Signal Analysis by Expansion Over the Squared Eigenfunctions of an Associated Schrödinger Operator

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Abstract

This article introduces a new signal analysis method. The main idea consists in interpreting a pulse-shaped signal, after multiplying it by a positive parameter, as a potential of a Schrödinger operator and representing this signal with the discrete spectrum of this operator. We present some results obtained in the analysis of the arterial blood pressure with this method.

Introduction

Let $H(V)$ be a Schrödinger operator in $L^2(\mathbb{R})$:

$$H(V)\psi = -\frac{d^2}{dx^2}\psi + V\psi, \quad \psi \in \mathcal{D}(H(V)) = H^2(\mathbb{R}), \quad (1)$$

where V, called a potential, satisfies:

$$V \in L_1^1(\mathbb{R}), \quad \frac{\partial^m}{\partial x^m}V \in L^1(\mathbb{R}), \quad m = 1, 2, \quad (2)$$

with

$$L_1^1(\mathbb{R}) = \{V|\int_{-\infty}^{+\infty}|V(x)|(1+|x|)dx < \infty\}. \quad (3)$$

$H^2(\mathbb{R})$ refers to the two order Sobolev space. The spectral problem of $H(V)$ is given by:

$$-\frac{d^2\psi}{dx^2} + V(x,t)\psi = k^2\psi, \quad k \in \overline{\mathbb{C}}^+, \quad x \in \mathbb{R}, \quad (4)$$

where k^2 and ψ are respectively the eigenvalues of $H(V)$ and the associate eigenfunctions. Under hypothesis (2), $H(V)$ is self adjoint and its spectrum consists of:

- a continuous spectrum equal to $[0, +\infty)$,
- a discrete spectrum composed of negative eigenvalues of multiplicity 1.

Let y be a real valued function representing the signal to be analyzed such that:

$$y \in L_1^1(\mathbb{R}), \quad y(x) \geq 0, \quad \forall x \in \mathbb{R},$$
$$\frac{\partial^m y}{\partial x^m} \in L^1(\mathbb{R}), \quad m = 1, 2, \quad (5)$$

then $H(-\chi y)$ is defined in $L^2(\mathbb{R})$ for all $\chi > 0$ by:

$$H(-\chi y) = -\frac{d^2}{dx^2} - \chi y, \quad \mathcal{D}(H(-\chi y)) = H^2(\mathbb{R}). \quad (6)$$

Under hypothesis (5), there is a non-zero ([1], corollary 2.4.4), finite number N_χ ([2], theorem 1) of negative eigenvalues of the operator $H(-\chi y)$.

1 Semi-classical signal analysis (SCSA)

We define the SCSA approximation by:

Definition 1.1. *Let y be a real valued function satisfying hypothesis (5) and χ a positive parameter, then the SCSA approximation is defined by:*

$$y_\chi(x) = \frac{4}{\chi}\sum_{n=1}^{N_\chi}\kappa_{n\chi}\psi_{n\chi}^2(x), \quad x \in \mathbb{R}, \quad (7)$$

where $-\kappa_{n\chi}^2$ are the negative eigenvalues of $H(-\chi y)$ with $\kappa_{n\chi} > 0$ and $\kappa_{1\chi} > \kappa_{2\chi} > \cdots > \kappa_{n\chi}$, $n = 1, \cdots, N_\chi$ and $\psi_{n\chi}$, $n = 1, \cdots, N_\chi$ are the associate L^2-normalized eigenfunctions.

The values $\frac{\kappa_{n\chi}^2}{\chi}$ can be interpreted as particular values of a signal that can not be extracted using usual sampling methods: values on the discretisation points and extremal values. The semi-classical interpretation related to the Bohr-Sommerfeld quantification seems to be natural in this case [3]. Let us illustrate this idea briefly. Suppose that y, $\forall x \in \mathbb{R}$ is a positive signal that vanishes rapidly. We denote $h = \frac{1}{\sqrt{\chi}}$ and $\lambda_h = \frac{k^2}{\chi}$. Then the spectral problem of $H(-\chi y)$ can be written in the following form:

$$-h^2\frac{d^2\psi}{dx^2}(\sqrt{\lambda_h\chi}, x) - y(x)\psi(\sqrt{\lambda_h\chi}, x) = \lambda_h\psi(\sqrt{\lambda_h\chi}, x). \quad (8)$$

When $\chi \to +\infty$, hence $h \to 0$, the problem is equivalent to a semi-classical problem [4]. If $0 \leq y(x) \leq y_{max}$, $\forall x \in \mathbb{R}$, then it is well-known that the negative
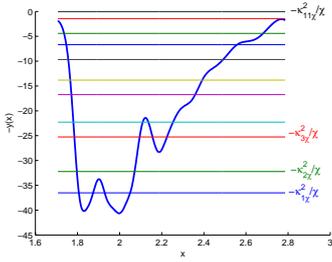
Figure 1: The negative eigenvalues of the Schrödinger operator $-h^2 \dfrac{d^2}{dx^2} - y(x)$

Figure 2: ABP estimation with the SCSA

eigenvalues of the operator $-h^2 \dfrac{d^2}{dx^2} - y(x)$, namely $-\dfrac{\kappa_{n\chi}^2}{\chi}$, $n = 1, \cdots, N_\chi$ are comprised between $-y_{max}$ and 0 as it is illustrated in figure 1. For a fixed value of χ, they correspond to particular values of $-y$.

Therefore, we can associate to a signal, taken as a potentiel well, the values $\dfrac{\kappa_{n\chi}^2}{\chi}$, $n = 1, \cdots, N_\chi$ which are used as some representative values of the signal. This is a new quantification approach: a semi-classical quantification. It is based on some standard results from semi-classical analysis gathered in the following proposition (for more details, see [3]):

Proposition 1.1. *i) Let y be a function satisfying hypothesis (5). Let $\chi > 0$ and $-\kappa_{n\chi}^2$, $n = 1, \cdots, N_\chi$ with $-\kappa_{1\chi}^2 < -\kappa_{2\chi}^2 < \cdots < 0$ the negative eigenvalues of $H(-\chi y)$. We suppose that $0 \leq y(x) \leq y_{max}$, $\forall x \in \mathbb{R}$ then, $\dfrac{\kappa_{n\chi}^2}{\chi} \leq y_{max}$, $n = 1, \cdots, N_\chi$.*

ii) Moreover if $y \in C^\infty(\mathbb{R})$ such that for one $\gamma_0 \in \mathbb{R}$, $\min_{\mathbb{R}} (-y + \gamma_0) > 0$ and for all $\alpha \in \mathbb{N}$, there is a constant $C_\alpha > 0$ such that $|\frac{\partial^\alpha y}{\partial x^\alpha}| \leq C_\alpha(-y + \gamma_0)$, then every regular value of y is an accumulation point of the set $(\frac{\kappa_{n\chi}^2}{\chi}, \chi > 0, n = 1, \cdots, N_\chi)$ (v is a regular value if $0 < v < y_{max}$ and if $y(x) = v$ then $|\frac{dy(x)}{dx}| > 0$).

2   Numerical results

Figure 2 shows the reconstruction of one beat of an Arterial Blood Pressure (ABP) signal with the SCSA. We noticed that only 5 to 10 negative eigenvalues are sufficient for a good reconstruction of an ABP signal. 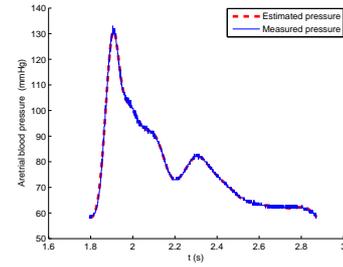The SCSA was also applied for the separation of the systolic and the diastolic pressures which describe fast and slow phenomena respectively [5]. We also point out that the SCSA introduces some interesting parameters that give relevant physiological information. These parameters are the negative eigenvalues and the so called invariants that consist in some momentums of $\kappa_{n\chi}$, $n = 1, \cdots, N_\chi$ [6]. For example, these new cardiovascular indices allow the discrimination between healthy patients and heart failure subjects [6].

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