Traffic Grooming in Unidirectional Wavelength-Division Multiplexed Rings with Grooming Ratio C = 6

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Abstract

SONET/WDM networks using wavelength add-drop multiplexing can be constructed using certain graph decompositions used to form a grooming, consisting of unions of primitive rings. The cost of such a decomposition is the sum, over all graphs in the decomposition, of the number of vertices of nonzero degree in the graph. The existence of such decompositions with minimum cost, when every pair of sites employs no more than $\frac{1}{6}$ of the wavelength capacity, is determined with a finite number of possible exceptions. Indeed, when the number N of sites satisfies $N \equiv 1 \pmod 3$, the determination is complete, and when $N \equiv 2 \pmod 3$, the only value left undetermined is $N = 17$. When $N \equiv 0 \pmod 3$, a finite number of values of N remain, the largest being $N = 2580$. The techniques developed rely heavily on tools from combinatorial design theory.

Keywords: traffic grooming, combinatorial designs, block designs, group-divisible designs, optical networks, wavelength-division multiplexing

1 Traffic grooming in wavelength-division multiplexed rings

Many current network infrastructures are based on the synchronous optical network (SONET). A SONET ring typically consists of a set of nodes connected by an optical fiber in a unidirectional ring topology. Nodes of the network insert and/or extract the data streams on a wavelength by means of an add drop multiplexer (ADM). A *wavelength-division multiplexed* (WDM) or *dense WDM* (DWDM) optical network can handle many wavelengths, each with large bandwidth available. On the other hand, a single user seldom needs such large bandwidth. Therefore, by using multiplexed access such as time-division multiple access (TDMA) or code-division multiple access (CDMA), different users can share the same wavelength, thereby optimizing the bandwidth usage of the network.

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Traffic grooming is the generic term for packing low rate signals into higher speed streams (see [17, 32, 34]). By using traffic grooming, not only is the bandwidth usage optimized, but also the cost of the network can be reduced by lessening the total number of ADMs. If traffic grooming is used, one node may or may not use the same wavelength (and therefore the same ADM device) in the communication
with several nodes. Depending on these choices the total number of ADMs in the network may be reduced. Minimizing the number of ADMs is different from minimizing the number of wavelengths. Indeed, even for the unidirectional ring, the number of wavelengths and the number of ADMs cannot always be simultaneously minimized (see [11, 25] for uniform traffic), although in many cases both parameters can be minimized simultaneously. Both minimization problems have been considered by many authors. See [1, 15] for minimization of the number of wavelengths and [25, 26, 28, 36, 40] for minimization of ADMs. Numerical results, heuristics, and tables have also been given (see, for example, [37]). We consider the particular case of unidirectional rings, so that the routing is unique. There is static uniform symmetric all-to-all traffic, i.e., there is exactly one request of a given size from i to j for each pair (i, j), and no wavelength conversion. With a pair of nodes, $\{i, j\}$, is associated a circle, $C_{\{i,j\}}$, containing both the request from i to j and from j to i. We assume that both requests use the same wavelength. For uniform symmetric traffic in an unidirectional ring, this assumption is not an important restriction and it allows us to focus on the grooming phase independent of the routing. A circle is then a reservation of a fraction of the bandwidth in the whole ring network corresponding to a communication between two nodes. (It is also possible to consider more general classes other than circles containing two symmetric requests packed into the same wavelength. These components are known as circles [11, 40], circuits [37], or primitive rings [13, 14].) If each circle requires only $\frac{1}{C}$ of the bandwidth of a wavelength, we can groom C circles on the same wavelength. C is the *grooming ratio* (or grooming factor). For example, if the request from i to j (and from j to i) is packed in an OC-12 and a wavelength can carry up to an OC-48, the grooming factor is 4. Given the grooming ratio C and the size N of the ring, the objective is to minimize the total number of (SONET) ADMs used, denoted $A(C, N)$. This lowers the network cost by eliminating as many ADMs as possible compared to the no-grooming case.

The problem of minimizing the number of ADMs in a unidirectional ring with uniform traffic can be modeled by graphs, as shown in [5]. Given a unidirectional SONET ring with N nodes, $\overrightarrow{C}_N$, and grooming ratio C, consider the complete graph K_N, i.e., the graph with N vertices in which there is an edge (i, j) for every pair of vertices i and j. The number of edges of K_N equals the number of circles $R = \frac{N(N-1)}{2}$. Moreover, there is a one-to-one mapping between the circles of $\overrightarrow{C}_N$, $C_{\{i,j\}}$ and the edges of K_N, (i, j). Let $\mathcal{S}$ be an assignment of wavelengths and time slots for all requirements among all possible pairs of nodes requiring A ADMs. Let B_ℓ be a subgraph of K_N representing the usage of a given wavelength ℓ in the assignment $\mathcal{S}$. To be precise, let the edges in $E(B_\ell)$ correspond to the circles $C_{\{i,j\}}$ groomed onto the wavelength ℓ, and let the vertices in $V(B_\ell)$ correspond to the nodes of $\overrightarrow{C}_N$ using wavelength ℓ. The number of vertices of B_ℓ, $|V(B_\ell)|$ is the number of nodes using wavelength ℓ or, alternatively, the number of ADMs required for wavelength ℓ. Evidently the total number of edges of B_ℓ, $E(B_\ell)$ is at most the grooming ratio C. With these correspondences the original problem of finding the minimum number of ADMs, $A(C, N)$, required in a ring $\overrightarrow{C}_N$ with grooming ratio C, is equivalent to the following problem in graphs.

Problem 1.1 *Given a number of nodes N and a grooming ratio C, find a partition of the edges of K_N into subgraphs B_ℓ, $\ell = 1, \dots, W$, with $|E(B_\ell)| \le C$ such that $\sum_{1\le\ell\le W} |V(B_\ell)|$ is minimum.*

In this paper we develop techniques for solving the unidirectional wavelength assignment when the grooming ratio is 6. We determine the exact values of $A(6,N)$ for all values of N except for a finite number of cases.

The paper is organized as follows. In section 2 we introduce some notation and previous results. Section 3 is devoted to the lower bound; in that section we also determine the structure of a decomposition that realizes the lower bound. In section 4, we give constructions that achieve the lower bound for most values of N. That section is divided into three parts. In section 4.1 we show some results from design theory that will be needed later. Section 4.2 is devoted to showing constructions for small cases. Finally, in section 4.3, we give general constructions for all values of N with few exceptions.

2 Previous results

Optimal constructions for given grooming ratio C have been obtained using tools of graph and design theory [12]. In particular, results are available for grooming ratio $C = 3$ [3], $C = 4$ [6, 28], $C = 5$ [4], and $C \geq N(N-1)/6$ [6]. The problem is also solved for large values of C [6]. Related problems have been studied in both the context of variable traffic requirements [11, 16, 27, 36, 39] and the case of fixed traffic requirements [3, 4, 5, 6, 17, 25, 26, 28, 29, 32, 37, 40].

We now present some results to be used in later sections, leaving specific results on design theory until section 4.1.

Let $\rho(B_\ell)$ denote the ratio for the subgraph B_ℓ, $\rho(B_\ell) = \frac{|E(B_\ell)|}{|V(B_\ell)|}$, and $\rho(m)$ be the maximum ratio of a subgraph with m edges. Let $\rho_{\max}(C)$ denote the maximum ratio of subgraphs with $m \leq C$ edges. We have $\rho_{\max}(C) = \max\{\rho(B_\ell) \mid |E(B_\ell)| \leq C\} = \max_{m \leq C} \rho(m)$. For the sake of illustration, Table 1 gives the values of $\rho_{\max}(C)$ for small values of C. For example, for $C = 6$, $\rho_{\max}(6) = \frac{3}{2}$, the bound being attained for K_4.

Theorem 2.1 (see [5]) *Any grooming of R circles with a grooming factor C needs at least $\frac{R}{\rho_{\max}(C)}$ ADMs, i.e., $A(C,N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}$.*

The grooming problem is closely connected to problems in combinatorial design theory. Indeed, an $(N,k,1)$-design is exactly a partition of the edges of K_N into subgraphs isomorphic to K_k (these are the *blocks* of the design). That corresponds to requiring in our partitioning problem that all the subgraphs B_ℓ be isomorphic to K_k. The classical equivalent definition is, given a set of N elements, find a set of blocks such that each block contains k elements and each pair of elements appears in exactly one block (see [12]). More generally, a G-design of order N (see [12, section IV.22], [7, 8]) consists of a partition of the edges of K_N into subgraphs isomorphic to a given graph G. Our interest in the existence of a G-design is shown by the following proposition.

Proposition 2.2 *If there exists a G-design of order N, where G is a graph with at most C edges and ratio $\rho_{\max}(C)$, then $A(C,N) = \frac{N(N-1)}{2\rho_{\max}(C)}$.*

Necessary conditions 2.3 (existence of a G-design) *If there exists a G-design, then*

(i) *$\frac{N(N-1)}{2}$ is a multiple of $E(G)$,*

(ii) *$N-1$ is a multiple of the greatest common divisor of the degrees of the vertices of G.*

Table 1: Values of $\rho_{\max}(C)$ for small C.

C	1	2	3	4	5	6	7	8	9	10
$\rho_{\max}(C)$	$\frac{1}{2}$	$\frac{2}{3}$	1	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{9}{5}$	2
C	11	12	13	14	15	16	24	32	48	64
$\rho_{\max}(C)$	2	2	$\frac{13}{6}$	$\frac{14}{6}$	$\frac{5}{2}$	$\frac{5}{2}$	3	$\frac{32}{9}$	$\frac{9}{2}$	$\frac{64}{11}$

Wilson's theorem [31, 38] establishes that these necessary conditions are also sufficient for large N. From that, given any value of C, for an infinite number of values of N, $A(C, N) = \frac{N(N-1)}{2\rho_{\max}(C)}$. Unfortunately, the values of N for which Wilson's theorem applies are very large. Nevertheless, for small values of C, we can use exact results from design theory. For example, from the existence of G-designs for $G = K_4$ we obtain the following result.

Theorem 2.4 $A(6, N) = \frac{N(N-1)}{3}$ *when* $N \equiv 1$ *or* $4 \pmod{12}$.

The nonexistence of certain G-designs for some values of C and N implies that K_N cannot be optimally decomposed by using isomorphic copies of the same subgraph. This lack of regularity in the decomposition makes it harder to find optimal decompositions and thus to find the value of $A(C, N)$. Furthermore, the solution may be very different for different values of C and N, and Proposition 2.3 suggests that the solutions depend on the congruence class of N.

Theorem 2.1 suggests that the minimum number of ADMs can be achieved by choosing subgraphs such that the average ratio is maximized, or roughly speaking, by choosing subgraphs with a ratio equal to $\rho_{\max}(C)$ whenever possible. Although this last sentence is not to be taken literally, we do show in section 3 that most of the subgraphs in optimal decompositions for $C = 6$ must be isomorphic to K_4.

Even if G-designs do not give a direct solution to our problem, related combinatorial structures assist in the solution. For instance, some types of designs may give a decomposition for a part of the graph or may help constructing solutions by composition from smaller cases.

We introduce specific concepts and results from design theory in section 4.1 in order not to make the presentation overly technical at the outset. See [9, 12] for undefined terms and for a general overview of design theory.

In the remainder of the paper we use standard terms from graph theory. However, let us introduce some notation and terminology that may not be standard. Let $v_1, v_2, \ldots, v_l$ be nonnegative integers; the *complete multipartite graph with class sizes* $v_1, v_2, \ldots, v_l$, denoted $K_{v_1,v_2,\ldots,v_l}$ is the graph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_l$, where $|V_i| = v_i$, and two vertices $x \in V_i$ and $y \in V_j$ are adjacent if and only if $i \neq j$. For $u > 0$, we write $K_{g\times u}$ (resp., $K_{g\times u,m}$) $K_{g,g,\ldots,g}$ (resp., $K_{g,g,\ldots,g,m}$) when g occurs u times.

Given a complete graph K_n, the graph $K_n - e$ is the result of removing one edge. In this paper we also use names for given graphs that are given in Table 2.

3 Lower bound for grooming ratio $C = 6$

In this section we first give the lower bound for grooming factor $C = 6$ (Theorem 3.1), and then we discuss the possible structure of any decomposition attaining the lower bound.

Theorem 3.1 *Let $R = \frac{N(N-1)}{2}$ denote the number of edges of K_N and A the number of ADMs.*

- *If $N \equiv 1 \pmod 3$, then $A \geq \frac{2R}{3} + \epsilon$, where $\epsilon = 2$ if $N \equiv 7$ or $10 \pmod{12}$ and 0 otherwise.*
- *If $N \equiv 2 \pmod 3$, then $A \geq \frac{2R+N+2}{3}$.*
- *If $N \equiv 0 \pmod 3$, then $A \geq \lceil \frac{6R+2N}{9} \rceil + \epsilon$, where $\epsilon = 1$ if $N \equiv 18, 27 \pmod{36}$, and $\epsilon = 0$ otherwise.*

Proof: Let $G_{i,j}$ denote a graph with i edges and j vertices. In Table 2 are indicated all the possible degree sequences of the connected graphs with $i \leq 6$ (at most six edges) and one example of such a graph. Consider a decomposition of K_N and let $\alpha_{i,j}$ be the number of graphs of type $G_{i,j}$ appearing in the decomposition. We have the two following equations:

$$R = \frac{N(N-1)}{2} = \sum_{i,j} i \cdot \alpha_{i,j}, \tag{1}$$

$$A = \sum_{i,j} j \cdot \alpha_{i,j}. \tag{2}$$

From (1) and (2) and the fact that $C = 6$ implies $i \leq 6$, we deduce

$$\begin{aligned} 3A &= 2R + 3\alpha_{6,5} + 6\alpha_{6,6} + 9\alpha_{6,7} + 2\alpha_{5,4} + 5\alpha_{5,5} + 8\alpha_{5,6} \\ &\quad + 4\alpha_{4,4} + 7\alpha_{4,5} + 3\alpha_{3,3} + 6\alpha_{3,4} + 5\alpha_{2,3} + 4\alpha_{1,2}. \end{aligned} \tag{3}$$

So we always have $A \geq 2R/3$, equality being attained only if there exists a $(N, 4, 1)$-design, which is true only for $N \equiv 1$ or $4 \pmod{12}$ (Theorem 2.4).

Case 1. $N \equiv 1 \pmod 3$.

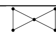
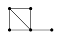
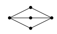
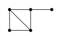
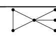
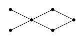
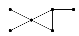
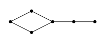
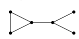
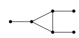
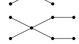
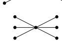
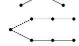
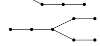
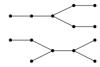
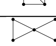
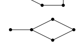
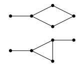
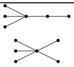
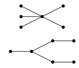
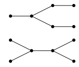
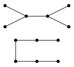
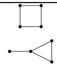
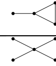
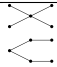
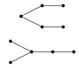
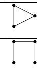
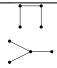
If $N \equiv 7$ or $10 \pmod{12}$, then $R \equiv 3 \pmod 6$ and the decomposition must contain some graphs having strictly less than six edges. Thus, either it contains at least two subgraphs having less than six edges and then $3A \geq 2R + 4$ or only one graph, which is necessarily a C_3; but that is impossible as $K_N - C_3$ cannot be partitioned into K_4, as the three nodes of the C_3 have degree $N - 2 \equiv 2 \pmod 3$ (Condition 2.3). Thus we have $A \geq 2R/3 + 2$.

Case 2. $N \equiv 2 \pmod 3$.

The degree of a vertex of K_N is $\equiv 1 \pmod 3$ and so in each vertex we have to use at least either a graph $G_{i,j}$ having a vertex of degree $\equiv 1 \pmod 3$ or two graphs $G_{i,j}$ each having a vertex of degree $\equiv 2 \pmod 3$.

For a graph $G_{i,j}$, let $g^1_{i,j}$ denote its number of vertices of degree $\equiv 1 \pmod 3$ and $g^2_{i,j}$ denote its number of vertices of degree $\equiv 2 \pmod 3$. Write $g_{i,j} = g^1_{i,j} + \frac{1}{2}g^2_{i,j}$. For example, for $A_{6,5}$ (two triangles with a common vertex) the sequence of degrees is 42222 and so $a^1_{6,5} = 1$, $a^2_{6,5} = 4$, and $a_{6,5} = 3$, and for $B_{6,5}$ with degree sequence 43221, $b^1_{6,5} = 2$, $b^2_{6,5} = 2$, and $b_{6,5} = 3$. Values of $g_{i,j}$ are given in Table 2.

Table 2: Graphs with v vertices and $e \leq 6$ edges. $g_{i,j}$ is the average contribution to degree $\equiv 1 \pmod 3$, $g'_{i,j}$ the average contribution to degree $\equiv 2 \pmod 3$, $\delta_{i,j} = \max_g g_{i,j}$, and $\delta'_{i,j} = \max_{g'} g'_{i,j}$.

Graph		e	v	deg. seq.	$g_{i,j}$		$g'_{i,j}$	
$A_{6,4} = K_4$		6	4	3333	0		0	
$A_{6,5}$		6	5	42222	3	$= \delta_{6,5}$	4.5	$= \delta'_{6,5}$
$B_{6,5}$		6	5	43221	3		3	
$C_{6,5} = K_{3,2}$		6	5	33222	1.5		3	
$D_{6,5}$		6	5	33321	1.5		1.5	
$A_{6,6}$		6	6	522111	4.5	$= \delta_{6,6}$	4.5	$= \delta'_{6,6}$
$B_{6,6}$		6	6	422211	4.5		4.5	
$C_{6,6}$		6	6	432111	4.5		3	
$D_{6,6}$		6	6	322221	3		4.5	
$E_{6,6}$		6	6	332211	3		3	
$F_{6,6}$		6	6	333111	3		1.5	
$A_{6,7}$		6	7	5211111	6	$= \delta_{6,7}$	4.5	
$B_{6,7}$		6	7	4221111	6		4.5	
$C_{6,7}$		6	7	4311111	6		3	
$D_{6,7}$		6	7	6111111	6		3	
$E_{6,7}$		6	7	2222211	4.5		6	$= \delta'_{6,7}$
$F_{6,7}$		6	7	3222111	4.5		4.5	
$H_{6,7}$		6	7	3321111	4.5		3	
$A_{5,4} = K_4 - e$		5	4	3322	1	$= \delta_{5,4}$	2	$= \delta'_{5,4}$
$A_{5,5}$		5	5	42211	4	$= \delta_{5,5}$	3.5	
$B_{5,5}$		5	5	22222	2.5		5	$= \delta'_{5,5}$
$C_{5,5}$		5	5	32221	2.5		3.5	
$D_{5,5}$		5	5	33211	2.5		2	
$A_{5,6}$		5	6	421111	5.5	$= \delta_{5,6}$	3.5	
$B_{5,6}$		5	6	511111	5.5		2.5	
$C_{5,6}$		5	6	322111	4		3.5	
$D_{5,6}$		5	6	331111	4		2	
$E_{5,6}$		5	6	222211	3		5	$= \delta'_{5,6}$
$A_{4,4} = C_4$		4	4	2222	2	$= \delta_{4,4}$	4	$= \delta'_{4,4}$
$B_{4,4}$		4	4	3221	2		2.5	
$A_{4,5}$		4	5	41111	5	$= \delta_{4,5}$	2.5	
$B_{4,5}$		4	5	22211	3.5		4	$= \delta'_{4,5}$
$C_{4,5}$		4	5	32111	3.5		2.5	
$A_{3,3} = C_3$		3	3	222	1.5	$= \delta_{3,3}$	3	$= \delta'_{3,3}$
$A_{3,4}$		3	4	2211	3	$= \delta_{3,4}$	3	$= \delta'_{3,4}$
$B_{3,4}$		3	4	3111	3		1.5	
$A_{2,3}$		2	3	211	2.5	$= \delta_{2,3}$	2	$= \delta'_{2,3}$
$A_{1,2} = K_2$		1	2	11	2	$= \delta_{1,2}$	1	$= \delta'_{1,2}$

Now, the condition that the sum of the degrees of a given vertex is $N-1 \equiv 1 \pmod 3$ implies that

$$\sum_{G_{i,j}} g_{i,j} \quad \geq \quad N. \tag{4}$$

Let $\delta_{i,j} = \max_g g_{i,j}$, with the maximum taken over all the graphs with i edges and j vertices. For example, $\delta_{6,5} = 3$ (attained for $A_{6,5}$ and $B_{6,5}$), $\delta_{6,6} = 4.5$ (attained for $A_{6,6}$, $B_{6,6}$, and $C_{6,6}$), and so on.

Equation (4) becomes

$$\sum_{i,j} \alpha_{i,j}\delta_{i,j} \quad \geq \quad N. \tag{5}$$

That is by using the values of $\delta_{i,j}$

$$\begin{aligned}3\alpha_{6,5} + 4.5\alpha_{6,6} + 6\alpha_{6,7} + \alpha_{5,4} + 4\alpha_{5,5} + 5.5\alpha_{5,6} + 2\alpha_{4,4} + 5\alpha_{4,5}& \\ + 1.5\alpha_{3,3} + 3\alpha_{3,4} + 2.5\alpha_{2,3} + 2\alpha_{1,2} \quad &\geq \quad N.\end{aligned} \tag{6}$$

Now (3) plus inequality (6) gives

$$\begin{aligned}3A \quad \geq \quad & 2R + N + 1.5\alpha_{6,6} + 3\alpha_{6,7} + \alpha_{5,4} + \alpha_{5,5} + 2.5\alpha_{5,6} \\ & + 2\alpha_{4,4} + 2\alpha_{4,5} + 1.5\alpha_{3,3} + 3\alpha_{3,4} + 2.5\alpha_{2,3} + 2\alpha_{1,2}\end{aligned} \tag{7}$$

and so $A \geq \lceil \frac{2R+N}{3} \rceil$. But, as $N \equiv 2 \pmod 3$ and $R \equiv 1 \pmod 3$, we have $\lceil \frac{2R+N}{3} \rceil = \frac{2R+N+2}{3}$ and finally $A \geq \frac{2R+N+2}{3}$.

Case 3. $N \equiv 0 \pmod 3$.

In this case each vertex of K_N has degree $\equiv 2 \pmod 3$. Thus we have to use in each vertex at least either a graph $G_{i,j}$ having a vertex of degree $\equiv 2 \pmod 3$ or two graphs $G_{i,j}$ each having a vertex of degree $\equiv 1 \pmod 3$.

For a given graph $G_{i,j}$, let us define $g'_{i,j} = g^2_{i,j} + \frac{1}{2}g^1_{i,j}$. For example, for $A_{6,5}$, $a'_{i,j} = 4.5$, but for $B_{6,5}$, $b'_{i,j} = 3$ (values of $g'_{i,j}$ are indicated in Table 2).

The condition that the sum of the degrees of a vertex is $N-1 \equiv 2 \pmod 3$ implies that

$$\sum_{G_{i,j}} g'_{i,j} \quad \geq \quad N. \tag{8}$$

Let $\delta'_{i,j} = \max_{g'} g'_{i,j}$, with the maximum taken over all graphs with i edges and j vertices. For example, $\delta'_{6,5} = 4.5$ (attained only for $A_{6,5}$).

Equation (8) becomes

$$\sum_{i,j} \alpha_{i,j}\delta'_{i,j} \quad \geq \quad N \tag{9}$$

or, replacing by the values of $\delta'_{i,j}$,

$$\begin{aligned}4.5\alpha_{6,5} + 4.5\alpha_{6,6} + 6\alpha_{6,7} + 2\alpha_{5,4} + 5\alpha_{5,5} + 5\alpha_{5,6} + 4\alpha_{4,4} + 4\alpha_{4,5}& \\ + 3\alpha_{3,3} + 3\alpha_{3,4} + 2\alpha_{2,3} + \alpha_{1,2} \quad &\geq \quad N.\end{aligned} \tag{10}$$

Now (3) with both sides multiplied by 3 and inequality (10) with both sides multiplied by 2 give

$$\begin{aligned} 9A \ \geq\ & 6R + 2N + 9\alpha_{6,6} + 15\alpha_{6,7} + 2\alpha_{5,4} + 5\alpha_{5,5} + 14\alpha_{5,6} + 4\alpha_{4,4} + 13\alpha_{4,5} \\ & + 3\alpha_{3,3} + 12\alpha_{3,4} + 11\alpha_{2,3} + 10\alpha_{1,2}. \end{aligned} \tag{11}$$

As $N \equiv 0 \pmod 3$, we have $6R \equiv 0 \pmod 9$ and we obtain $\lceil \frac{6R+2N}{9} \rceil = \frac{6R+2N+\beta}{9}$, where $\beta = 0$ when $N \equiv 0 \pmod 9$, $\beta = 3$ when $N \equiv 3 \pmod 9$, and $\beta = 6$ when $N \equiv 6 \pmod 9$.

Furthermore, if $N \equiv 3$ or $6 \pmod{12}$, $R \equiv 3 \pmod 6$ and so we cannot use only graphs with six edges. In that case, $9A > 6R + 2N$, in particular if $N \equiv 18$ or $27 \pmod{36}$, we have $A \geq \frac{6R+2N+9}{9}$. □

Let us now examine the possible structure for a decomposition of K_N in order to match the lower bound of Theorem 3.1.

The following remarks are obtained by checking carefully the graphs in Table 2 and the equations in the proof of Theorem 3.1.

Remark 3.2 *When $N \equiv 7$ or $10 \pmod{12}$, the only way to match the lower bound ($A = 2R/3+2$) with $R \equiv 3 \pmod 6$ and degree $N-1 \equiv 0 \pmod 3$ is by using three subgraphs $G_{5,4}$ (that is, $K_4 - e$) sharing the two vertices with degree 3. It corresponds to a covering of K_N by K_4 in which an edge is covered four times.*

When $N \equiv 2 \pmod 3$ we distinguish two possible subcases, depending on the congruence class of R. If $N \equiv 2$ or $11 \pmod{12}$, that is, $R \equiv 1 \pmod 6$, the only possibility is $\alpha_{1,2} = 1$ and therefore, we have the next remark.

Remark 3.3 *Any decomposition attaining the lower bound with $N \equiv 2$ or $11 \pmod{12}$ must contain one K_2, $\frac{N-2}{3}$ graphs of type $A_{6,5}$ or $B_{6,5}$, and the remaining subgraphs being K_4.*

If $N \equiv 5$ or $8 \pmod{12}$, that is, $R \equiv 4 \pmod 6$, there are different possibilities.

Remark 3.4 *Any decomposition attaining the lower bound with $N \equiv 5$ or $8 \pmod{12}$ must contain K_4 and*

- *either $\alpha_{4,4} = 1$, that is, one $A_{4,4}$ or $B_{4,4}$ and $\frac{N-2}{3}$ $A_{6,5}$ or $B_{6,5}$;*
- *or $\alpha_{4,5} = 1$, that is, one $A_{4,5}$ ($B_{4,5}$ and $C_{4,5}$ do not work as $b_{4,5} = c_{4,5} = 3.5 < 5$) and $\frac{N-5}{3}$ $A_{6,5}$ or $B_{6,5}$;*
- *or $\alpha_{5,4} = 2$, that is, two $A_{5,4}$ ($K_4 - E$) and $\frac{N-2}{3}$ $A_{6,5}$ or $B_{6,5}$;*
- *or $\alpha_{5,4} = 1$ and $\alpha_{5,5} = 1$, that is, one $A_{5,4}$, one $A_{5,5}$, and $\frac{N-5}{3}$ $A_{6,5}$ or $B_{6,5}$;*
- *or $\alpha_{5,5} = 2$, that is, two $A_{5,5}$ and $\frac{N-8}{3}$ $A_{6,5}$ or $B_{6,5}$.*

When $N \equiv 0 \pmod 3$, (3), (10), and (11) can be used to determine the structure of any decomposition attaining the lower bound. Denote by F_4 the graph consisting of two $A_{6,5}$ sharing the same vertex of degree 4 (equivalently, F_4 consists of 4 C_3 having a common vertex). A graph F_4 is decomposed into two $A_{6,5}$ and therefore despite having 9 vertices must be attributed a cost of 10.

The decomposition depends on the congruence class modulo 36 as follows.

Remark 3.5 *Any decomposition attaining the lower bound must satisfy*

- $N \equiv 0$ *or* $9 \pmod{36}$*: the graph is decomposed into* $\frac{N}{9}$ *vertex disjoint* F_4 *plus* K_4*;*
- $N \equiv 3$ *or* $30 \pmod{36}$*:* $R \equiv 3 \pmod{6}$ *implies that* $\alpha_{3,3} = 1$*, and therefore the decomposition contains one* C_3*,* $\frac{N-3}{9}$ *vertex disjoint* F_4 *plus* K_4*.*

To obtain the possible decompositions in the remaining cases we use the parameter $g'_{i,j}$ in the inequalities (10) and (11) to obtain

$$9A \;\geq\; \begin{array}{c} 6R + 2N + 3(b'_{6,5} + c'_{6,5}) + 6d'_{6,5} + 2a'_{5,4} \\ + 5b'_{5,5} + 4a'_{4,4} + 3a'_{3,3} + 7\sum g'_{i,j}. \end{array} \tag{12}$$

(The last term $\sum g'_{i,j}$ corresponds to graphs $G_{i,j}$ different from K_4, $G_{6,5}$, $A_{5,4}$, $B_{5,5}$, $A_{4,4}$, and $A_{3,3}$.) When $N \equiv 12$ or $21 \pmod{36}$, $R \equiv 0 \pmod{6}$ and all subgraphs must contain six edges. Precisely, (12) shows the following.

Remark 3.6 *Any decomposition with* $N \equiv 12$ *or* $21 \pmod{36}$ *that meets the lower bound must contain* K_4 *plus*

- *either a* $C_{6,5}$ *(*$K_{3,2}$*) and* $\frac{N-3}{9}$ F_4 *all vertex disjoint;*
- *or a* $B_{6,5}$ *sharing its vertex of degree* 4 *with an* $A_{6,5}$ *and its vertex of degree* 1 *with another* $A_{6,5}$ *and* $\frac{N-12}{9}$ F_4 *(all these graphs having no other vertices in common);*
- *or five* $A_{6,5}$ *sharing the vertex of degree* 4 *and then* $\frac{N-21}{9}$ F_4*;*
- *or a vertex belonging to four* F_4 *with degree* 2 *in each of them;*
- *or a vertex belonging to three* F_4*, once with degree* 4 *and twice with degree two.*

Similarly, for the remaining cases, we have the next remark.

Remark 3.7 *Any decomposition with* $N \equiv 6$ *or* $15 \pmod{36}$ *that meets the lower bound must contain* K_4 *plus*

- *either one* C_3 *and same as above (one* $C_{6,5}$ *or* $B_{6,5}$ *or some vertex belonging to five, four, or three* F_4*);*
- *or one* $A_{4,4}$*, one* $A_{5,4}$*, and* $\frac{N-6}{9}$ F_4 *disjoint except for two vertices of degree* 3 *in* $A_{5,4}$*;*
- *or three* $A_{5,4}$ *and* $\frac{N-6}{9}$ F_4*, vertex disjoint except for the six vertices of degree* 3 *in the three* $A_{5,4}$*.*

Remark 3.8 *Any decomposition with* $N \equiv 24$ *or* $33 \pmod{36}$ *that meets the lower bound must contain* K_4 *plus*

- *either two* C_3 *and* $\frac{N-6}{9}$ F_4 *vertex disjoint;*
- *or only graphs with six edges like*
  - *one* $D_{6,5}$*,*
  - *two* $B_{6,5}$ *or* $C_{6,5}$*,*

– *one* $B_{6,5}$ *or* $C_{6,5}$ *with some vertex belonging to five, four, or three* F_4*,*
– *a vertex in eight* $A_{6,5}$ *or two vertices each in four* $A_{6,5}$ *or other combinations with same vertex (or two vertices) belonging to three or more subgraphs.*

Remark 3.9 *Any decomposition with* $N \equiv 18$ *or* $27 \pmod{36}$ *that meets the lower bound must contain* K_4 *plus*

- *either* 3 C_3*,*
- *or one* C_3 *(and some subgraphs as in the preceding case),*
- *or one* $A_{4,4}$ *and one* $B_{5,5}$*,*
- *or* 3 $A_{5,4}$ *and some vertex belonging to three or more subgraphs.*

4 Upper bounds and optimal constructions

4.1 Some results from design theory

4.1.1 Definitions and previous results

A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a set of points, $\mathcal{G}$ is a partition of X into *groups*, and $\mathcal{B}$ is a collection of subsets of X called *blocks* such that any pair of distinct points from X occur together either in one group or in exactly one block, but not both. A K-GDD of type $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ is a GDD in which every block has size from the set K and in which there are u_i groups of size g_i for $i = 1, 2, \dots, s$.

Remark 4.1 *The existence of a decomposition of* $K_{g\times u,m}$ *into* K_4 *is equivalent to the existence of a* 4*-GDD of type* $g^u m^1$*.*

A *transversal design* $\mathrm{TD}(k, g)$ is a k-GDD of type g^k.

A *pairwise balanced design* (PBD) with parameters $(K; v)$ is a K-GDD of type 1^v. In particular, if $K = k$, a PBD is a G-design with G being the complete graph K_k.

A group divisible design $(X, \mathcal{G}, \mathcal{B})$ is *resolvable* (and referred to as an RGDD) if its block set $\mathcal{B}$ admits a partition into *parallel classes*, each parallel class being a partition of the point set X. A *double group divisible design* (DGDD) is a quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{B})$, where X is a set of points, $\mathcal{H}$ and $\mathcal{G}$ are partitions of X (into holes and groups, respectively), and $\mathcal{B}$ is a collection of subsets of X (blocks) such that

(i) for each block $B \in \mathcal{B}$ and each hole $H \in \mathcal{H}$, $|B \cap H| \leq 1$, and

(ii) any pair of distinct points from X which are not in the same hole occur either in some group or in exactly one block, but not both.

A K-DGDD of type $(g_1, h_1^v)^{u_1} (g_2, h_2^v)^{u_2} \dots (g_s, h_s^v)^{u_s}$ is a double group-divisible design in which every block has size from the set K and in which there are u_i groups of size g_i, each of which intersects each of the v holes in h_i points. Thus $g_i = v \cdot h_i$ for $i = 1, 2, \dots, s$. Not every DGDD can be expressed this way, of course, but this is the most general type that we require. One special case, a *modified group divisible design* K-MGDD of type g^u, is a K-DGDD of type $(g, 1^g)^u$. A k-DGDD of type $(g, h^v)^k$ is an incomplete transversal design (ITD) $(k, g; h^v)$ and is equivalent to a set of $k-2$ holey MOLS of type h^v (see, e.g., [12]).

We recall some known results on designs to be used in subsequent sections.

Theorem 4.2 (see Theorem 1.27 of [12]) *The multipartite graph $K_{2\times u}$ can be partitioned into $\frac{u(u-1)}{3}$ K_4 when $u \equiv 1 \pmod 3$, $u > 4$. Equivalently there exists a 4-GDD of type 2^u.*

Theorem 4.3 (see [10] and Chapter 7 of [9]) *The multipartite graph $K_{g\times 4}$ can be partitioned into K_4 if and only if $g \neq 2, 6$. Equivalently there exists a $TD(4, g)$ if and only if $g \neq 2, 6$.*

The primary recursive construction that we use is Wilson's fundamental construction (WFC) for GDDs (see, e.g., [12]).

Construction 4.4 *Let $(X, \mathcal{G}, \mathcal{B})$ be a GDD, and let $w : X \to \mathbb{Z}^+ \cup \{0\}$ be a weight function on X. Suppose that for each block $B \in \mathcal{B}$, there exists a K-GDD of type $\{w(x) : x \in B\}$. Then there is a K-GDD of type $\{\sum_{x\in G} w(x) : G \in \mathcal{G}\}$.*

We make use of the following existence result.

Theorem 4.5 (see [24]) *There exists a 4-DGDD of type $(mt, m^t)^n$ if and only if $t, n \geq 4$ and $(t-1)(n-1)m \equiv 0 \pmod 3$ except for $(m, n, t) = (1, 4, 6)$ and except possibly for $m = 3$ and $(n, t) \in \{(6, 14), (6, 15), (6, 18), (6, 23)\}$.*

We also make use of the following simple construction for 4-GDDs, which was stated in [23].

Construction 4.6 *If there is a 4-DGDD of type $(g_1, h_1^v)^{u_1}$ $(g_2, h_2^v)^{u_2} \dots (g_s, h_s^v)^{u_s}$, and for each $i = 1, 2, \dots, s$ there is a 4-GDD of type $h_i^v a^1$, where a is a fixed nonnegative integer, then there is a 4-GDD of type $h^v a^1$, where $h = \sum_{i=1}^{s} u_i h_i$.*

The following results on transversal designs are known (see, for example, [12]).

Theorem 4.7 *A $TD(k, g)$ exists if*

1. *$k = 5$ and $g \geq 4$ and $g \notin \{6, 10\}$;*
2. *$k = 6$ and $g \geq 5$ and $g \notin \{6, 10, 14, 18, 22\}$;*
3. *$k = 7$ and $g \geq 7$ and $g \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60, 62\}$.*

Finally, we make use of the following results on 4-GDDs (see, e.g., [12, 21, 22, 23, 33]).

Theorem 4.8 (see [12, III.1.3, Theorem 1.27]) *Let u and t be positive integers. Then there exists a 4-GDD of type t^u if and only if the conditions in the following table are satisfied:*

Existence of 4-GDDs of Type t^u		
t	u	Necessary and Sufficient Conditions
$1, 5 \pmod 6$	$1, 4 \pmod{12}$	$u \geq 4$
$2, 4 \pmod 6$	$1 \pmod 3$	$u \geq 4,\ (t, u) \neq (2, 4)$
$3 \pmod 6$	$0, 1 \pmod 4$	$u \geq 4$
$0 \pmod 6$	no constraint	$u \geq 4, (t, u) \neq (6, 4)$

Theorem 4.9 (see [12, III.1.3, Theorem 1.28]) *A 4-GDD of type $3^u m^1$ exists if and only if either $u \equiv 0 \mod 4$ and $m \equiv 0 \mod 3$, $0 \leq m \leq (3u-6)/2$; or $u \equiv 1 \mod 4$ and $m \equiv 0 \mod 6$, $0 \leq m \leq (3u-3)/2$; or $u \equiv 3 \mod 4$ and $m \equiv 3 \mod 6$, $0 < m \leq (3u-3)/2$.*

Theorem 4.10 (see [21, Theorem 1.7]) *There exists a 4-GDD of type g^4m^1 with $m > 0$ if and only if $g \equiv m \equiv 0 \mod 3$ and $0 < m \leq \frac{3g}{2}$.*

Theorem 4.11 (see [22, Theorem 1.6]) *There exists a 4-GDD of type 6^um^1 for every $u \geq 4$ and $m \equiv 0 \mod 3$ with $0 \leq m \leq 3u - 3$ except for $(u, m) = (4, 0)$ and except possibly for $(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$.*

Theorem 4.12 (see [18, Theorem 3.16]) *There exists a 4-GDD of type 12^um^1 for each $u \geq 4$ and $m \equiv 0 \mod 3$ with $0 \leq m \leq 6(u - 1)$.*

We also employ current existence results on 4-RGDDs.

Theorem 4.13 (see [19, 20]) *The necessary conditions for the existence of a 4-RGDD(t^u), namely, $u \geq 4$, $tu \equiv 0 \pmod 4$ and $t(u - 1) \equiv 0 \pmod 3$, are also sufficient except for $(t, u) \in \{(2, 4), (2, 10), (3, 4), (6, 4)\}$ and possibly excepting*

1. $t \equiv 2, 10 \pmod{12}$*:* $t = 2$ *and* $u \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 142, 178, 184, 202, 214, 238, 250, 334, 346\}$*;* $t = 10$ *and* $u \in \{4, 34, 52, 94\}$*;* $t \in [14, 454] \cup \{478, 502, 514, 526, 614, 626, 686\}$ *and* $u \in \{10, 70, 82\}$*;*
2. $t \equiv 6 \pmod{12}$*:* $t = 6$ *and* $u \in \{6, 54, 68\}$*;* $t = 18$ *and* $u \in \{18, 38, 62\}$*;*
3. $t \equiv 9 \pmod{12}$*:* $t = 9$ *and* $u = 44$*;*
4. $t \equiv 0 \pmod{12}$*:* $t = 12$ *and* $u = 27$*;* $t = 36$ *and* $u \in \{11, 14, 15, 18, 23\}$*.*

4.1.2 Existence of 4-GDDs of type 36^um^1, for small values of m

Here we consider 4-GDDs of type 36^um^1 with $m \in \{3, 6, 9, \ldots, 33\}$. Whenever we refer to a 4-RGDD of type g^u, the existence of such RGDDs comes from Theorem 4.13.

Lemma 4.14 *There exists a 4-GDD of type 36^um^1 for each $u \geq 4$, $u \equiv 0, 1, 3 \mod 4$ and $m \in \{3, 6, 9, \ldots, 33\}$.*

Proof: Start with a TD$(5, u)$ and adjoin an infinite point ∞ to the groups, then delete a finite point so as to form a $\{5, u+1\}$-GDD of type 4^uu^1. Each block of size $u+1$ intersects the group of size u in the infinite point ∞ and each block of size 5 intersects the group of size u, but certainly not in ∞. Now, in the group of size u, we give ∞ weight 0 (when $u \equiv 0, 1 \mod 4$) or 3 (when $u \equiv 3 \mod 4$) and give the remaining points weight 0, 3, 6, 9, or 12. Give all other points in the $\{5, u+1\}$-GDD weight 9. Replace the blocks in the $\{5, u+1\}$-GDD by 4-GDDs of types 9^u, 9^u3^1, or $9^4(3i)^1$ (from Theorem 4.10) with $i \in \{0, 1, 2, 3, 4\}$ to obtain the 4-GDDs. Here, the input designs that are 4-GDDs of type 9^u3^1 when $u \equiv 3 \mod 4$ come from [23]. □

This leaves only the case for $u \equiv 2 \mod 4$ to consider.

Lemma 4.15 *There exists a 4-GDD of type 36^6m^1 for each $m \in \{3, 6, 9, \ldots, 33\}$.*

Proof: For $m \in \{3, 6, 9, 12, 15\}$, starting from a 4-DGDD of type $(36, 6^6)^6$ from Theorem 4.5 and applying Construction 4.6 with 4-GDDs of type 6^6m^1 to fill in holes, we obtain the designs. For other values of m, start from a TD$(7, 9)$ and apply WFC with weight 4 to the points in the first six groups and weight 1 or 4 to the remaining points. The 4-GDD of type 4^61^1 is from [30, 23]. □

Lemma 4.16 *There exists a 4-GDD of type $36^{10}m^1$ for each $m\in\{3,6,9,\ldots,144\}$.*

Proof: Complete the 12 parallel classes of a 4-RGDD of type 4^{10} to obtain a 5-GDD of type $4^{10}12^1$. Apply WFC and give weight 9 to the points in the groups of size 4 and weight 0, 3, 6, 9, or 12 to the remaining points. The result follows from Theorem 4.10. □

Lemma 4.17 *There exists a 4-GDD of type $36^{14}m^1$ for each $m \in \{3,6,9,\ldots,48\}$.*

Proof: Take a 5-GDD of 4^{15} and apply WFC with weight 9 to the points in the first 14 groups and weight 0, 3, 6, 9, or 12 to the remaining points. □

Lemma 4.18 *There exists a 4-GDD of type $36^{18}m^1$ for each $m \in \{3,6,9,\ldots,48\}$.*

Proof: Take a $(77,\{5,9^*\},1)$-PBD (the existence of such a PBD follows from [2]) and remove a point not in the single block of size 9 to obtain a $\{5,9\}$-GDD of type 4^{19}. The single block of size 9 can hit only 9 groups of the GDD. Apply WFC with weight 9 to the points in the first 18 groups such that the single block of size 9 is covered by them and weight 0, 3, 6, 9, or 12 to the remaining points. □

Lemma 4.19 *There exists a 4-GDD of type $36^{22}m^1$ for each $m\in\{3,6,9,\ldots,336\}$.*

Proof: Complete the 28 parallel classes of a 4-RGDD of type 4^{22} to obtain a 5-GDD of type $4^{22}28^1$. Apply WFC and give weight 9 to the points in the groups of size 4 and weight 0, 3, 6, 9, or 12 to the remaining points. □

Lemma 4.20 *There exists a 4-GDD of type $36^u m^1$ for each $u \geq 26$ and $u \equiv 2 \mod 4$ with $m \in \{3,6,9,\ldots,33\}$.*

Proof: Suppose that $u = 4s+2$ and $s \geq 6$. Take a 4-GDD of type $(36s-36)^4(216+m)^1$ from Theorem 4.10 and fill in 4-GDDs of type $36^{(s-1)}$ and 4-GDDs of type 36^6m^1 to obtain the 4-GDDs. □

Combining Lemmas 4.14–4.20, we have the following.

Theorem 4.21 *There exists a 4-GDD of type $36^u m^1$ for each $u \geq 4$ with $m \in \{3,6,9,\ldots,33\}$.*

4.1.3 Existence of 4-GDDs of type $36^u m^1$, for large values of m and other types

Now we consider 4-GDDs of type $36^u m^1$ with $m\in\{117,822,840,846,852\}$.

Lemma 4.22 *There exists a 4-GDD of type $36^u m^1$ for each $u \geq 7$,*

$$u \notin U = \{10,14,15,18,20,22,26,30,34,38,46,60,62\}$$

and $m \equiv 0 \mod 3$ with $0 \leq m \leq 18u-18$.

Proof: Start with a TD$(7,u)$ and adjoin an infinite point ∞ to the groups, then delete a finite point so as to form a $\{7,u+1\}$-GDD of type $6^u u^1$. Each block of size $u+1$ intersects the group of size u in the infinite point ∞ and each block of size 7 intersects the group of size u, but certainly not in ∞. Now, in the group of size u, we give ∞ weight 0 or $3u-3$ and give the remaining points weight 0, 3, 6, 9, 12, or 15. Give all other points in the $\{7,u+1\}$-GDD weight 6. Replace the blocks in the $\{7,u+1\}$-GDD by 4-GDDs of types 6^u, $6^u(3u-3)^1$ or $6^6(3i)^1$ with $i \in \{0,1,2,3,4,5\}$ to obtain the 4-GDDs. Here, the input 4-GDDs all come from Theorem 4.11. □

Recall that a necessary condition for the existence of a 4-GDD of type $g^u m^1$ is that $u >= 2m/g+1 > 0$ (see [12]). This leaves the cases for $m = 117$ and $u \in U$ as well as $m = 822, 840, 846, 852$ and $u = 60, 62$ to treat.

Lemma 4.23 *There exists a 4-GDD of type $36^u 117^1$ for each $u \in U$.*

Proof: For $u = 10$, the proof follows from Lemma 4.16. For other values of u, start from a 4-RGDD of type 12^u and complete all the parallel classes to obtain a 5-GDD of type $12^u(4u-4)^1$. Give weight 0 or 3 to the points in the group of size $4u-4$ and weight 3 to the remaining points. □

Lemma 4.24 *There exists a 4-GDD of type $36^u m^1$ for each $u = 60, 62$ and $m = 822, 840, 846$.*

Proof: Take a 4-GDD of type $6^{\frac{u}{2}} 69^1$ from Theorem 4.11 and adjoin an infinite point ∞ to the groups, then delete a finite point in the group of size 69 so as to form a $\{4,7\}$-GDD of type $3^u 69^1$. Each block of size 7 intersects the group of size 69 in the infinite point ∞, while each block of size 4 does not. Now, we give ∞ weight $0, 3, \ldots, 27$ or 30 and give all the remaining points weight 12. Replace the blocks in the $\{4,7\}$-GDD by 4-GDDs of types 12^4 or $12^6 i^1$ with $i \in \{0, 3, 6, \ldots, 30\}$ from Theorem 4.12 to obtain the 4-GDDs. □

We still have $m = 852$ and $u \in \{60, 62\}$ to handle.

Lemma 4.25 *There exists a 4-GDD of type $36^u 852^1$ for each $u \in \{60, 62\}$.*

Proof: For $u = 60$, the proof is similar to that of Lemma 4.23. Here, we employ a 4-RGDD of type 6^{60}. For $u = 62$, take a resolvable 3-RGDD of type 12^{62} and apply weight 3, using resolvable 3-MGDDs of type 3^3 to obtain a resolvable 3-DGDD of type $(36, 12^3)^{62}$. Adjoin 732 infinite points to complete the parallel classes and then adjoin a further 120 ideal points, filling in 4-GDDs of type $12^{62} 120^1$ from Theorem 4.12, to obtain a 4-GDD of type $36^{62}(732+120)^1$. □

Combining Lemmas 4.22–4.25, together with the fact that a necessary condition for the existence of a 4-GDD of type $g^u m^1$ is that $u >= 2m/g + 1 > 0$ (see [12]), we obtain the following result.

Theorem 4.26

1. *There exists a 4-GDD of type $36^u 117^1$ if and only if $u \geq 8$.*
2. *There exists a 4-GDD of type $36^u 822^1$ if and only if $u \geq 47$.*
3. *There exists a 4-GDD of type $36^u m^1$ with $m = 840, 846$ if and only if $u \geq 48$.*
4. *There exists a 4-GDD of type $36^u 852^1$ if and only if $u \geq 49$.*

Here we collect some partial results with $g = 117$ to be used later.

Lemma 4.27 *There exists a 4-GDD of type $117^7 m^1$ for $m \in \{3, 21, 27, 33\}$.*

Proof: A 4-GDD of type $117^7 3^1$ appears in [23]. A 4-GDD of type $9^7 27^1$ appears in [23]. So fill one set of groups in a 4-DGDD of type $(117, 9^{13})^7$ from [24] to obtain a 4-GDD of type $117^7 27^1$.

For $117^7 33^1$, start from a 4-GDD of type $12^7 33^1$ and give weight 7 to each point, using 4-MGDDs of type 7^4. This gives a 4-DGDD of type $(84, 12^7)^7 (231, 33^7)^1$. Adjoining 33 infinite points and filling in 4-GDDs of type $12^7 33^1$ and a 4-GDD of type 33^8, we obtain a 4-GDD of type $117^7 33^1$. Similarly, we can start from a 4-GDD of type $12^8 21^1$ to obtain a 4-GDD of type $117^7 21^1$. □

4.2 Optimal constructions for small cases

We include in this section constructions for small cases to be used in the general theorems. In this discussion, we denote the graph $A_{6,5}$ as $\{A,B,C,D,E\}$, where A is the vertex of degree 4 and where $\{B,C\}$ and $\{D,E\}$ are edges; we denote the graph $B_{6,5}$ as $\{A,B,C,D,E\}$, where A is the vertex of degree 4, C is the vertex of degree 3, B and D the vertices of degree 2 (joined to A and C), and E is the vertex of degree 1.

Let us start this section with a trivial result.

Lemma 4.28 *The lower bound is attained for $N \leq 6$, i.e., $A(6,2) = 2$, $A(6,3) = 3$, $A(6,4) = 4$, $A(6,5) = 9$, and $A(6,6) = 12$.*

Let us recall that the lower bound also holds for $N \equiv 1$ or $4 \pmod{12}$ by Theorem 2.4.

We have the following results for small values of N.

Lemma 4.29 *The lower bound is not attained for $N = 7$. Moreover, $A(6,7) = 17$.*

Proof: The partition is obtained using the two K_4 $\{0,1,2,3\}$ and $\{0,4,5,6\}$, the $K_{2,3}$ between nodes $1,2$ and $4,5,6$, and the $K_{1,3}$ between node 3 and nodes $4,5,6$.

An exhaustive search establishes that no decomposition exists with cost 16. □

Lemma 4.30 *The lower bound is realized for $N = 8$, i.e., $A(6,8) = 22$.*

Proof: Let the vertices of K_8 be $V_8 = \{i,\ i \in \mathbb{Z}_8\}$. The decomposition consists of two K_4 $\{0,1,2,3\}$ and $\{0,4,5,6\}$, two $B_{6,5}$ $\{\{1,4\},\{1,5\},\{1,6\},\{1,7\},\{4,7\},\{5,7\}\}$ and $\{\{0,7\},\{2,7\},\{3,7\},\{6,7\},\{2,6\},\{3,6\}\}$, and the C_4 $(2,4,3,5)$. □

Lemma 4.31 *The lower bound is not attained for $N = 9$. Moreover, $A(6,9) = 27$.*

Proof: The general lower bound gives $A(6,9) \geq 26$. However, to obtain $A(6,9) = 26$, K_9 can be partitioned into one F_4 and four K_4, but $K_9 - F_4$ is $K_{2,2,2,2}$, which cannot be decomposed into K_4. Thus $A(6,9) \geq 27$.

Furthermore, a partition of K_9 is obtained using the three K_4 with vertex sets

$$\{0,4,5,6\}, \{0,3,7,8\}, \{1,2,3,6\},$$

plus the three $K_{2,3}$ $\{3i+1, 3i+2 | 3i, 3(i+1)+1, 3(i+1)+2\}$, $i = 0,1,2$, indices taken modulo 9. So altogether $A(6,9) = 27$. □

Lemma 4.32 *The lower bound is not attained for $N = 10$. Moreover, $A(6,10) = 34$.*

Proof: First we establish that $A(6,10) \leq 34$. Form three K_4 meeting in the element 9.

The remaining edges form $K_{3,3,3}$ on vertex set $\{0,\ldots,8\}$. Suppose that $\{0,1,2\}$ is one class of the tripartition. Choose a matching $\{a_1,b_1\}, \{a_2,b_2\}, \{a_3,b_3\}$ on the vertices $\{3,\ldots,8\}$ and for $i = 1,2,3$, form a $K_4 - e$ on $\{0,1,a_i,b_i\}$ omitting the edge $\{0,1\}$. The remaining 12 edges form a 6-wheel (a 6-cycle with a seventh vertex attached to each of the six). This can be decomposed into two copies of $D_{6,5}$.

There are three 4-vertex 6-edge graphs, three 4-vertex 5-edge graphs, and two 5-vertex 6-edge graphs in this partition, for a total of 34.

Any solution of cost less than 34 must have at least four K_4 by (3), and there is a unique way up to isomorphism to place four K_4. An exhaustive examination establishes that no such decomposition has cost less than 34. □

Lemma 4.33 *The lower bound is realized for $N = 11$, i.e., $A(6, 11) = 41$.*

Proof: Let the vertices of K_{11} be $V_{11} = \{\alpha\} \cup \{\beta\} \cup \{x_i^j,\ i, j \in \mathbb{Z}_3\}$. The decomposition consists of the K_2 $\{\alpha, \beta\}$, plus the three $A_{6,5}$ $\left\{x_i^0, x_{i+1}^1, x_{i+2}^2, x_{i+1}^2, x_{i+2}^1\right\}$, $i = 0, 1, 2$, plus the three K_4 $\left\{\alpha, x_i^0, x_i^1, x_i^2\right\}$, $i = 0, 1, 2$, plus the three K_4 $\left\{\beta, x_0^j, x_1^j, x_2^j\right\}$, $j = 0, 1, 2$. □

Lemma 4.34 *The lower bound is not attained for $N = 12$. Moreover, $A(6, 12) = 48$.*

Proof: The general lower bound gives $A(6, 12) \geq 47$. However, to obtain $A(6, 12) = 47$, there must be 11 6-vertex graphs in the decomposition. The only way in which nine of these can be K_4 leaves four K_3, so we need only consider situations with eight K_4 and three 6-edge graphs on five vertices. An exhaustive search establishes that no such decomposition exists. Thus $A(6, 12) \geq 48$.

Let $V = \sum_{i=1}^{4} V_i$ with $|V_i| = 3$; then $K_{3\times 4}$ can be partitioned into nine K_4 (Theorem 4.3). Thus a partition of K_{12} uses nine K_4 and four C_3. So altogether $A(6, 12) = 48$. □

Lemma 4.35 *The lower bound is realized for $N = 14$, i.e., $A(6, 14) = 66$.*

Proof: Let the vertices of K_{14} be $V_{14} = \{\alpha\} \cup \{\beta\} \cup \{x_i^j,\ i \in \mathbb{Z}_4, j \in \mathbb{Z}_3\}$. The decomposition consists of the K_2 $\{\alpha, \beta\}$, plus the four $B_{6,5}$ $\left\{x_i^0, x_{i+2}^1, x_{i+1}^2, x_{i+3}^1, x_{i+3}^2\right\}$, $i = 0, 1, 2, 3$, plus the 11 K_4 $\{x_0^j, x_1^j, x_2^j, x_3^j\}$, $j = 0, 1, 2$, $\left\{\alpha, x_i^0, x_i^1, x_i^2\right\}$, $i = 0, 1, 2, 3$, $\left\{\beta, x_i^0, x_{i+1}^1, x_{i+2}^2\right\}$, $i = 0, 1, 2, 3$. □

The next lemma enables us to determine that the lower bound is attained for several values of N.

Lemma 4.36 *When $N = 2t + 1$, $t \equiv 1 \pmod 3$, $t > 4$, then $A(6, N) \leq 4\frac{t(t-1)}{3} + 5\left\lfloor \frac{t}{2} \right\rfloor + \epsilon$, where $\epsilon = 3$ if t is odd and 0 otherwise.*

Proof: Let the vertices be α and x_i^j for $i \in \mathbb{Z}_2$ and $j \in \mathbb{Z}_t$. A partition of K_{2t+1} consists of a partition of $K_{2\times t}$ into $\frac{t(t-1)}{3}$ K_4 (Theorem 4.2), plus $\left\lfloor \frac{t}{2} \right\rfloor$ G_5, each one formed as the union of the two C_3 $\left\{\alpha, x_0^{2k}, x_1^{2k}\right\}$ and $\left\{\alpha, x_0^{2k+1}, x_1^{2k+1}\right\}$, $k = 0, 1, \ldots, \left\lfloor \frac{t}{2} \right\rfloor / 2 - 1$, and plus the C_3 $\left\{\alpha, x_0^{t-1}, x_1^{t-1}\right\}$ when t is odd. So altogether we have $A(6, 2t + 1) \leq 4\frac{t(t-1)}{3} + 5\left\lfloor \frac{t}{2} \right\rfloor + \epsilon$, where $\epsilon = 3$ if t is odd and 0 otherwise. □

Corollary 4.37 *The lower bound is realized for $N \in \{15, 21, 27, 33\}$: $A(6, 15) = 74$, $A(6, 21) = 145$, $A(6, 27) = 241$ and $A(6, 33) = 360$.*

Proof: Application of Lemma 4.36 and Theorem 3.1. □

Lemma 4.38 *The lower bound is not attained for $N = 19$. Moreover, $A(6, 19) = 119$.*

Proof: We first establish that $A(6, 19) \leq 119$. Partition K_{19} into 25 K_4,

{{0,1,2,4}, {0,3,5,6}, {0,7,8,9}, {0,10,11,12}, {0,13,14,15},
{0,16,17,18}, {1,3,7,10}, {1,5,8,11}, {1,6,13,16}, {1,9,14,17},
{1,12,15,18}, {2,3,8,15}, {2,5,9,18}, {2,6,10,17}, {2,7,12,13},
{2,11,14,16}, {3,4,14,18}, {3,9,12,16}, {3,11,13,17}, {4,5,12,17},
{4,6,9,15}, {5,10,15,16}, {6,7,11,18}, {6,8,12,14}, {8,10,13,18}},

and four other graphs,

$$D_{6,5}\text{: } \{\{4,7\},\{4,8\},\{4,16\},\{7,16\},\{8,16\},\{8,17\}\},$$
$$C_{6,5}\text{: } \{\{4,10\},\{4,11\},\{4,13\},\{9,10\},\{9,11\},\{9,13\}\},$$
$$D_{5,5}\text{: } \{\{5,7\},\{5,13\},\{5,14\},\{7,14\},\{10,14\}\}, \text{ and}$$
$$C_4\text{: } \{\{7,15\},\{7,17\},\{11,15\},\{15,17\}\}.$$

A maximum packing of K_4 in K_{19} has 25 K_4, but the example in [35] does not leave edges having a partition with cost 19, as this example does. Indeed, exhaustive computation showed that there are 249 nonisomorphic graphs that can be left by taking 25 K_4 from K_{19}. None yields a graph with cost less than 19. The only remaining possibility is to choose 24 K_4, three 5-edge 4-vertex graphs, and two 6-edge 5-vertex graphs, but a further exhaustive computation yielded no such partition. □

Lemma 4.39 *The lower bound is realized for $N = 20$, i.e., $A(6, 20) = 134$.*

Proof: Let the vertices of K_{20} be $V = V_1 \cup V_2$ with $|V_1| = 5$ and $|V_2| = 15$, and let the vertices of V_1 be $\{i,\ i \in \mathbb{Z}_5\}$.

The K_{15} on V_2 can be partitioned into seven parallel classes $\mathcal{C}_j,\ j \in \mathbb{Z}_7$, each consisting of five triangles $\mathcal{C}_{j,k},\ k \in \mathbb{Z}_5$, by the existence of a resolvable $(15, 3, 1)$-design.

For $i \in \mathbb{Z}_5$, we construct five K_4 built on node i and class $\mathcal{C}_{i,k}$, so altogether we have 25 K_4. Furthermore, the 10 triangles of the classes $\mathcal{C}_5$ and $\mathcal{C}_6$ can be joined in pairs to form five graphs isomorphic to $A_{6,5}$ (since there exist five vertices each belonging to exactly one triangle of $\mathcal{C}_5$ and one of $\mathcal{C}_6$). Finally, the K_5 on V_1 can be decomposed into one C_4 and one $A_{6,5}$. Altogether we have decomposed K_{20} into 1 C_4, 6 $A_{6,5}$, and 25 K_4. □

Lemma 4.40 *The lower bound is realized for $N = 23$, i.e., $A(6, 23) = 177$.*

Proof: Let the vertices of K_{23} be $\{\alpha\} \cup \{\beta\} \cup \{x_i^j,\ i \in \mathbb{Z}_7, j \in \mathbb{Z}_3\}$. The decomposition consists of the K_2 $\{\alpha, \beta\}$, plus the 7 $A_{6,5}$ $\left\{x_i^0, x_i^1, x_i^2, x_{i+1}^1, x_{i+2}^2\right\}$, $i \in \mathbb{Z}_7$, and the 35 K_4,

$$\left\{\alpha, x_i^0, x_{i+2}^1, x_{i+4}^2\right\}, \left\{\beta, x_i^0, x_{i+4}^1, x_{i+1}^2\right\}, \left\{x_i^0, x_{i+3}^1, x_{i+5}^1, x_{i+6}^1\right\},$$
$$\left\{x_i^1, x_{i+3}^2, x_{i+5}^2, x_{i+6}^2\right\}, \text{ and } \left\{x_i^2, x_{i+1}^0, x_{i+2}^0, x_{i+4}^0\right\} \text{ for } i \in \mathbb{Z}_7.$$

□

Lemma 4.41 *The lower bound is realized for $N = 26$, i.e., $A(6, 26) = 226$.*

Proof: Let the vertices of K_{26} be $\{\alpha\} \cup \{\beta\} \cup \{x_i^j,\ i \in \mathbb{Z}_8, j \in \mathbb{Z}_3\}$. The decomposition consists of the K_2 $\{\alpha, \beta\}$, plus the 8 $A_{6,5}$ $\left\{x_i^0, x_{i+5}^2, x_{i+6}^1, x_{i+2}^2, x_{i+7}^1\right\}$, plus the 16 K_4 $\left\{\alpha, x_i^0, x_i^1, x_i^2\right\}$ and $\left\{\beta, x_i^0, x_{i+1}^2, x_{i+3}^1\right\}$, $i \in \mathbb{Z}_8$, plus the 24 K_4 $\{x_i^j, x_{i+1}^j, x_{i+2}^{j+1}, x_{i+5}^{j+1}\}$, $i \in \mathbb{Z}_8$ and $j \in \mathbb{Z}_3$, and plus the 6 K_4 $\{x_i^j, x_{i+2}^j, x_{i+4}^j, x_{i+6}^j\}$, $i = 0, 1$ and $j \in \mathbb{Z}_3$. □

Lemma 4.42 *The lower bound is realized for $N = 29$, i.e., $A(6, 29) = 281$.*

Proof: Let $V = V_1 \cup V_2$ with $|V_1| = 8$ and $|V_2| = 21$, and let the vertices of V_1 be $\{i,\ i \in Z_8\}$.

The K_8 on V_1 can be decomposed into one C_4, 2 $B_{6,5}$, and 2 K_4. The K_{21} on V_2 can be partitioned into 10 parallel classes $\mathcal{C}_j,\ j \in \mathbb{Z}_{10}$, each consisting of 7 triangles $\mathcal{C}_{j,k},\ k \in \mathbb{Z}_7$, by the existence of a resolvable $(21, 3, 1)$-design. Finally, like for $N = 20$ (Lemma 4.39), we build for each $i \in \mathbb{Z}_8$, 7 K_4 on node i and class $\mathcal{C}_{i,k}$, so altogether 56 K_4; then we pair two by two the triangles of the last two classes $\mathcal{C}_8$ and $\mathcal{C}_9$ to obtain 7 $A_{6,5}$. Altogether we have decomposed K_{29} into 1 C_4, 9 graphs of type $A_{6,5}$ or $B_{6,5}$, and 58 K_4. □

Lemma 4.43 *The lower bound is realized for $N = 32$, i.e., $A(6, 32) = 342$.*

Proof: Let the vertices of K_{32} be $\{\alpha, \beta, \gamma, \delta, \epsilon\} \cup V_1 \cup V_2 \cup V_3$, where $|V_j| = 9$, $J = 0, 1, 2$, and $V_j = \{x_i^j,\ i \in \mathbb{Z}_9\}$. The K_9 on V_j can be partitioned into four parallel classes $\mathcal{C}_k^j$, $k \in \mathbb{Z}_4$, each consisting of three triangles $\mathcal{C}_{k,l}^j$, $k \in \mathbb{Z}_4$, by the existence of a resolvable $(9, 3, 1)$-design. Let $\mathcal{C}_3^j = \{\{x_i^j, x_{3+i}^j, x_{6+i}^j\},\ i \in \mathbb{Z}_3\}$.

As for $N = 20$ (Lemma 4.39), we build for α 9 K_4 with classes $\mathcal{C}_0^j$, $j = 0, 1, 2$, for β 9 K_4 with classes $\mathcal{C}_1^j$, $j = 0, 1, 2$, and for γ 9 K_4 with classes $\mathcal{C}_2^j$, $j = 0, 1, 2$, so altogether 27 K_4. We also build the 45 K_4 $\left\{x_i^0, x_{i+3}^0, x_{i+4}^1, x_{i+4}^2\right\}$, $\left\{x_i^0, x_{i+2}^1, x_{i+8}^1, x_i^2\right\}$, $\left\{x_i^0, x_i^1, x_{i+2}^2, x_{i+5}^2\right\}$, $\left\{\delta, x_i^0, x_{i+3}^1, x_{i+7}^2\right\}$, and $\left\{\epsilon, x_i^0, x_{i+5}^1, x_{i+8}^2\right\}$, $i \in \mathbb{Z}_9$, and the 9 $A_{6,5}$ $\left\{x_i^0, x_{i+6}^1, x_{i+3}^2, x_{i+7}^1, x_{i+6}^2\right\}$, $i \in \mathbb{Z}_9$. Finally the K_5 on $\{\alpha, \beta, \gamma, \delta, \epsilon\}$ can be decomposed into a C_4 and one $A_{6,5}$. Altogether we have decomposed K_{32} into 1 C_4, 10 $A_{6,5}$, and 72 K_4. $\square$

Lemma 4.44 *The lower bound is realized for $N = 35$, i.e., $A(6, 35) = 409$.*

Proof: Let the vertices of K_{35} be $\{\alpha\} \cup \{\beta\} \cup \{x_i^j,\ i \in \mathbb{Z}_{11}, j \in \mathbb{Z}_3\}$. The decomposition consists of the K_2 $\{\alpha, \beta\}$, plus the 11 $A_{6,5}$ $\left\{x_i^0, x_{i+3}^1, x_{i+5}^2, x_{i+6}^1, x_{i+6}^2\right\}$, $i \in \mathbb{Z}_{11}$, plus the 88 K_4,

$$\left\{\alpha, x_i^0, x_{i+1}^1, x_{i+2}^2\right\}, \left\{\beta, x_i^0, x_{i+2}^1, x_{i+7}^2\right\}, \left\{x_i^0, x_{i+7}^1, x_{i+8}^1, x_{i+10}^1\right\},$$

$$\left\{x_i^1, x_{i+6}^2, x_{i+7}^2, x_{i+9}^2\right\}, \left\{x_i^2, x_i^0, x_{i+2}^0, x_{i+10}^0\right\}, \left\{x_i^0, x_{i+4}^0, x_{i+4}^1, x_{i+9}^1\right\},$$

$$\left\{x_i^1, x_{i+4}^1, x_{i+3}^2, x_{i+8}^2\right\}, \left\{x_i^0, x_{i+5}^0, x_{i+4}^2, x_{i+8}^2\right\}$$

for $i \in \mathbb{Z}_{11}$. $\square$

Lemma 4.45 *The lower bound is realized for $N = 36$, i.e., $A(6, 36) = 428$.*

Proof: First recall that K_{12} can be partitioned into four disjoint C_3 plus nine K_4. Thus let the vertices of K_{12} be labeled α_i, $i \in \mathbb{Z}_4$, and x_j, $j \in \mathbb{Z}_8$, such that $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ is one K_4 and the four C_3 are $\{\alpha_i, x_{2i}, x_{2i+1}\}$, $i \in \mathbb{Z}_4$.

Now let the 36 vertices be α_i, $i \in \mathbb{Z}_4$, and x_j^k, $j \in \mathbb{Z}_8$, and $k \in \mathbb{Z}_4$, and let $V_k = \left\{x_j^k,\ j \in \mathbb{Z}_8\right\}$. A partition of K_{36} uses

- the K_4 $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$;
- eight $A_{6,5}$, each the union of two C_3 $\left\{\alpha_i, x_{2i}^{2k}, x_{2i+1}^{2k}\right\}$ and $\left\{\alpha_i, x_{2i}^{2k+1}, x_{2i+1}^{2k+1}\right\}$, $i = 0, 1, 2, 3$ and $k = 0, 1$;
- the 8 remaining K_4 of the partition of the K_{12} on the vertices α_i, $i \in \mathbb{Z}_4 \cup \left\{x_j^k,\ j \in \mathbb{Z}_8\right\}$, removing the K_4 $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ and 4 C_3 $\left\{\alpha_i, x_{2i}^k, x_{2i+1}^k\right\}$, to obtain a total of 32 K_4;
- the 64 K_4 of the partition of the multipartite graph $K_{8\times 4}$ with vertex set $V_0 \cup V_1 \cup V_2 \cup V_3$.

Altogether the partition uses 8 $A_{6,5}$ and 97 K_4 and we have $A(6, 36) = 428$. $\square$

The following corollary facilitates a kind of induction in general constructions.

Corollary 4.46 *When $N = 36u + m$, $m = 0, 3, 6, 15, 21, 27, 33$, and $u \geq 4$, then $A(6, N) = 432u^2 + 24um - 4u + A(6, m)$.*

Proof: From Theorem 4.21 there exists a 4-GDD of type $36^u m^1$ for each $u \geq 4$ and $m \in \{3, 6, 9, \ldots, 33\}$. Thus $A(6, 36u + m) \geq uA(6, 36) + \frac{4 \cdot 36^2 \cdot u(u-1)}{6 \cdot 2} + \frac{4 \cdot 36 \cdot m \cdot u}{6} + A(6, m) = 432u^2 - 4u + 24um + A(6, m)$. □

We did not find decompositions for 18, 24, or 30 nor were we able to prove that the lower bound cannot be realized for those values.

For this reason, we need decompositions for larger values of N in order to compose them and obtain results for the whole congruence class (modulo 36). Moreover, since the bound cannot be realized for $N = 12$ we employ another result for the same class (see Theorem 4.51).

Lemma 4.47 *The lower bound is realized for $N = 117$, i.e., $A(6, 117) = 4550$.*

Proof: The design is based on $\mathbb{Z}_{104}$ with 13 infinite points to be added. Consider the blocks

$$\mathcal{B}_1 = \{\{1,50,51,92\}, \{1,5,26,63\}, \{2,29,55,56\}, \{2,6,25,31\}, \{2,40,49,62\}, \{2,28,71,89\}, \{1,30,60,66\}, \{2,59,92,103\}, \{2,69,78,83\}, \{1,56,74,77\}, \{1,11,93,98\}\},$$

and $\mathcal{B}_2 = \{\{8,49,102\}, \{5,16,89\}, \{3,77,92\}, \{6,71,90\}, \{4,62,74\}, \{7,41,43\}, \{1,70,72\}, \{2,61,99\}\}$.

Each block in $\mathcal{B}_1$ generates 52 blocks, by adding $2a$ to each element for $a \in \mathbb{Z}_{52}$ and reducing modulo 104. The differences covered by $\mathcal{B}_1 \cup \mathcal{B}_2$ form the set $\mathbb{Z}_{104} \setminus (\{8a : a \in \mathbb{Z}_{13}\} \cup \{52\})$. To be precise, a difference d that occurs actually occurs twice, once in a pair $\{a, a+d\}$ with a even, and once in a pair $\{b, b+d\}$ with b odd, so that all 104 pairs in the cyclic orbit comprising the pairs of difference d arise once. Adding the block $\{0, 8a, 24a, 72a\}$ covers the differences $\{8a : a \in \mathbb{Z}_{13} \setminus \{0\}\}$, and 104 blocks are generated by adding each element of $\mathbb{Z}_{104}$ and reducing modulo 104. The blocks in $\mathcal{B}_2$ together contain 24 entries whose residues modulo 26 are $\mathbb{Z}_{26} \setminus \{0, 13\}$. The blocks $\{\{b_1 + 26x, b_2 + 26x, b_3 + 26x\} : \{b_1, b_2, b_3\} \in \mathcal{B}_2, x \in \mathbb{Z}_4\}$ form a partial parallel class missing the elements $\{13a : a \in \mathbb{Z}_8\}$. Now add the infinite point ∞_0 to each block of this partial parallel class to form $\mathcal{B}_{2,0}$. Form a new partial parallel class $\mathcal{B}_{2,a}$ for $1 \leq a \leq 12$ by adding $2a$ to each noninfinite point (modulo 104) and replacing ∞_0 by ∞_a. Now place a (13,4,1)-design on the 13 infinite points.

Finally, form 13 F_4 as follows. For $0 \leq a < 13$, form an F_4 with center ∞_a and containing the triangles $\{\infty_a, a + 13x, a + 13x + 52\}$ for $x \in \mathbb{Z}_4$. □

Lemma 4.48 *The lower bound is met with equality for $N = 7 \cdot 117 + m$ for $m \in \{3, 21, 27, 33\}$, i.e., for $N \in \{822, 840, 846, 852\}$.*

Proof: Form a 4-GDD of type $117^7 m^1$, and place a decomposition with cost $A(6, 117)$ on each of the seven groups of size 117 and a decomposition with cost $A(6, m)$ on the last. □

4.3 Optimal general constructions

The following three results give constructions that meet the lower bound. Therefore they determine the value of $A(6, N)$ for all values of N with few exceptions.

Theorem 4.49 *The value of $A(6, N)$ for $N \equiv 1 \pmod 3$ is given by $A(6, N) = \lceil \frac{2R}{3} \rceil + \epsilon$, where $\epsilon = 2$ if $N \equiv 7$ or $10 \pmod{12}$ and 0 otherwise, except for $A(6, 7) = 17$, $A(6, 10) = 34$, $A(6, 19) = 119$.*

Proof: For $N \notin \{7, 10, 19\}$, by a result of Mills on covering K_N by K_4 (see Theorem 8.9 of [12]), there exists a covering of K_N with $\lceil \frac{N(N-1)}{12} \rceil$ K_4 and therefore $A = \lceil \frac{2R}{3} \rceil + \epsilon$. Lemmas 4.29, 4.32, and 4.38 give the result for the remaining values of N. □

Theorem 4.50 *If $N \equiv 2 \pmod 3$, then $A(6,N) = \frac{2R+N+2}{3}$, except possibly for $N = 17$.*

Proof: *Case* 1. $N \equiv 2$ or $11 \pmod{12}$.

To prove the theorem for $N \equiv 2$ or $11 \pmod{12}$, we show that K_N can be decomposed into one K_2, $\frac{N-2}{3}$ $A_{6,5}$ or $B_{6,5}$ and K_4.

- The result is true for $N = 2, 11, 14$ (Lemmas 4.33 and 4.35); for $N = 11$ the decomposition uses 1 K_2, 3 $A_{6,5}$, and 6 K_4; for $N = 14$ the decomposition uses 1 K_2, 4 $B_{6,5}$, and 11 K_4.
- If $N = 12u+2$, $u \geq 4$, then $K_{12u+2} - K_2$ can be decomposed into u $K_{14} - K_2$ and $K_{12\times u}$. Furthermore, each $K_{14} - K_2$ can be decomposed into 4 $B_{6,5}$ and 11 K_4 (Lemma 4.35), and $K_{12\times u}$ can be decomposed into $12u(u-1)$ K_4 (existence of a 4-GDD of type 12^u by Theorem 4.12).
- If $N = 12u+11$, $u \geq 4$, then $K_{12u+11} - K_2$ can be decomposed into u $K_{14} - K_2$, one $K_{11} - K_2$, and $K_{12\times u,9}$. Furthermore $K_{14} - K_2$ and $K_{11} - K_2$ can be decomposed into $A_{6,5}$ or $B_{6,5}$ and K_4 (Lemmas 4.33 and 4.35), and $K_{12\times u,9}$ can be decomposed into K_4 (existence of a 4-GDD of type $12^u 9$ by Theorem 4.12).
- The theorem is also true for $N = 23, 26, 35$ by Lemmas 4.40, 4.41, and 4.44 and for $N = 38, 47$; for $N = 38$ (resp., 47), $K_{38} - K_2$ (resp., $K_{47} - K_2$) can be decomposed into four (resp., 5) $K_{11} - K_2$ plus $K_{9\times 4}$ (resp., $K_{9\times 5}$). Each $K_{11} - K_2$ can be decomposed into $A_{6,5}$ (resp., $B_{6,5}$) and K_4 (Lemmas 4.33 and 4.35), and $K_{9\times 4}$ (resp., $K_{9\times 5}$) can be decomposed into K_4 (existence of a 4-GDD of type 9^4 and 9^5).

Case 2. $N \equiv 5$ or $8 \pmod{12}$.

In this case, we prove that K_N can be decomposed into one C_4, $\frac{N-2}{3}$ $A_{6,5}$, or $B_{6,5}$ and K_4.

- That is true for $N = 5, 8$ (Lemmas 4.28 and 4.30); for $N = 5$, the decomposition uses one C_4 and one $A_{6,5}$; for $N = 8$ the decomposition uses one C_4, two $B_{6,5}$, and two K_4.
- If $N = 12u + 5$ (resp., $12u + 8$), $u \geq 4$, then K_N can be decomposed into u $K_{14} - K_2$, one K_5 (resp., K_8), and one $K_{12\times u,3}$ (resp., $K_{12\times u,6}$). Furthermore, each $K_{14} - K_2$ can itself be decomposed into $B_{6,5}$ and K_4, and $K_{12\times u,3}$ (resp., $K_{12\times u,6}$) into K_4 (existence of a 4-GDD of type $12^u 3$ and $12^u 6$).
- The theorem is also true for $N = 20, 29, 32$ by Lemmas 4.39, 4.42, and 4.43 and for $N = 41, 44$; for $N = 41$ (resp., 44), we use the decomposition of K_N into four $K_{11} - K_2$, one K_5 (resp., K_8), and $K_{9\times 4,3}$ (resp., $K_{9\times 4,6}$).
- It remains for us to solve the case $N = 17$.

$\square$

Theorem 4.51 *If $N \equiv 0 \pmod 3$, then $A(6,N) = \lceil \frac{6R+2N}{9} \rceil + \epsilon$, where $\epsilon = 1$ if $N \equiv 18, 27 \pmod{36}$, and $\epsilon = 0$ otherwise, except for $N \in \{9, 12\}$ and possibly when*

$$\begin{array}{rcl}
N \equiv 0 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \in \{2,3\}, \\
N \equiv 3 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \in \{1,2,3\}, \\
N \equiv 6 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \in \{1,2,3\}, \\
N \equiv 9 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \in \{1,2,4,5,6,7,8,9,10\}, \\
N \equiv 12 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \leq 70, \neq 23,
\end{array}$$

$$
\begin{array}{lll}
N \equiv 15 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \in \{1,2,3\}, \\
N \equiv 18 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \leq 70, \neq 23, \\
N \equiv 21 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \in \{1,2,3\}, \\
N \equiv 24 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \leq 71, \neq 23, \\
N \equiv 27 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \in \{1,2,3\}, \\
N \equiv 30 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \leq 68, \neq 22, \\
N \equiv 33 \pmod{36} & \textit{and} & \lfloor N/36 \rfloor \in \{1,2,3\}.
\end{array}
$$

Proof: First we treat cases when $N \equiv 0, 3, 6, 15, 21, 27, 33 \pmod{36}$. By Lemma 4.28 we have $A(6,3) = 3$ and $A(6,6) = 12$. By Corollary 4.37 we have $A(6,15) = 74$, $A(6,21) = 145$, $A(6,27) = 241$, and $A(6,33) = 360$. From Lemma 4.45 we have $A(6,36) = 428$. Then applying Corollary 4.46 we have $A(6, 36u + m) = 432u^2 + 24um - 4u + A(6,m)$, $u \geq 4$, and $m = 0, 3, 6, 15, 21, 27, 33$.

To treat $N \equiv 9 \pmod{36}$, use Lemmas 4.47 and 4.45, together with a 4-GDD of type $36^u 117^1$ from Theorem 4.26 to establish that $A(6, 36u+117) = u \cdot A(6,36) + A(6,117) + 432u(u-1) + 2808u$ for $u \geq 8$.

Finally, to handle $N = 36u + m$ with $m \in \{822, 840, 846, 852\}$ (corresponding to cases when $N \equiv 30, 12, 18, 24 \pmod{36}$), use Lemmas 4.45 and 4.48, together with a 4-GDD of type $36^u m^1$ from Theorem 4.26 to obtain the result. The ingredient designs are available when $u \geq 47$ for $m = 822$, $u \geq 48$ for $m \in \{840, 846\}$, and $u \geq 49$ for $m = 852$. □

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