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Constrained extremal problems in the Hardy space H^2 and Carleman's formulas

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Abstract: We study some approximation problems on a strict subset of the circle by analytic functions of the Hardy space H^2 of the unit disk (in $\mathbb{C}$), whose modulus satisfy a pointwise constraint on the complentary part of the circle. Existence and uniqueness results, as well as pointwise saturation of the constraint, are established. We also derive a critical point equation which gives rise to a dual formulation of the problem. We further compute directional derivatives for this functional as a computational means to approach the issue. We then consider a finite-dimensional polynomial version of the bounded extremal problem.

Key-words: Hardy spaces, analytic functions, approximation, bounded extremal problems, Carleman formulas.

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Problèmes extrémaux contraints dans l'espace de Hardy H^2 et formules de Carleman

Résumé : Nous étudions des problèmes d'approximation sur un strict sous-ensemble du cercle par des fonctions analytiques de l'espace de Hardy H^2 du disque unité (de $\mathbb{C}$), soumises en module à une contrainte ponctuelle sur la partie complémentaire du cercle. Des résultats d'existence, d'unicité et de saturation de la contrainte sont établis, ainsi qu'une équation aux points critiques qui permet de proposer une formule de dualité. Nous considérons enfin une version polynômiale (de dimension finie) de ces problèmes extrémaux bornés.

Mots-clés : Espaces de Hardy, fonctions analytiques, approximation, problèmes extrémaux bornés, formules de Carleman.

1 Introduction

If D is a finitely connected plane domain with rectifiable boundary ∂D, a holomorphic function f in the Smirnov class $\mathcal{E}^1(D)$ can be recovered from its boundary values by the Cauchy formula [17]. When the boundary values are only known on a strict subset I of ∂D having positive linear measure, they still define f uniquely but the recovery cannot be achieved in closed form. In fact, it becomes a special case of a classical ill-posed issue namely the Cauchy problem for the Laplace equation. This issue is quite important in physics and engineering [21, 26, 31].
Following an original idea of Carleman, one approach to the recovery of f from its knowledge on I is to introduce an auxiliary "quenching" function φ, holomorphic and bounded in D, such that $|\varphi| \equiv 1$ a.e. on $\partial D \setminus I$ and $|\varphi| > 1$ in D; such a function is easily constructed by solving a Dirichlet problem for $\log|\varphi|$. In [20], it was proven by Goluzin and Krylov that

$$f(z) = \lim_{n\to\infty} f_n(z), \quad \text{where} \quad f_n(z) \triangleq \frac{1}{2\pi}\int_I \left(\frac{\varphi(\xi)}{\varphi(z)}\right)^n \frac{f(\xi)}{\xi - z}\, d\xi, \qquad z \in D, \tag{1}$$

the convergence being locally uniform in D. Cauchy integrals like those defining f_n in (1) are called *Carleman's formulas* [2]. On the unit disk $\mathbb{D}$ where $\mathcal{E}^p(\mathbb{D})$ coincides with the Hardy class H^p, see [17], it is proved in [32] that if $f \in H^p$ with $1 < p < \infty$, then the convergence actually holds in H^p .
Two questions arise naturally, namely what is the meaning of f_n for *fixed* n, and what is its asymptotic behaviour if $f \in L^p(I)$ is *not* the trace of a Hardy function? On $\mathbb{D}$, when $f \in L^2(I)$ and φ is a quenching function with the additional property that $|\varphi|$ is constant a.e. on I, it was proven in [7] that f_n is closest to f in $L^2(I)$-norm among all $g \in H^2$ such that $\|g\|_{L^2(\mathbb{T}\setminus I)} \le \|f_n\|_{L^2(\mathbb{T}\setminus I)}$, where $\mathbb{T}$ denotes the unit circle. In the present paper, among other things, we will see that if φ is holomorphic and bounded on $\mathbb{D}$ together with its inverse, then f_n is closest to f w.r.t. the weighted $L^2(|\varphi_{|_I}|^2, I)$-norm among all $g \in H^2$ such that $|g| \le |f_n|$ a.e. on $\mathbb{T} \setminus I$. These extremal properties of f_n are all the more remarkable than Carleman's formulas were originally defined without reference to optimization.
Still one point is unsatisfactory, namely the extremal properties of f_n we just mentioned are implicit in that the level of the pointwise constraint on $\mathbb{T} \setminus I$ is $|f_n|$ itself. This is why we make a slight twist and we rather investigate on $\mathbb{D}$ the following extremal problem. Let $I \subset \mathbb{T}$ be a subset of positive Lebesgue measure and set $J = \mathbb{T} \setminus I$ for the complementary subset. The question we raise is the following.

BEP: Given $f \in L^2(I)$ and $M \in L^2(J)$, $M \ge 0$, find $g_0 \in H^2$ such that $|g_0(e^{i\theta})| \le M(e^{i\theta})$ a.e. on J and

$$\|f - g_0\|_{L^2(I)} = \min_{\substack{g\in H^2 \\ |g|\le M \text{ a.e. on } J}} \|f - g\|_{L^2(I)} \,. \tag{2}$$

This should be compared with the so-called *bounded extremal problems* BEP_p. studied in [3, 7, 8] for $1 \le p \le \infty$.

BEP_p: Given $f \in L^p(I)$, $\psi \in L^p(J)$ and a positive constant C, find $g_0 \in H^p$ such that

$$\|g_0 - \psi\|_{L^p(J)} \leq C \quad \text{and}$$

$$\|f - g_0\|_{L^p(I)} = \min_{\substack{g \in H^p \\ \|g-\psi\|_{L^p(J)} \leq C}} \|f - g\|_{L^p(I)} \,. \tag{3}$$

Note that in problem BEP (2), we did not introduce a reference function ψ on J as was done in BEP_p (3). While it is straightforward to handle such a generalization when ψ is the trace on J of a H^2 function, the general case conceals further difficulties that are left here for further research.
When I is of full measure, both problems (2) and (3) reduce to classical extremal problems, see *e.g.* [17, 19]. Therefore we limit our discussion to the case where J has positive measure.
The first reference dealing with bounded extremal problems seems to be [24], where BEP_2 is studied for $f = 0$ and I an interval on the half-plane rather than the disk. The case $\psi = 0$ is solved in [3] using Toeplitz operators, and error rates when C goes large and I is an arc can be found in [6]. Weighted versions of BEP_2 for $L^2(|\varphi|^2)$-norms as those discussed above were also solved in [28].
The general version BEP_p in the range $1 \leq p < \infty$ is taken up in [7] where the link with Carleman's formulas is pointed out, while existence and uniqueness results are also presented. Reformulations of BEP_p in abstract Hilbert or smooth Banach space settings were carried out in [14, 15, 29, 39], leading to the construction of backward minimal vectors and hyperinvariant subspaces for certain classes of operators that need not be compact nor quasinilpotent, thereby generalizing [4]. Versions of BEP_2 where the constraint bears on the imaginary part rather than the modulus, useful among other things to approach inverse Dirichlet-Neumann problems, are presented in [22, 27]. Together with meromorphic generalizations, problem BEP_p was studied in [10] for $p \geq 2$, while problem BEP_∞ was studied in [8, 9], with related completion issues.
A major incentive to study BEP_p came from engineering problems, more precisely from questions pertaining to system identification and design. This motivation is quite explicit in [24], and all-pervasive in [3, 6, 8, 9, 38] whose results have been used effectively to identify hyperfrequency filters [5]. The connection with identification is more transparent on the half plane, where f represents the so-called transfer-function of a linear dynamical system as measured in the frequency bandwidth I using harmonic identification techniques. Recall that a linear dynamical system is just a convolution operator, and that its transfer function is the Fourier-Laplace transform of its kernel [16]. Now, by the Paley-Wiener and Hausdorff-Young theorems, causality and $L^r \to L^s$ stability of the system cause f to belong to the Hardy class $\mathcal{H}^p$ of the half plane with $1/p = 1/r - 1/s$, as soon as the latter is less than or equal to $1/2$. Because f is only known up to modelling and measurement errors, one is led to approximate the data on I by a $\mathcal{H}^p$ function while controlling its deviation from some reference behaviour ψ outside I, which is precisely the analog of (3) on the half-plane. It is mapped to BEP_p *via* the isometry $g \mapsto (1+w)^{-2/p} g((w-1)/(w+1))$ from H^p onto $\mathcal{H}^p$. More on the relations between Hardy spaces, system identification and control can be found in [18, 30, 31]. Note that in BEP_p, it is indeed essential to bound the behaviour of g_0 on J, for traces of Hardy functions are dense

in $L^p(I)$ (in $C(I)$ if $p = \infty$) so that BEP_p has no solution if $C = \infty$ unless f is already the trace of a Hardy function. In practice, since modelling and measurement errors will prevent this from ever happening, the error $\|f - g\|_{L^p(I)}$ can be made arbitrarily small at the cost of $\|g\|_{L^p(J)}$ becoming arbitrarily large, which does not make for a valid identification scheme.

The present paper deals with a mixed situation, where an integral criterion is minimized on I under a pointwise constraint on J. Here again, the motivation of the authors stems from system identification. Indeed, the L^2 norm on I has a probabilistic interpretation, being the variance of the output when the system is fed by noise whose spectrum is uniformly distributed in the bandwidth, that suits some classical framework. On another hand, it is often the case that the transfer function has to meet uniform bounds for physical reasons. For instance when identifying a passive device, it should be less than 1 at all frequencies. This way one is led to consider problem (2) with $M \equiv 1$ (which is $BEP_{2,\infty}$ below).

Such issues and motivations are also the topics of the recent work [36, 37], where BEP-like problems are considered in $L^p(I)$, for $1 \leq p \leq \infty$, with a pointwise constraint acting on the whole $\mathbb{T}$, when the approximated function f and the constraint M are assumed to be continuous functions (on $\mathbb{T}$ and I, respectively), with $M > 0$ and $|f| \leq M$ on I.

Problem (2) is considerably more difficult to analyze than BEP_2, due to the fact that pointwise evaluation is not smooth –actually not even defined– in $L^2(J)$. Its solution depends in a rather deep fashion on the multiplicative structure of Hardy functions and all our results, beyond existence and uniqueness, will hold under the extra-assumption that the boundary of I has measure zero. We do not know the extend to which this assumption can be relaxed.

The organization of the paper is as follows. In section 2 we set up some notation and recall standard properties of H^p-spaces, BMO and conjugate functions. Section 3 deals with existence and uniqueness issues, as well as pointwise saturation of the constraint. In section 4 we establish an analog, in this nonsmooth and infinite-dimensional context, of the familiar critical point equation from convex analysis. It gives rise to a saddle-point characterization of the optimal value that yields a dual formulation of the problem. The latter is connected in section 5 to Toepliz operators and Carleman's formulas, and used to compute the concave dual functional whose maximization is tantamount to solve the problem. We further compute directional derivatives for this functional as a means to approach the issue from a computational point of view. Section 6 is devoted to a finite-dimensional polynomial version of (2), valid when I is a union of arcs, which is of interest in its own right and provides an alternative way to constructively approximate the solution to the original problem.

2 Notations and preliminaries

Let $\mathbb{T}$ be the unit circle endowed with the normalized Lebesgue measure ℓ, and I a subset of $\mathbb{T}$ such that $\ell(I) > 0$ with complementary subset $J = \mathbb{T} \setminus I$. To avoid dealing with trivial instances of problem (2) *we assume throughout that* $\ell(J) > 0$.

If h_1 (resp. h_2) is a function defined on a set containing I (resp. J), we use the notation $h_1 \vee h_2$ for the concatenated function, defined on the whole of $\mathbb{T}$, which is h_1 on I and h_2 on J.

For $E \subset \mathbb{T}$, we let ∂E and $\overset{\circ}{E}$ denote respectively the boundary and the interior of E when viewed as a subset of $\mathbb{T}$; we also write χ_E for the characteristic function of E and $h_{|_E}$ to mean the restriction to E of a function h defined on a set containing E.

When $1 \leq p \leq \infty$, we write $L^p(E)$ for the familiar Lebesgue space of (equivalence classes of a.e. coinciding) complex-valued measurable functions on E with finite L^p norm, and we indicate by $L^p_{\mathbb{R}}(E)$ the real subspace of real-valued functions. Likewise $C(E)$ stands for the space of complex-valued continuous functions on E, while $C_{\mathbb{R}}(E)$ indicates real-valued continuous functions. The norm on $L^p(E)$ is denoted by $\| \ \|_{L^p(E)}$, and if h is defined on a set containing E we write for simplicity $\|h\|_{L^p(E)}$ to mean $\|h_{|_E}\|_{L^p(E)}$. When E is compact the norm of $C(E)$ is the *sup* norm.

Recall that the Hardy space H^p is the closed subspace of $L^p(\mathbb{T})$ consisting of functions whose Fourier coefficients of strictly negative index do vanish. These are the nontangential limits of functions analytic in the unit disk $\mathbb{D}$ having uniformly bounded L^p means over all circles centered at 0 of radius less than 1. The correspondence is one-to-one and, using this identification, we alternatively regard members of H^p as holomorphic functions in the variable $z \in \mathbb{D}$. This extension is obtained from the values on $\mathbb{T}$ through a Cauchy as well as a Poisson integral [35, ch. 17, thm 11], namely if $g \in H^p$ then, for $z \in \mathbb{D}$:

$$g(z) = \frac{1}{2\,i\,\pi} \int_{\mathbb{T}} \frac{g(\xi)}{\xi - z}\, d\xi \text{ and } g(z) = \frac{1}{2\,\pi} \int_{\mathbb{T}} \mathrm{Re}\left\{\frac{e^{i\theta} + z}{e^{i\theta} - z}\right\} g(e^{i\theta})\, d\theta\,. \qquad (4)$$

Because of this Poisson representation, $g(re^{i\theta})$ converges to $g(e^{i\theta})$ in $L^p(\mathbb{T})$ as soon as $1 \leq p < \infty$. Moreover, (4) entails that, for $1 \leq p \leq \infty$, a Hardy function g is uniquely determined, up to a purely imaginary constant, by its real part h on $\mathbb{T}$:

$$g(z) = i\mathrm{Im}g(0) + \frac{1}{2\,\pi} \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, h(e^{i\theta})\, d\theta\,, \qquad z \in \mathbb{D}. \qquad (5)$$

The integral in the right-hand side of (5) is called the *Riesz-Herglotz transform* of h and, whenever $h \in L^1_{\mathbb{R}}(\mathbb{T})$, it defines a holomorphic function in $\mathbb{D}$ which is real at 0 and whose nontangential limit exists a.e. on $\mathbb{T}$ with real part equal to h. However, only if $1 < p < \infty$ is it guaranteed that $g \in H^p$ when $h \in L^p_{\mathbb{R}}(\mathbb{T})$. In fact, the Riesz-Herglotz transform assumes the form $h(e^{i\theta}) + i\widetilde{h}(e^{i\theta})$ a.e. on $\mathbb{T}$, where the real-valued function $\widetilde{h}$ is said to be *conjugate* to h, and the property that $\widetilde{h} \in L^p_{\mathbb{R}}(\mathbb{T})$ whenever $h \in L^p_{\mathbb{R}}(\mathbb{T})$ holds true for $1 < p < \infty$ but not for $p = 1$ nor $p = \infty$. The map $h \to \widetilde{h}$ is called the *conjugation operator*, and for $1 < p < \infty$ it is bounded $L^p_{\mathbb{R}}(\mathbb{T}) \to L^p_{\mathbb{R}}(\mathbb{T})$ by a theorem of M. Riesz [19, chap. III, thm 2.3]; in this range of exponents, we will denote its norm by K_p. It

follows easily from Parseval's relation that $K_2 = 1$, but it is rather subtle that $K_p = \tan(\pi/(2p))$ for $1 < p \leq 2$ while $K_p = \cot(\pi/(2p))$ for $2 \leq p < \infty$ [33].
A sufficient condition for $\widetilde{h}$ to be in $L^1(\mathbb{T})$ is that h belongs to the the so-called *Zygmund class* $L\log^+ L$, consisting of measurable functions ϕ such that $\phi \log^+ |\phi| \in L^1(\mathbb{T})$ where we put $\log^+ t = \log t$ if $t \geq 1$ and 0 otherwise. More precisely, if we denote by m_h the *distribution function* of h defined on $\mathbb{R}^+$ with values in $[0, 1]$ according to the formula

$$m_h(\tau) = \ell\left(\{\xi \in \mathbb{T};\ |h(\xi)| > \tau\}\right),$$

and if we further introduce the non-increasing rearrangement of h given by

$$h^*(t) = \inf\{\tau;\ m_h(\tau) \leq t\}, \qquad t \geq 0,$$

it turns out that $h \in L\log^+ L$ if and only if the quantity

$$\|h\|_{L\log^+ L} \triangleq \int_0^1 h^*(t)\log(1/t)\,dt \tag{6}$$

is finite [11, lem. 6.2.], which makes $L\log^+ L$ into a Banach function space. Then, it is a theorem of Zygmund [11, cor. 6.9.] that

$$\|\widetilde{h}\|_{L^1(\mathbb{T})} \leq C_0 \|h\|_{L\log^+ L} \tag{7}$$

for some universal constant C_0. A partial converse, due to M. Riesz, asserts that if a real-valued h is bounded from below and if moreover $\widetilde{h} \in L^1(\mathbb{T})$, then $h \in L\log^+ L$ [11, cor. 6.10].
We mentioned already that $\widetilde{h}$ needs not be bounded if $h \in L^\infty_{\mathbb{R}}(\mathbb{T})$. In this case all one can say in general is that $\widetilde{h}$ has *bounded mean oscillation*, meaning that $\widetilde{h} \in L^1(\mathbb{T})$ and

$$\|\widetilde{h}\|_{BMO} \triangleq \sup_E \frac{1}{\ell(E)} \int_E |\widetilde{h} - \widetilde{h}_E|\,d\theta < \infty, \qquad \text{with } \widetilde{h}_E \triangleq \frac{1}{\ell(E)} \int_E \widetilde{h}\,d\theta,$$

where the *supremum* is taken over all subarcs $E \subset \mathbb{T}$. Actually [19, chap. VI, thm 1.5], there is a universal constant C_1 such that

$$\|\widetilde{h}\|_{BMO} \leq C_1 \|h\|_{L^\infty(\mathbb{T})}.$$

The subspace of $L^1(\mathbb{T})$ consisting of functions whose BMO-norm is finite is called BMO for short. Notice that $\|\ \|_{BMO}$ is a genuine norm modulo additive constants only. A theorem of F. John and L. Nirenberg [19, ch. VI, thm. 2.1] asserts there are positive constants C, c, such that, for each real-valued $\varphi \in BMO$, every arc $E \subset \mathbb{T}$, and any $x > 0$,

$$\frac{\ell\left(\{t \in E:\ |\varphi(t) - \varphi_E| > x\}\right)}{\ell(E)} \leq C \exp\left(\frac{-cx}{\|\varphi\|_{BMO}}\right). \tag{8}$$

Conversely, if (8) holds for some finite $A > 0$ in place of $\|\varphi\|_{BMO}$, every arc E and any $x > 0$, then $\varphi \in BMO$ and $A \sim \|\varphi\|_{BMO}$. The John-Nirenberg theorem easily implies that $BMO \subset L^p$ for all $p < \infty$. The space of H^1-functions whose boundary values lie in BMO will be denoted by $BMOA$, and $BMOA/\mathbb{C}$ is a Banach space equipped with the BMO-norm. Clearly $BMOA \subset H^p$ for $1 \leq p < \infty$, and $h + i\widetilde{h} \in BMOA$ whenever $h \in L^\infty(\mathbb{T})$. A sufficient condition for the boundedness of $\widetilde{h}$ is that h be Dini-continuous; recall that a function h defined on $\mathbb{T}$ is said to be Dini-continuous if $\omega_h(t)/t \in L^1([0,\pi])$, where

$$\omega_h(t) = \sup_{|\theta_1-\theta_2|\leq t} \left|h\left(e^{i\theta_1}\right) - h\left(e^{i\theta_2}\right)\right|, \qquad t \in [0,\pi],$$

is the modulus of continuity of h. Specifically [19, chap. III, thm 1.3], it holds that

$$\omega_{\widetilde{h}}(\rho) \leq C_2 \left(\int_0^\rho \frac{\omega_h(t)}{t}\,dt \,+\, \rho\int_\rho^\pi \frac{\omega_h(t)}{t^2}\,dt\right) \tag{9}$$

where C_2 is a constant independent of f. From (9) it follows easily that $\widetilde{h}$ is continuous if h is Dini-continuous, and moreover that

$$\|\widetilde{h}\|_{L^\infty(\mathbb{T})} \leq \omega_{\widetilde{h}}(\pi) \leq C_2 \int_0^\pi \frac{\omega_h(t)}{t}\,dt, \tag{10}$$

where the first inequality comes from the fact that $\widetilde{h}$ is continuous on $\mathbb{T}$ and therefore vanishes at some point since it has zero-mean.

We now turn to multiplicative properties of Hardy functions. It is well-known (see *e.g.* [17, 19, 23]) that a nonzero $f \in H^p$ can be uniquely factored as $f = jw$ where

$$w(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|\,d\theta\right\} \tag{11}$$

belongs to H^p and is called the *outer factor* of f, while $j \in H^\infty$ has modulus 1 a.e. on $\mathbb{T}$ and is called the *inner factor* of f. The latter may be further decomposed as $j = bS_\mu$, where

$$b(z) = e^{i\theta_0} z^k \prod_{z_l\neq 0} \frac{-\bar{z}_l}{|z_l|}\,\frac{z-z_l}{1-\bar{z}_l z} \tag{12}$$

is the *Blaschke product*, with order $k \geq 0$ at the origin, associated to the sequence $z_l \in \mathbb{D}\setminus\{0\}$ and to the constant $e^{i\theta_0} \in \mathbf{T}$, while

$$S_\mu(z) = \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\right\} \tag{13}$$

is the *singular inner factor* associated with μ, a positive measure on $\mathbb{T}$ which is singular with respect to Lebesgue measure. The z_l are of course the zeros of f in $\mathbb{D}\setminus\{0\}$, counted with their multiplicities, while k is the order of the zero at 0. If there are infinitely many zeros, the convergence of the product $b(z)$ in $\mathbb{D}$ is ensured by the condition $\sum_l(1-|z_l|) < \infty$ which holds automatically when $f \in H^p\setminus\{0\}$. If there are only finitely many z_l, we say that (12) is a finite Blaschke product; note that a finite Blaschke product may alternatively be defined as a rational function of the form q/q^R, where q is an algebraic polynomial whose roots lie in $\mathbb{D}$ and q^R indicates the *reciprocal polynomial* given by $q^R(z) = z^n\overline{q(1/\bar{z})}$ if n is the degree of q. The integer n is also called the degree of the Blaschke product.

That $w(z)$ in (11) is well-defined rests on the fact that $\log|f| \in L^1$ if $f \in H^1\setminus\{0\}$; this also entails that a H^p function cannot vanish on a subset of strictly positive

Lebesgue measure on $\mathbb{T}$ unless it is identically zero. For simplicity, we often say that a function is outer (resp. inner) if it is equal to its outer (resp. inner) factor.
Intimately related to Hardy functions is the Nevanlinna class N^+ consisting of holomorphic functions in $\mathbb{D}$ that can be factored as jE, where j is an inner function and E an outer function of the form

$$E(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\rho(e^{i\theta})\,d\theta\right\}, \tag{14}$$

with ρ a positive function such that $\log\rho \in L^1(\mathbb{T})$ (although ρ itself need not be summable). Such a function again has nontangential limits of modulud ρ a.e. on $\mathbb{T}$ that serve as definition of its boundary values. The Nevanlinna class will be instrumental to us in that $N^+ \cap L^p(\mathbb{T}) = H^p$, see for example [17, thm 2.11] or [19, 5.8, ch.II]. Thus formula (14) defines a H^p-function if, and only if $\rho \in L^p(\mathbb{T})$. A useful consequence is that, whenever $g_1 \in H^{p_1}$ and $g_2 \in H^{p_2}$, we have $g_1g_2 \in H^{p_3}$ if, and only if $g_1g_2 \in L^{p_3}$. In particular $g_1g_2 \in H^{p_3}$ if $1/p_1 + 1/p_2 = 1/p_3$.
It is a classical fact [19, ch. II, sec. 1] that a function f holomorphic in the unit disk belongs to H^p if, and only $|f|^p$, which is subharmonic in $\mathbb{D}$, has a harmonic majorant there. This makes for a conformally invariant definition of Hardy spaces over general domains in $\overline{\mathbb{C}}$. In this connection, the Hardy space $\bar{H}^p$ of $\overline{\mathbb{C}} \setminus \mathbb{D}$ can be given a treatment parallel to H^p using the conformal map $z \mapsto 1/z$. Specifically, $\bar{H}^p$ consists of L^p functions whose Fourier coefficients of strictly positive index do vanish; these are, a.e. on $\mathbb{T}$, the complex conjugates of H^p-functions, and they can also be viewed as nontangential limits of functions analytic in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ having uniformly bounded L^p means over all circles centered at 0 of radius bigger than 1. We also set $\overline{BMOA} = \bar{H}^1 \cap BMO$. We further single out the subspace $\bar{H}^p_0$ of $\bar{H}^p$, consisting of functions vanishing at infinity or, equivalently, having vanishing mean on $\mathbb{T}$. Thus, a function belongs to $\bar{H}^p_0$ if, and only if, it is a.e. on $\mathbb{T}$ of the form $e^{-i\theta}\overline{g(e^{i\theta})}$ for some $g \in H^p$. For $G \in \bar{H}^p_0$, the Cauchy formula assumes the form:

$$G(z) = \frac{1}{2\,i\,\pi}\int_{\mathbb{T}} \frac{G(\xi)}{z-\xi}\,d\xi\,, \qquad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}. \tag{15}$$

If E is a measurable subset of $\mathbb{T}$, we set

$$< f, g >_E = \frac{1}{2\pi}\int_E f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta \tag{16}$$

whenever $f \in L^p(E)$ and $g \in L^q(E)$ with $1/p+1/q = 1$. If f and g are defined on a set containing E, we often write for simplicity $< f, g >_E$ to mean $< f_{|_E}, g_{|_E} >$. The duality product $< \ , \ >_{\mathbb{T}}$ makes H^p and $\bar{H}^q_0$ orthogonal to each other, and reduces to the familiar scalar product on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$. We note in particular the orthogonal decomposition:

$$L^2(\mathbb{T}) = H^2 \oplus \bar{H}^2_0. \tag{17}$$

For $f \in C(\mathbb{T})$ and $\nu \in \mathcal{M}$, the space of complex Borel measures on $\mathbb{T}$, we set

$$\nu.f = \int_{\mathbb{T}} f(e^{i\theta})\,d\nu(\theta) \tag{18}$$

and this pairing induces an isometric isomorphism between $\mathcal{M}$ (endowed with the norm of the total variation) and the dual of $C(\mathbb{T})$ [35, thm 6.19]. If we let $\mathcal{A} \subset H^\infty$ designate the disk algebra of functions analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, and if $\mathcal{A}_0$ indicates those functions in $\mathcal{A}$ vanishing at zero, it is easy to see that $\mathcal{A}_0$ is the orthogonal space under (18) to those measures whose Fourier coefficients of strictly negative index do vanish. Now, it is a fundamental theorem of F. and M. Riesz that such measures have the form $d\nu(\theta) = g(e^{i\theta})\, d\theta$ with $g \in H^1$, so the Hahn-Banach theorem implies that H^1 is dual *via* (18) to the quotient space $C(\mathbb{T})/\mathcal{A}_0$ [19, chap. IV, sec. 1]. Equivalently, $\bar{H}_0^1$ is dual to $C(\mathbb{T})/\overline{\mathcal{A}}$ under the pairing arising from the line integral:

$$(\dot{f}, F) = \frac{1}{2i\pi} \int_{\mathbb{T}} f(\xi) F(\xi)\, d\xi\,, \tag{19}$$

where F belongs to $\bar{H}_0^1$ and $\dot{f}$ indicates the equivalence class of $f \in C(\mathbb{T})$ modulo $\overline{\mathcal{A}}$. This entails that, contrary to $L^1(\mathbb{T})$, the spaces H^1 and $\bar{H}_0^1$ enjoy a weak-* compactness property of their unit ball.

Finally, we define the analytic and anti-analytic projections $\mathbf{P}_+$ and $\mathbf{P}_-$ on Fourier series by:

$$\mathbf{P}_+\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=0}^{\infty} a_n e^{in\theta}, \quad \mathbf{P}_-\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=-\infty}^{-1} a_n e^{in\theta}\,.$$

Equivalent to the M. Riesz theorem is the fact that $\mathbf{P}_+ : L^p \to H^p$ and $\mathbf{P}_- : L^p \to \bar{H}_0^p$ are bounded for $1 < p < \infty$, in which case they coincide with the Cauchy projections:

$$\mathbf{P}_+(h)(z) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{h(\xi)}{\xi - z}\, d\xi, \mathbf{P}_-(h)(s) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{h(\xi)}{s - \xi}\, d\xi, \tag{20}$$

for $z \in \mathbb{D}$, $s \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. When restricted to $L^2(\mathbb{T})$, the projections $\mathbf{P}_+$ and $\mathbf{P}_-$ are just the orthogonal projections onto H^2 and $\bar{H}_0^2$ respectively. Likewise $\mathbf{P}_+ : L^\infty \to BMOA$ and $\mathbf{P}_- : L^\infty \to \overline{BMOA}$ are also bounded.

Although $\mathbf{P}_\pm(h)$ needs not be the Fourier series of a function when h is merely in $L^1(\mathbb{T})$, it is nevertheless Abel summable almost everywhere to a function lying in $L^s(\mathbb{T})$ for $0 < s < 1$, and it can still be interpreted as the trace of an analytic function in some Hardy space of exponent s that we did not introduce [17, cor. to thm 3.2]. To us it will be sufficient, when $h \in L^1$, to regard $\mathbf{P}_\pm(f)$ as the Fourier series of a distribution. Finally, we record for further reference the following elementary fact:

Lemma 1 *Let $v \in L^1(J)$ be such that $\mathbf{P}_+(0 \vee v) \in L^2(\mathbb{T})$. Then, whenever $g \in H^2$ is such that $g \in L^2(I) \vee L^\infty(J)$, it holds that*

$$< \mathbf{P}_+(0 \vee v)\,,\, g >_{\mathbb{T}} = < v\,,\, g >_J\,.$$

Proof. Since by hypothesis $\mathbf{P}_+(0 \vee v)$ is a *square summable function* on $\mathbb{T}$, we can define a function $u \in L^1(\mathbb{T})$ by the formula:

$$u = (0 \vee v) - \mathbf{P}_+(0 \vee v)\,,$$

and by the very definition of u all its Fourier coefficients of non-negative index do vanish hence $u \in \bar{H}^1_0$. In addition it is clear that $u_{|_I} \in L^2(I)$ and consequently, if $g \in H^2$ is such that $g \in L^2(I) \vee L^\infty(J)$, we have upon checking summability on I and J separately that $u\overline{g} \in \bar{H}^1_0$. Therefore we get:

$$< v\,,\, g >_J = < v\overline{g}\,,\, 1 >_J = < (0 \vee v)\,\overline{g}\,,\, 1 >_{\mathbb{T}}$$

$$= < u\,\overline{g}\,,\, 1 >_{\mathbb{T}} + < \mathbf{P}_+(0 \vee v)\,\overline{g}\,,\, 1 >_{\mathbb{T}}$$

$$= < \mathbf{P}_+(0 \vee v)\,\overline{g}\,,\, 1 >_{\mathbb{T}} = < \mathbf{P}_+(0 \vee v)\,,\, g >_{\mathbb{T}}$$

where the next-to-last equality uses that the mean of the $\bar{H}^1_0$-function $u\overline{g}$ is zero. ∎

3 Well-posedness of the bounded extremal problem BEP

We first reduce problem BEP (2) to a standard form $BEP_{2,\infty}$ where $M \equiv 1$. As the log-modulus of a nonzero Hardy function is integrable, we will safely assume that $\log M \in L^1(J)$ for otherwise the zero function is the only candidate approximant. Then, letting w_M be the outer function with modulus 1 on I and M on J, we have that g belongs to H^2 and satisfies $|g| \leq M$ a.e. on J if, and only if g/w_M lies in H^2 and satisfies $g/w_M \leq 1$ a.e. on J; it is so because g/w_M lies by construction in the Nevanlinna class N^+ whose intersection with $L^2(\mathbb{T})$ is H^2. Altogether, upon replacing f by f/w_M and g by g/w_M, we see that Problem (2) is equivalent to the following normalized case which is the one we shall really work with.

$BEP_{2,\infty}$: Given $f \in L^2(I)$, find $g_0 \in H^2$ such that $|g_0(e^{i\theta})| \leq 1$ a.e. on J and

$$\|f - g_0\|_{L^2(I)} = \min_{\substack{g \in H^2 \\ |g| \leq 1 \text{ a.e. on } J}} \|f - g\|_{L^2(I)} \,. \tag{21}$$

Let us begin with a basic existence and uniqueness result:

Theorem 1 *Problem $BEP_{2,\infty}$ (21) has a unique solution g_0, and necessarily $\|g_0\|_{L^2(I)} \leq \|f\|_{L^2(I)}$. Moreover $\|g_0\|_{L^\infty(J)} = 1$ unless $f = g_{|_I}$ for some $g \in H^2$ such that $\|g\|_{L^\infty(J)} < 1$.*

Corollary 1 *Problem BEP (2) has a unique solution.*

Proof of Theorem 1. Define a convex subset of $L^2(I)$ by putting $\mathcal{C} := \{g_{|_I};\ \ g \in H^2,\ \|g\|_{L^\infty(J)} \leq 1\}$. *We claim* that $\mathcal{C}$ is closed. Indeed, let $\{g_n\}$ be a sequence in H^2, with $\|g_n\|_{L^\infty(J)} \leq 1$, that converges in $L^2(I)$ to some ϕ. Clearly $\{g_n\}$ is bounded in $L^2(\mathbb{T})$, therefore some subsequence g_{k_n} converges weakly to $g \in H^2$. Since $|g_{k_n}| \leq 1$ on J, we may assume upon refining the subsequence further that it converges weak-* in $L^\infty(J)$ to a limit which can be none but $g_{|_J}$. By weak-* compactness of balls in $L^\infty(J)$, we get $\|g\|_{L^\infty(J)} \leq 1$, hence $g_{|_I} \in \mathcal{C}$. But ${g_{k_n}}_{|_I}$ *a fortiori* converges weakly to $g_{|_I}$ in $L^2(I)$, thus $\phi = g_{|_I} \in \mathcal{C}$ *as claimed*. By standard properties of the projection on a non-empty (for $0 \in \mathcal{C}$) closed convex set in a Hilbert space, we now deduce that the solution g_0 to (21) uniquely exists, and is characterized by the variational inequality [13, thm V.2.]

$$g_{0_{|_I}} \in \mathcal{C} \quad \text{ and } \quad \mathrm{Re} < f - g_0 \,, \, \phi - g_0 >_I \leq 0, \quad \forall \phi \in \mathcal{C}. \tag{22}$$

Using $\phi = 0$ in (22) and applying the Schwarz inequality yields $\|g_0\|_{L^2(I)} \leq \|f\|_{L^2(I)}$.
Assume finally that $\|g_0\|_{L^\infty(J)} < 1$. Given $h \in H^\infty$, $g_0 + th$ is a candidate approximant for small $t \in \mathbb{R}$ hence the map $t \mapsto \|f - g_0 - th\|^2_{L^2(I)}$ has a minimum at $t = 0$. Differentiating under the integral sign and equating the derivative to zero yields $2\mathrm{Re} < f - g_0, h >_I = 0$ whence $< f - g_0, h >_I = 0$ upon replacing h by ih. Letting $h = e^{ik\theta}$ for $k \in \mathbb{N}$ we see that $(f - g_0) \vee 0$ lies in $\bar{H}^2_0$, hence it is identically zero because it vanishes on J. Thus $f = g_{0_{|_I}}$ as was to be shown. ■

Theorem 1 entails that the constraint $\|g\|_{L^\infty(J)} \leq 1$ in Problem (21) is saturated unless $f = g_{0|_I}$. If the boundary of I has measure zero, much more in fact is true:

Theorem 2 *Assume that $\ell(\partial I) = 0$ and let g_0 be the solution to Problem* (21)*. Then $|g_0| = 1$ a.e. on J unless $f = g_{|_I}$ for some $g \in H^2$ such that $\|g\|_{L^\infty(J)} \leq 1$.*

It would be interesting to know how much the assumption $\ell(\partial I) = 0$ can be relaxed in the above statement. Reducing Problem BEP (2) to Problem $BEP_{2,\infty}$ (21) as before, we obtain as a corollary:

Corollary 2 *Assume that $\ell(\partial I) = 0$ and let g_0 be the solution to Problem* (2)*. If $\log M \in L^1(J)$, then $|g_0(e^{i\theta})| = M(e^{i\theta})$ a.e. on J unless $f = g_{|_I}$ for some $g \in H^2$ such that $|g(e^{i\theta})| \leq M(e^{i\theta})$ a.e. on J.*

To prove Theorem 2 we establish three lemmas, the second of which will be of repeated use in the paper.

Lemma 2 *Let $E \subset \mathbb{T}$ be infinite and $K_1 \subset \mathbb{T}$ be a compact set such that $\overline{E} \cap K_1 = \emptyset$. If we define a collection $\mathcal{R}$ of rational functions in the variable z by*

$$\mathcal{R} = \left\{ c_0 + i \sum_{k=1}^{n} c_k \frac{e^{i\,\psi_k} + z}{e^{i\,\psi_k} - z}\,; \right. \tag{23}$$

$$\left. c_0\,,\ c_k \in \mathbb{R}\,,\quad e^{i\,\psi_k} \in E\,,\quad 1 \leq k \leq n,\quad n \in \mathbb{N} \right\}\,,$$

then $\mathcal{R}_{|_{K_1}}$ is uniformly dense in $C_{\mathbb{R}}(K_1)$.

Proof. It is elementary to check that members of $\mathcal{R}$ are real-valued a.e. on $\mathbb{T}$. Also, it is enough to assume that E consists of a sequence $\{e^{i\psi_k}\}_{k\in\mathbb{N}}$ that converges in $\mathbb{T}$ to some $e^{i\psi_\infty}$. We work over the real axis where computations are slightly simpler, and for this we consider the Möbius transform:

$$\varphi(z) = i\,\frac{e^{i\psi_\infty} + z}{e^{i\psi_\infty} - z}\,,$$

that maps $\mathbb{T}$ onto $\mathbb{R} \cup \{\infty\}$ with $\varphi(e^{i\psi_\infty}) = \infty$. Set $K_2 = \varphi(K_1)$, and note that it is compact in $\mathbb{R}$ since $e^{i\psi_\infty} \notin K_1$. Let $\mathcal{R}_{\mathbb{R}}$ denote the collection of all functions $r \circ \varphi^{-1}$ as r ranges over $\mathcal{R}$. We are now left to prove that the restrictions to K_2 of functions in $\mathcal{R}_{\mathbb{R}}$ are uniformly dense in $C_{\mathbb{R}}(K_2)$. For this, we put $t_k = \varphi(e^{i\psi_k})$ and, denoting by $t = \varphi^{-1}(z)$ the independent variable in $\mathbb{R}$, we compute from (23) that

$$\mathcal{R}_{\mathbb{R}} = \{a_0 + \sum_{k=1}^{n} \frac{b_k}{t - t_k}\,,\ a_0\,,\ b_k \in \mathbb{R}\,,\quad 1 \leq k \leq n\,,\quad n \in \mathbb{N}\},$$

that is to say $\mathcal{R}_{\mathbb{R}}$ is the set of real rational functions bounded at infinity, each pole of which is simple and coincides with some t_k. Thus if $P_{\mathbb{R},n}$ stands for the space of real polynomials of degree at most n, we get

$$\mathcal{R}_{\mathbb{R}} = \left\{ \frac{p_n(t)}{\prod_{k=1}^{n}(t - t_k)}\,,\ p_n \in P_{\mathbb{R},n},\quad 1 \leq k \leq n,\quad n \in \mathbb{N} \right\}\,,$$

where the empty product is 1. We claim that to each $\epsilon > 0$ and $p \in P_{\mathbb{R},n}$ there exists $r \in \mathcal{R}_{\mathbb{R}}$ such that
$$||r - p||_{L^\infty(K_2)} \le \epsilon,$$
and this will achieve the proof since $P_{\mathbb{R},n}$ is dense in $C_{\mathbb{R}}(K_2)$ by the Stone-Weierstrass theorem. To establish the claim, let U be a neighborhood of 0 in $\mathbb{R}^n$ such that

$$\forall (x_1 \dots x_n) \in U, \quad \left| 1 - \frac{1}{\prod_{k=1}^n (1 - x_k)} \right| \le \frac{\epsilon}{1 + ||p||_{L^\infty(K_2)}}.$$

Next, pick n distinct numbers $t_{k_1}, \dots, t_{k_n}$ so large in modulus that $t/t_{k_j} \in U$ for $t \in K_2$ and $1 \le j \le n$; this is certainly possible since K_2 is compact whereas $|t_k|$ tends to ∞ because $e^{i\psi_k} \to e^{i\psi_\infty}$. Finally, set

$$r(t) = \frac{p(t)}{\prod_{j=1}^n (1 - \frac{t}{t_{k_j}})}.$$

Clearly r belongs to $\mathcal{R}_{\mathbb{R}}$, and

$$||p - r||_{L^\infty(K_2)} \le ||p||_{L^\infty(K_2)} \left\| 1 - \frac{1}{\prod_{j=1}^n (1 - \frac{t}{t_{k_j}})} \right\|_{L^\infty(K_2)} \le \epsilon$$

as claimed. ■

Lemma 3 *Let $f \in L^2(I)$ and g_0 be the solution to problem* (21)*. For h a real-valued Dini-continuous function on* $\mathbb{T}$ supported on the interior $\mathring{I}$ of I, *let*

$$b(z) = \frac{1}{2\pi} \int_I \frac{e^{it} + z}{e^{it} - z} h(e^{it})\, dt, \qquad z \in \mathbb{D}, \tag{24}$$

be the Riesz-Herglotz transform of h. Then b is continuous on $\overline{\mathbb{D}}$, and moreover

$$\mathrm{Re} < (f - g_0)\,\overline{g}_0\,,\, b >_I = 0\,. \tag{25}$$

Proof. It follows from (9) that b continuous on $\overline{\mathbb{D}}$. For $\lambda \in \mathbb{R}$, consider the function
$$\omega_\lambda(z) = \exp \lambda\, b(z)\,, \qquad z \in \mathbb{D},$$
which is the outer function in H^∞ whose modulus is equal to $\exp \lambda\, h$. Since $|\omega_\lambda| = 1$ on J, the function $g_0\, \omega_\lambda$ is a candidate approximant in problem (21) thus $\lambda \to \|f - g_0\, \omega_\lambda\|^2_{L^2(I)}$ reaches a minimum at $\lambda = 0$. By the boundedness of b, we may differentiate this function with respect to λ under the integral sign, and equating the derivative to 0 at $\lambda = 0$ yields (25). ■

Lemma 4 *Let $f \in L^2(I)$ and g_0 be the solution to Problem* (21)*. Then $(f - g_0)\,\overline{g}_0$ has real mean on I:*

$$\mathrm{Re} < (f - g_0)\,\overline{g}_0\,,\, i >_I = 0. \tag{26}$$

Proof. For each $\alpha \in [-\pi, \pi]$, the function $g_0\, e^{i\alpha}$ belongs to H^2 and is a candidate approximant in (21) since it has the same modulus as g_0. Hence the function $\alpha \to \|f - g_0\, e^{i\alpha}\|_{L^2(I)}$ reaches a minimum at $\alpha = 0$, and differentiating under the integral sign yields (26). ∎

Proof of Theorem 2. Since $\partial J = \partial I$ has measure zero, it is equivalent to show that $|g_0| = 1$ a.e. on $\overset{\circ}{J}$. Let

$$E = \{e^{i\theta} \in \overset{\circ}{J}\,,\ |g_0(e^{i\theta})| < 1\}\,,$$

and assume for a contradiction that $\ell(E) > 0$. By countable additivity, there is $\varepsilon > 0$ such that

$$E_\varepsilon = \{e^{i\theta} \in \overset{\circ}{J}\,,\ |g_0(e^{i\theta})| \le 1 - \varepsilon\}$$

has strictly positive measure. Hence by inner regularity of Lebesgue measure, there is a compact set $K \subset E_\varepsilon$ such that $\ell(K) > 0$, and since $K \subset \overset{\circ}{J}$ it is at positive distance from I, say, η. For $\lambda \in \mathbb{R}$ and F a measurable subset of K, let $w_{\lambda,F}$ be the outer function whose modulus is $\exp\lambda$ on F, and 1 on $\mathbb{T} \setminus F$. By definition $w_{\lambda,F}(z) = \exp\{\lambda\, A_F(z)\}$, where

$$A_F(z) = \frac{1}{2\,\pi} \int_F \frac{e^{it} + z}{e^{it} - z}\, dt\,, \qquad z \in \mathbb{D} \tag{27}$$

is the Riesz-Herglotz transform of χ_F. For $\lambda < \log(1/(1-\varepsilon))$ the function $g_0\, w_{\lambda,F}$ belongs to H^2 and satisfies $|g_0\, w_{\lambda,F}| \le 1$ a.e. on J so that, by definition of g_0, the function $\lambda \to \|f - g_0\, w_{\lambda,F}\|_{L^2(I)}$ reaches a minimum at $\lambda = 0$. From (27), we see that A_F is uniformly bounded on I because $|e^{it} - e^{i\theta}| \ge \eta > 0$ whenever $e^{it} \in F$ and $e^{i\theta} \in I$. Therefore we may differentiate under the integral sign to compute the derivative of $\|f - g_0\, w_{\lambda,F}\|^2_{L^2(I)}$ with respect to λ, which gives us

$$-2\mathrm{Re} < f - g_0\, \exp\{\lambda\, A_F\}\,,\, g_0\, A_F\, \exp\{\lambda\, A_F\} >_I\ .$$

Since the latter must vanish at $\lambda = 0$ we obtain

$$\mathrm{Re} < f - g_0\,,\, g_0\, A_F >_I = \mathrm{Re} < (f - g_0)\,\overline{g}_0\,,\, A_F >_I = 0\,. \tag{28}$$

Let e^{it_0} be a density point of K and I_l denote the arc centered at e^{it_0} of length l, so that $\ell(I_l \cap K)/l \to 1$ as $l \to 0$. In particular $\ell(I_l \cap K) \ne 0$ for sufficiently small l. Noting that

$$\left| \frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} - \frac{e^{it_0} + e^{i\theta}}{e^{it_0} - e^{i\theta}} \right| \le 2l/\eta^2 \quad \text{for } e^{it} \in I_l \cap K,\ \ e^{i\theta} \in I, \tag{29}$$

and observing that $(f - g_0)\overline{g}_0 \in L^1(I)$, we get from (28)-(29) that

$$\mathrm{Re} < (f - g_0)\overline{g}_0\,,\, \frac{e^{it_0} + e^{i\theta}}{e^{it_0} - e^{i\theta}} >_I \tag{30}$$

$$= \lim_{l \to 0} \mathrm{Re} < (f - g_0)\,\overline{g}_0\,,\, \frac{2\pi}{\ell(I_l \cap K)} A_{I_l \cap K} >_I = 0\,.$$

Thus, if we let $\mathcal{D}_K$ denote the set of density points of K, we may capsulize (30) and (26) by saying that $(f-g_0)\,\overline{g}_0$ is orthogonal to the *real* vector space

$$\mathcal{S}_K = \{i\,c_0 + \sum_{k=1}^{n} c_k \frac{e^{i\,\phi_k}+z}{e^{i\,\phi_k}-z}\,,\ c_0\,,\ c_k \in \mathbb{R}\,,\ e^{i\,\phi_k} \in \mathcal{D}_K\,,\quad 1 \le k \le n,\quad n \in \mathbb{N}\}$$

for the *real* scalar product Re $< \ , \ >_I$. Since $\ell(\partial I) = 0$ we can replace I by $\bar{I}$ in this product:

$$\text{Re} < (f-g_0)\,\overline{g}_0,\, r >_{\bar{I}} = 0\,, \qquad \forall\, r \in \mathcal{S}_K\,. \tag{31}$$

As $\ell(K) > 0$ and almost every point of K is a density point by Lebesgue's theorem [35, sec. 7.12], the set $\mathcal{D}_K$ is certainly infinite. Moreover, since $K \subset \overset{\circ}{J}$, we have that $\overline{I} \cap \overline{\mathcal{D}}_K = \emptyset$. Now, Lemma 2 with $E = \mathcal{D}_K$ and $K_1 = \overline{I}$ implies in view of (31) that

$$\text{Re} < (f-g_0)\,\overline{g}_0\,,\, i\phi >_{\bar{I}} = 0\,, \qquad \forall \phi \in C_{\mathbb{R}}(\bar{I}). \tag{32}$$

By Riesz duality it follows that $(f-g_0)\,\overline{g}_0$ is real-valued a.e. on $\bar{I}$. In particular, if h is a Dini-continuous real function supported on $\overset{\circ}{I}$, (32) holds with $\phi = \widetilde{h}_{|\bar{I}}$. Hence by Lemma 3, where I can be replaced by $\bar{I}$,

$$< (f-g_0)\,\overline{g}_0\,,\, h >_{\bar{I}} = 0. \tag{33}$$

However, by regularization, Dini-continuous –even smooth– functions are uniformly dense in the space of continuous functions with compact support on $\overset{\circ}{I}$ [25, chap. 1, prop. 8]. Therefore (33) in fact holds for every continuous h supported on $\overset{\circ}{I}$. Consequently $(f-g_0)\,\overline{g}_0$ must vanishes a.e. on $\overset{\circ}{I}$ thus also on I. This implies that either $g_0 = f$ a.e. on I or $g_0 = 0$ on a set of positive measure, in which case $g_0 = 0$. In any case, by Theorem 1, f is the trace on I of a H^2-function with modulus at most 1 on J. ■

We now turn to the continuity of the solution to problem (21) with respect to the data.

Theorem 3 *Let $f \in L^2(I)$ and g_0 be the solution to problem* (21)*. Assume that $f^{\{n\}}$ converges to f in $L^2(I)$ as $n \to \infty$, and let $g_0^{\{n\}}$ indicate the corresponding solution to problem* (21)*. Then ${g_0^{\{n\}}}_{|I}$ converges to ${g_0}_{|I}$ in $L^2(I)$ and ${g_0^{\{n\}}}_{|J}$ converges weak-* to ${g_0}_{|J}$ in $L^\infty(J)$. If moreover $\ell(\partial I) = 0$ and f is not the trace on I of a H^2-function less than 1 in modulus a.e. on J, then $g_0^{\{n\}}$ converges to g_0 in $L^2(\mathbb{T})$.*

Proof. By definition $\|g_0^{\{n\}}\|_{L^\infty(J)} \le 1$, and by Theorem 1

$$\|g_0^{\{n\}}\|_{L^2(I)} \le \|f^{\{n\}}\|_{L^2(I)},$$

hence $g_0^{\{n\}}$ is a bounded sequence in H^2. Let g_∞ be a weak accumulation point and $g_0^{\{k_n\}}$ a subsequence converging weakly to g_∞ in H^2; *a fortiori* ${g_0^{\{k_n\}}}_{|I}$

converges weakly to $g_{\infty|_I}$ in $L^2(I)$. By weak (resp. weak-*) compactness of balls in $L^2(I)$ (resp. $L^\infty(J)$), we get $|g_\infty| \leq 1$ a.e. on J and

$$\|f - g_\infty\|_{L^2(I)} \leq \liminf_{n\to\infty} \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)}.$$

In particular g_∞ is a candidate approximant, so one has the series of inequalities:

$$\|f - g_0\|_{L^2(I)} \leq \|f - g_\infty\|_{L^2(I)} \leq \liminf_{n\to\infty} \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)} \tag{34}$$
$$\leq \limsup_{n\to\infty} \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)}.$$

If one of these were strict, there would exist $\varepsilon > 0$ such that

$$\|f - g_0\|_{L^2(I)} + \varepsilon \ \leq\ \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)} \tag{35}$$

for infinitely many n. But $\|f - f^{\{k_n\}}\|_{L^2(I)} < \varepsilon/2$ for large n, thus for infinitely many n (35) yields

$$\|f^{\{k_n\}} - g_0\|_{L^2(I)} + \varepsilon/2 \ \leq\ \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)}$$

contradicting the definition of $g_0^{\{k_n\}}$. Therefore equality holds throughout in (34), whence $g_\infty = g_0$ by the uniqueness part of Theorem 1. Equality in (34) is also to the effect that

$$\lim_{n\to\infty} f^{\{k_n\}} - g_0^{\{k_n\}} = f - g_0 \quad \text{in} \ \ L^2(I)$$

because the norm of the weak limit is not less than the limit of the norms. Refining k_n if necessary, we can assume in addition that ${g_0^{\{k_n\}}}_{|_J}$ converges weak-* to some h in $L^\infty(J)$, and since we already know that it converges weakly to $g_{0|_J}$ in $L^2(J)$ we get $h = g_{0|_J}$. Finally if $\ell(\partial I) = 0$, we deduce from Theorem 2 that $|g_0| = 1$ a.e. on J hence ${g_0^{\{k_n\}}}_{|_J}$ converges to $g_{0|_J}$ in $L^2(J)$ for again the norm of the weak limit is not less than the limit of the norms. Altogether we have shown that any sequence meeting the assumptions contains a subsequence satisfying the conclusions, which is enough to prove the theorem. ■

To conclude this section, we prove that if f has more summability than required, then so does g_0.

Proposition 1 *Assume that $f \in L^p(I)$ for some finite $p > 2$. If g_0 denotes the solution to problem (21) and if $\ell(\partial I) = 0$, then $g_0 \in H^p$ and $\|g_0\|_{L^p(I)} \leq (1 + K_{p/2})\|f\|_{L^p(I)}$.*

Proof. Let h be a Dini-continuous real-valued function supported in $\overset{\circ}{I}$, and b his Riesz-Herglotz transform. Since b has real part h on $\mathbb{T}$, Lemma 3 gives us

$$< |g_0|^2 \,,\, h >_I = \mathrm{Re}\ < f\overline{g}_0 \,,\, b >_I . \tag{36}$$

Using Hölder's inequality in (36) and observing that $\|g_0\|_{L^2(I)} \leq \|f\|_{L^2(I)} \leq \|f\|_{L^p(I)}$ in view of Theorem 1 and the fact that $p > 2$ while $\ell(I) < 1$, we obtain, with $1/p + 1/2 + 1/s_0 = 1$:

$$\left|< |g_0|^2 \,,\, h >_I\right| \ \leq\ \|f\|_{L^p(I)}\, \|g_0\|_{L^2(I)}\, \|b\|_{L^s(I)} \leq \|f\|^2_{L^p(I)}\, \|b\|_{L^{s_0}(I)} .$$

Thus, because the conjugation operator has norm K_{s_0} on $L^{s_0}(\mathbb{T})$ while h is supported on I, we get *a fortiori*

$$\left|< |g_0|^2 \,,\, h >_I\right| \;\leq\; (1+K_{s_0})\|f\|^2_{L^p(I)}\, \|h\|_{L^{s_0}(I)}. \tag{37}$$

Now, Dini-continuous functions supported on $\overset{\circ}{I}$ are dense in $L^{s_0}(\overset{\circ}{I})$, hence also in $L^{s_0}(I)$ as $\ell(\partial I)=0$. Therefore (37) implies by duality

$$\|g_0\|_{L^{p_1}(I)} \leq (1+K_{s_0})^{1/2}\|f\|_{L^p(I)}, \qquad 1/p_1 = (1/p+1/2)/2. \tag{38}$$

Hölder's inequality in (36), using this time (38) instead of $\|g_0\|_{L^2(I)} \leq \|f\|_{L^p(I)}$, strengthens (37) to

$$\left|< |g_0|^2 \,,\, h >_I\right| \;\leq\; (1+K_{s_0})^{1/2}(1+K_{s_1})\|f\|^2_{L^p(I)}\, \|h\|_{L^{s_1}(I)},\ 1/p+1/p_1+1/s_1 = 1,$$

which gives us by duality

$$\|g_0\|_{L^{p_2}(I)} \leq (1+K_{s_0})^{1/4}\,(1+K_{s_1})^{1/2}\,\|f\|_{L^p(I)}, \qquad 1/p_2 = (1/p+1/p_1)/2.$$

Set $1/p_k = (1/p+1/p_{k-1})/2$ and $1/p+1/p_k+1/s_k = 1$. Iterating this reasoning yields by induction

$$\|g_0\|_{L^{p_k}(I)} \leq \|f\|_{L^p(I)}\ \Pi_{j=0}^{k-1}(1+K_{s_j})^{1/2^{k-j}}\,. \tag{39}$$

As k goes large p_k increases to p and $K_{s_k} = K_{p_{k+1/2}}$ decreases to $K_{p/2}$. Hence the product on the right of (39) becomes arbitrarily close to $1+K_{p/2}$, and the result now follows on letting $k \to +\infty$. ∎

In problem (21), it would be interesting to know whether $g_0 \in BMOA$ when $f \in L^\infty(I)$ and $\ell(\partial I) = 0$.

4 The critical point equation

In any convex minimization problem, the solution is characterized by a variational inequality saying that the *criterium* increases with admissible increments of the variable. If the problem is smooth, infinitesimal increments span a half-space whose boundary hyperplane is tangent to the admissible set, and the variational inequality becomes an equality asserting that the derivative of the objective function is zero on that hyperplane. This equality, sometimes called a *critical point equation*, expresses that the vector gradient of the objective function in the ambient space lies orthogonal to the constraint; this vector is an implicit parameter of the critical point equation, known as a *Lagrange parameter*.
In problem (21) the variational inequality is (22). However, the non-smoothness of the L^∞-norm makes it *a priori* unclear whether a critical point equation exists. It turns out that it does, at least when $\ell(\partial I) = 0$.

Theorem 4 *Assume that $f \in L^2(I)$ is not the trace on I of a H^2-function of modulus less than or equal to 1 a.e on J, and suppose further that $\ell(\partial I) = 0$. Then, $g_0 \in H^2$ is the solution to problem (21) if, and only if, the following two conditions hold:*

(i) $|g_0(e^{i\theta})| = 1$ for a.e. $e^{i\theta} \in J$,

(ii) there exists a non-negative function $\lambda \in L^1_{\mathbb{R}}(J)$ such that,

$$(g_{0|_I} - f) \vee \lambda\, g_{0|_J} \in \bar{H}^1_0. \tag{40}$$

Moreover, if $f \in L^p(I)$ for some p such that $2 < p < \infty$, then $\lambda \in L^p_{\mathbb{R}}(J)$.

Remark: Note that (40) is equivalent to saying that $(g_{0|_I} - f) \vee \lambda\, g_{0|_J} \in L^1(\mathbb{T})$ and

$$\mathbf{P}_+ \left((g_{0|_I} - f) \vee \lambda\, g_{0|_J} \right) = 0 \tag{41}$$

which is the critical point equation proper, with Lagrange parameter λ. Observe that $\log \lambda \in L^1_{\mathbb{R}}(J)$, otherwise the $\bar{H}^1_0$-function $(g_{0|_I} - f) \vee (\lambda\, g_{0|_J})$ would be zero hence $f = g_{0|_I}$, contrary to the hypothesis.

To prove Theorem 4, we need two lemmas the first of which stands somewhat dual to Lemma 3:

Lemma 5 *Let $f \in L^2(I)$ and g_0 be the solution to problem (21). If h is a non-negative function in $L^\infty(\mathbb{T})$ which is supported on $\overset{\circ}{J}$, and if*

$$a(z) = \frac{1}{2\pi} \int_J \frac{e^{i\theta} + z}{e^{i\theta} - z}\, h(e^{i\theta})\, d\theta\,, \qquad z \in \mathbb{D}, \tag{42}$$

denotes its Riesz-Herglotz transform, then a is continuous on $\overline{I}$ and we have that

$$\mathrm{Re} < (f - g_0)\, \overline{g}_0\,,\, a >_I \;\; \geq \; 0\,. \tag{43}$$

Proof. Since h is supported in $\overset{\circ}{J}$, it is clear from the definition that a is continuous on $\overline{I}$. For $t \in \mathbb{R}$, let us put

$$w_t(z) = \exp t\, a(z)\,, \qquad z \in \mathbb{D},$$

which is the outer function in H^∞ whose modulus is equal to $\exp\{t\, h\}$. As $h \geq 0$, the function $g_0\, w_t$ is a candidate approximant in problem (21) when $t \leq 0$. Since $t \to \|f - g_0\, w_t\|^2_{L^2(I)}$ can be differentiated with respect to t under the integral sign by the boundedness of a on I, its derivative at $t = 0$ must be non-positive by the minimizing property of g_0. But this derivative is just $-2\mathrm{Re} < (f - g_0)\, \overline{g}_0\,,\, a >_I$. ■

Our second preparatory result is of technical nature:

Lemma 6 *Assume that $f \in L^2(I)$ and let g_0 be the solution to problem* (21)*. If $f \neq g_{0|_I}$ and $\ell(\partial I) = 0$, then there exists a unique $\lambda \in L^1_{\mathbb{R}}(J)$ such that*

$$(g_{0|_I} - f)\, \overline{g}_{0|_I} \vee \lambda \;\in \bar{H}^1_0\,. \tag{44}$$

Necessarily $\lambda \geq 0$ a.e. on J, and if $f \in L^\infty(I)$ then $\lambda \in L^p(J)$ for $1 < p < \infty$. If $f^{\{n\}} \in L^\infty(I)$ converges to f in $L^2(I)$ while $g_0^{\{n\}}$ is the corresponding solution to problem (21)*, and if we write by* (44)

$$\left(g_{0|_I}^{\{n\}} - f^{\{n\}}\right) \overline{g}_{0|_I}^{\{n\}} \vee \lambda^{\{n\}} \;\in \bar{H}^1_0, \qquad \text{with} \;\; \lambda^{\{n\}} \in L^1_{\mathbb{R}}(J), \tag{45}$$

then the sequence of concatenated functions in (45) *converges weak-* in $\bar{H}^1_0$ to the function* (44)*.*

Proof. The uniqueness of λ is clear because if $\lambda' \in L^1_{\mathbb{R}}(J)$ satisfies (44), then $0 \vee (\lambda - \lambda') \in \bar{H}^1_0$ so that $\lambda = \lambda'$. To prove the existence of λ, assume first that $f \in L^\infty(I)$ and fix $p \in (2, \infty)$. By proposition 1 and Hölder's inequality, we know that $(g_0 - f)\, \overline{g}_0 \in L^p(I)$. For h a real-valued function in $L^q(J)$ where $1/q = 1 - 1/p$, let a be the Riesz-Herglotz transform of $0 \vee h$ given by (42) and put

$$\mathcal{L}(h) = \mathrm{Re} < (f - g_0)\, \overline{g}_0\,,\, a >_I\,. \tag{46}$$

As $0 \vee h$ vanishes on I by construction, it is clear that

$$\mathcal{L}(h) = \mathrm{Re} < (f - g_0)\, \overline{g}_0\,,\, \widetilde{0 \vee h} >_I,$$

and since the conjugation operator is bounded by K_q on $L^q_{\mathbb{R}}(\mathbb{T})$, we obtain from Hölder's inequality

$$|\mathcal{L}(h)| \;\leq\; K_q\, \|(f - g_0)\, \overline{g}_0\|_{L^p(I)}\, \|h\|_{L^q(J)}\,.$$

Thus $\mathcal{L}$ is a continuous linear form on $L^q_{\mathbb{R}}(J)$ and there exists $\lambda \in L^p_{\mathbb{R}}(J)$ such that

$$\mathcal{L}(h) =< \lambda\,,\, h >_J, \qquad h \in L^q(J). \tag{47}$$

By Lemma 5, $\mathcal{L}$ is a positive functional on bounded functions supported on $\overset{\circ}{J}$. Hence $\lambda \geq 0$ a.e. on $\overset{\circ}{J}$ thus also on J since $\ell(\partial J) = \ell(\partial I) = 0$. As $\mathrm{Re}\, a = h$ and λ is real-valued, equation (47) gives us

$$\mathcal{L}(h) = \mathrm{Re} < \lambda\,,\, a >_J, \qquad h \in L^q(J), \tag{48}$$

and therefore, substracting (46) from (48), we get

$$\mathrm{Re} < (g_{0|_I} - f)\,\overline{g}_{0|_I} \vee \lambda\ ,\ a >_{\mathbb{T}} = 0 \tag{49}$$

whenever a is the Riesz-Herglotz transform of some $h \in L^q_{\mathbb{R}}(J)$.
By regularization Dini-continuous functions are dense in continuous functions with compact support in $\mathring{I}$, so they are dense in $L^q(I)$ since $\ell(\partial I) = 0$. Hence it follows from Lemma 3 and the boundedness of the conjugation operator in $L^q_{\mathbb{R}}(\mathbb{T})$ that

$$\mathrm{Re}\ < (g_0 - f)\,\overline{g}_0\ ,\ b >_I = 0\,. \tag{50}$$

whenever b is the Riesz-Herglotz transform of some $\phi \in L^q_{\mathbb{R}}(I)$. As λ is real-valued and $\mathrm{Re}\, b = 0$ a.e. on J, we may rewrite (50) in the form

$$\mathrm{Re} < (g_{0|_I} - f)\,\overline{g}_{0|_I} \vee \lambda\ ,\ b >_{\mathbb{T}} = 0. \tag{51}$$

Now, by (5), every H^q-function is the sum of three terms: a pure imaginary constant, the Riesz-Herglotz transform of $\phi \vee 0$ for some $\phi \in L^q_{\mathbb{R}}(I)$, and the Riesz-Herglotz transform of $0 \vee h$ for some $h \in L^q_{\mathbb{R}}(J)$. Therefore by (51), (49), (26) and the realness of λ, we obtain

$$\mathrm{Re} < (g_{0|_I} - f)\,\overline{g}_{0|_I} \vee \lambda\ ,\ g >_{\mathbb{T}} = 0\,, \quad \forall g \in H^q.$$

Changing g into ig we see that the real part is superfluous and letting $g(e^{i\theta}) = e^{ik\theta}$ for $k \in \mathbb{N}$ we get

$$(g_{0|_I} - f)\,\overline{g}_{0|_I} \vee \lambda\ \in \bar{H}^p_0\,. \tag{52}$$

If f is now an arbitrary function in $L^2(I)$ and $f^{\{n\}}$, $g_0^{\{n\}}$ are as indicated in the statement of the lemma, we know from (52), since $f^{\{n\}} \in L^\infty(I)$, that there is a unique $\lambda^{\{n\}}$ meeting (45). By Theorem 3 we have that $g_0^{\{n\}} \to g_0$ in H^2, hence by the Schwarz inequality

$$\lim_{n\to\infty} \left\| \left(g_0^{\{n\}} - f^{\{n\}}\right)\, \overline{g}_0^{\{n\}} - (g_0 - f)\,\overline{g}_0 \right\|_{L^1(I)} = 0. \tag{53}$$

Besides, since $\lambda^{\{n\}} \geq 0$ and the mean on $\mathbb{T}$ of a $\bar{H}^1_0$-function is zero, (45) implies

$$\begin{aligned}\left\|\lambda^{\{n\}}\right\|_{L^1(J)} &= \int_J \lambda^{\{n\}}(t)\,dt = \int_I \left(f^{\{n\}} - g_0^{\{n\}}\right)\, \overline{g}_0^{\{n\}}(t)\,dt \\ &\leq \left\| \left(g_0^{\{n\}} - f^{\{n\}}\right)\, \overline{g}_0^{\{n\}} \right\|_{L^1(I)}\,,\end{aligned}$$

and in view of (53) we deduce that $\left\|\lambda^{\{n\}}\right\|_{L^1(J)}$ is bounded independently of n. Consequently the sequence

$$\left(g_0^{\{n\}}{}_{|_I} - f^{\{n\}}\right)\, \overline{g}_0^{\{n\}}{}_{|_I} \vee \lambda^{\{n\}} \tag{54}$$

has a weak-* convergent subsequence to some F in $\bar{H}^1_0$, regarding the latter as dual to $C(\mathbb{T})/\mathcal{A}$ under the pairing $<\ \ ,\ \ >_{\mathbb{T}}$. Checking this convergence on continuous functions supported on the interior of I, we conclude from (53) that $F_{|_{\mathring{I}}} = (g_{0|_I} - f)\,\overline{g}_{0|_I}$ a.e. on $\mathring{I}$ thus also on I. Therefore if we let $\lambda = F_{|_J}$, we meet (44). Checking the same convergence on positive functions supported on $\overset{\circ}{J}$, we deduce since $\lambda^{\{n\}} \geq 0$ that $F_{|_J}$ is non-negative. Finally, since F is determined by its trace $(g_{0|_I} - f)\,\overline{g}_{0|_I}$ on I, there is a unique weak-* accumulation point of the bounded sequence (54) which is thus convergent. ■

Proof of Theorem 4. To prove sufficiency, assume that $g_0 \in H^2$ satisfies $(i)-(ii)$, and let $u \in H^2$ be such that $\|u\|_{L^\infty(J)}\| \leq 1$. From (41) we get

$$\mathbf{P}_+\left(0 \vee \lambda\, g_{0|_J}\right) = \mathbf{P}_+\left((f - g_{0|_I}) \vee 0\right) \in H^2,$$

thus applying Lemma 1 with $v = \lambda\, g_{0|_J}$ and $g = u - g_0$, we obtain

$$\begin{aligned} < \lambda g_0\,,\, u - g_0 >_J &= - < \mathbf{P}_+\left((f - g_{0|_I}) \vee 0\right)\,,\, u - g_0 >_{\mathbb{T}} \qquad (55) \\ &= - < f - g_0\,,\, u - g_0 >_I\,. \end{aligned}$$

Since $\mathrm{Re} < \lambda g_0\,,\, u - g_0 >_J = \mathrm{Re} < \lambda\,,\, u\bar{g}_0 - 1 >_J$ is non-negative because $\lambda \geq 0$ and $\mathrm{Re}(u\bar{g}_0) \leq |u| \leq 1$, we see from (55) that (22) is met.
Proving necessity is a little harder. For this, let g_0 solve problem 21 and observe from Theorem 2 that (i) holds. Thus we are left to prove (ii); in fact, we will show that the function λ from Lemma 6 meets (40).
Assume first that $f \in L^\infty(I)$. From Proposition 1 we get in particular $g_0 \in H^4$, and by Lemma 6 there is $\lambda \geq 0$ in $L^2_{\mathbb{R}}(J)$ such that (44) holds with $\bar{H}^1_0$ replaced by $\bar{H}^2_0$. Using (i), we may rewrite this as

$$\left((g_{0|_I} - f) \vee \lambda\, g_{0|_j}\right)\overline{g}_0 \;=\; F, \qquad F \in \bar{H}^2_0\,. \qquad (56)$$

Let $g_0 = jw$ be the inner-outer factorization of g_0. We will show that $F \in \bar{j}\bar{H}^2_0$, *and this will achieve the proof when* $f \in L^\infty(I)$. Indeed, dividing (56) by $\bar{g}_0$ then yields

$$(g_{0|_I} - f) \vee \lambda\, g_{0|_j} \;\in\; \bar{w}^{-1}\bar{H}^2_0 \qquad (57)$$

which means that the concatenated function in (57) is of the form:

$$e^{-i\theta}\overline{g(e^{i\theta})/w(e^{i\theta})}$$

for some $g \in H^2$. However, g/w belongs to the Nevanlinna class N^+ by definition, and it also lies in $L^2(\mathbb{T})$ because so does the function on the left-hand side of (57) (recall $|g_0| = 1$ a.e. on J). Hence $g/w \in H^2$, implying that $e^{-i\theta}\overline{g(e^{i\theta})/w(e^{i\theta})} \in \bar{H}^2_0 \subset \bar{H}^1_0$, as desired.
Let $j = bS_\mu$ where b is the Blaschke product defined by (12) and S_μ the singular inner factor defined by (13). To prove that $F \in \bar{j}\bar{H}^2_0$, it is enough by uniqueness of the inner-outer factorization to establish separately that $F \in \bar{b}\bar{H}^2_0$ and $F \in \bar{S}_\mu\bar{H}^2_0$. To establish the former, it is sufficient to show that $F \in \bar{b}_1\bar{H}^2_0$ whenever b_1 is a finite Blaschke product dividing b, *i.e.* such that $b = b_1 b_2$ with b_2 a Blaschke product. Pick such a b_1 and put for simplicity $\gamma_0 = b_2 S_\mu w$, so that $g_0 = b_1\gamma_0$. We can write $b_1 = q/q^R$, where q is an algebraic polynomial and $q^R = z^n\overline{q(1/\bar{z})}$ its reciprocal. We may assume that q is monic and $\deg q > 0$:

$$q(z) = z^n + \alpha_{n-1}z^{n-1} + \alpha_{n-2}z^{n-2} + \ldots + \alpha_0\,, \quad \text{for some } n \in \mathbb{N}\setminus\{0\}.$$

When the set of monic polynomials of degree n gets identified with $\mathbb{C}^n$, taking as coordinates all the coefficients except the leading one, the subset Ω of those polynomials whose roots lie in $\mathbb{D}$ is open. Now, if $Q \in \Omega$ and $b_Q = Q/Q^R$ denotes the associated Blaschke product, the function $g = b_Q\gamma_0$ is a candidate approximant in Problem (21) since $|g| = |g_0|$ on $\mathbb{T}$, thus the map

$$Q \to \|f - \gamma_0\, b_Q\|^2_{L^2(I)} \tag{58}$$

reaches a minimum on Ω at $Q = q$. Let us write a generic $Q \in \Omega$ as

$$Q(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0.$$

Because $b_Q(e^{i\theta})$ is a rational function in the variables a_j whose denominator is locally uniformly bounded away from 0 on $\mathbb{T}$, we may differentiate (58) under the integral sign with respect to $\mathrm{Re}\, a_j$, $\mathrm{Im}\, a_j$. Since q is a minimum point, equating these partial derivatives to zero at $(a_l) = (\alpha_l)$ yields

$$-2\mathrm{Re} < (f - g_0)\,\overline{\gamma}_0\,, \left(x_j\,\frac{\partial b_Q(e^{i\theta})}{\partial \mathrm{Re}\, a_j} + y_j\,\frac{\partial b_Q(e^{i\theta})}{\partial \mathrm{Im}\, a_j}\right)\Bigg|_{\substack{a_l\, =\, \alpha_l\\ 0\le l\le n-1}} >_I = 0\,,$$

for all $x_j\,,\ y_j \in \mathbb{R}$ and every $j \in \{0, \ldots, n-1\}$. After a short computation, this gives us

$$\mathrm{Re} < (f - g_0)\,\overline{\gamma}_0\ ,\ \frac{z_j e^{ij\theta}}{q^R(e^{i\theta})} - \frac{(x_j - iy_j)e^{i(n-j)\theta}q(e^{i\theta})}{\left(q^R(e^{i\theta})\right)^2} >_I =\ 0\,,$$

for all $z_j = x_j\,, + i\,y_j \in \mathbb{C}$, where the second argument in the above scalar product is a function of $e^{i\theta} \in I$. Multiplying both arguments of this product by the unimodular function $\overline{b_1(e^{i\theta})} = q^R/q(e^{i\theta})$ does not affect its value, thus

$$\mathrm{Re} < (f - g_0)\,\overline{g}_0\ ,\ \frac{z_j e^{ij\theta}}{q(e^{i\theta})} - \frac{\bar{z}_j e^{i(n-j)\theta}}{q^R(e^{i\theta})} >_I\ . \in \mathbb{R}\,, \tag{59}$$

for all $z_j \in \mathbb{C}$. In another connection, by the very definition of q^R, we have that

$$\frac{e^{i(n-j)\theta}}{q^R(e^{i\theta})} = \frac{e^{i(n-j)\theta}}{e^{in\theta}\overline{q(e^{i\theta})}} = \overline{\left(\frac{e^{ij\theta}}{q(e^{i\theta})}\right)}$$

hence the second argument of $<\ ,\ >_I$ in (59) is pure imaginary on $\mathbb{T}$, and since λ is real a.e. on J

$$\mathrm{Re} < \lambda\ ,\ \frac{z_j e^{ij\theta}}{q(e^{i\theta})} - \frac{\bar{z}_j e^{i(n-j)\theta}}{q^R(e^{i\theta})} >_J =\ 0\,, \quad \forall\, z_j \in \mathbb{C}\,. \tag{60}$$

Therefore, substracting (59) from (60), we obtain from (i) and (56) that

$$\mathrm{Re} < F\ ,\ \frac{z_j e^{ij\theta}}{q(e^{i\theta})} - \frac{\bar{z}_j e^{i(n-j)\theta}}{q^R(e^{i\theta})} >_{\mathbb{T}} =\ 0\,, \quad \forall\, z_j \in \mathbb{C}\,. \tag{61}$$

The roots of q^R are reflected from those of q across $\mathbb{T}$, thus lie outside $\overline{\mathbb{D}}$. Hence $e^{i(n-j)\theta}/q^R(e^{i\theta}) \in H^2$, and since $F \in \bar{H}^2_0$ we see from (17) that (61) simplifies to

$$\mathrm{Re} < F\ ,\ \frac{z_j e^{ij\theta}}{q(e^{i\theta})} >_{\mathbb{T}} =\ 0\,, \quad \forall\, z_j \in \mathbb{C}\,.$$

As z_j is an arbitrary complex number, the symbol "Re" is redundant in this equation, therefore $< F\,,\, e^{ij\theta}/q(e^{i\theta}) >_{\mathbb{T}} = 0$ for all $j \in \{0, \ldots, n-1\}$ and combining linearly these n equations gives us

$$< F\,,\, \frac{p(e^{i\theta})}{q(e^{i\theta})} >_{\mathbb{T}} = 0\,, \qquad \forall p \in P_{n-1}\,, \tag{62}$$

where P_{n-1} is the space of algebraic polynomials of degree at most $n-1$. Now, it is elementary that

$$\bar{b}_1 \bar{H}_0^2 = \frac{q^R}{q}\,\bar{H}_0^2 = \left(\frac{P_{n-1}}{q}\right)^{\perp} \qquad \text{in} \quad \bar{H}_0^2, \tag{63}$$

and consequently from (62) and (63), we see that $F \in \bar{b}_1 \bar{H}_0^2$ as desired.
We turn to the proof that $F \in \bar{S}_\mu \bar{H}_0^2$, assuming that μ is not the zero measure otherwise it is trivial. We need introduce the inner divisors of S_μ which, by uniqueness of the inner-outer factorization, are just the singular factors S_{μ_0} where μ_0 is a positive measure on $\mathbb{T}$ such that $\mu - \mu_0$ is still positive. Pick such a μ_0, and set $\beta_0 = bS_{\mu-\mu_0}w$ so that $g_0 = S_{\mu_0}\beta_0$. For $a \in \mathbb{D}$, consider the function

$$j_a(z) \;=\; \frac{S_{\mu_0}(z) + a}{1 + \bar{a}S_{\mu_0}(z)}\,, \qquad z \in \mathbb{D}.$$

It is elementary to check that j_a is inner, so that $\beta_0 j_a$ is a candidate approximant in problem (21) because $|\beta_0 j_a| = |g_0|$ a.e. on $\mathbb{T}$. Therefore the map

$$a \to \|f - \beta_0\, j_a\|^2_{L^2(I)} \tag{64}$$

reaches a minimum on $\mathbb{D}$ at $a = 0$. Since

$$\frac{\partial j_a(z)}{\partial \mathrm{Re}\, a} \;=\; \frac{1}{1 + \bar{a}S_{\mu_0}(z)} - \frac{S_{\mu_0}(z)(S_{\mu_0}(z) + a)}{\left(1 + \bar{a}S_{\mu_0}(z)\right)^2}\,,$$

$$\frac{\partial j_a(z)}{\partial \mathrm{Im}\, a} \;=\; \frac{i}{1 + \bar{a}S_{\mu_0}(z)} + \frac{iS_{\mu_0}(z)(S_{\mu_0}(z) + a)}{\left(1 + \bar{a}S_{\mu_0}(z)\right)^2},$$

are bounded for $z \in \mathbb{T}$, locally uniformly with respect to $a \in \mathbb{D}$, we may differentiate (64) under the integral sign with respect to $\mathrm{Re}\, a$ and $\mathrm{Im}\, a$, and equating both partial derivatives to zero at $a = 0$ yields

$$\mathrm{Re}\, < (f - g_0)\,\overline{\beta}_0\,,\; (x + iy) - (x - iy)S^2_{\mu_0} >_I = 0\,, \qquad \forall x\,,\; y \in \mathbb{R}\,.$$

Multiplying both arguments of $<\;,\;>_I$ by the unimodular function $\overline{S}_{\mu_0}$ we get

$$\mathrm{Re}\, < (f - g_0)\,\overline{g}_0\,,\; (x + iy)\,\overline{S}_{\mu_0} - (x - iy)S_{\mu_0} >_I = 0\,, \qquad \forall x\,,\; y \in \mathbb{R}\,. \tag{65}$$

In another connection, as $(x+iy)\,\overline{S}_{\mu_0} - (x-iy)S_{\mu_0}$ is pure imaginary on $\mathbb{T}$ while λ is real-valued,

$$\mathrm{Re}\, < \lambda\,,\; (x + iy)\,\overline{S}_{\mu_0} - (x - iy)S_{\mu_0} >_J = 0\,, \qquad \forall x\,,\; y \in \mathbb{R}\,. \tag{66}$$

Substracting (65) from (66), we deduce from (i) and (56) that

$$\mathrm{Re}\, < F\,,\; (x + iy)\,\overline{S}_{\mu_0} - (x - iy)S_{\mu_0} >_{\mathbb{T}} = 0\,, \qquad \forall x\,,\; y \in \mathbb{R}.$$

Since $F \in \bar{H}^2_0$ while $S_{\mu_0} \in H^2$, this simplifies to

$$\text{Re} < F \ , \ (x+iy)\,\overline{S}_{\mu_0}) >_{\mathbb{T}} = 0\,, \qquad \forall x\,,\ y \in \mathbb{R}\,.$$

But $x + iy$ is arbitrary in $\mathbb{C}$, so the symbol "Re" is redundant in the above equation and we obtain

$$< F \ , \ \overline{S}_{\mu_0}) >_{\mathbb{T}} = 0. \tag{67}$$

Put $F(e^{i\theta}) = e^{-i\theta}\overline{g(e^{i\theta})}$ with $g \in H^2$, and conjugate (67) after multiplying both arguments by $e^{i\theta}$:

$$< g \, , \, e^{-i\theta} S_{\mu_0} >_{\mathbb{T}} = 0. \tag{68}$$

As S_μ is a nontrivial singular inner factor, it follows from [1, cor. 6.1.] that the closed linear span of the functions $\mathbf{P}_+(e^{-i\theta}S_{\mu_0})$ when S_{μ_0} ranges over all inner divisors of S_μ is equal to $(S_\mu H^2)^\perp$ in H^2. Hence (68) implies that $g \in S_\mu H^2$, and therefore $F \in \overline{S}_\mu\, \bar{H}^2_0$ as announced.

Having completed the proof of necessity when $f \in L^\infty(I)$, we now remove this restriction. Let $f \in L^2(I)$ and $f^{\{n\}} \in L^\infty(I)$ converge to f in $L^2(I)$. Adding to $f^{\{n\}}$ a small $L^2(I)$-function that goes to zero with n if necessary, we may assume that $f^{\{n\}} \notin H^2_{|_I}$. With the notations of Lemma 6, let us put for simplicity

$$F^{\{n\}} \triangleq \left({g_0}^{\{n\}}_{|_I} - f^{\{n\}} \right) \overline{g}_0{}^{\{n\}}_{|_I} \vee \lambda^{\{n\}}, \qquad F \triangleq \left({g_0}_{|_I} - f \right) \overline{g}_{0|_I} \vee \lambda. \tag{69}$$

By the first part of the proof, we can write

$$F^{\{n\}} = \bar{g}_0^{\{n\}} G^{\{n\}}\,, \text{ where } G^{\{n\}} \triangleq \left(({g_0}^{\{n\}}_{|_I} - f^{\{n\}}) \vee \lambda^{\{n\}}\, {g_0}^{\{n\}}_{|_J} \right) \in \bar{H}^1_0. \tag{70}$$

Note that $\|G^{\{n\}}\|_{L^1(\mathbb{T})}$ is bounded since $\|f^{\{n\}} - g_0^{\{n\}}\|_{L^2(I)} \leq \|f^{\{n\}}\|_{L^2(I)}$ (for the zero function is a candidate approximant) and $\|\lambda^{\{n\}} g_0^{\{n\}}\|_{L^1(J)} = \|\lambda^{\{n\}}\|_{L^1(J)}$ is bounded by Lemma 6. Thus, extracting a subsequence if necessary, we may assume that $G^{\{n\}}$ converges weak-* to some $G \in \bar{H}^1_0$, and then $G^{\{n\}}(z) \to G(z)$ for fixed $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by (15). Moreover, still from Lemma 6, we know that $F^{\{n\}}$ converges to F weak-* in $\bar{H}^1_0$, so we get by (15) again that $F^{\{n\}}(z) \to F(z)$ for fixed $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Finally Theorem 3 entails that $\overline{g}_0^{\{n\}} \to \overline{g}_0$ in $\bar{H}^2$, hence using (15) once more we get that $\bar{g}_0^{\{n\}}(z) \to \bar{g}_0(z)$ for fixed $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Altogether, in view of (70), this implies

$$F(z) = \lim_{n\to\infty} F^{\{n\}}(z) = \bar{g}_0(z)\, G(z)\,, \qquad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}\,,$$

showing that $F/\bar{g}_0 = G \in \bar{H}^1_0$. By (i) and the definition (69) of F, this yields (40) and achieves the proof. ■

Using Theorem 4 it is easy to characterize the solution to problem (2). For this, we write $L^1(M^2 d\theta, J)$ to mean those functions h on J such that $hM^2 \in L^1(J)$.

Corollary 3 *Assume that $M \in L^2(J)$ is non-negative with $\log M \in L^1(J)$, and that $f \in L^2(I)$ is not the trace on I of an H^2-function of modulus less than or equal to M a.e on J; suppose further that $\ell(\partial I) = 0$. Then, for $g_0 \in H^2$ to be the solution to problem (2), it is necessary and sufficient that the following two properties hold:*

(i) $|g_0(e^{i\theta})| = M(e^{i\theta})$ *for a.e.* $e^{i\theta} \in J$,

(ii) there exists a non-negative measurable function $\lambda \in L^1(M^2 d\theta, J)$, *such that:*

$$(g_{0|_I} - f) \vee \lambda\, g_{0|_J} \in \bar{w}_M^{-1} \bar{H}_0^1, \tag{71}$$

where w_M *designates the outer function with modulus* 1 *a.e. on* I *and modulus* M *a.e. on* J. *In particular if* $1/M \in L^\infty(J)$ *(more generally if* $\lambda M \in L^1(J)$*), then (71) amounts to:*

$$(g_{0|_I} - f) \vee \lambda\, g_{0|_J} \in \bar{H}_0^1. \tag{72}$$

Remark: We observe that, of necessity, $\log \lambda \in L^1(J)$.

Proof. Clearly (i) is equivalent to $|g_0/w_M| = 1$ a.e. on J, and since $|w_M|^2 = 1 \vee M^2$ we see on multiplying (71) by $\bar{w}_M$ that it is equivalent to

$$\left(\frac{g_{0|_I}}{w_M} - \frac{f}{w_M}\right) \vee \left(\lambda M^2\right) \frac{g_{0|_J}}{w_M} \in \bar{H}_0^1.$$

The conclusion now follows from Theorem 4 and the reduction of problem (2) to problem (21) given in section 3. If $\lambda M \in L^1(J)$ so does $\lambda g_{0|_J}$ by (i), and the function (71) lies in $e^{-\imath\theta}\overline{N^+} \cap L^1(\mathbb{T}) = \bar{H}_0^1$. ■

Relation (72) can be recast as a spectral equation for a Toeplitz operator, which should be compared with those in [3, 8] that form the basis of a constructive approach to BEP_2. There, λ is a constant and the operators involved are continuous. In our case we consider the Toeplitz operator $\phi_{0\vee(\lambda-1)}$

$$\phi_{0\vee(\lambda-1)}(g) = \mathbf{P}_+\left(0 \vee (\lambda-1) g_{|_J}\right),$$

having symbol $0 \vee (\lambda - 1)$, with values in H^2 and domain

$$\mathcal{D} = \{g \in H^2;\ \lambda g_{|_J} \in L^1(J),\ \mathbf{P}_+(0 \vee \lambda g_{|_J}) \in H^2\}.$$

By Beurling's theorem [19, chap. II, cor. 7.3] $\phi_{0\vee(\lambda-1)}$ is densely defined, for $\mathcal{D}$ contains $w_\rho H^2$ where w_ρ is the outer function with modulus $1 \vee \min(1, 1/\lambda)$. Note also that $I + \phi_{0\vee(\lambda-1)}$ is injective, because if $g_{|_I} \vee \lambda g_{|_J} \in \bar{H}_0^2$ for some $g \in \mathcal{D}$ we may multiply it by $\bar{g}$ to obtain a $\bar{H}_0^1$-function h which is real-valued on $\mathbb{T}$ and thus identically zero by Poisson representation of $\overline{h(1/\bar{z})} \in e^{i\theta} H^1$.

Corollary 4 *Let* $M \in L^2(J)$ *be non-negative and* $1/M \in L^\infty(J)$. *Assume* $f \in L^2(I)$ *is not the trace on* I *of a* H^2*-function of modulus less than or equal to* M *a.e on* J*; suppose further that* $\ell(\partial I) = 0$. *If* g_0 *is the solution to problem* (2) *and* λ *is as in (71), then*

$$g_0 = \left(I + \phi_{0\vee(\lambda-1)}\right)^{-1} \mathbf{P}_+(f \vee 0). \tag{73}$$

Proof. From (72) we see that $\lambda g_{0|_J} \in L^1(J)$ and that

$$\mathbf{P}_+(0 \vee \lambda g_{0|_J}) = \mathbf{P}_+((f - g_{0|_I}) \vee 0) \in H^2,$$

hence $g_0 \in \mathcal{D}$. Using that $g_0 = \mathbf{P}_+(g_0)$, we now obtain (73) on rewriting (72) as

$$\mathbf{P}_+\Big(g_0 \,+\, 0 \vee (\lambda - 1) g_{0|_J} \,-\, f \vee 0\Big) = 0.$$

■

Further smoothness properties of $\lambda M^2 \in L^1(J)$ follow from the next representation formula.

Proposition 2 *Let $M \in L^2(J)$ be non-negative with $\log M \in L^1(J)$, and assume that $f \in L^2(I)$ is not the trace on I of an H^2-function of modulus less than or equal to M a.e. on J. Suppose also that $\ell(\partial I) = 0$. If g_0 denotes the solution to problem* (2) *and $\lambda \in L^1(M^2 d\theta, J)$ is the non-negative function such that (71) holds, then λM^2 extends across $\overset{\circ}{J}$ to a holomorphic function F on $\overline{\mathbb{C}} \setminus \overline{I}$ satisfying*

$$F\left(1/\bar{z}\right) \,=\, \overline{F(z)}, \qquad z \in \overline{\mathbb{C}} \setminus \overline{I}. \tag{74}$$

Moreover, we have the Herglotz-type representation:

$$F(z) \,=\, \frac{1}{2i\pi} \int_I \frac{e^{i\theta} + z}{e^{i\theta} - z} \operatorname{Im}\left\{ f(e^{i\theta})\, \overline{g_0(e^{i\theta})} \right\} d\theta\,, \qquad z \in \overline{\mathbb{C}} \setminus \overline{I}. \tag{75}$$

Proof. By (i) of Corollary 3 we know that $|g_0| = M$ a.e. on J, hence multiplying (71) by $\bar{g}_0$ we get

$$\left(|g_{0|_I}|^2 - f\, \bar{g}_{0|_I} \right) \,\vee\, \lambda\, M^2 \,\in\, e^{-i\theta}\, \overline{N}^+ \cap L^1(\mathbb{T}) = \bar{H}^1_0. \tag{76}$$

Call F the concatenated function on the left of (76), so that $H(z) = i\, \overline{F(1/\bar{z})}$ lies in H^1 and vanishes at zero since it has zero mean on $\mathbb{T}$. Clearly H has real part $\operatorname{Im} f\, \bar{g}_{0|_I} \vee 0$ on $\mathbb{T}$, so the Riesz-Herglotz representation (5) yields:

$$i\, \overline{F(1/\bar{z})} \,=\, \frac{1}{2\pi} \int_I \frac{e^{i\theta} + z}{e^{i\theta} - z} \operatorname{Im}\left\{ f(e^{i\theta})\, \overline{g_0(e^{i\theta})} \right\} d\theta\,, \qquad z \in \mathbb{D}\,,$$

and upon conjugating and changing z into $1/\bar{z}$ we obtain (75) for $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. As the right-hand side extends analytically to $\mathbb{D}$ across $\overset{\circ}{J}$ by reflection, (74) follows.

■

The interpretation of λ as a Lagrange parameter is justified by the duality relation below. For convenience, we write $L^1_+(M^2 d\theta, J)$ for the set of non-negative functions in $L^1(M^2 d\theta, J)$ whose logarithm lies in $L^1(J)$.

Proposition 3 *Assume that $M \in L^2(J)$ is non-negative with $\log M \in L^1(J)$, and that $f \in L^2(I)$ is not the trace on I of an H^2-function of modulus less than or equal to M a.e on J. Suppose further that $\ell(\partial I) = 0$, and let $g_0 \in H^2$ be the solution to Problem 2 with λ as in (71). Then, it holds that*

$$\|f - g_0\|^2_{L^2(I)} = \max_{\mu \in L^1_+(M^2 d\theta, J)} \; \min_{g \in H^2} \; \|f - g\|^2_{L^2(I)} + \int_J \mu\, (|g|^2 - M^2)\, d\theta \tag{77}$$

$$= \min_{g\in H^2} \max_{\mu\in L^1_+(M^2 d\theta,J)} \|f-g\|^2_{L^2(I)} + \int_J \mu\,(|g|^2 - M^2)\,d\theta.$$

Moreover, the max min *and the* min max *are simultaneously met for* $g = g_0$ *and* $\mu = \lambda$.

Proof. Let A, B respectively denote the max min and the min max in (77). Setting $g = g_0$ for each μ, we get $\|f-g_0\|^2_{L^2(I)} \geq A$ from Corollary 3-(i). For the reverse inequality, we fix $\mu = \lambda$ and we show that

$$\min_{g\in H^2} \|f-g\|^2_{L^2(I)} + \int_J \lambda\,(|g|^2 - M^2)\,d\theta$$

is attained at g_0. Clearly, it is enough to minimize over those $g \in H^2$ such that $\lambda|g|^2 \in L^1(J)$. Pick such a g, and for $t \in \mathbb{R}$ let $g_t = g_0 + t(g-g_0)$. The function

$$\Psi(t) = \|f-g_t\|^2_{L^2(I)} + \int_J \lambda\,(|g_t|^2 - M^2)\,d\theta,$$

is convex and continuously differentiable on $\mathbb{R}$. Differentiating under the integral sign, we get

$$\Psi'(t) = 2\mathrm{Re}\,(< g_t - f,\, g-g_0 >_I + < \lambda g_t,\, g - g_0 >_J)\,,$$

and in particular

$$\Psi'(0) = 2\mathrm{Re}\left(< (g_{0|_I} - f) \vee \lambda g_{0|_J}\,,\, g - g_0 >_{\mathbb{T}}\right) \tag{78}$$

$$= 2\mathrm{Re}\left(< \big((g_{0|_I} - f) \vee \lambda g_{0|_J}\big)(\overline{g} - \overline{g_0})\,,\, 1 >_{\mathbb{T}}\right).$$

Now $(g_0 - f) \vee \lambda g_0 \in e^{-i\theta}\overline{N^+}$ by (71), and since $g - g_0 \in H^2$ it also holds that $\overline{g} - \overline{g_0} \in \overline{N^+}$. Therefore

$$\big((g_{0|_I} - f) \vee \lambda g_{0|_J}\big)(\overline{g} - \overline{g_0}) \in e^{-i\theta}\overline{N^+},$$

and since it belongs to $L^1(\mathbb{T})$ because $\lambda^{1/2} g_{0|_J}$ and $\lambda^{1/2} g_{|_J}$ both lie in $L^2(J)$, we deduce that it is also in $\bar{H}^1_0$. Consequently it has zero mean on $\mathbb{T}$, and we see from (78) that $\Psi'(0) = 0$, hence Ψ meets a minimum at 0 by convexity. Expressing that $\|f-g_0\|^2_{L^2(I)} = \Psi(0) \leq \Psi(1)$ for each $g \in H^2$ such that $\lambda|g|^2 \in L^1(J)$ leads us to $\|f-g_0\|^2_{L^2(I)} \leq A$, as desired. Thus we have proven the first equality in (77) and we also have shown it is an equality for $g = g_0$ and $\mu = \lambda$.
To establish that $\|f-g_0\|^2_{L^2(I)} = B$, observe first that

$$\max_{\mu\in L^1_+(M^2 d\theta,J)} \|f-g\|^2_{L^2(I)} + \int_J \mu\,(|g|^2 - M^2)\,d\theta = +\infty$$

unless $|g| \leq M$ a.e. on J; indeed if $|g| > M$ on a set $E \subset J$ of strictly positive measure, we can set $\mu = \rho\chi_E + \varepsilon$ for fixed $\varepsilon > 0$ and arbitrarily large ρ. Thus we may restrict the minimization in the second line of (77) to those g such that $|g| \leq M$ a.e. on J. For such g the maximum is attained when $\mu = 0$, and by definition g_0 minimizes $\|f-g_0\|^2_{L^2(I)}$ among them. Moreover, when $g = g_0$, we see from Corollary 3-(i) that μ is irrelevant in the criterion and can be chosen to be λ. This achieves the proof. ■

Note that Proposition 3 would still hold if we dropped the log-integrability requirement in the definition of $L^1(M^2 d\theta, J)$, for the latter was never needed in the proof. However, this requirement conveniently restricts the maximization space in (77) to a class of μ for which one can form the outer function w_μ, and this will be of use in what follows.

5 The dual functional and Carleman's formulas

For $M \in L^2(J)$ a non-negative function such that $\log M \in L^1(J)$ and $f \in L^2(I)$ which is not the trace on I of a H^2-function of modulus less than or equal to M a.e. on J, we denote by Φ_M the *dual functional* in problem (2) acting on $L^1_+(M^2 d\theta, J)$ as follows (compare [12, sec. 4.3]):

$$\Phi_M(\mu) = \min_{g \in H^2} \; \|f - g\|^2_{L^2(I)} + \int_J \mu\,(|g|^2 - M^2)\, d\theta, \qquad \mu \in L^1_+(M^2 d\theta, J). \quad (79)$$

As an *infimum* of affine functions, Φ_M is concave and upper semi-continuous with respect to μ. In view of (77), solving problem (2) amounts to maximize Φ_M over the convex set $L^1_+(M^2 d\theta, J)$. As we shall see momentarily (*cf.* Proposition 4), the true nature of Carleman-type formulas in this context is that they solve for the optimal g in (79) whenever the *min* is attained. We begin with a theorem showing how Carleman's formula solves for g_0 in (71) as a function of f and λ.

Theorem 5 *Let $M \in L^2(J)$ be non-negative with $\log M \in L^1(J)$, and assume that $f \in L^2(I)$ is not the trace on I of a H^2-function of modulus less than or equal to M a.e. on J. Suppose that $\ell(\partial I) = 0$, and let g_0 be the solution to problem (2) while $\lambda \in L^1(M^2 d\theta, J)$ denotes the non-negative function such that (71) holds. Write $w_{\lambda^{1/2}}$ for the outer function with modulus $\lambda^{1/2}$ a.e. on J and modulus 1 a.e. on I. Then*

$$g_0(z) \; = \; \frac{1}{2i\pi\, w_{\lambda^{1/2}}(z)} \int_I \frac{w_{\lambda^{1/2}}(\xi)\, f(\xi)}{\xi - z}\, d\xi, \qquad z \in \mathbb{D}. \quad (80)$$

Conversely, if λ is a positive function on J such that $\log \lambda \in L^1(J)$ and if g_0 defined by (80) lies in H^2, then g_0 is the solution to problem (2) where $M = |g_{0|_J}|$. In this case λ is the function appearing in (71).

Proof. Assume g_0 is the solution to problem (2) so that (i) and (ii) of Corollary 3 hold. Dividing (71) by $\bar{w}_{\lambda^{1/2}}$ and using that $|w_{\lambda^{1/2}}|^2 = 1 \vee \lambda$, we deduce

$$w_{\lambda^{1/2}}(g_0 - (f \vee 0)) \; \in \; \bar{w}^{-1}_{\lambda^{1/2}}\, \bar{w}_M^{-1}\, \bar{H}^1_0.$$

Since $\lambda \in L^1(M^2 d\theta, J)$, the left-hand side lies in $L^2(\mathbb{T})$ and therefore it belongs to $\bar{H}^2_0$ because the right-hand side is in $e^{-i\theta}\overline{N^+}$ by construction. In particular

$$\mathbf{P}_+ \left(w_{\lambda^{1/2}}(g_0 - (f \vee 0)) \right) = 0. \quad (81)$$

But $w_{\lambda^{1/2}} g_0 \in H^2$ because it clearly belongs to $N^+ \cap L^2(\mathbb{T})$, so that (81) implies

$$w_{\lambda^{1/2}} g_0 = \mathbf{P}_+(w_{\lambda^{1/2}} g_0) = \mathbf{P}_+ \left(w_{\lambda^{1/2}}(f \vee 0) \right).$$

Now (80) follows from this and (20). Conversely, assume that g_0 defined by (80) lies in H^2 and set $M = |g_0|_{|_J}$. Since $f w_{\lambda^{1/2}} \in L^2(I)$, we see from (80) and (20) that $g_0 w_{\lambda^{1/2}} \in H^2$ and that

$$g_0 w_{\lambda^{1/2}} = \mathbf{P}_+ \left(f w_{\lambda^{1/2}} \vee 0 \right)$$

which implies (81). Thus $w_{\lambda^{1/2}}(g_0 - (f \vee 0)) \in \bar{H}^2_0$ and multiplying by $\bar{w}_M \bar{w}_{\lambda^{1/2}} \in \bar{H}^2$ yields

$$\bar{w}_M\, |w_{\lambda^{1/2}}|^2 \,(g_0 - (f \vee 0)) = \bar{w}_M \big((g_{0|_I} - f) \vee \lambda g_{0|_J} \big) \in \bar{H}^1_0$$

from which (71) follows. As (i) of Corollary 3 is met by definition, g_0 indeed solves for problem (2). ■

Theorem 5 can be used as follows to compute the function $\Phi_M(\mu)$ introduced in (79).

Proposition 4 *Let $M \in L^2(J)$ be non-negative with $\log M \in L^1(J)$, and assume that $f \in L^2(I)$ is not the trace on I of an H^2-function of modulus less than or equal to M a.e. on J. Suppose further that $\ell(\partial I) = 0$ and let $\mu \in L^1_+(M^2 d\theta, J)$. Then, the function $\Phi_M(\mu)$ defined by (79) can be expressed as*

$$\Phi_M(\mu) = \left\|\mathbf{P}_-\left(f w_{\mu^{1/2}} \vee 0\right)\right\|^2_{L^2(\mathbb{T})} - \left\|\mu^{1/2} M\right\|^2_{L^2(J)}. \tag{82}$$

Moreover, if we set

$$g_\mu(z) = \frac{1}{2i\pi\, w_{\mu^{1/2}}(z)} \int_I \frac{w_{\mu^{1/2}}(\xi)\, f(\xi)}{\xi - z}\, d\xi, \qquad z \in \mathbb{D}, \tag{83}$$

then the infimum in the right-hand side of (79) is attained at $g = g_\mu$ whenever the latter belongs to H^2. In particular, this is the case when $1/\mu \in L^\infty(J)$.

Proof. Assume first that μ is such that $g_\mu \in H^2$; this holds in particular when $1/\mu \in L^\infty(J)$, because then $1/w_{\mu^{1/2}} \in H^\infty$ while (20) shows that the integral in (83) lies in H^2. From Theorem 5 it follows that g_μ is the solution to problem (2) where M gets replaced by $|g_\mu|$, and μ plays the role of λ in (71). Hence Proposition 3 implies that g_μ is an infimizer of

$$\min_{g\in H^2} \|f - g\|^2_{L^2(I)} + \int_J \mu\left(|g|^2 - |g_\mu|^2\right) d\theta,$$

and since μ is kept fixed g_μ is clearly also an infimizer of

$$\min_{g\in H^2} \|f - g\|^2_{L^2(I)} + \int_J \mu\left(|g|^2 - M^2\right) d\theta$$

which is just the right-hand side of (79). This proves the second assertion of the proposition.

By (83)) and (20), taking into account that $|w_{\mu^{1/2}}| = 1 \vee \mu^{1/2}$, what precedes can be reformulated as

$$\begin{aligned}
\Phi_M(\mu) &= \|f - g_\mu\|^2_{L^2(I)} + \int_J \mu\left(|g_\mu|^2 - M^2\right) d\theta \\
&= \left\|(w_{\mu^{1/2}} f \vee 0) - w_{\mu^{1/2}} g_\mu\right\|^2_{L^2(\mathbb{T})} - \int_J \mu\, M^2\, d\theta \\
&= \left\|P_{\bar{H}^2_0}\left(f w_{\mu^{1/2}} \vee 0\right)\right\|^2_{L^2(\mathbb{T})} - \left\|\mu^{1/2} M\right\|^2_{L^2(J)}.
\end{aligned}$$

This proves (82) when $g_\mu \in H^2$. To get it in general we apply what we just did to the sequence $\mu_n = \mu + 1/n$, observing that $g_{\mu_n} \in H^2$ because $1/\mu_n \in L^\infty(J)$. By monotone convergence we obtain

$$\lim_{n\to\infty} \left\|\mu_n^{1/2} M - \mu^{1/2} M\right\|_{L^2(J)} = 0. \tag{84}$$

Moreover, as $\log \mu_n$ decreases to $\log \mu$, we certainly have on putting $\log^-(x) = \max\{-\log x, 0\}$ and $\log^+(x) = \max\{\log x, 0\}$ that

$$\begin{array}{rcl} \log^- \mu_n & \leq & \log^- \mu \leq |\log \mu| \in L^1(J), \\ \log^+ \mu_n & \leq & \log^+ \left(\mu_n M^2\right) + \left|\log M^2\right| \leq \left|\mu_n M^2 - 1\right| + 2|\log M| \\ & \leq & (\mu+1)M^2 + 1 + 2|\log M| \in L^1(J), \end{array}$$

and therefore, by dominated convergence as applied to $\log \mu_n = \log^+ \mu_n - \log^- \mu_n$, we obtain

$$\lim_{n\to\infty} \exp\left\{\frac{1}{4\pi}\int_J \frac{e^{it}+z}{e^{it}-z}\log \mu_n dt\right\} = \exp\left\{\frac{1}{4\pi}\int_J \frac{e^{it}+z}{e^{it}-z}\log \mu dt\right\}, \qquad z \in \mathring{I},$$

in other words $w_{\mu_n^{1/2}}$ converges pointwise to $w_{\mu^{1/2}}$ on $\mathring{I}$ and therefore almost everywhere on I since $\ell(\partial I) = 0$. Thus, appealing to dominated convergence once more, we get

$$\lim_{n\to\infty} \left\| f w_{\mu_n^{1/2}} - f w_{\mu^{1/2}} \right\|_{L^2(I)} = 0, \tag{85}$$

and from (84), (85), and (82) which is known to hold with μ replaced by μ_n, we see that

$$\lim_{n\to\infty} \Phi_M(\mu_n) = \left\| P_{\bar{H}_0^2}\left(f w_{\mu^{1/2}} \vee 0\right)\right\|^2_{L^2(\mathbb{T})} - \left\|\mu^{1/2} M\right\|^2_{L^2(J)}. \tag{86}$$

In another connection, it is plain that

$$\limsup_{n\to\infty} \Phi_M(\mu_n) \leq \Phi_M(\mu) \leq \liminf_{n\to\infty} \Phi_M(\mu_n), \tag{87}$$

where the first inequality comes from (84) and the upper semi-continuity of Φ_M in $L^1_+(M^2 d\theta, J)$ while the second inequality is obvious from (79), (84), and the fact that $\mu \leq \mu_n$. Now (82) follows from (86) and (87). ■

Being concave on the convex set $L^1_+(M^2 d\theta, J)$, the functional Φ_M has a directional derivative at every point in each admissible direction. Here, a direction h is said to be admissible at $\mu \in L^1_+(M^2 d\theta, J)$ if $\mu + th \in L^1_+(M^2 d\theta, J)$ as soon as $t \geq 0$ is small enough. From a constructive viewpoint, computing this derivative is important when designing ascent algorithms to maximize Φ_M and thus numerically solve for problem (2). The next proposition does it, under mild assumptions on f, in those directions h such that $h/\mu \in L^\infty(J)$. Note since $\mu \neq 0$ a.e.(for $\log \mu \in L^1(J)$) that such directions are dense in the set of all admissible directions, hence this result allows one indeed to find a direction of ascent for Φ_M.

Proposition 5 *Assumptions and notations being as in Proposition 4, suppose in addition that $|f|^2$ lies in the Zygmund class $L\log^+ L$. Let further h be a real function on J such that $\|h/\mu\|_{L^\infty(J)} < 1$. Then $\mu + h \in L^1_+(M^2 d\theta, J)$ and $h \in L^1(M^2 d\theta, J)$. Moreover, defining g_μ as in (83), it holds that $h|g_\mu|^2 \in L^1(J)$ and that*

$$\left|\Phi_M(\mu+h) - \Phi_M(\mu) - \int_J h(|g_\mu|^2 - M^2)\, d\theta\right| = \mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right), \tag{88}$$

where the function $\mathbf{o}$, which depends on f and μ only, is a little o of its argument near 0.

Proof. Clearly $\mu + h = \mu(1 + h/\mu) \in L^1_+(M^2 d\theta, J)$ whenever $\|h/\mu\|_{L^\infty(J)} < 1$, which in turn entails $h \in L^1(M^2 d\theta, J)$. In another connection, we see from (20) that (83) can be rewritten as

$$w_{\mu^{1/2}} g_\mu = \mathbf{P}_+(f w_{\mu^{1/2}} \vee 0), \tag{89}$$

and since $|w_{\mu^{1/2}}|^2 = 1 \vee \mu$ we see that $w_{\mu^{1/2}} g_\mu \in H^2$ hence

$$h|g_{\mu|_J}|^2 = (h/\mu)\mu|g_{\mu|_J}|^2 \in L^1(J)$$

for $h/\mu \in L^\infty(J)$. Thus the integral in the left-hand side of (88) is well-defined. Next, multiplying the $\bar{H}^2_0$-function $w_{\mu^{1/2}} g_\mu - (w_{\mu^{1/2}} f \vee 0)$ by the $\bar{H}^2$-function $\overline{w_{\mu^{1/2}} g_\mu}$ yields

$$(|g_{\mu|_I}|^2 - f\bar{g}_{\mu|_I}) \vee \mu|g_{\mu|_J}|^2 \in \bar{H}^1_0.$$

Therefore the conjugate function of $(|g_{\mu|_I}|^2 - \mathrm{Re}(\bar{f} g_{\mu|_I})) \vee \mu|g_{\mu|_J}|^2$ lies in $L^1(\mathbb{T})$, and by Zygmund's theorem so does the conjugate function of $|f|^2 \vee 0$ since the latter lies in $L\log^+ L$ by assumption. Adding up yields

$$\widetilde{\overbrace{\left(\frac{|g_\mu|^2_{|_I} + |f|^2}{2} + \frac{|g_{\mu|_I} - f|^2}{2}\right) \vee \mu|g_\mu|^2_{|_J}}} \in L^1(\mathbb{T}),$$

and since the function under brace is positive it lies in $L\log^+ L$ by the M. Riesz theorem. *A fortiori* then,

$$\begin{aligned}\left|\mathbf{P}_-(f w_{\mu^{1/2}|_I} \vee 0)\right|^2 &= \left|(f w_{\mu^{1/2}|_I} \vee 0) - w_{\mu^{1/2}} g_\mu\right|^2 \\ &= |g_{\mu|_I} - f|^2 \vee \mu|g_\mu|^2_{|_J} \in L\log^+ L.\end{aligned} \tag{90}$$

Now, let us write

$$\begin{aligned}w_{(\mu+h)^{1/2}}(z) &= w_{\mu^{1/2}}(z) \exp\left\{\frac{1}{4\pi}\int_J \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(1+h/\mu)(e^{i\theta})\,d\theta\right\} \\ &= w_{\mu^{1/2}}(z)\, e^{\Delta_h(z)},\end{aligned}$$

where we have put for simplicity

$$\Delta_h(z) = \frac{1}{4\pi}\int_J \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(1+h/\mu)(e^{i\theta})\,d\theta, \qquad z \in \mathbb{D}. \tag{91}$$

Note that $\Delta_h \in BMOA$ since $\log(1+h/\mu) \in L^\infty(J)$. With this notation, it is straightforward that

$$\begin{aligned}&\left\|\mathbf{P}_-\left(f w_{(\mu+h)^{1/2}} \vee 0\right)\right\|^2_{L^2(\mathbb{T})} - \left\|\mathbf{P}_-\left(f w_{\mu^{1/2}} \vee 0\right)\right\|^2_{L^2(\mathbb{T})} \\ &= \left\|\mathbf{P}_-\left(f w_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right)\right\|^2_{L^2(\mathbb{T})} \\ &+2\mathrm{Re} < \mathbf{P}_-\left(f w_{\mu^{1/2}} \vee 0\right), \mathbf{P}_-\left(f w_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right) >_{\mathbb{T}},\end{aligned} \tag{92}$$

and our next goal is to prove that

$$\left| 2\mathrm{Re} < \mathbf{P}_-\left(f w_{\mu^{1/2}} \vee 0\right) \;,\; \mathbf{P}_-\left(f w_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right) >_{\mathbb{T}} - \int_J h|g_\mu|^2\,d\theta \right|$$

$$= \mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right). \tag{93}$$

For this, since $\mathbf{P}_+ + \mathbf{P}_- = \mathrm{id}$, we first observe from (17) that

$$\begin{aligned}
&< \mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right), \mathbf{P}_-\left(fw_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right) >_{\mathbb{T}} \\
&=< \mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right), (e^{\Delta_h}-1)(fw_{\mu^{1/2}} \vee 0) >_{\mathbb{T}} \\
&=< \mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right), (e^{\Delta_h}-1)\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right) >_{\mathbb{T}} \\
&=< \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, e^{\Delta_h}-1 >_{\mathbb{T}}
\end{aligned}$$

where we used in the second equality that $(e^{\Delta_h}-1)\mathbf{P}_+\left(fw_{\mu^{1/2}} \vee 0\right) \in H^2$ for $e^{\Delta_h}-1 \in H^\infty$. Besides,

$$\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right) + \mathbf{P}_+\left(fw_{\mu^{1/2}} \vee 0\right) = 0 \quad a.e. \text{ on } J$$

which implies in view of (89) that

$$\int_J h|g_\mu|^2 = \int_J \frac{h}{\mu}\left|\mathbf{P}_+\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2 =< \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, 0 \vee h/\mu >_{\mathbb{T}}.$$

Altogether, the expression inside absolute values on the left-hand side of (93) is thus

$$< \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, \mathrm{Re}\left(2(e^{\Delta_h}-1) - (0 \vee h/\mu)\right) >_{\mathbb{T}}.$$

Now, if we remark from (91) that, on $\mathbb{T}$, we have $2\Delta_h = 0 \vee \log(1+h/\mu) + i\varphi$ where φ denotes the conjugate function of $0 \vee \log(1+h/\mu)$, the above quantity becomes $Q_1 + Q_2$ with

$$\begin{aligned}
Q_1 &\triangleq 2 < \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, (\cos(\varphi/2)-1)\left(1 \vee (1+h/\mu)^{1/2}\right) >_{\mathbb{T}}, \\
Q_2 &\triangleq 2 < \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, (1+h/\mu)^{1/2} - 1 - h/(2\mu) >_J.
\end{aligned}$$

We prove separately that Q_1 and Q_2 are both $\mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right)$; here and thereafter, we use the same symbol $\mathbf{o}$ for different functions as this causes no confusion. On the one hand, since there is an absolute constant C such that $\left|(1+h/\mu)^{1/2} - 1 - h/(2\mu)\right| < C\|h/\mu\|^2_{L^\infty(J)}$ for $\|h/\mu\|_{L^\infty(J)} < 1$, we have that

$$|Q_2| \leq 2C\|f\|^2_{L^2(\mathbb{T})}\|h/\mu\|^2_{L^\infty(J)} \tag{94}$$

which is indeed $\mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right)$, where "$\mathbf{o}$" is independent of μ. On the other hand, as $\cos(\varphi/2) - 1 \leq 0$, it holds for $\|h/\mu\|_{L^\infty(J)} < 1$ that

$$|Q_1| \leq 2\sqrt{2} < \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, 1 - \cos(\varphi/2) >_{\mathbb{T}}. \tag{95}$$

Put for simplicity

$$B_h \triangleq (1-\cos(\varphi/2))/\varphi \text{ and } u_n \triangleq \min\left\{\left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, n\right\},$$

so that B_h is uniformly bounded (independently of h) and so is u_n for fixed n. By monotone convergence we get

$$< \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, 1 - \cos(\varphi/2) >_{\mathbb{T}} \tag{96}$$

$$= \lim_{n\to\infty} < u_n\ ,\ 1-\cos(\varphi/2) >_{\mathbb{T}} = \lim_{n\to\infty} < B_h u_n\ ,\ \varphi >_{\mathbb{T}}\ .$$

Being the product of two functions in $BMOA$, the function

$$(B_h u_n + i\widetilde{B_h u_n})((0 \vee \log(1+h/\mu)) + i\varphi)$$

certainly lies in H^1 and since it is real at 0 we deduce from the Cauchy formula and (7) that

$$|< B_h u_n\ ,\ \varphi >_{\mathbb{T}}| = \left| < \widetilde{B_h u_n}\ ,\ \log(1+h/\mu) >_J \right| \tag{97}$$

$$\leq \|B_h u_n\|_{L\log^+ L} \|\log(1+h/\mu)\|_{L^\infty(J)}.$$

As $|B_h u_n| \leq \left| B_h \left(\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right)^2 \right|$ the same is true of their decreasing rearrangements, thus by (6) and the inequality $|\log(1+h/\mu)| \leq 2|h/\mu|$ which is valid for $|h/\mu| \leq 1/2$, we obtain from (96)-(97) that

$$< \left|\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right|^2\ ,\ 1-\cos(\varphi/2) >_{\mathbb{T}}$$

$$\leq\ 2 \left\| B_h \left(\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right)^2 \right\|_{L\log^+ L} \|h/\mu\|_{L^\infty(J)}$$

as soon as $\|h/\mu\|_{L^\infty(J)} < 1/2$. Therefore, to prove that (95) is $\mathbf{o}(\|h/\mu\|_{L^\infty(J)})$, it is enough to show that

$$\lim_{\|h/\mu\|_{L^\infty(J)}\to 0} \left\| B_h \left(\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right)^2 \right\|_{L\log^+ L} = 0. \tag{98}$$

It is easily checked that $\|B_h\|_{L^\infty(\mathbb{T})} \leq 8$, say, which entails for the decreasing rearrangements the inequality

$$\left(B_h \left(\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right)^2 \right)^* \ \leq\ 8 \left(\left(\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right)^2 \right)^*. \tag{99}$$

Since the right-hand side of (99) is independent of h and lies in $L\log^+ L$ by (90), we deduce from the definition (6) of the $L\log^+ L$-norm that to each $\varepsilon > 0$ there is $\eta > 0$ such that

$$E \subset [0,1] \text{ and } \ell(E) < \eta \Longrightarrow \int_E \left(B_h \left(\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right)^2 \right)^* (t)\ \log(1/t)\, dt <\ \varepsilon.$$

Thus, (98) will hold if only $\left(B_h \left(\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right)^2 \right)^*$ tends to 0 *in measure* as $\|h_n/\mu\|_{L^\infty(J)} \to 0$. Since a function and its decreasing rearrangement have the same distribution function, this is equivalent to

$$\lim_{\|h/\mu\|_{L^\infty(J)}\to 0} B_h \left(\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right)^2 = 0 \quad \text{in measure on } \mathbb{T}. \tag{100}$$

Because $|w_{\mu^{1/2}}| = 1$ on I and $\mathbf{P}_-$ is a contraction in $L^2(\mathbb{T})$, we have the Kolmogorov estimate

$$\ell\{\xi \in \mathbb{T};\ \left|\mathbf{P}_- \left(f w_{\mu^{1/2}} \vee 0\right)\right|^2 > x\} \leq \frac{\|f\|^2_{L^2(I)}}{x},$$

hence (100) will hold if B_h alone converges to 0 in measure. As $|B_h| \leq C'|\varphi|$ for some absolute constant C' it is sufficient to establish that φ in turn converges to 0 in measure on $\mathbb{T}$. But this follows from the fact that φ tends to 0 in $L^p(\mathbb{T})$ when $\|h/\mu\|_{L^\infty(J)} \to 0$ for $1 < p < \infty$, since by the M. Riesz theorem

$$\|\varphi\|_{L^p(\mathbb{T})} \leq K_p \|\log(1 + h/\mu)\|_{L^p(\mathbb{T})} \leq 2K_p \|h/\mu\|_{L^\infty(\mathbb{T})}$$

as soon as $\|h/\mu\|_{L^\infty(J)} < 1/2$. This completes the proof of (93). In the same vein we show that

$$\left\|\mathbf{P}_- \left(f w_{\mu^{1/2}}(e^{\Delta_h} - 1) \vee 0\right)\right\|^2_{L^2(\mathbb{T})} = \mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right). \tag{101}$$

Indeed, since $\mathbf{P}_-$ is a contraction in $L^2(\mathbb{T})$ and $|w_{\mu^{1/2}}| \equiv 1$ on I, we have that

$$\begin{aligned}\left\|\mathbf{P}_- \left(f w_{\mu^{1/2}}(e^{\Delta_h} - 1) \vee 0\right)\right\|^2_{L^2(\mathbb{T})} &\leq \; < |f|^2 \,,\; \left|e^{\Delta_h} - 1\right|^2 >_{L^2(I)} \\ &= 2 < |f|^2 \,,\; (1 - \cos(\varphi/2)) >_{L^2(I)}\end{aligned}$$

which can be treated like the right-hand side of (95) to obtain (101), granted that $|f|^2 \vee 0 \in L\log^+ L$. In view of (82), (92), (93) and (101), the proof is complete once we have observed that

$$\left\|(\mu + h)^{1/2} M\right\|^2_{L^2(J)} - \left\|(\mu)^{1/2} M\right\|^2_{L^2(J)} = \int_J hM^2. \tag{102}$$

■

Remark: It would be interesting to know whether Proposition 5 holds true as soon as $f \in L^2(I)$, without having to assume that $|f|^2 \in L\log^+ L$. In this case, it is easy to check using (10), (94), (95), and (102) that

$$\begin{aligned}&\left|\Phi_M(\mu + h) - \Phi_M(\mu) - \int_J h(|g_\mu|^2 - M^2)\, d\theta\right| \\ &= \mathbf{O}\left(\left(\|h/\mu\|^2_{L^\infty(J)} + \int_0^\pi \frac{\omega_{0\vee h/\mu}(t)}{t}\, dt\right)^2\right),\end{aligned}$$

which is a weak substitute to (88) under the (much stronger) assumption that $0 \vee h/\mu$ is Dini-continuous.

6 A constructive polynomial approach

We now establish a finite dimensional (polynomial) analogous of Theorem 4 in order to constructively approach problem $BEP_{2,\infty}$ (21), assuming I to be a finite union of closed disjoint sub-arcs of $\mathbb{T}$. This is done from the point of view of convex optimization theory and somehow independently of the previous results, and allows us to get an alternative proof of Theorem 4.
Let T_n be the space of algebraic polynomials in the variable $z = e^{i\theta}$ of degree less or equal to n with coefficients in $\mathbb{C}$. We introduce the following finite dimensional bounded extremal problem $FBEP^n_{2,\infty}$.

$FBEP^n_{2,\infty}$**:** For $f \in L^2(I)$, find $k_n \in T_n$ such that $|k_n(e^{i\theta})| \leq 1$ for a.e $e^{i\theta} \in J$ and such that

$$\|f - k_n\|_{L^2(I)} = \min_{\substack{g \in T_n \\ |g| \leq 1 \text{ a.e. in } J}} \|f - g\|_{L^2(I)} . \tag{103}$$

The existence of k_n is ensured by the compactness of the approximation set $\{g \in T_n, \, ||g||_{L^\infty(J)} \leq 1\}$ whereas uniqueness follows from the convexity of this set combined with the strict convexity of the L^2 norm.
For $g \in T_n$ define

$$E(g) = \{x \in \overline{J}, \ |g(x)| = ||g||_{L^\infty(J)}\}$$

the so called set of critical points of g. The solution k_n is caracterised by the following result.

Theorem 6 *The element $g \in T_n$ is the optimal solution of $FBEP^n_{2,\infty}$ (103) if, and only if, the following two conditions hold:*

- $||g||_{L^\infty(J)} \leq 1$,

- *there exists a set of at most $2(n+1)$ distinct points $x_i \in E(g)$ and associated positive real numbers λ_i such that, for $r \leq 2(n+1)$:*

$$\forall h \in T_n, < g - f, h >_I + \sum_{i=1}^{r} \lambda_i g(x_i) \overline{h(x_i)} = 0 . \tag{104}$$

Moreover the (λ_i) verify the following boundedness equation:

$$\sum_{i=1}^{r} |\lambda_i| \leq 2||f||^2_{L^2(I)}. \tag{105}$$

Remark: The subset of extremal points $\{x_i, \ i = 1, \ldots, r\}$ is possibly empty (i.e. $r = 0$).

Proof. Suppose g verifies the two latter conditions and differs from k_n. Set $h = k_n - g$ and observe that,

$$\text{Re}\left(g(x_i)\overline{h(x_i)}\right) = \text{Re}\left(g(x_i)\overline{k_n(x_i)} - 1\right) \leq 0 \ \ i = 1 \ldots r. \tag{106}$$

From the uniqueness and optimality of k_n we deduce,

$$\begin{aligned} ||k_n - f||^2_{L^2(I)} &= ||g - f + h||^2_{L^2(I)} \\ &= ||g - f||^2_{L^2(I)} + ||h||^2_{L^2(I)} + 2\mathrm{Re} < g - f, h >_I \\ &< ||g - f||^2_{L^2(I)} \end{aligned}$$

The latter leads to:

$$\mathrm{Re} < g - f, h >_I < 0$$

which combined with (106) contradicts (104).

Conversely, suppose that g solves $FBEP^n_{2,\infty}$ and let ϕ_0 be the $\mathbb{R}-linear$ forms defined on T_n by

$$\forall h \in T_n,\ \phi_0(h) = \mathrm{Re} < g - f, h >_I .$$

For each critical point x of g define the following $\mathbb{R}-linear$ form ϕ_x,

$$\forall h \in T_n,\ \phi_x(h) = \mathrm{Re}\left(g(x)\overline{h(x)}\right).$$

Finally define K as follows:

$$K = \{\phi_0\} \cup \{\phi_x,\ x \in E(g)\}.$$

The set K can be seen seen as a subset of the dual of $T_n^{\mathbb{R}}$ which is defined to be the real vector space formed by the elements of T_n. Simple inspection shows that K is a closed bounded set of $(T_n^{\mathbb{R}})^*$, hence compact, as well as its convex hull denoted by $\hat{K}$ (note that the finite dimensional setting is crucial here). In order to get a contradiction, suppose that $0 \notin \hat{K}$, so that by the Hahn Banach theorem there exists $h_0 \in T_n$ such that:

$$\forall \phi \in \hat{K},\ \phi(h_0) \geq \tau > 0.$$

The latter and the continuity of g and h_0 ensure the existence of a neighborhood V of $E(g)$ such that for x in $U = J \cap V$, then $\mathrm{Re}\left(g(x)\overline{h_0(x)}\right) \geq \frac{\tau}{2}$, and for x in $J \backslash U$, then $|g(x)| \leq 1 - \delta$ ($\delta > 0$). Observe first that for $\epsilon > 0$ such that $\epsilon ||h_0||_{L^\infty(J)} < \delta$ we have

$$\sup_{x \in J\backslash U} |g(x) - \epsilon h_0(x)| \leq 1. \tag{107}$$

For all $x \in U$:

$$\begin{aligned} |g(x) - \epsilon h_0(x)|^2 &= |g(x)|^2 - 2\mathrm{Re}\left(g(x)\overline{h(x)}\right) + \epsilon^2 |h_0(x)|^2 \\ &\leq |g(x)|^2 - 2\mathrm{Re}\left(\epsilon g(x)\overline{h(x)}\right) + \epsilon^2 |h_0(x)|^2 \\ &\leq 1 - \epsilon\tau + \epsilon^2 ||h_0||^2_{L^\infty(J)} , \end{aligned}$$

which combined with (107) shows that for ϵ sufficiently small we have,

$$||g - \epsilon h||_{L^\infty(J)} \leq 1. \tag{108}$$

But

$$\begin{aligned} ||f-g-\epsilon h||^2_{L^2(J)} &= ||f-g||^2_{L^2(J)} - 2\epsilon\phi_0(h) + \epsilon^2||h||^2_{L^2(J)} \\ &\leq ||f-g||^2_{L^2(J)} - 2\epsilon\tau + \epsilon^2||h||^2_{L^2(J)}\,, \end{aligned}$$

which, with (108), indicates that for ϵ small enough, the function $(g - \epsilon h)$ provides a better candidate for (103) than g. Hence $0 \in \hat{K}$.
Carathéodory's theorem [34] is now to the effect that there exists r' elements γ_k of K with $r' \leq 2(n+1)+1$ such that:

$$\sum_{i=1}^{r'} \alpha_i \gamma_i = 0\,, \tag{109}$$

where the α_i are positive real numbers such that $\sum \alpha_i = 1$. Now suppose that $\phi_0 \neq \gamma_k$, $k = 1, \cdots, r'$. Evaluating the sum (109) on the element g yields,

$$\sum_{i=1}^{r'} \alpha_i \gamma_i(g) = \sum_{i=1}^{r'} \alpha_i |g(x_i)|^2 = 1.$$

Equation (109) can therefore be written as:

$$\forall h \in T_n, \quad \alpha_1 \mathrm{Re} < f-g, h >_I + \sum_{i=2}^{r'} \alpha_i \mathrm{Re}(g(x_i)\overline{h(x_i)}) = 0.$$

Dividing the latter by α_1 and noting that the latter equation is also true for the element (ih) leads to (104). Finally replacing h by g in (104) we obtain:

$$\begin{aligned} \sum_{i=1}^{r} \lambda_i = \sum_{i=1}^{r} |\lambda_i| &= < f-g, g >_I \\ &\leq < f-g, f-g >_I + |< f-g, f >_I| \\ &\leq ||f||^2_{L^2(I)} + ||f-g||_{L^2(I)} ||f||_{L^2(I)} \leq 2||f||^2_{L^2(I)}\,, \end{aligned}$$

where the last two inequalities are obtained by observing that 0 is a valid candidate for (103). ∎

The next result is mainly concerned with the behavior of $FBEP^n_{2,\infty}$ when n goes towards infinity.

Theorem 7 *The sequence (k_n) of polynomials solutions to $FBEP^n_{2,\infty}$ (103) converges as $n \to \infty$ towards the solution $g_0 \in H^2$ to $BEP_{2,\infty}$ (21) with respect to the $L^2(\mathbb{T})$ norm. On J, the sequence (k_n) converges w.r.t. the weak-* topology of $L^\infty(J)$ and w.r.t. the $L^p(J)$ norm, for $1 \leq p < \infty$. In other words we have:*

$$\lim_{n\to\infty} ||g_0 - k_n||_{L^2(\mathbb{T})} = 0\,,$$

$$\forall h \in H^1, \ \lim_{n\to\infty} < k_n, h >_J = < g_0, h >_J\,,$$

$$\forall p\,,\ 1 \leq p < \infty, \quad \lim_{n\to\infty} ||g_0 - k_n||_{L^p(J)} = 0\,.$$

Proof. Our first objective is to show (constructively, below) that g_0 can be approximated in the L^2 sense on I by polynomials that remain bounded by 1 on J (thus belong to the approximating class of $FBEP^n_{2,\infty}$ for large enough n). By hypothesis, I is the union of N disjoint closed sub-arcs of $\mathbb{T}$ and can therefore be written as,

$$I = \bigcup_{i=1}^{N} (e^{ia_i}, e^{ib_i})$$

where without loss of generality we can impose

$$0 = a_1 \leq b_1 \leq a_2 \cdots \leq b_N \leq 2\pi.$$

The inner-outer decomposition of g_0 therefore takes the form [17, 19]:

$$g_0(z) = B(z) \exp\left(\sum_{i=1}^{N} \frac{1}{2\pi} \int_{a_i}^{b_i} \frac{e^{it}+z}{e^{it}-z} \log(|g_0|)dt \right).$$

Let (ϵ_n) be a decreasing sequence of positive real numbers converging towards 0. We define a sequence (v_n) of H^2 as:

$$v_n(z) = B(z) \exp\left(\frac{1}{2\pi} \sum_{i=1}^{N} \int_{a_i+\epsilon_n}^{b_i-\epsilon_n} \frac{e^{it}+z}{e^{it}-z} \log(|g_0|)dt \right) = B(z)\times$$

$$\exp\left(\frac{-1}{2\pi} \sum_{i=1}^{N} \int_{a_i}^{a_i+\epsilon_n} \frac{e^{it}+z}{e^{it}-z} \log(|g_0|)dt + \int_{b_i}^{b_i+\epsilon_n} \frac{e^{it}+z}{e^{it}-z} \log(|g_0|)dt \right).$$

We claim that (v_n) converges to g_0 in the L^2 sense on I. To prove this we first show that v_n converges pointwise to g_0 a.e. in I. Let $e^{i\psi}$ be a point of the interior of I and such that g_0 admits a radial limit at $e^{i\psi}$. For n sufficiently large $e^{i\psi}$ is contained in none of the sub-arcs $(a_i, a_i+\epsilon_n)$ nor $(b_i, b_i+\epsilon_n)$. This is to the effect that

$$\left| \int_{a_i}^{a_i+\epsilon_n} \frac{e^{it}+e^{\psi}}{e^{it}-e^{i\psi}} \log(|g_0|)dt \right| \leq \int_{a_i}^{a_i+\epsilon_n} \frac{|e^{it}+e^{\psi}|}{|e^{it}-e^{i\psi}|} |\log(|g_0|)|dt$$

$$= \int_{a_i}^{a_i+\epsilon_n} |\mathrm{cotg}(\frac{t-\psi}{2}) \log(|g_0|)|dt$$

$$\leq \max(|\mathrm{cotg}(\frac{a_i-\psi}{2})|\,,\ |\mathrm{cotg}(\frac{a_i+\epsilon_n-\psi}{2})|) \int_{a_i}^{a_i+\epsilon_n} |\log(|g_0|)|dt. \qquad (110)$$

The same is true with b_i in place of a_i. As the last term of (110) can be set arbitrarily small for n sufficiently large, the pointwise convergence of v_n to g_0 is ensured. Finally remark that by construction $|v_n| \leq |g_0|+1$, which leads to the majoration $|v_n - g_0|^2 \leq (2|g_0|+1)^2$. Hence Lebesgue's dominated convergence theorem applies and

$$\lim_{n\to\infty} ||g_0 - v_n||_{L^2(I)} = 0.$$

Now let $\epsilon > 0$ and $0 < \alpha < 1$ such that $||g_0 - \alpha g_0||_{L^2(I)} \leq \frac{\epsilon}{4}$. Let n_0 sufficiently large such that $||v_{n_0} - g_0||_{L^2(I)} \leq \frac{\epsilon}{4}$. For $r < 1$ we define u_r belonging to disk algebra in the following way,

$$\forall\theta \in [0, 2\pi],\ u_r(e^{i\theta}) = \int_{\mathbb{T}} P_r(\theta - t) v_{n_0}(re^{it})dt,$$

where P_r is the Poisson kernel.
Let $e^{i\phi} \in J$. Observe that by construction $|v_n| = 1$ a.e on the sub-arc $(e^{i(\phi-\epsilon_{n_0})}, e^{i(\phi+\epsilon_{n_0})})$. This is to the effect that

$$\begin{aligned} |u_r(e^{i\phi})| &\leq \int_{\mathbb{T}} P_r(\phi - t)|v_{n_0}(re^{it})|dt \\ &\leq P_r(\epsilon_{n_0}) \int_{\mathbb{T}} |v_{n_0}(re^{it})|dt + \int_{-\epsilon_{n_0}}^{+\epsilon_{n_0}} P_r(t)dt \\ &\leq P_r(\epsilon_{n_0})||v_{n_0}||_{L^1(\mathbb{T})} + 1. \end{aligned}$$

Hence for r sufficiently close to 1, we have $|u_r| \leq 1/\alpha^2$ on J and $||u_r - v_{n_0}||^2_{L^2(I)} \leq \frac{\epsilon}{4}$. Finally, we call q the truncated Taylor expansion of u_r (which converges uniformly on $\mathbb{T}$), where the truncation order has been chosen large enough so as to ensure that $|q| \leq 1/\alpha$ on J and $||q - u_r||^2_{L^2(I)} \leq \frac{\epsilon}{4}$. We have:

$$\begin{gathered} ||\alpha q - g_0||_{L^2(I)} \\ \leq \alpha \left(||q - u_r||_{L^2(I)} + ||u_r - v_{n_0}||_{L^2(I)} + ||v_{n_0} - g_0||_{L^2(I)} \right) + ||g_0 - \alpha g_0||_{L^2(I)} \\ \leq \epsilon\,. \end{gathered}$$

Hence, the αq furnish the desired polynomials.
Because they belong to the approximating class in $FBEP^n_{2,\infty}$, for large enough n, the above inequality is to the effect that:

$$\lim_{n\to\infty} ||f - k_n||_{L^2(I)} = ||f - g_0||_{L^2(I)}. \tag{111}$$

As a bounded sequence of elements of H^2, (k_n) admits a weak convergent sub-sequence. The traces on J of this sub-sequence are bounded in the L^∞ sense on J, hence up to another sub-sequence we obtain a sequence (k'_n) converging in addition in the weak-* sense on J. Let g be the weak limit (H^2 sense) of k'_n. As the balls are weak-* closed in L^∞ we have $||g||_{L^\infty(J)} \leq 1$, and it follows from (111) that $||f - g||_{L^2(I)} = ||f - g_0||_{L^2(I)}$. The uniqueness of g_0 leads to $g = g_0$. Now (111) and the constraint's saturation are to the effect that $\limsup ||k'_n||_{L^2(\mathbb{T})} \leq ||g_0||_{L^2(\mathbb{T})}$ which in turn proves that k'_n converges strongly in the L^2 sense to g_0. The same kind of remark on the trace on J of k'_n leads to the strong convergence in the L^p (for the reflexive cases $1 < p < \infty$) sense on J. Finally we remark that the preceding arguments are also true when k_n is replaced by any subsequence of the latter; hence k_n contains no sub-sequence not converging to g_0. ■

Remark: A discretization on T_n is also taken up in [37], for approximation issues of BEP type in $L^p(I)$ and constrained on $\mathbb{T}$, with smooth data [36]. This issues might themselves be normalized and formulated as $BEP_{p,\infty}$ type problems, g being this time a Schur function.

When I is a finite union of closed disjoint arcs of $\mathbb{T}$, Theorem 4 may now be viewed as a corollary to Theorem 6, of which it is a infinite dimensional analogous. We detail below this alternative proof.

We define $H^{2,\infty}$ and $H^{2,1}$ to be the following vector spaces:

$$H^{2,\infty} = \{h \in H^2,\ ||h||_{L^\infty(J)} < \infty\},$$

$$H^{2,1} = \{h \in H^1,\ ||h||_{L^2(I)} < \infty\}.$$

We begin with a technical lemma.

Lemma 7 *Let $v \in L^1(J)$ such that $\mathbf{P}_+(0 \vee v) \in H^{2,1}$, then the following holds:*

$$\forall h \in H^{2,\infty},\ < \mathbf{P}_+(0 \vee v), h >_{\mathbb{T}} = < v, h >_J .$$

Proof. Let u be the function defined on $\mathbb{T}$ by

$$u = (0 \vee v) - \mathbf{P}_+(0 \vee v).$$

By this very definition all the Fourier coefficients of u of non-negative index vanish, u is L^2 integrable on I and L^1 integrable on J. Hence we conclude that $\overline{u} \in H^{2,1}$ and that $\overline{u}(0) = 0$ ($\overline{u}$ has now a canonical extension to the disc). Let $h \in H^{2,\infty}$. We have:

$$\begin{aligned} < v\chi_J, h >_{\mathbb{T}} &= < u, h >_{\mathbb{T}} + < \mathbf{P}_+(0 \vee v), h >_{\mathbb{T}} \\ &= \overline{u}(0)h(0) + < \mathbf{P}_+(0 \vee v), h >_{\mathbb{T}} \\ &= < \mathbf{P}_+(0 \vee v), h >_{\mathbb{T}}, \end{aligned} \tag{112}$$

where the second equality occurs because $(\overline{u}h) \in H^1$. ∎

Proof of Theorem 4. In view of (41), point (ii) of Theorem 4 and equation (40) can be equivalently stated as:
there exists a non-negative function $\lambda \in L^1_{\mathbb{R}}(J)$ such that,

$$\forall h \in H^{2,\infty}, < g - f, h >_I + < \lambda g, h >_J = 0. \tag{113}$$

Suppose that $g \in H^2$ verifies $|g(e^{i\theta})| = 1$ for a.e. $e^{i\theta} \in J$ and that (113) holds, while $g \neq g_0$, the solution to $BEP_{2,\infty}$. Set $h = (g_0 - g) \in H^{2,\infty}$ and observe that,

$$\text{Re} < \lambda g, h >_J = \frac{1}{2\pi} \int_J \lambda(\text{Re}(\overline{g}g_0) - 1) \leq 0 \tag{114}$$

Uniqueness and optimality of g_0 lead (as in the proof of Theorem 6)) to

$$\text{Re} < g - f, h >_I < 0 ,$$

which combined with (114) contradicts (113).
Suppose now that g is the optimal solution of $BEP_{2,\infty}$. The property $|g| = 1$ on J has already been proved in Theorem 2. In order to let n go to infinity rewrite (104) with self explaining notations as:

$$\forall m \in \{0 \ldots n\}, < k_n - f, e^{im\theta} >_I + \sum_{i=1}^{r(n)} \lambda_i^n k_n(e^{i\theta_i^n})\overline{e^{im\theta_i^n}} = 0. \tag{115}$$

We define (Λ_n) to be a family of linear forms on $C(J)$ defined in the following way:

$$\forall u \in C(J),\ \ \Lambda_n(u) = \sum_{i=1}^{r(n)} \lambda_i^n k_n(e^{i\theta_i^n})u(e^{\theta_i^n}).$$

Equation (105) shows now that (Λ_n) is a bounded sequence of elements in $C(J)^*$ which by the Banach-Alaoglu theorem [13] admits a weak-* converging subsequence whose limit we call Λ. Now the Riesz representation theorem ensures the existence of a complex measure μ associated to Λ, so that appealing to Theorem 7 we obtain

$$\forall m \in \mathbb{N}, < g_0 - f, e^{im\theta} >_I + \int_J \overline{e^{im\theta}} d\mu = 0 \tag{116}$$

by taking the limit in (115). Now F. and M. Riesz Theorem asserts that μ is absolutely continuous with respect to the Lebesgue measure so that there exists $v \in L^1(J)$ such that:

$$\forall m \in \mathbb{N}, < g_0 - f, e^{im\theta} >_I + < v, e^{im\theta} >_J = 0\,,$$

which is equivalent to

$$\forall m \in \mathbb{N}, < g_0 - f, e^{im\theta} >_I + < \lambda g_0, e^{im\theta} >_J = 0\,, \tag{117}$$

where we have defined $\forall z \in J,\ \ \lambda(z) = v(z)\overline{g_0(z)}$. Equation (117) is to the effect that,

$$\mathbf{P}_+((g_0 - f)\chi_I) = -\mathbf{P}_+(0 \vee \lambda g_0)$$

which indicates that $\mathbf{P}_+(0 \vee \lambda g_0)$ is in H^2 (note that this is not trivial, since v occurred till now as an L^1 function). Now thanks to Lemma 7 we obtain,

$$\forall u \in H^{2,\infty} < g_0 - f, u >_I + < \lambda g_0, u >_J = 0. \tag{118}$$

In order to prove that $\lambda \in \mathbb{R}^+$, consider the valid variation $h = g_0 b$ where b is defined as in (24),

$$b(z) = \frac{1}{2\,\pi} \int_I \frac{e^{it} + z}{e^{it} - z}\, h(e^{it})\, dt = \frac{1}{2\,\pi} \int_{\mathbb{T}} \frac{e^{it} + z}{e^{it} - z}\, \chi_I(e^{it})\, h(e^{it})\, dt\,, \tag{119}$$

with $h \in C^\infty_{c,\mathbb{R}}(I)$. We already now (as h is a valid variation) that

$$\mathrm{Re} < (f - g_0)\overline{g_0}, b >_I = 0,$$

which yields

$$\forall h \in C^\infty_{c,\mathbb{R}}(I), \quad < \mathrm{Im}(\lambda), \mathrm{Im}(b) >_J = 0$$

by remarking that b is pure imaginary on J. Now using the same technique as in the proof of the constraint's saturation we obtain

$$\forall u \in C_{\mathbb{R}}(J), \quad < \mathrm{Im}(\lambda), u >_J = 0$$

which proves that λ takes real values.
Finally using the fact that $BEP_{2,\infty}$ is a convex problem we obtain by derivating one more time that:

$$\mathrm{Re} < (g_0 - f), b^2 >_I \geq 0$$

which leads to

$$\forall u \in C_{\mathbb{R}}(J), \quad < \lambda, u^2 >_J \geq 0.$$

Hence $\lambda \geq 0$. Because (118) implies that (40) holds, the function $(f - g_0) \vee \lambda g_0$ cannot vanish on a measurable set of positive measure unless it is the zero function. But this would imply $f = g_0$ a.e on I which contradicts the assumptions on f. This yields $\lambda > 0$ a.e on J. ■

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