the problem is to seek for those cases in which the decomposition can be performed by isomorphic graphs, i.e. all the subgraphs in which we decompose K_N look the same. Such a decomposition is a classical problem in graph theory and also in design theory. Namely, a G-design of order N (see [21] chap. 22 or [13] or [14]) consists on a partition of the edges of K_N into subgraphs isomorphic to a given graph G. Due to Theorem 3.2 and given the definition of a G-design, the next result is immediate:

Proposition 3.4 *If there exists a G-design of order N, where G is a graph with at most C edges and ratio $\rho_{\max}(C)$, then*

$$A(C, N) = \frac{N(N-1)}{2\rho_{\max}(C)}.$$

It is only a matter of looking in the existing bibliography to check the conditions for the existence of a G–design. Some basic necessary conditions are given by the following result:

Proposition 3.5 (Existence of a G-design) *If there exists a G-design, then*

(i) $\frac{N(N-1)}{2}$ *should be a multiple of* $|E(G)|$

(ii) $N-1$ *should be a multiple of the greatest common divisor of the degrees of the vertices of G.*

Wilson showed in [45] that those necessary conditions are alsosufficient if N is large enough. This result implies that the lower bound given in Theorem 3.2 is attained for infinite values of N, and thus it is a tight bound:

Theorem 3.6 *Given C, for an infinite number of values of N, $A(C, N) = \frac{N(N-1)}{2\rho_{\max}(C)}$.*

Unfortunately, the values of N for which Wilson's Theorem[45] applies are very large. However, for small values of C,it is possible to use other exact results from design theory. Precisely, Table 3 shows the value of $A(C, N)$ for some values of N, obtained from the existence of G-designs for small graphs with at most C edges and with ratio $\rho_{\max}(C)$.

Optimal results for large values of C The non existence of G–designs for some values of C and N implies that K_N cannot be optimally decomposed by using isomorphic copies of the same subgraph. This lack of regularity in the decomposition makes it harder to find optimal decompositions, and thus to find the value of $A(C,N)$. Even more, the solution maybe very different for different values of C and N. However, if C is large compared to N, it is not difficult to find the value of $A(C,N)$. Indeed, if $C \geq R = \frac{N(N-1)}{2}$, i.e. when the number of requests that can be groomed in one wavelength (C) is larger than or equal to the total number of requests (R), all requests can be groomed in one wavelength, and one ADM per node is enough. In other words $A(C,N) = N$ whenever $C \geq R$.Besides this trivial case, some results can easily be obtained for large values of C. In order to show the kind of reasonings that can be used in this problem, let us consider the case $\frac{R}{2} \leq C < R$: If $A(C,N)$ is supposed to be less than $2N$ when $\frac{R}{2} \leq C < R$ (we will prove that), there must be vertices that appear in only one subgraph (if all vertices appear in at least two subgraphs, the total sum of vertices would be larger than or equal to $2N$. Therefore at least one of the subgraphs in the decomposition must have exactly N vertices (to cover all edges incident to the vertices appearing only in that subgraph). On the other hand at least two subgraphs are needed to perform such a decomposition. According to the previous paragraph, one of the subgraphs has N vertices and at most C edges. The remaining subgraph (or subgraphs) in the decomposition, will contain at least $R - C$ edges. Since the number of vertices of a simple graph (connected or not) with m edges is at least $\varphi(m) = \min\left\{k \mid \frac{k(k-1)}{2} \geq m\right\}$, that is $\varphi(m) = \left\lceil \frac{1+\sqrt{1+8m}}{2} \right\rceil$, the number of ADMs will be at least $N+\varphi(R-C)$.Finally decomposition of K_N using such number of ADMs is trivially obtained by taking, as one subgraph, $R - C$ edges from a complete subgraph with $\varphi(R-C)$ vertices; and another subgraph with N vertices and the remaining edges.

These paragraphs are in fact a proof for the following Theorem:

Theorem 3.7 ([10]) *Let $R = N(N-1)/2$ and let $\varphi(m) = \left\lceil \frac{1+\sqrt{1+8m}}{2} \right\rceil$. For all $R/2 \leq C < R$ the number of ADMs required in a unidirectional ring with N nodes and grooming ratio C, is given by the expression*

$$A(C,N) = N + \varphi(R-C)$$

Following the same ideas as above but with a more sophisticated analysis it is possible to find the minimum number of ADMs for smaller values of C, as shown in the next theorem:

Theorem 3.8 ([10]) *Let $R = N(N-1)/2$ and let $\varphi(m) = \left\lceil \frac{1+\sqrt{1+8m}}{2} \right\rceil$. The number of ADMs required in a unidirectional ring with N nodes and grooming ratio C, for $R/3 \leq C < R/2$, is given by*

$$\min \begin{cases} 2N, \\ N + \varphi(C) + \varphi(R-2C), \\ \varphi\left(R - C - \frac{(\varphi(C)-1)(\varphi(C)-2)}{2}\right) + \\ \quad +N + \varphi(C) - 1. \end{cases}$$

except for $N = 4$ and $C = 2$, and for $N = 7$ and $C = 7$. In those cases, it holds $A(2,4) = 9$ and $A(7,7) = 15$

Table 3: Values for $A(C,N)$ obtained from the existence of G–designs

C	N	$A(C,N)/R$
3	1 or 3 (mod 6)	1
4	0 or 1 (mod 8)	1
5	0 or 1 (mod 10)	4/5
6, 7	1 or 4 (mod 12)	2/3
8	0 or 1 (mod 16)	5/8
9	0 or 1 (mod 18)	5/9
10	1 or 5 (mod 20)	1/2
16	1 (mod 30)	2/5

Optimal results for given values of C Given a fixed value of C the problem has ben solved in the previous section for small values of N. The technique might be pushed till maybe $C \geq R/5$ but will not be useful for large values of N. For larger values of N there might be some cases for which there exists a G–design, but for other cases K_N cannot be optimally decomposed by using isomorphic copies of the same subgraph. An Optimal decomposition will contain different kind of subgraphs instead. Even more, the solution maybe very different for different values of C and N and Proposition 3.5 suggests that the solutions will depend on some congruence relationship on N.

Moreover, also the lower bounds are needed to be more carefully recomputed, using different reasonings for different values of N.

Theorem 3.9 ([7]) *The number of ADMs required in a unidirectional ring with N nodes and grooming ratio $C = 3$ is given by the expression*

$$A(3, N) = \frac{N(N-1)}{2} + \epsilon_3(N)$$

where $\epsilon_3(N)$ is a function of N given by

$$\epsilon_3(N) = \begin{cases} 0 & \textit{if } N \equiv 1, 3 \ (\textit{mod } 6) \\ 2 & \textit{if } N \equiv 5 \ (\textit{mod } 6) \\ \lceil N/4 \rceil + 1 & \textit{if } N \equiv 8 \ (\textit{mod } 12) \\ \lceil N/4 \rceil & \textit{otherwise} \end{cases}$$

The proof uses techniques inspired in design theory. In the even case,the optimal solutions use a lot of K_3's and some $K_{1,3}$ (complete bipartite graphs)or P_4 (paths). When the degree of K_N is odd, subgraphs with odd degree must be used. For example, if $N \equiv 0$ or 4 (mod 12), the optimal solution consists of $\frac{N(N-1)}{6} - \frac{N}{4}$ K_3'sand $\frac{N}{4}$ $K_{1,3}$.The following theorem has been first demonstrate in [36] and ashorter proof using design theory is given in [10]. Bothproofs consist on a partition of the edges of K_N into C_4 (cycles),$K_3 + e$ (complete graph plus one vertex joined by an edge) and K_3.

Theorem 3.10 ([36]) *The number of ADMs required in a unidirectional ring with N nodes and grooming ratio $C = 4$ is given by the expression $A(4, N) = N(N-1)/2$.*

When $C = 5$, Theorem 3.11 gives a partition of K_N into$K_4 - e$ (complete graph minus one edge) plus some K_2, K_3 and C_4.

Theorem 3.11 ([8]) *The number of ADMs required in a unidirectional ring with N nodes and grooming ratio $C = 5$ is given by the expression*

$$A(5, N) = 4 \left\lfloor \frac{N(N-1)}{10} \right\rfloor + \epsilon_5(N)$$

with

$$\epsilon_5(N) = \begin{cases} 0 & \textit{if } N \equiv 0, 1 \ (\textit{mod } 5), N \neq 5 \\ \textit{and } 1 & \textit{if } N = 5 \\ 2 & \textit{if } N \equiv 2, 4 \ (\textit{mod } 5), N \neq 7 \\ 3 & \textit{if } N = 7 \\ 3 & \textit{if } N \equiv 3 \ (\textit{mod } 5), N \neq 8 \\ 4 & \textit{if } N = 8 \end{cases}$$

Note that the partitions given in the proofs of Theorems 3.9,3.10 and 3.11 use the minimum number of wavelengths,which is not the case for the following result:

Proposition 3.12 ([10]) *The number of ADMs required in a unidirectional ring with N nodes and grooming ratio $C = 12$ and $N = 4h + 1$ is given by the expression*

$$A(12, 4h + 1) = (4h + 1)h.$$

This proposition gives a partition of K_{4h+1} into K_5´s and$K_{2,2,2}$´s, both having the maximum ratio 2. Therefore it has also been shown that the lower bound given by Theorem 3.2 can be attained even if the decomposition cannot be performed with isomorphic copies of the same subgraph (G–designs).By the date of submission of this paper, the optimal value of$A(C, N)$ for all values of N is still an open problem for other values of C, $C \leq N(N-1)/6$.

3.4 Other models

In Section 3, we have presented the problem of traffic grooming inunidirectional WDM rings with uniform unitary traffic. We have alsogiven the optimal solution for various values of N and C. We haveshown how to use graph theory and design tools to solve the optimallythe problem for practical values and infinite congruence classes ofvalues for a given C. The tools can be easily extended to uniformbut non unitary traffic. Indeed, if we have a request of size r

from$i$ to j, it suffices to consider decomposition of the edges of thecomplete multipartite graph rK_N. We can also extend the ideas tothe case of arbitrary traffic, but it requires to partition generalgraphs and this is known to be a difficult problem in graph theory(see [33] for an approximation algorithm). We can alsoconsider networks different from the unidirectional ring, if we arefirst able to group the requests into circles (that is the way used in[22, 23] for bidirectional rings). Finally, the tools canalso be used to groom traffic in a slightly different context, forexample when the traffic is expressed in terms of STM-1 (eachone needed one wavelength) and we grouped them into bands or fibers,typically a fiber containing 8 bands of 4 wavelengths (see[38]).

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