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# Fault tolerant on-board networks with priorities

J.-C. Bermond\* and F. Havet\*,  
Projet Mascotte, CNRS/INRIA/UNSA,  
INRIA Sophia-Antipolis,  
2004 route des Lucioles BP 93,  
06902 Sophia-Antipolis Cedex, France  
bermond, fhavet@sophia.inria.fr  
and

C. D. Tóth  
Department of Mathematics  
MIT, Room 2-336  
Cambridge, MA 02139, USA.  
toth@math.mit.edu

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## Abstract

We consider on-board networks in satellites interconnecting entering signals (inputs) to amplifiers (outputs). The connections are made via expensive switches, each of which has four available links. The paths connecting inputs to outputs should be link-disjoint. Some of the input signals, called priorities, must be connected to the amplifiers which provide the best quality of service (that is, to some specific outputs). In practice, amplifiers are prone to fail and the faults cannot be repaired. Therefore, extra outputs have to be built into the network to ensure that every input can be routed to operational outputs. Given three integers,  $n$ ,  $p$ , and  $f$ , we would like to design a low cost network (where the network cost is proportional to the total number of switches) such that it is possible to route all  $n$  inputs to  $n$  operational amplifiers, and to route the  $p$  priorities to the  $p$  best quality amplifiers for any set of  $f$  faulty and  $p$  best-quality amplifiers. Let  $R(n, p, f)$  be the minimum number of switches of such a network. We prove here that  $R(n, p, f) \leq \frac{n+f}{2} \lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f)$  with  $g$  a function depending only on  $f$ . We then compute  $R(n, p, f)$  exactly for a few small values of  $p$  and  $f$ .

**keywords:** network design, on-board network, fault tolerance, vulnerability

## 1 Introduction

Modern telecommunication satellites are very complex to design. Components are often prone to failure, and so providing robustness at the lowest possible cost is an important issue for the

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manufacturers. A key component of telecommunication satellites is an interconnection network which allows one to redirect signals received by the satellite to a set of amplifiers where the signals will be retransmitted. The network is made of expensive switches; so we want to minimize their number subject to the following conditions: each input (respectively output) is adjacent to exactly one link; each switch is adjacent to at most four links; there are  $n$  inputs (signals) and  $n + f$  outputs (amplifiers); among the  $n + f$  outputs,  $f$  are allowed to fail (all failures are irreversible). All the input signals should be sent to operational outputs via link-disjoint paths. Designing such networks is a complex problem that was proposed by Alcatel Space Industries and partial solutions are given in [4] and [6]. Alcatel Space Industries proposed also to consider the case where each signal needs a specific amplifier; that leads to design a network realizing any permutation from the inputs to the outputs (see [2, 3]). Recently it posed the following new problem. Out of the total of  $n$  inputs,  $p$  are called priorities and must be connected to the amplifiers providing the best quality of service (that is, to some specific outputs) and the other signals should be sent to other amplifiers. Note that the priority signals are given, but the amplifiers providing the best quality service change with the position of the satellite and so for any set of  $f$  failed outputs, the network should be able to route the signals to operational outputs such that it routes the  $p$  priorities to any given subset of  $p$  operational outputs.

This problem can be restated more formally as follows:

**Definition 1** An  $(n, p, f)$ -network  $G$  is a graph  $(V, E)$  where the vertex set  $V$  is partitioned into four subsets  $P, I, O$  and  $S$  called respectively the *priorities*, the *ordinary inputs*, the *outputs* and the *switches*, satisfying the following constraints:

- there are  $p$  priorities,  $n - p$  ordinary inputs and  $n + f$  outputs;
- each priority, each ordinary input and each output is connected to exactly one switch;
- switches have degree at most 4.

An  $(n, p, f)$ -network is a *repartitor* if for any disjoint subsets  $F$  and  $Q$  of  $O$  with  $|F| = f$  and  $|Q| = p$ , there exist  $n$  edge-disjoint paths in  $G$  such that  $p$  of them connect  $P$  to  $Q$  and the  $n - p$  others connects  $I$  to  $O \setminus (Q \cup F)$ . The set  $F$  corresponds to the set of failures and  $Q$  to the set of amplifiers providing the best quality of service. We denote by  $R(n, p, f)$  the minimum number of switches (i.e., the cardinality of  $S$ ) of a  $(n, p, f)$ -repartitor. An  $(n, p, f)$ -repartitor with  $R(n, p, f)$  switches will be called a *minimum repartitor*.

**Problem 2** Determine  $R(n, p, f)$  and construct minimum (or almost minimum) repartitors.

The problem is interesting for every integer  $n$  and  $f$  and for every  $p$ ,  $0 \leq p \leq n$ . In this paper, we focus on fixed values of  $p$  and  $f$ ; we give asymptotic bounds on  $R(n, p, f)$  if  $n$  goes to infinity. In current application networks, the number of input signals  $n$  is around 15 and the numbers of priorities  $p$  and extra amplifiers  $f$  range from 0 to  $n/2$ . As technology progresses, satellite parts are getting smaller and more reliable, and so the number of inputs is expected to increase in the future followed by a modest increase in the number of priorities and tolerable amplifier failures.

As indicated above, the problem with no priority (that is,  $p = 0$ ) has been considered in [4] and partially solved in [6]. In [4], it is shown that  $R(n, 0, 1) = R(n, 0, 2) = n$ . In [6], it is proved that  $\frac{3n}{2} - O(\frac{n}{f}) \leq R(n, 0, f) \leq \frac{3n}{2} + g(f)$  with  $g$  a function depending only on  $f$ . The minimum repartitor size is also bounded for small values of  $f$ :  $R(n, 0, 4) = n + \lceil \frac{n}{4} \rceil$ ,

$R(n, 0, 6) = n + \frac{n}{4} + \sqrt{\frac{n}{8}} + O(1)$ ,  $R(n, 0, 8) = n + \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(\sqrt[4]{n})$ ,  $R(n, 0, 10) = n + \frac{3n}{8} + \Theta(\sqrt{n})$  and  $R(n, 0, 12) = n + \frac{3n}{7} + O(\sqrt{n})$ .

Repartitors with no priorities, also called *selectors*, are somewhat similar to concentrators [7]. An  $(n, m)$ -concentrator,  $m \leq n$ , is a directed acyclic graph with maximal degree 4 that has  $n$  distinguished *input* vertices and a disjoint set of  $m$  distinguished *output* vertices such that for any subset  $A$  of  $m$  inputs there exists a set of  $m$  vertex-disjoint paths connecting  $A$  to the outputs. Researchers have studied the minimum number of *edges* of an  $(n, m)$ -concentrator. In our problem, we minimize the number of *switches* of selectors, which is similar to minimizing the number of edges since every switch has degree at most 4, and so the number of edges of a minimum  $(p, n)$ -selector is at most twice the number of its switches. Havet [8] constructed repartitors based on techniques used for designing concentrators [1, 9]. However, these repartitors are far from optimal, especially when the number of inputs or priority inputs is small, because one step of this method emulates high degree vertices by small subnetworks with switches of degree 4.

In this paper, we study  $(n, p, f)$ -repartitors **when  $p$  is not zero**. We first give a general upper bound for  $R$ :

$$R(n, p, f) \leq \frac{n+f}{2} \lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f),$$

where  $g$  is a function of  $f$ . We then give optimal or near optimal bounds on  $R(n, p, f)$  for small values of  $p$  and  $f$ :

$R(n, p, f)$	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$f = 0$	$n - 1$	$\lceil \frac{3n}{2} - 3 \rceil$	$\frac{3n}{2} + \frac{\sqrt{n}}{2} + \Theta(1)$	$\frac{7n}{4} + \Theta(1)$
$f = 1$	$\lceil \frac{3n-1}{2} \rceil$	$\frac{3n}{2} + \frac{\sqrt{n}}{2} + \Theta(1)$	$\frac{7n}{4} + \Theta(1)$	
$f = 2$	$\lceil \frac{3n-1}{2} \rceil$			

We present our general bounds in Section 2. Our lower bounds in Section 3 are based on a minimum cut maximum flow type criterion, a sensible classification of connected subgraphs, and elementary graph theory. In Section 4, we present a few explicit constructions and prove that they are indeed repartitors.

## 2 General upper bounds

In this section, we give an upper bound on  $R(n, p, f)$ . We first give an inductive construction of  $(n, p, 0)$ -networks and then construct an  $(n, p, f)$ -repartitor from an  $(n-p, 0, f)$ -repartitor and an  $(n+f, p, 0)$ -repartitor.

Clearly,  $R(n, p, f)$  grows as a function of  $f$  since an  $(n, p, f)$ -repartitor may be obtained from an  $(n, p, f')$ -repartitor (with  $f' \geq f$ ) by deleting any set of  $f' - f$  outputs. Moreover, priority signals and ordinary signals play symmetric roles: Swapping the ordinary inputs and the priorities is a one-to-one mapping between  $(n, p, f)$ -repartitors and  $(n, n-p, f)$ -repartitors.

**Proposition 3** (i) If  $f \leq f'$  then  $R(n, p, f) \leq R(n, p, f')$ ;  
(ii)  $R(n, p, f) = R(n, n - p, f)$ .

**Proposition 4**

$$R(n, 1, 0) \leq n - 1.$$

**Proof.** Let  $G$  be the graph with one priority  $p_1$ , ordinary input set  $\{i_1, i_2, \dots, i_{n-1}\}$ , output set  $\{o_1, o_2, \dots, o_n\}$  and switch set  $\{s_1, s_2, \dots, s_{n-1}\}$  such that  $(p_1, s_1, s_2, \dots, s_{n-1}, o_n)$  is a path and for  $l$ ,  $1 \leq l \leq n - 1$ ,  $i_l$  and  $o_l$  are adjacent to  $s_l$ . (See Figure 1.) We claim that  $G$  is

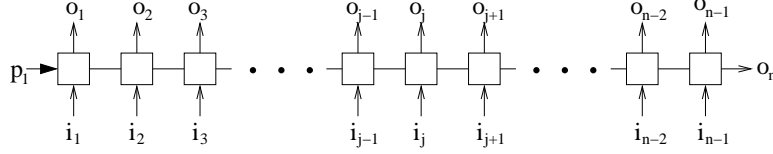


Figure 1: Minimum  $(n, 1, 0)$ -repartitor.

an  $(n, 1, 0)$ -repartitor. Indeed, suppose that  $o_j$  is the priority output. Since  $o_{n-1}$  and  $o_n$  are equivalent, we may suppose that  $j \neq n$ . The desired paths are  $P_0 = (p_1, s_1, s_2, \dots, s_j, o_j)$ ,  $P_j = (i_j, s_j, s_{j+1}, \dots, s_{n-1}, o_n)$  and  $P_l = (i_l, s_l, o_l)$ , for  $l \neq j$ ,  $1 \leq l \leq n - 1$ . ■

**Lemma 5** For  $n \geq p \geq 1$ ,

- (i)  $R(2n, 2p, 0) \leq 2R(n, p, 0) + n - 1$ ,
- (ii)  $R(2n + 1, 2p, 0) \leq R(n + 1, p, 0) + R(n, p, 0) + n$ ,
- (iii)  $R(2n, 2p + 1, 0) \leq R(n, p + 1, 0) + R(n, p, 0) + n$ ,
- (iv)  $R(2n + 1, 2p + 1, 0) \leq R(n + 1, p + 1, 0) + R(n, p, 0) + n$ .

**Proof.** (i) Let  $G^1$  and  $G^2$  be two  $(n, p, 0)$ -repartitors. For  $i = 1, 2$ , let  $I^i, P^i, O^i = \{o_1^i, o_2^i, \dots, o_n^i\}$  and  $S^i$  be the ordinary input set, priority set, output set and switch set of  $G^i$ . Let  $H$  be a graph defined as follows: its ordinary input set is  $I = I^1 \cup I^2$ , its priority set is  $P = P^1 \cup P^2$ , its switch set is  $S^1 \cup S^2 \cup S$  with  $S = \{s_1, s_2, \dots, s_{n-1}\}$  and its output set is  $O = \{o_1, o_2, \dots, o_{n-1}\} \cup \{o'_1, o'_2, \dots, o'_{n-1}\} \cup \{o_n^1, o_n^2\}$ ;  $H$  contains the edges of  $G^1$  and  $G^2$  except the edges incident to  $o_j^1$  and  $o_j^2$  for  $1 \leq j \leq n - 1$ . Furthermore, for every  $1 \leq j \leq n - 1$ ,  $s_j$  is linked to  $o_j, o'_j$ , the switch adjacent to  $o_j^1$  in  $G_1$  and the switch adjacent to  $o_j^2$  in  $G_2$ . See Figure 2 (i) for an illustration.

We claim that  $H$  is a  $(2n, 2p, 0)$ -repartitor. Indeed, let  $Q$  be a  $2p$ -subset of  $O$ . Clearly, one can partition  $O$  into two  $n$ -sets  $O_1$  and  $O_2$  such that for  $i = 1, 2$ ,  $o_n^i \in O_i$ ,  $|Q \cap O_i| = p$  and  $\forall 1 \leq j \leq n - 1, |O_i \cap \{o_j, o'_j\}| = 1$ . For  $i = 1, 2$ , let  $Q_i = Q \cap O_i$ . Since  $G^1$  and  $G^2$  are  $(n, p, 0)$ -repartitors, there are edge-disjoint paths joining  $P^i$  to  $Q_i$  and  $I^i$  to  $O_i \setminus Q_i$ . Hence, there are edge-disjoint paths joining  $P$  to  $Q$  and  $I$  to  $O \setminus Q$ .

The proofs of (ii), (iii) and (iv) are analogous and are omitted. Figure 2 (ii)—(iv) shows the schematic construction of  $H$  in these three cases. ■

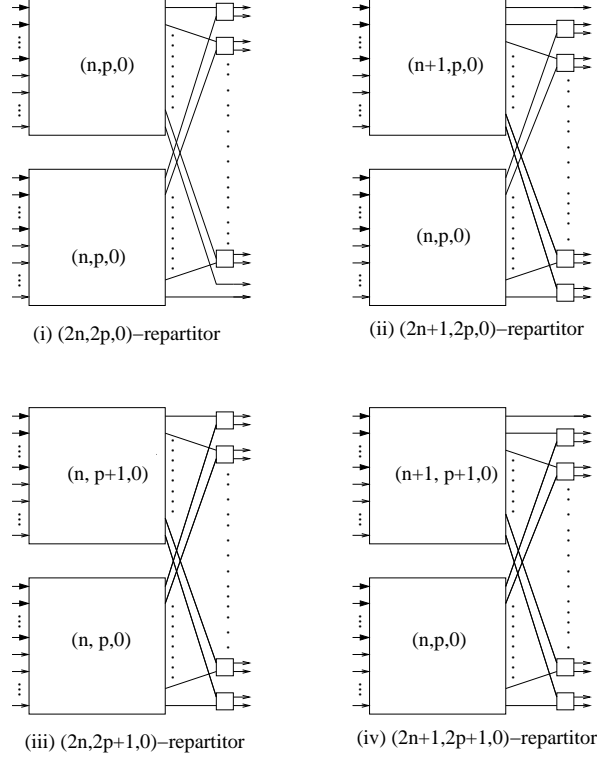


Figure 2: Construction of repartitors from smaller ones.

**Theorem 6**

$$R(n, p, 0) \leq \left\lfloor \frac{n}{2} \right\rfloor \lceil \log_2 p \rceil + n - p.$$

**Proof.** We proceed by induction on  $p$ . The inequality holds for  $p = 1$  by Proposition 4.

Suppose that the inequality holds for every  $p' < p$ . Assume first that  $p$  and  $n$  are even. By Lemma 5 (i),

$$R(n, p, 0) \leq 2R(n/2, p/2, 0) + n/2 - 1.$$

By the induction hypothesis,

$$R(n, p, 0) \leq 2 \left\{ \left\lfloor \frac{n}{4} \right\rfloor \lceil \log_2(p/2) \rceil + n/2 - p/2 \right\} + n/2 - 1.$$

Since  $2 \lfloor \frac{n}{4} \rfloor \leq \lfloor \frac{n}{2} \rfloor$ , we obtain

$$R(n, p, 0) \leq \left\lfloor \frac{n}{2} \right\rfloor \lceil \log_2(p/2) \rceil + n - p + n/2 - 1.$$

Because  $\lceil \log_2(p/2) \rceil + 1 = \lceil \log_2(p) \rceil$ , we obtain

$$R(n, p, 0) \leq \left\lfloor \frac{n}{2} \right\rfloor \lceil \log_2 p \rceil + n - p - 1.$$

If  $p$  or  $n$  is odd, we obtain the result analogously by Lemma 5 (ii), (iii) and (iv). ■

**Lemma 7** For  $p \leq n$ ,

$$R(n, p, f) \leq R(n - p, 0, f) + R(n + f, p, 0).$$

**Proof.** Let  $G_1$  be an  $(n - p, 0, f)$ -repartitor with output set  $O^1 = \{o_1^1, o_2^1, \dots, o_{n+f-p}^1\}$  and let  $G_2$  be an  $(n + f, p, 0)$ -repartitor with ordinary input set  $I^2 = \{i_1^2, i_2^2, \dots, i_{n+f-p}^2\}$ . Let  $G$  be the network obtained from the union of  $G_1$  and  $G_2$  by replacing each pair  $\{o_j^1, i_j^2\}$  by an edge between their neighboring switches. (See Figure 3.) We show that  $G$  is an  $(n, p, f)$ -repartitor.

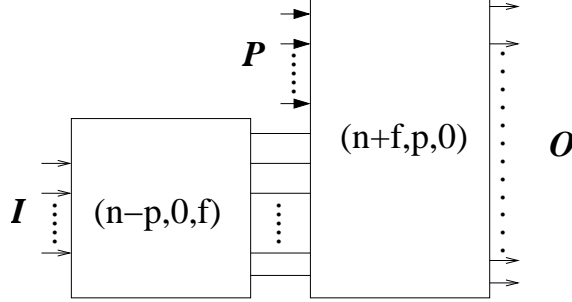


Figure 3: Construction of an  $(n, p, f)$ -repartitor.

Let  $P = \{m_1, m_2, \dots, m_p\}$  the set of priorities of  $G_2$  be that of  $G$ ,  $I = \{i_1, i_2, \dots, i_{n-p}\}$  the input set of  $G_1$  be the ordinary input set of  $G$ , and  $O = \{o_1, o_2, \dots, o_{n+f}\}$  the output set of  $G_2$  be that of  $G$ . Let  $Q$  and  $F$  be two disjoint subsets of  $O$  with cardinalities  $p$  and  $f$  respectively. Since  $G_2$  is a repartitor there exists a set of edge-disjoint paths  $\{P_1, P_2, \dots, P_p\} \cup \{Q_1, Q_2, \dots, Q_{n+f-p}\}$  such that for  $1 \leq j \leq p$ ,  $P_j$  links  $m_j$  to an element of  $Q$  and for  $1 \leq j \leq n + f - p$ ,  $Q_j$  links  $i_j^2$  to an element of  $O \setminus Q$ . Let  $J$  be the set of indices  $j$  such that  $Q_j$  ends in a vertex of  $F$ . Set  $F^1 = \{o_j^1, j \in J\}$ . Since  $G_1$  is an  $(n - p, 0, f)$ -repartitor, there are edge-disjoint paths  $R_l$  in  $G_1$   $1 \leq l \leq n - p$  such that  $R_l$  links  $i_l$  to a vertex  $o_{\phi(l)}^1$  that is not in  $F^1$  for some function  $\phi$ . Now the union of the paths  $R_l$  and  $Q_{\phi(l)}$  induces a path  $P'_l$  in  $G$  which joins  $i_l$  to  $o_{\phi(l)} \in O \setminus (Q \cup F)$ . The paths  $P'_l$ ,  $1 \leq l \leq n - p$ , and the paths  $P_j$ ,  $1 \leq j \leq p$ , are obviously edge-disjoint. ■

**Theorem 8**

$$R(n, p, f) \leq \frac{n+f}{2} \lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f),$$

where  $g(f)$  is a function depending on  $f$  only.

**Proof.** By Lemma 7,  $R(n, p, f) \leq R(n - p, 0, f) + R(n + f, p, 0)$ . By Theorem 6,  $R(n + f, p, 0) \leq n + f - p + \frac{n+f}{2} \lceil \log_2 p \rceil$  and it is proved in [6] that  $R(n - p, 0, f) \leq \frac{3}{2}(n - p) + g(f)$ . ■

### 3 Lower bounds

Before we proceed with the lower bounds on  $R(n, p, f)$ , we make two observations on the structure of minimum repartitors. We first show that we can assume without loss of generality that

in a minimum repartitor all switches, with the exception of at most one, have degree 4. Let  $\epsilon(f) = 1$  if  $f$  is odd and 0 otherwise.

**Proposition 9** *There is a minimum  $(n, p, f)$ -repartitor with  $\epsilon(f)$  switches of degree 3 and all the others with degree 4.*

**Proof.** Let  $R$  be a minimum  $(n, p, f)$ -repartitor with the minimum number of switches with degree less than 4.

Obviously,  $R$  has no switches of degree 0 or 1 otherwise the network obtained by removing them is also a repartitor, which contradicts the minimality of  $R$ . Similarly,  $R$  has no switch of degree two. If a switch  $S$  had degree two, then we obtain a smaller valid repartitor by removing  $S$  and connecting its neighbours by an edge.

$R$  has at most one switch of degree 3, otherwise the network obtained from  $R$  by adding an edge between two switches of degree 3 is an  $(n, p, f)$ -repartitor with fewer switches of degree less than 4. Since there is an even number of odd degree vertices in every graph,  $R$  has  $\epsilon(f)$  switches of degree 3. ■

In the remainder of Section 3, we assume that every  $(n, p, f)$ -repartitor has exactly  $\epsilon(f)$  switches of degree 3. Next, we show that every switch is connected to at most one input.

**Proposition 10** *In a minimum  $(n, p, f)$ -repartitor, a switch is connected to at most one ordinary input and at most one priority.*

**Proof.** Let  $R$  be an  $(n, p, f)$ -repartitor containing a switch  $s$  connected to at least two ordinary inputs  $i_1$  and  $i_2$ . Let  $v_1$  and  $v_2$  be the two neighbours of  $s$  distinct from  $i_1$  and  $i_2$ . Then the  $(n, p, f)$ -network obtained from  $R$  by removing  $s$  and adding the two edges  $(i_1, v_1)$  and  $(i_2, v_2)$  is also a repartitor and  $R$  is not minimum, a contradiction. The same proof works if both  $i_1$  and  $i_2$  are priority inputs. ■

It follows that there are more switches than ordinary inputs.

**Corollary 11**

$$R(n, p, f) \geq n - p.$$

### 3.1 Cut criterion

One main tool to obtain lower bounds is to use the following cut criterion which gives necessary conditions for a network to be a repartitor. Let  $W$  be a set of vertices in a graph  $G$ . We denote by  $in(W)$  (resp.  $out(W)$ ,  $pr(W)$ ) the cardinality of the set  $In(W)$  (resp.  $Out(W)$ ,  $Pr(W)$ ) of ordinary inputs (resp. outputs, priorities) of  $W$ . We denote by  $deg(W)$  the number of edges incident to exactly one vertex in  $W$ .

**Proposition 12 (Cut criterion)** *Let  $G$  be an  $(n, p, f)$ -repartitor and  $W$  be a set of vertices of  $G$ .*

(i) *If  $p + f \leq out(W)$  then  $deg(W) \geq in(W) - out(W) + 2p + f - pr(W)$ .*



(ii) If  $p \leq \text{out}(W) \leq p + f$  then  $\text{deg}(W) \geq \text{in}(W) + p - \text{pr}(W)$ .

(iii) If  $\text{out}(W) \leq p$  then  $\text{deg}(W) \geq \text{in}(W) + |\text{out}(W) - \text{pr}(W)|$ .

(iv)  $\text{deg}(W) \geq \text{in}(W) + \text{pr}(W) - \max(0, \text{out}(W) - f)$ .

**Proof.**

(i) Suppose that  $p$  outputs of  $\text{Out}(W)$  are in  $Q$  and  $f$  others are faulty. Then  $\text{in}(W) - \text{out}(W) + p + f$  paths from an ordinary input to an output of  $O \setminus Q$  leave  $W$ , and  $p - \text{pr}(W)$  paths from priorities enter  $W$ . Since all these paths are edge-disjoint they go through different edges with an end in  $W$  and the other not in  $W$ . Thus,  $\text{deg}(W) \geq \text{in}(W) - \text{out}(W) + 2p + f - \text{pr}(W)$ .

(ii) Suppose that  $p$  outputs of  $\text{Out}(W)$  are in  $Q$  and  $\text{out}(W) - p$  others are faulty. Then  $\text{in}(W)$  paths from an ordinary input to an output leave  $W$  and  $p - \text{pr}(W)$  paths from priorities enter  $W$ . Thus,  $\text{deg}(W) \geq \text{in}(W) + p - \text{pr}(W)$ .

(iii) Suppose that  $\text{out}(W)$  outputs of  $\text{Out}(W)$  are in  $Q$ . Then  $\text{in}(W)$  paths from an ordinary input to an output leave  $W$ . If  $\text{out}(W) \geq \text{pr}(W)$  then  $\text{out}(W) - \text{pr}(W)$  paths from priorities enter  $W$  and if  $\text{out}(W) \leq \text{pr}(W)$  then  $\text{pr}(W) - \text{out}(W)$  paths from priorities leave  $W$ . Thus,  $\text{deg}(W) \geq \text{in}(W) + |\text{out}(W) - \text{pr}(W)|$ .

(iv) Suppose that  $\min(\text{out}(W), f)$  outputs of  $\text{Out}(W)$  are faulty. Then  $\text{in}(W) + \text{pr}(W) - \max(0, \text{out}(W) - f)$  paths leave  $W$ . ■

### 3.2 A bipartite graph on blocks and switches

We express the size of a minimum repartitor in terms of the number of switches with no inputs. By Proposition 10, we can distinguish two kinds of switches in a minimum  $(n, p, f)$ -repartitor: An *ordinary switch* is a switch adjacent to an ordinary input. A *usual switch* is a switch that is not an ordinary switch. Let  $\mathcal{S}_o$  (resp.  $\mathcal{S}_u$ ) denote the set of ordinary (resp. usual) switches and  $s_o$  (resp.  $s_u$ ) their cardinality. Counting the ordinary inputs, we have  $s_o = n - p$ . Hence the total number of switches is

$$N = n - p + s_u.$$

Thus, a minimum  $(n, p, f)$ -repartitor is a repartitor with the fewest possible *usual switches*.

In order to obtain lower bounds for  $s_u$ , we consider the total number  $\Sigma$  of all edges incident to usual switches such that we count twice the edges between two vertices of  $\mathcal{S}_u$ . To avoid this weight problem, we insert a *link vertex*  $b_e$  in the middle of every edge  $e$  incident to two vertices of  $\mathcal{S}_u$ . Let  $\tilde{G}$  be the graph obtained from  $G$  by replacing each edge  $e = (s, s')$  between two usual switches by the path  $(s, b_e, s')$ . After this transformation,  $\Sigma$  is the number of edges of  $\tilde{G}$  between  $\mathcal{S}_u$  and  $G - \mathcal{S}_u$ . Note that our *cut criterion* (Proposition 12) holds for  $\tilde{G}$ , too.

Following a definition of [6], let us call the connected components of  $\tilde{G} - \mathcal{S}_u$  *blocks*. A crucial observation in our argument is that  $\Sigma$  is the number of edges of a bipartite graph between  $\mathcal{S}_u$  and the blocks. We distinguish two kinds of blocks, the *principal blocks* correspond to connected components of  $G - \mathcal{S}_u$  and *link blocks* correspond to a link vertex. Note that by the definition of block, every input or output incident to a switch in a block is also part of the block. We can deduce the following easy bound on the number of edges between a block  $B$  and  $\mathcal{S}_u$ .

**Proposition 13** *For every block  $B$ ,  $\text{deg}(B) \leq \text{in}(B) + 2 - \text{out}(B) - \text{pr}(B)$ .*

**Proof.** The statement holds trivially for link blocks, since all of them are of degree 2. By definition, a principal block  $B$  has  $in(B)$  switches. Let  $e(B)$  be the number of edges connecting two switches of  $B$ . There are  $in(B)$  (resp.  $out(B)$ ,  $pr(B)$ ) edges linking inputs (resp. outputs, priorities) to switches. Thus,  $deg(B) = [\sum_{v \text{ switch}} deg(v)] - 2e(B) - in(B) - out(B) - pr(B)$ . Since  $B$  is connected then  $e(B) \geq in(B) - 1$ . Since the degree of every switch is at most 4, we obtain  $deg(B) \leq 4in(B) - 2e(B) - in(B) - out(B) - pr(B) \leq in(B) + 2 - out(B) - pr(B)$ . ■

**Remark 14** *Note that if equality holds in Proposition 13, then every switch has degree four.*

We list a few immediate consequences of the cut criterion (Proposition 12) for a block  $B$ .

**Proposition 15** *Let  $B$  be a block of  $\tilde{G}$ .*

- (i)  $pr(B) \leq 1$ .
- (ii) If  $p \geq 2$  then  $out(B) \leq 1$ .
- (iii) If  $f \geq 1$ , then  $out(B) + pr(B) \leq 1$ .

**Proof.** (i) If  $B$  has two or more priorities, then by Proposition 13,  $deg(B) \leq in(B) - out(B)$ . This contradicts Proposition 12 (iv).

(ii) Suppose that  $B$  contains two or more outputs. We distinguish three cases: If  $out(B) \geq p + f$ , then  $deg(B) \geq in(B) + 4 - out(B) - pr(B)$  by Proposition 12 (i) contradicting Proposition 13. If  $p \leq out(B) \leq p + f$ , then  $deg(B) \geq in(B) + 2 - pr(B)$  by Proposition 12 (ii) contradicting Proposition 13. Finally, if  $p \geq out(B)$ , then  $deg(B) \geq in(B) + out(B) - pr(B)$  by Proposition 12 (iii) contradicting Proposition 13.

(iii) If  $B$  has one priority and one output then by Proposition 13,  $deg(B) \leq in(B)$ . This contradicts Proposition 12.(iv) if  $f \geq 1$ . ■

In the remainder of this section, **we assume that either  $p \geq 2$  or  $p = 1$  and  $f \geq 1$** . Indeed the case  $p = 1$ ,  $f = 0$  is already solved by Proposition 4 and Corollary 11.

We partition the blocks into the following four sets:

- $\mathcal{B}'_p$ , the set of blocks having one priority and one output;
- $\mathcal{B}_p$ , the set of blocks having one priority and no output;
- $\mathcal{B}_1$ , the set the blocks having no priority and one output;
- $\mathcal{B}_0$ , the set of blocks having no priority and no output (including the link blocks).

**Remark 16** *Note that if  $f \geq 1$ , then  $\mathcal{B}'_p$  is empty.*

**Proposition 17** *Every block  $B \in \mathcal{B}'_p$  satisfies  $deg(B) = in(B)$ . Every block  $B \in \mathcal{B}_1 \cup \mathcal{B}_p$  satisfies  $deg(B) = in(B) + 1$ .*

**Proof.** It follows directly from Propositions 12 (iii) and 13. ■

**Remark 18** Note that in a block of  $\mathcal{B}'_p \cup \mathcal{B}_p \cup \mathcal{B}_1$  every switch has degree four by Remark 14.

**Proposition 19** *A block  $B \in \mathcal{B}_0$  of a minimum  $(n, p, f)$ -repartitor satisfies  $\deg(B) = \text{in}(B) + 2$  unless it contains the only vertex switch of degree 3 in which case  $\deg(B) = \text{in}(B) + 1$ .*

**Proof.** If  $B$  is a link block, then  $\deg(B) = 2 = \text{in}(B) + 2$ . Assume now that  $B$  contains switches. Let  $e(B)$  be the number of edges between pairs of switches of  $B$ . By Proposition 12:

$$\deg(B) \geq \text{in}(B). \quad (\text{a})$$

In the proof of Proposition 13, we had  $\deg(B) = \sum_{v \text{ switch}} \deg(v) - 2e(B) - \text{in}(B)$ . Thus, if one switch has degree 3 then

$$\deg(B) = 3\text{in}(B) - 2e(B) - 1. \quad (\text{b})$$

Since  $e(B) \geq \text{in}(B) - 1$ , Equations (a) and (b) yield  $\deg(B) = \text{in}(B) + 1$ .

If every switch has degree four we obtain

$$\deg(B) = 3\text{in}(B) - 2e(B). \quad (\text{c})$$

Because  $e(B) \geq \text{in}(B) - 1$ , Equations (a) and (c) yield  $\deg(B) = \text{in}(B) + 2$  or  $\deg(B) = \text{in}(B)$ . Now if  $\deg(B) = \text{in}(B)$ , then the graph obtained by removing the switches of  $B$  and linking one to one the ordinary inputs of  $B$  to the neighbours of  $B$  is also an  $(n, p, f)$ -repartitor. This contradicts the minimality of  $G$ . So  $\deg(B) = \text{in}(B) + 2$ . ■

Let  $b_0, b_1, b_p$  and  $b'_p$  be the cardinalities of  $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_p$  and  $\mathcal{B}'_p$  respectively. We denote the number of ordinary inputs which are in a block of  $\mathcal{B}_0$  (resp.,  $\mathcal{B}_1, \mathcal{B}_p$ , and  $\mathcal{B}'_p$ ) by  $n_0$  (resp.,  $n_1, n_p$ , and  $n'_p$ ).

**Let  $\epsilon'(f) = 0$  if  $f$  is even, let  $\epsilon'(f) = 1$  if  $f$  is odd and the switch of degree 3 is usual, and let  $\epsilon'(f) = -1$  otherwise.**

We summarize a few equations for further reference.

**Proposition 20** *We have the following equalities:*

$$b_1 + b'_p = n + f. \quad (1)$$

$$b'_p = 0 \quad \text{if } f \geq 1. \quad (2)$$

$$b_p + b'_p = p. \quad (3)$$

$$n_1 + n_0 + n_p + n'_p = n - p. \quad (4)$$

$$4s_u = 2n + f + 2b_0 - 2b'_p + \epsilon'(f). \quad (5)$$

**Proof.** (1) there is a one-to-one correspondence between blocks of  $\mathcal{B}_1 \cup \mathcal{B}'_p$  and the outputs they contain.

(2) is Remark 16.

(3) there is a one-to-one correspondence between blocks of  $\mathcal{B}_p \cup \mathcal{B}'_p$  and the priorities they contain.

(4) is obtained by noting that the  $n - p$  ordinary inputs are in blocks.

(5) double counts the edges between usual switches and blocks. On one side, there are  $4s_u$  edges (minus  $\epsilon'(f)$  if the switch of degree 3 is a usual one). On the other side, we have by Proposition 17,  $n'_p$  edges incident to blocks of  $\mathcal{B}'_p$ ,  $n_p + b_p$  edges incident to blocks of  $\mathcal{B}_p$  and  $n_1 + b_1$  edges incident to blocks of  $\mathcal{B}_1$  and by Proposition 19,  $n_0 + 2b_0$  edges incident to blocks of  $\mathcal{B}_0$  (or  $n_0 + 2b_0 - 1$  if the switch of degree 3 is in a block of  $\mathcal{B}_0$ ). Thus  $4s_u = n'_p + n_p + b_p + n_1 + b_1 + n_0 + 2b_0 + \epsilon'(f)$ . Then by (1), (3) and (4), we obtain the result. ■

### 3.3 The lower bounds

**Theorem 21** *If  $f \geq 1$ , then*

$$R(n, p, f) \geq \frac{3n}{2} - p + \frac{f + \epsilon(f)}{4}.$$

**Proof.** By (5) and (2),  $s_u = n/2 + (f + 2b_0 + \epsilon'(f))/4$ . Now if  $\epsilon'(f) = -1$ , by Remark 18, the switch of degree 3 is in a block of  $\mathcal{B}_0$ ; hence  $b_0 \geq 1$ . Therefore  $s_u \geq n/2 + (f + \epsilon(f))/4$ . As  $N = n - p + s_u$ , we obtain  $R(n, p, f) \geq \frac{3n}{2} - p + \frac{f + \epsilon(f)}{4}$ . ■

**Theorem 22**

$$R(n, 2, 0) \geq \frac{3n}{2} - 3.$$

**Proof.** By (5),  $s_u \geq n/2 - b'_p/2 \geq n/2 - 1$ . Then  $R(n, 2, 0) \geq \frac{3n}{2} - 3$ . ■

In the remainder of this section, **we will suppose  $p \geq 3$  or ( $p = 2$  and  $f \geq 1$ )**.

**Lemma 23** *For  $p \geq 3$  or ( $p = 2$  and  $f \geq 1$ ), a usual switch is adjacent to at most two elements of  $\mathcal{B}_1$ .*

**Proof.** Suppose to the contrary that a usual switch  $S$  is adjacent to three blocks  $B_1$ ,  $B'_1$  and  $B''_1$  of  $\mathcal{B}_1$ . Then  $W = S \cup B_1 \cup B'_1 \cup B''_1$  satisfies  $out(W) = 3$  and contradicts Proposition 12 (ii) if  $p = 2$  and  $f \geq 1$ , or Proposition 12 (iii) if  $p \geq 3$ , as  $deg(W) = in(W) + 1$  and  $pr(W) = 0$ . ■

For  $0 \leq i \leq 2$ , let  $\mathcal{S}_i$  be the set of usual switches adjacent to exactly  $i$  elements of  $\mathcal{B}_1$  and  $s_i$  its cardinality. By Lemma 23,  $(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2)$  is a partition of  $\mathcal{S}_u$ . Moreover, from Proposition 20, one obtains the following corollary:

**Corollary 24** *For  $p \geq 3$  or ( $p = 2$  and  $f \geq 1$ ),*

$$\sum_{B \in \mathcal{B}_1} deg(B) = n_1 + b_1 = n_1 + n + f - b'_p = 2s_2 + s_1 \leq s_u + s_2. \quad (6)$$

**Lemma 25** *If  $p \geq 3$ , then a block of  $\mathcal{B}'_p$  and a switch of  $\mathcal{S}_2$  cannot be adjacent.*

**Proof.** Suppose that a block  $B'$  of  $\mathcal{B}'_p$  and a switch  $S_2$  of  $\mathcal{S}_2$  are adjacent. Let  $B_1$  and  $B'_1$  be the blocks of  $\mathcal{B}_1$  adjacent to  $S_2$ . Then  $W = B' \cup S_2 \cup B_1 \cup B'_1$  contradicts Proposition 12 (iii) as  $\deg(W) = \text{in}(W)$ ,  $\text{out}(W) = 3$  and  $\text{pr}(W) = 1$ . ■

It follows from Remark 16 and Lemma 25 that a switch of  $\mathcal{S}_2$  is adjacent to two blocks of  $\mathcal{B}_1$  and two blocks of  $\mathcal{B}_0 \cup \mathcal{B}_p$ . Let  $H$  be the multigraph whose vertices correspond to the blocks of  $\mathcal{B}_0 \cup \mathcal{B}_p$  and where two vertices are joined by  $\mu$  edges if the corresponding blocks share  $\mu$  neighbours in  $\mathcal{S}_2$ .

**Lemma 26** *If  $p \geq 2$  and  $f \geq 1$ , then  $H$  has no cycle of length at most  $\lfloor \frac{f+3}{2} \rfloor$ . If  $p \geq 3$ , then  $H$  is a simple graph (has no cycle of length 2).*

**Proof.** Suppose that in  $H$  there is a cycle  $C$  of length at most  $\lfloor \frac{f+3}{2} \rfloor$  or a cycle of length 2 if  $p \geq 3$ . Let  $W$  be the union of the blocks  $\mathcal{B}_0 \cup \mathcal{B}_p$  corresponding to the vertices of  $C$ , plus the switches of  $\mathcal{S}_2$  corresponding to edges of  $C$  and the blocks of  $\mathcal{B}_1$  adjacent to these switches. Then  $W$  contradicts Proposition 12 (i), (ii) or (iii), as  $\deg(W) = \text{in}(W) - \text{pr}(W)$ . ■

Since a simple graph on  $n$  vertices has at most  $n(n-1)/2$  edges, we have the following:

**Corollary 27** *If ( $p \geq 2$  and  $f \geq 1$ ) or  $p \geq 3$ ,*

$$s_2 \leq \frac{(b_0 + b_p)(b_0 + b_p - 1)}{2}. \quad (7)$$

Turán's theorem asserts that a simple graph on  $n$  vertices without a cycle of length 3 has at most  $n^2/4$  edges. So we get:

**Corollary 28** *For  $p \geq 2$  and  $f \geq 3$ ,*

$$s_2 \leq (b_0 + b_p)^2/4. \quad (8)$$

**Theorem 29**

$$R(n, 2, 2) \geq R(n, 2, 1) \geq \frac{3n}{2} + \frac{\sqrt{n+3}}{2} - 3.$$

$$R(n, 3, 0) \geq \frac{3n}{2} + \frac{\sqrt{n-3}}{2} - \frac{9}{2}.$$

**Proof.** Suppose  $p = 2$  and  $f = 1$ . From (7) and (6), as  $b'_p = 0$  thus  $b_p = p = 2$ , we obtain:

$$\frac{(b_0 + 2)(b_0 + 1)}{2} \geq s_2 \geq n_1 + n + 1 - s_u.$$

Replacing  $s_u$  by its value in (5),

$$\begin{aligned}\frac{b_0^2 + 3b_0 + 2}{2} &\geq n_1 + n + 1 - n/2 - 1/4 - b_0/2 - \epsilon'(1)/4, \\ b_0^2 + 4b_0 + 4 &\geq n + 3, \\ b_0 + 2 &\geq \sqrt{n + 3}.\end{aligned}$$

Using this bound on  $b_0$  in (5), we obtain  $s_u \geq \frac{n}{2} + \frac{\sqrt{n+3}}{2} - 1$ . Thus

$$R(n, 2, 1) \geq \frac{3n}{2} + \frac{\sqrt{n+3}}{2} - 3.$$

The proof is analogous if  $p = 3$  and  $f = 0$ . We obtain  $\frac{(b_0+b_p)(b_0+b_p-1)}{2} \geq n_1 + n - b'_p - s_u$  and  $(b_0 + b_p)^2 \geq n - b'_p + b_p$ . Hence  $b_0 \geq \sqrt{n - b'_p + b_p} - b_p$ . So  $s_u \geq \frac{n}{2} + \frac{\sqrt{n-b'_p+b_p}}{2} - \frac{b_p+b'_p}{2} \geq \frac{n}{2} + \frac{\sqrt{n-3}}{2} - \frac{3}{2}$ .

Finally  $R(n, 3, 0) \geq \frac{3n}{2} + \frac{\sqrt{n-3}}{2} - \frac{9}{2}$ . ■

**Theorem 30**

If  $f \geq 3$ , then  $R(n, 2, f) \geq \frac{3n}{2} + \frac{\sqrt{n}}{\sqrt{2}} + \Omega(f)$ .

**Proof.** Suppose that  $p = 2$  and  $f \geq 3$ . In the same way as above, it follows from (5), (6) and (8) that

$$\begin{aligned}(b_0 + 2)^2 + 2b_0 &\geq 2n + 3f - \epsilon'(f), \\ b_0 + 3 &\geq \sqrt{2n + 3f - \epsilon'(f) + 5}.\end{aligned}$$

Using this bound on  $b_0$  in (5), we obtain

$$s_u \geq \frac{n}{2} + \frac{1}{2}\sqrt{2n + 3f - \epsilon'(f) + 5} + \frac{f + \epsilon'(f)}{4} - \frac{3}{2}.$$

Thus

$$R(n, 2, f) \geq \frac{3n}{2} + \frac{1}{2}\sqrt{2n + 3f - \epsilon'(f) + 5} + \frac{f + \epsilon'(f)}{4} - \frac{7}{2}.$$
■

**Lemma 31** *If  $p \geq 4$  or ( $p = 3$  and  $f \geq 1$ ), then a block of  $\mathcal{B}_0 \cup \mathcal{B}_p$  is adjacent to at most one switch of  $\mathcal{S}_2$ .*

**Proof.** Suppose that a block  $B \in \mathcal{B}_0 \cup \mathcal{B}_p$  is adjacent to two switches  $S_2$  and  $S'_2$  of  $\mathcal{S}_2$ . Let  $B_1$  and  $C_1$  and  $B'_1$  and  $C'_1$  be the elements of  $\mathcal{B}_1$  adjacent to  $S_2$  and  $S'_2$ , respectively. (See Figure 4, left.) Then the set  $S_2 \cup S'_2 \cup B \cup B_1 \cup C_1 \cup B'_1 \cup C'_1$  contradicts Proposition 12 (iii) if  $p \geq 4$ , or Proposition 12 (ii) if  $p = 3$  and  $f \geq 1$ . ( $\deg(W) = \text{in}(W) + 2 - pr(W)$ ,  $\text{out}(W) = 4$  and  $pr(W) \leq 1$ .) ■

Let  $\mathcal{B}'_0$  (resp.  $\mathcal{B}''_0$ ) be the set of elements of  $\mathcal{B}_0$  adjacent to exactly one (resp. no) switch in  $\mathcal{S}_2$  and let  $b'_0$  (resp.  $b''_0$ ) be its cardinality.

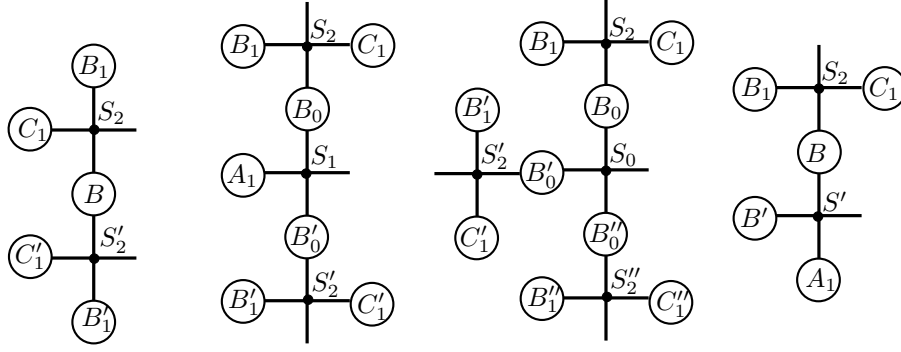


Figure 4: Illustration for Lemmas 31, 34, 35, and 38.

**Corollary 32** For  $p \geq 4$  or ( $p = 3$  and  $f \geq 1$ ),

$$b_0 + b_p \geq b'_0 + b_p \geq 2s_2. \quad (9)$$

**Theorem 33** If  $p \geq 4$  or ( $p = 3$  and  $f \geq 1$ ),

$$R(n, p, f) \geq \frac{7n}{4} - \frac{7p}{4} + \frac{5f - \epsilon(f)}{8}.$$

**Proof.** By (6) and (9), we have  $s_u \geq n + f - b'_p - (b_0 + b_p)/2$ . By (5),  $s_u = n/2 + (f + \epsilon'(f))/4 + (b_0 - b'_p)/2$ . Adding these two inequalities, we obtain:

$$2s_u \geq \frac{3n}{2} + \frac{5f + \epsilon'(f)}{4} - \frac{b_p + 3b'_p}{2}.$$

Hence,  $R(n, p, f) \geq \frac{7n}{4} - \frac{7p}{4} + \frac{5f + \epsilon'(f)}{8}$ . ■

**Lemma 34** If  $p \geq 5$  or ( $p = 4$  and  $f \geq 1$ ), every switch of  $\mathcal{S}_1$  is adjacent to at most one block of  $\mathcal{B}'_0$ .

**Proof.** Suppose that  $S_1 \in \mathcal{S}_1$  is a switch adjacent to two blocks  $B_0$  and  $B'_0$  in  $\mathcal{B}'_0$ . Let  $S_2$  (resp.  $S'_2$ ) be the switch of  $\mathcal{S}_2$  adjacent to  $B_0$  (resp.  $B'_0$ ), and  $B_1$  and  $C_1$  (resp.  $B'_1$  and  $C'_1$ ) the two blocks of  $\mathcal{B}_1$  adjacent to  $S_2$  (resp.  $S'_2$ ). Let  $A_1$  be the block of  $\mathcal{B}_1$  adjacent to  $S_1$ . (See Figure 4, middle left.) Then  $W = A_1 \cup S_1 \cup S_2 \cup B_0 \cup B_1 \cup C_1 \cup S'_2 \cup B'_0 \cup B'_1 \cup C'_1$  contradicts Proposition 12 as  $\deg(W) \leq \text{in}(W) + 3$  and  $\text{out}(W) = 5$ . ■

**Lemma 35** If  $p \geq 6$  or ( $p = 5$  and  $f \geq 1$ ), every switch of  $\mathcal{S}_0$  is adjacent to at most two blocks of  $\mathcal{B}'_0$ .

**Proof.** Suppose that  $S_0$  is a switch adjacent to three blocks  $B_0$ ,  $B'_0$  and  $B''_0$  in  $\mathcal{B}'_0$ . Let  $S_2$  (resp.  $S'_2$ ,  $S''_2$ ) be the switch of  $\mathcal{S}_2$  adjacent to  $B_0$  (resp.  $B'_0$ ,  $B''_0$ ) and  $B_1$  and  $C_1$  (resp.  $B'_1$  and  $C'_1$ ,  $B''_1$  and  $C''_1$ ) the two blocks of  $\mathcal{B}_1$  adjacent to  $S_2$  (resp.  $S'_2$ ,  $S''_2$ ). (See Figure 4, middle right.) Then  $W = S_0 \cup S_2 \cup B_0 \cup B_1 \cup C_1 \cup S'_2 \cup B'_0 \cup B'_1 \cup C'_1 \cup S''_2 \cup B''_0 \cup B''_1 \cup C''_1$  contradicts Proposition 12 as  $\deg(W) \leq \text{in}(W) + 4$  and  $\text{out}(W) = 6$ . ■

**Proposition 36** *Let  $p \geq 2$ . A block  $B \in \mathcal{B}'_0$  has no switch of degree 3. Thus  $\deg(B) = \text{in}(B) + 2$ .*

**Proof.** Suppose that  $B$  has a switch of degree 3. Let  $S$  be the switch of  $\mathcal{S}_2$  adjacent to  $B$  and  $B_1$  and  $C_1$  be the two blocks of  $B_1$  adjacent to  $S$ . Then  $W = B \cup S \cup B_1 \cup C_1$  contradicts Proposition 12 (iii), as  $\deg(W) = \text{in}(W) + 1$ . ■

Let  $\mathcal{S}'_1$  be the set of elements of  $\mathcal{S}_1$  adjacent to exactly one block in  $\mathcal{B}'_0$  and let  $s'_1$  be its cardinality. Let  $n'_0$  denote the number of inputs in blocks of  $\mathcal{B}'_0$ .

**Corollary 37** *If  $p \geq 6$  or ( $p = 5$  and  $f \geq 1$ ),*

$$n'_0 + b'_0 \leq s'_1 + 2s_0. \quad (10)$$

**Proof.** Let us count the number  $e$  of edges between  $\mathcal{B}'_0$  and  $\mathcal{S}'_1 \cup \mathcal{S}_0$ .

On one side, every block  $B$  of  $\mathcal{B}'_0$  is adjacent to exactly one element of  $\mathcal{S}_2$  and no element in  $\mathcal{S}_1 \setminus \mathcal{S}'_1$ . Thus, by Proposition 36, there are  $\text{in}(B) + 1$  edges from  $B$  to  $\mathcal{S}'_1 \cup \mathcal{S}_0$ . Thus  $e = n'_0 + b'_0$ .

On the other side, there are  $s'_1$  (resp. at most  $2s_0$ ) edges joining  $\mathcal{S}'_1$  (resp.  $\mathcal{S}_0$ ) to  $\mathcal{B}'_0$  according to the definition of  $\mathcal{S}'_1$  and Lemma 34 (resp. Lemma 35). Thus  $e \leq s'_1 + 2s_0$ . ■

**Lemma 38** *If  $p \geq 4$ , a block of  $\mathcal{B}'_p$  is not adjacent to a switch of  $\mathcal{S}'_1$ .*

**Proof.** Suppose  $B'$  is a block of  $\mathcal{B}'_p$  adjacent to  $S'$  in  $\mathcal{S}'_1$ . Let  $A_1$  (resp.  $B$ ) be the block of  $\mathcal{B}_1$  (resp.  $\mathcal{B}'_0$ ) adjacent to  $S'$ . Let  $S_2$  be the switch of  $\mathcal{S}_2$  adjacent to  $B$ , and  $B_1$  and  $C_1$  the two blocks of  $\mathcal{B}_1$  adjacent to  $S_2$ . (See Figure 4, right.) Then  $W = B' \cup S' \cup A_1 \cup B \cup S_2 \cup B_1 \cup C_1$  contradicts Proposition 12 (iii), as  $\deg(W) = \text{in}(W) + 1$ ,  $\text{out}(W) = 4$  and  $\text{pr}(W) = 1$ . ■

**Lemma 39** *If  $p \geq 6$  or ( $p = 5$  and  $f \geq 1$ ), every element of  $\mathcal{B}''_0 \cup \mathcal{B}_p$  is adjacent to at most one element of  $\mathcal{S}'_1$ .*

**Proof.** Let block  $B \in \mathcal{B}''_0 \cup \mathcal{B}_p$  be adjacent to two switches  $S_1$  and  $S'_1$  of  $\mathcal{S}'_1$ . Let  $B_1$  and  $B_0$  (resp.  $B'_1$  and  $B'_0$ ) be the blocks of  $\mathcal{B}_1$  and  $\mathcal{B}'_0$  adjacent to  $S_1$  (resp.  $S'_1$ ). Let  $W_0$  (resp.  $W'_0$ ) be the union of the switch of  $\mathcal{S}_2$  adjacent to  $B_0$  (resp.  $B'_0$ ) and its two adjacent blocks of  $\mathcal{B}_1$ . Then  $W = B \cup B_1 \cup B_0 \cup W_0 \cup B'_1 \cup B'_0 \cup W'_0$  contradicts Proposition 12 (ii) or (iii), as  $\deg(W) = \text{in}(W) + 4 - \text{pr}(W)$ ,  $\text{out}(W) = 6$  and  $\text{pr}(W) \leq 1$ . ■

**Corollary 40** *If  $p \geq 6$  or ( $p = 5$  and  $f \geq 1$ ),*

$$b''_0 + b_p \geq 2s'_1. \quad (11)$$

**Theorem 41** *If  $p \geq 6$  or ( $p = 5$  and  $f \geq 1$ ),*

$$R(n, p, f) \geq \frac{15n}{8} - \frac{15p}{8} + \frac{13f - \epsilon(f)}{16}.$$



**Proof.** We have  $s_u = s_2 + s_1 + s_0$ , then by (6),

$$s_u = n + n_1 + f - b'_p - s_2 + s_0,$$

$$2s_u \geq 2n + 2f - 2b'_p - 2s_2 + 2s_0.$$

By (9) and (10),  $2s_0 \geq n'_0 + 2s_2 - s'_1 - b_p$ , thus

$$2s_u \geq 2n + 2f + n'_0 - s'_1 - b_p - 2b'_p.$$

Then by (11),

$$2s_u \geq 2n + 2f + n'_0 - b''_0/2 - 3b_p/2 - 2b'_p,$$

$$4s_u \geq 4n + 4f - b''_0 - 3b_p - 4b'_p.$$

Since  $b_0 = b'_0 + b''_0$ , by (5) and (9),  $2s_u \geq n + (f + \epsilon'(f))/2 - b'_p + b''_0 + 2s_2 - b_p$ , so by adding the two inequalities one has

$$6s_u \geq 5n + 2s_2 + \frac{9f + \epsilon'(f)}{2} - 4b_p - 5b'_p.$$

Therefore, by (6),

$$8s_u \geq 7n + \frac{13f + \epsilon'(f)}{2} - 4b_p - 7b'_p.$$

■

## 4 Constructions

### 4.1 Useful constructions

In this subsection, we give general constructions that show how to merge several repartitors into a larger one. The first one is in the same vein as Lemma 7. The second one constructs an  $(n-1, p, f)$ -repartitor from an  $(n, p, f)$ -repartitor under certain conditions.

**Lemma 42** For  $0 \leq f' \leq f$ ,

$$R(n, p, f) \leq R(n + f', p + f', f - f') + R(p, p, f').$$

**Proof.** Let  $G_1$  be an  $(p, p, f')$ -repartitor and  $G_2$  be a  $(n + f', p + f', f - f')$ -repartitor. For  $i = 1, 2$ , let  $I_i$  (resp.  $P_i, O_i, S_i$ ) be the sets of ordinary inputs (resp. priorities, outputs, switches) of  $G_i$  (note that  $I_1$  is empty); and let  $P_2 = \{p_1, p_2, \dots, p_{p+f'}\}$  and  $O_1 = \{o_1, o_2, \dots, o_{p+f'}\}$ . Let  $G$  be the network defined as follows:

Let  $V(G) = P_1 \cup S_1 \cup I_2 \cup S_2 \cup O_2$ ; two vertices of  $V(G)$  are joined by an edge if and only if  $(u, v) \in E(G_1) \cup E(G_2)$  or there exists an  $i \in \{0, 1\}$  such that  $(u, o_i) \in E(G_1)$  and  $(p_i, v) \in E(G_2)$ ; the ordinary input set of  $G$  is  $I_2$ , its priority set  $P_1$ , its output set  $O_2$  and its switch set  $S_1 \cup S_2$ .

The proof that  $G$  is an  $(n, p, f)$ -repartitor is similar to that of Lemma 7. ■

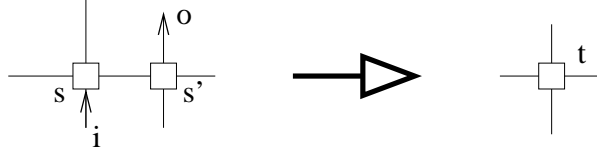


Figure 5: L-contraction

**Definition 43** Let  $G$  be a network. An edge  $(s, s')$  is said to be *linking* if there is an ordinary input  $i$  connected to  $s$  and an output  $o$  connected to  $s'$ .

The *L-contraction* of an edge  $(s, s')$  consists of contraction of the edge  $(s, s')$  into a vertex  $t$  and deletion of the input connected to  $s$  and the output connected to  $s'$ . (See Figure 5.)

**Lemma 44** Let  $G$  be an  $(n, p, f)$ -repartitor. The network obtained from  $G$  by the L-contraction of an edge is an  $(n - 1, p, f)$ -repartitor.

**Proof.** Let  $(s, s')$  be an edge of  $G$  and let  $i$  and  $o$  be the input and output linked to  $s$  and  $s'$ , respectively. Let  $G'$  be the network obtained by the L-contraction of  $(s, s')$  into  $t$ .

Consider two disjoint subsets of outputs  $Q$  and  $F$  in  $G'$  such that  $|Q| = p$  and  $|F| = k$ . Since  $G$  is a repartitor, there exists a set  $\mathcal{P}$  of  $n$  edge-disjoint paths in  $G$ , connecting the  $p$  priorities to  $Q$ , and the  $n - p$  ordinary inputs to  $O \setminus (Q \cup F)$ .

Case 1: A path  $P \in \mathcal{P}$  connects  $i$  to  $o$ . Then replacing in the  $n - 1$  paths of  $\mathcal{P} \setminus P$  the vertex  $s, s'$ , or the succession of these two by  $t$ , we obtain the required set of  $n - 1$  paths in  $G'$ .

Case 2: A path  $P_1$  of  $\mathcal{P}$  connects an ordinary input  $i' \neq i$  to  $o$ , and a path  $P_2$  of  $\mathcal{P}$  connects  $i$  to an output  $o'$ . Assume that these paths are  $P_1 = (i', s'_1, s'_2, \dots, s'_j, s', o)$  and  $P_2 = (i, s, s_1, s_2, \dots, s_l, o')$ . Let  $W'$  be the walk  $(i', s'_1, s'_2, \dots, s'_j, s', s, s_1, s_2, \dots, s_l, o')$  and let  $P'$  be the subpath of  $W'$  from  $i'$  to  $o'$ . We put  $\mathcal{P}' = (\mathcal{P} \setminus \{P_1, P_2\}) \cup \{P'\}$ . Then replacing in the  $n - 1$  paths of  $\mathcal{P}'$  the vertex  $s, s'$ , or the succession of these two by  $t$ , we obtain the required set of  $n - 1$  paths in  $G'$ . ■

## 4.2 Minimum $(n, 1, 0)$ -repartitors

**Theorem 45**

$$R(n, 1, 0) = n - 1.$$

**Proof.** By Proposition 4,  $R(n, 1, 0) \leq n - 1$ . By Corollary 11,  $R(n, 1, 0) \geq n - 1$ . ■

We can, in fact, completely describe the set of minimum  $(n, 1, 0)$ -repartitors.

Let  $T$  be a binary tree with  $n$  leaves and  $n - 1$  internal nodes, one being the root  $r$ . Let us define the graph  $G_T$  associated with  $T$  in the following way: every leaf of  $T$  is an output of  $G_T$ ; the internal nodes of  $T$  are the switches of  $G_T$ ; each internal node of  $T$  is adjacent to an ordinary input, and furthermore the root is adjacent to the priority. (See Figure 6.)

**Proposition 46** A  $(n, 1, 0)$ -repartitor is minimum if and only if it is associated with a binary tree with  $n$  leaves.

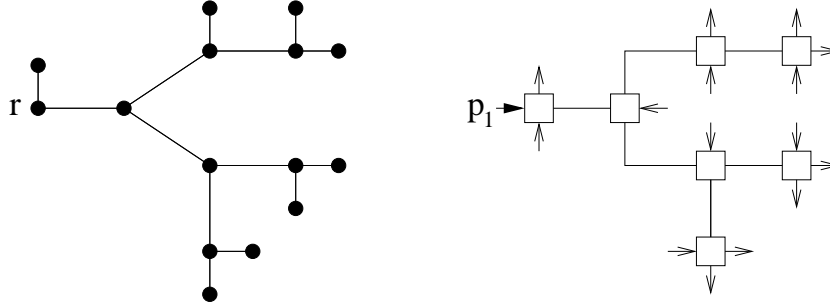


Figure 6: A binary tree with 8 leaves and the corresponding minimum  $(8, 1, 0)$ -repartitor.

**Proof.** It is easy to see that the graph associated with a binary tree is a minimum  $(n, 1, 0)$ -repartitor.

Let  $N$  be a minimum  $(n, 1, 0)$ -repartitor. It has  $n - 1$  switches and each of them is adjacent to exactly one ordinary input. Moreover  $n$  outputs and one priority are connected to the switches. Thus, there are exactly  $(4(n - 1) - 2n)/2 = n - 2$  edges connecting a switch to another. Since  $N$  is connected then it is a tree. Let  $T$  be the subtree of  $N$  induced by the switches and the outputs. It is clearly a binary tree. Indeed every output has degree one and every switch has degree 3 except the one that is connected to the priority that has degree 2 (and thus is the root).

■

### 4.3 Minimum $(n, 2, 0)$ -repartitors

**Theorem 47**

$$R(n, 2, 0) = \left\lceil \frac{3n}{2} - 3 \right\rceil.$$

**Proof.** Theorem 22 yields  $R(n, 2, 0) \geq \lceil \frac{3n}{2} - 3 \rceil$ . Since  $R(n, 1, 0) = n - 1$ , by the first inequality of Lemma 5,  $R(n, 2, 0) \leq \lceil \frac{3n}{2} - 3 \rceil$ . ■

**Remark 48** Note that minimum  $(n, 2, 0)$ -repartitors are not necessarily formed from two minimum  $(n/2, 1, 0)$ -repartitors with the construction of Lemma 5. Indeed, let  $R$  be a minimum  $(n, 2, 0)$ -repartitor and let  $o$  and  $o'$  be two distinct outputs incident to the switches  $s$  and  $s'$ , respectively. Let  $R'$  be the  $(n + 2, 2, 0)$ -network obtained by removing  $o$  and  $o'$  and adding three switches  $t$ ,  $t'$  and  $u$  such that  $t$  is adjacent to  $u$ ,  $s$  and an ordinary input and an output,  $t'$  is adjacent to  $u$ ,  $s'$  and an ordinary input and an output, and  $u$  is adjacent to two outputs. See Figure 7. It is easy to check that  $R'$  is a minimum  $(n + 2, 2, 0)$ -repartitor. Since  $t$  and  $t'$  are adjacent to an input and an output,  $R'$  may not be obtained via the previously mentioned construction.

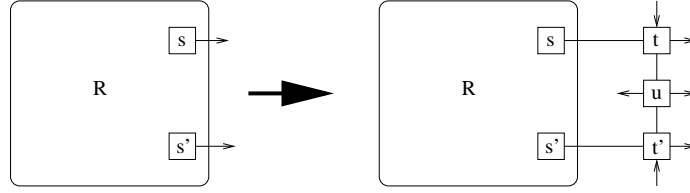


Figure 7: Extending a minimum  $(n, 2, 0)$ -repartitor to a minimum  $(n + 2, 2, 0)$ -repartitor.

#### 4.4 Minimum $(n, 1, 1)$ -repartitors and $(n, 1, 2)$ -repartitors

**Theorem 49**

$$R(n, 1, 1) = \left\lceil \frac{3n - 1}{2} \right\rceil.$$

**Proof.** Theorem 21 yields  $R(n, 1, 1) \geq \lceil \frac{3n-1}{2} \rceil$ . According to Lemma 42, we can bound  $R(n, 1, 1)$  by a sum:  $R(n, 1, 1) \leq R(n + 1, 2, 0) + R(1, 1, 1)$ . By Theorem 47,  $R(n + 1, 2, 0) = \lceil \frac{3n+3}{2} - 3 \rceil$ , and  $R(1, 1, 1) = R(1, 0, 1) = 1$ . Thus  $R(n, 1, 1) \leq \lceil \frac{3n}{2} - \frac{1}{2} \rceil$ . ■

**Theorem 50**

$$R(n, 1, 2) = \left\lceil \frac{3n - 1}{2} \right\rceil.$$

**Proof.** Theorem 21 yields  $R(n, 1, 2) \geq \lceil \frac{3n}{2} - \frac{1}{2} \rceil$ .

The network depicted in Figure 8 has  $\lceil \frac{3n}{2} - \frac{1}{2} \rceil$  switches. Checking that this network is an

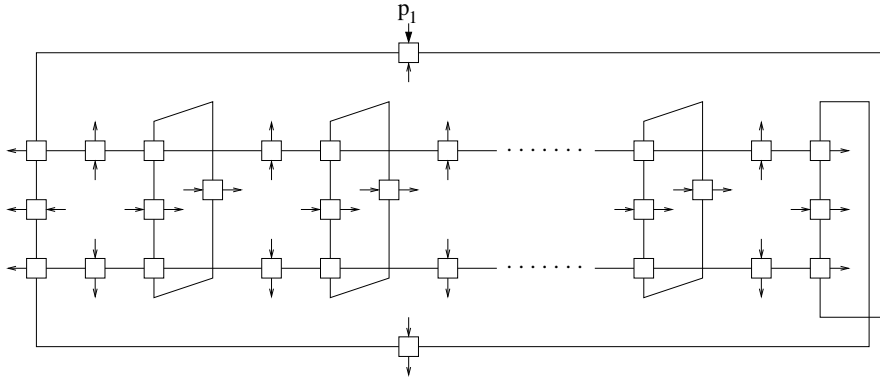


Figure 8: Minimum  $(n, 1, 2)$ -repartitor.

$(n, 1, 2)$ -repartitor is not difficult but tedious since we must investigate various configurations and exhibit each time the corresponding edge-disjoint paths. The complete proof is available in [5]. ■

## 4.5 $(n, 3, 0)$ -repartitors

### Theorem 51

$$R(n, 3, 0) = \frac{3n}{2} + \frac{\sqrt{n}}{2} + c,$$

where  $-5 \leq c \leq 9/4$ .

Let  $G_l$  be the graph with vertex set  $\{a_{i,j} | 1 \leq i \leq 3, 1 \leq j \leq l\}$  whose edge set is the union of the three paths  $P_i = (a_{i,1}, a_{i,2}, \dots, a_{i,l})$  for  $1 \leq i \leq 3$  and the  $l$  cycles  $C_j = (a_{1,j}, a_{2,j}, a_{3,j}, a_{1,j})$  for  $1 \leq j \leq l$ . See Figure 9. So  $G_l$  has  $3l$  vertices and  $6l - 3$  edges.

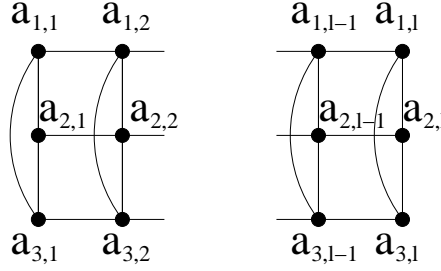


Figure 9: The graph  $G_l$ .

From  $G_l$ , we construct the network  $H_l$  as follows.

To each vertex  $a_{i,j}$ ,  $1 \leq i \leq 3$ ,  $1 \leq j \leq l$ , we associate a switch  $s_{i,j}$ . Each switch  $s_{i,1}$ ,  $1 \leq i \leq 3$ , is connected to a priority input  $p_i$  and each switch  $s_{i,l}$ ,  $1 \leq i \leq 3$ , is connected to an output  $o_i$ . We replace each edge  $e$  of  $G_l$  by a path  $P(e)$  consisting of  $n(e)$  switches with some ordinary input connected to each of them. In the bipartite graph  $\tilde{H}_l$  defined previously, this corresponds to association of each edge  $e$  with a block  $B_0(e)$  of  $\mathcal{B}_0$ . The size  $n(e)$  of the paths  $P(e)$  is  $in(B_0(e))$  and will be specified later. For any two non-adjacent edges  $e$  and  $f$  in  $G_l$ , we add a switch  $s_{e,f}$  in  $\tilde{H}_l$ : it is connected to an ordinary switch of  $P(e)$ , to an ordinary switch of  $P(f)$ , and two outputs  $o_{e,f}$  and  $o'_{e,f}$ . (In  $\tilde{H}_l$ ,  $s_{e,f}$  is in  $\mathcal{S}_2$ .)

If  $l \geq 3$ , an edge  $e$  of  $G_l$  is adjacent to 6 other edges except the edges of  $E_1 = \{(a_{i,1}, a_{i,2}) | 1 \leq i \leq 3\} \cup \{(a_{i,l-1}, a_{i,l}) | 1 \leq i \leq 3\}$ , which are adjacent to 5 other edges and the edges of  $E_2 = \{(a_{i,1}, a_{i+1,1}) | 1 \leq i \leq 3\} \cup \{(a_{i,l}, a_{i+1,l}) | 1 \leq i \leq 3\}$ , which are adjacent to 4 edges. Hence the number  $n(e)$  of switches of  $P(e)$  is defined as follows:  $n(e) = 6l - 8$  if  $e \in E_2$ ,  $n(e) = 6l - 9$  if  $e \in E_1$  and  $n(e) = 6l - 10$  otherwise.

In summary,  $H_l$  has for  $l \geq 3$ :

- 3 priority inputs,
- $n_l - 3 = 6(6l - 8) + 6(6l - 9) + (6l - 15)(6l - 10) = 36l^2 - 78l + 48$  ordinary inputs,
- $n_l$  outputs,
- $n_l - 3$  ordinary switches (those of the  $P(e)$ ),
- $(n_l - 3)/2$  switches of  $\mathcal{S}_2$  (that is,  $s_{e,f}$ ), and

-  $3l$  switches of  $\mathcal{S}_0$  (that is,  $s_{i,j}$ ,  $i = 1, 2, 3$ ,  $j = 1, 2, \dots, l$ ).

Hence  $H_l$  has  $N_l = \frac{3n_l}{2} + 3l - \frac{9}{2}$  switches. As  $n_l = 36l^2 - 78l + 51 = (6l - \frac{13}{2})^2 + \frac{35}{4}$ , then  $N_l = \frac{3n_l}{2} + \frac{1}{2}\sqrt{n_l - \frac{35}{4}} - \frac{5}{4} \leq \frac{3n_l}{2} + \frac{1}{2}\sqrt{n_l} - \frac{5}{4}$ .

Note that all the formulae are also valid for  $l = 2$ . Indeed in that case  $n(e) = 4$  for any of the 9 edges and so  $n_l - 3 = 36$ .

**Lemma 52**  $H_l$  is a repartitor.

We will reduce the proof of this lemma to the existence of some specific walks in the slightly modified graph  $G_l$ .

**Definition 53** Let  $G'_l$  be the graph obtained from  $G_l$  by adding the set  $R_3 = \{a_{1,l+1}, a_{2,l+1}, a_{3,l+1}\}$  and the three edges  $(a_{i,l}, a_{i,l+1})$ ,  $1 \leq i \leq 3$ .

Let  $E_3 = \{e_1, e_2, e_3\}$  be a set of distinct edges of  $G'_l$ . An  $E_3$ -good set of walks is a set of three edge-disjoint walks  $\{W_1, W_2, W_3\}$  such that  $W_i$  contains  $e_i$ , where the set of initial vertices of the  $W_i$ 's is  $\{a_{1,1}, a_{2,1}, a_{3,1}\}$ , and the set of terminal vertices of the  $W_i$ 's is  $R_3$ .

**Lemma 54** If  $G'_l$  admits an  $E_3$ -good set of walks for any set  $E_3$  of three distinct edges, two of which are non-adjacent, then  $H_l$  is a repartitor.

**Proof.** Let us define an *assignment* as a mapping  $\phi$  from the output set of  $H_l$  into the edges of  $G'_l$  such that:

- $\phi(o_i) = (a_{i,l}, a_{i,l+1})$  for  $i \in \{1, 2, 3\}$ ;
- for the two outputs  $o_{e,f}$  and  $o'_{e,f}$  adjacent to the switch  $s_{e,f}$ , then  $\phi(o_{e,f}) = e$  and  $\phi(o'_{e,f}) = f$ , or  $\phi(o_{e,f}) = f$  and  $\phi(o'_{e,f}) = e$ .

Let  $Q = \{q_1, q_2, q_3\}$  be any set of three outputs of  $H_l$  and let us denote  $\{o_1, o_2, o_3\}$  by  $O_3$ . To prove that  $H_l$  is a repartitor, we have to find the desired edge-disjoint paths in  $H_l$ .

First let us show that there exists an assignment  $\phi$  such that the three edges  $\phi(q_i)$  are distinct and two of them are non-adjacent.

- If  $|Q \cap O_3| \geq 2$ , assume that  $q_1 = o_{i_1}$  and  $q_2 = o_{i_2}$ . Then for any assignment  $\phi$ ,  $\phi(q_1) = e_1 = (a_{i_1,l}, a_{i_1,l+1})$  and  $\phi(q_2) = e_2 = (a_{i_2,l}, a_{i_2,l+1})$ . Hence  $e_1$  and  $e_2$  are not adjacent and distinct from  $\phi(e_3)$  since only  $q_1$  (resp.  $q_2$ ) may be mapped on  $e_1$  (resp.  $e_2$ ) by an assignment.
- If  $|Q \cap O_3| = 1$ , assume that  $q_1 = o_{i_1}$  and  $q_2$  (resp.  $q_3$ ) are connected to  $s_{e_2, f_2}$  (resp.  $s_{e_3, f_3}$ ). As all the edges adjacent to  $\phi(q_1) = e_1 = (a_{1,l}, a_{1,l+1})$  are pairwise adjacent, one of  $\{e_2, f_2\}$  is not adjacent to  $e_1$ . Let  $\phi(q_2)$  be this edge and let  $\phi(q_3)$  be an edge of  $\{e_3, f_3\} \setminus \{e_2\}$ . Then  $\phi$  satisfies the requirement.

- If  $|Q \cap O_3| = 0$ , assume that  $q_i$ ,  $1 \leq i \leq 3$ , are connected to  $s_{e_i, f_i}$ . By definition,  $|\{e_i, f_i\} \cap \{e_j, f_j\}| \leq 1$  for every  $1 \leq i, j \leq 3$ .

Suppose first that there are  $i, j$ ,  $i \neq j$ , such that  $|\{e_i, f_i\} \cap \{e_j, f_j\}| = 1$ . Without loss of generality, we may assume that  $e_1 = e_2$ . Then let  $\phi(q_1) = e_1$  and  $\phi(q_2) = f_2$ . It follows that  $e_1 = e_2$  and  $f_2$  are not adjacent. Let  $\phi(q_3)$  be an edge of  $\{e_3, f_3\} \setminus \{e_2, f_2\}$ . (Such an edge exists since  $|\{e_2, f_2\} \cap \{e_3, f_3\}| \leq 1$ .)

Suppose now that all the six edges  $e_1, f_1, e_2, f_2, e_3, f_3$  are distinct. If one edge of  $\{e_1, f_1\}$  is not adjacent to one edge of  $\{e_2, f_2\}$ , say  $e_1$  is not adjacent to  $e_2$ , then let  $\phi(q_i) = e_i$ . Otherwise the four edges  $e_1, f_1, e_2, f_2$  form a 4-cycle. Since there is no diagonal of a 4-cycle in  $G'_l$ ,  $e_3$  is adjacent to at most one edge of  $\{e_1, f_1\}$ , say  $e_3$  is not adjacent to  $e_1$ . Then setting  $\phi(q_i) = e_i$ , we obtain the desired assignment.

Let us now exhibit the  $n = 36l^2 - 78l + 51$  required edge-disjoint paths, 3 of them joining the priorities to  $Q$ , and the  $n - 3$  remaining joining the ordinary inputs to outputs.

Consider an assignment  $\phi$  as above and let  $E_3 = \{\phi(q_1), \phi(q_2), \phi(q_3)\}$ . By hypothesis, there exists an  $E_3$ -good set of walks  $\{W_1, W_2, W_3\}$  such that  $W_i$  contains  $\phi(q_i)$ .

Let  $o$  be an output not in  $O_3$ . Let  $e = \phi(o)$ ,  $s_{e,f}(o)$  be the switch adjacent to  $o$ ,  $s_e(o)$  the vertex of  $P(e)$  adjacent to  $s_{e,f}(o)$ , and  $i(o)$  the input adjacent to  $s_e(o)$ . The *default path* associated with  $o$  is  $P(o) = (i(o), s_e(o), s_{e,f}(o), o)$ .

If  $o \notin Q \cup O_3$ , we choose its default path for joining the ordinary input  $i(o)$  to  $o$ .

To each  $W_i$  corresponds a walk  $Q_i$  in  $H(l)$  obtained by replacing an edge of  $G'_l$  by the corresponding path  $P(e)$  and the edge  $(a_{i,l}, a_{i,l+1})$  by the output  $o_i$ .

If  $q_i \in O_3$  then choose  $Q_i$  which joins a priority to the priority output  $q_i$ .

If  $q_i \notin O_3$ , let  $s_i = s_e(q_i)$  be the switch that is in both  $Q_i$  and the default path of  $q_i$ . Let  $D_i$  (resp.  $F_i$ ) be the subpath of  $Q_i$  starting in  $a_{j_i,0}$  (resp. at  $s_i$ ) and ending at  $s_i$  (resp. in  $O_3$ ). Then choose  $(p_{j_i}, D_i, s_{e,f}(q_i), q_i)$  which joins a priority to a priority output, and  $(i_{q_i}, F_i)$  which joins an ordinary input to a non-priority output. All the paths we have constructed are edge-disjoint. ■

**Proof of Lemma 52.** By Lemma 54, it suffices to prove that for any set  $E_3$  of three edges of  $G'_l$  with two non-adjacent edges, there is an  $E_3$ -good set of walks.

For  $1 \leq i \leq 3$  and  $1 \leq j \leq k \leq l+1$ , we will denote by  $L_i[j, k]$  the path  $(a_{i,j}, a_{i,j+1}, \dots, a_{i,k})$ .

By symmetry of  $G'_l$ , it suffices to prove it for the following ten cases. All these cases and the corresponding covering walks are depicted in Figure 10. The walk  $W_1$  (resp.  $W_2, W_3$ ) is drawn as a dotted (resp. black, grey) line. The white (resp. black, grey) ellipses represent the possible positions of  $e_1$  (resp.  $e_2, e_3$ ).

- (1)  $e_1 = (a_{1,i_1}, a_{1,i_1+1})$ ,  $e_2 = (a_{1,i_2}, a_{1,i_2+1})$  and  $e_3 = (a_{1,i_3}, a_{1,i_3+1})$  with  $0 \leq i_1 < i_2 < i_3 \leq l$ . The covering walks are  $W_1 = (L_1[1, i_2], a_{3,i_2}, L_2[i_2, l+1])$ ,  $W_2 = (L_2[1, i_2], L_1[i_2, i_3], L_3[i_3, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{2,i_3}, L_1[i_3, l+1])$ .
- (2)  $e_1 = (a_{1,i_1}, a_{1,i_1+1})$ ,  $e_2 = (a_{1,i_2}, a_{1,i_2+1})$  and  $e_3 = (a_{3,i_3}, a_{3,i_3+1})$  with  $0 \leq i_1 < i_2 \leq l$ . The covering walks are  $W_1 = (L_1[1, i_2], L_2[i_2, l+1])$ ,  $W_2 = (L_2[1, i_2], a_{3,i_2}, L_1[i_2, l+1])$ , and  $W_3 = (L_3[1, l+1])$ .

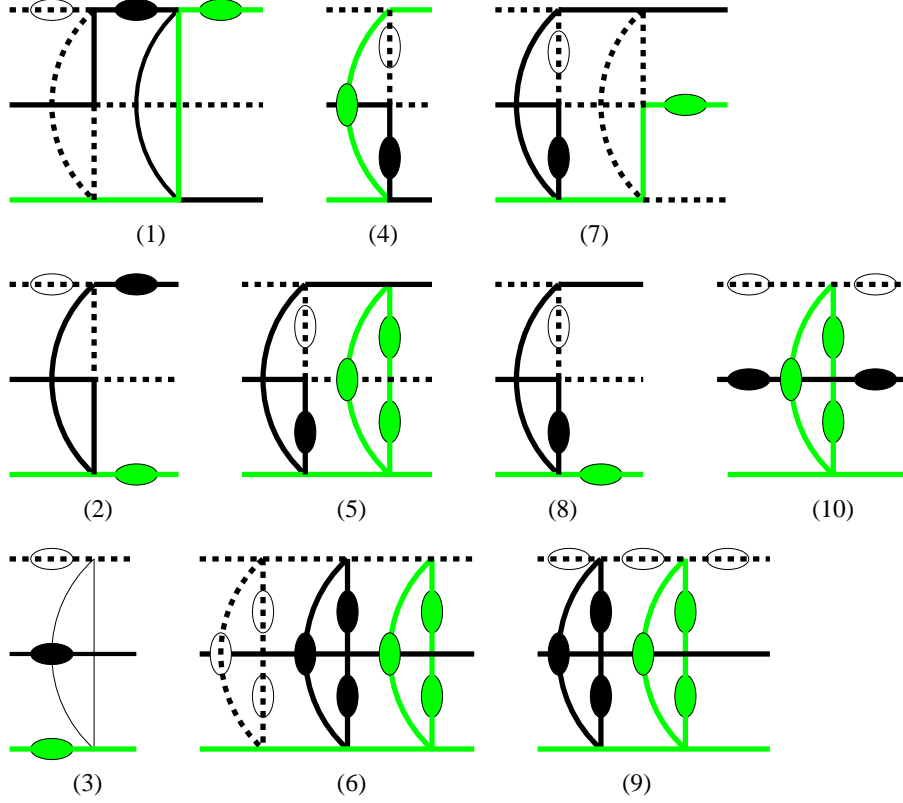


Figure 10: The covering walks in each case.

- (3)  $e_1 = (a_{1,i_1}, a_{1,i_1+1})$ ,  $e_2 = (a_{2,i_2}, a_{2,i_2+1})$  and  $e_3 = (a_{3,i_3}, a_{3,i_3+1})$ . The covering walks are  $W_1 = (L_1[1, l+1])$ ,  $W_2 = (L_2[1, l+1])$ , and  $W_3 = (L_3[1, l+1])$ .
- (4)  $e_1 = (a_{1,i}, a_{2,i})$ ,  $e_2 = (a_{2,i}, a_{3,i})$  and  $e_3 = (a_{3,i}, a_{1,i})$ .  $W_1 = (L_1[1, i], L_2[i, l+1])$ ,  $W_2 = (L_2[1, i], L_3[i, l+1])$ , and  $W_3 = (L_3[1, i], L_1[i, l+1])$ .
- (5)  $e_1 = (a_{1,i_1}, a_{2,i_1})$ ,  $e_2 = (a_{2,i_1}, a_{3,i_1})$  and  $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$  with  $i_3 \neq i_1$ .  $W_1 = (L_1[1, i_1], L_2[i_1, l+1])$ ,  $W_2 = (L_2[1, i_1], a_{3,i_1}, L_1[i_1, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$ .
- (6)  $e_1 \in \{(a_{1,i_1}, a_{2,i_1}), (a_{2,i_1}, a_{3,i_1}), (a_{3,i_1}, a_{1,i_1})\}$ ,  $e_2 \in \{(a_{1,i_2}, a_{2,i_2}), (a_{2,i_2}, a_{3,i_2}), (a_{3,i_2}, a_{1,i_2})\}$  and  $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$  with  $i_1 < i_2 < i_3$ .  $W_1 = (L_1[1, i_1], a_{2,i_1}, a_{3,i_1}, L_1[i_1, l+1])$ ,  $W_2 = (L_2[1, i_2], a_{3,i_2}, a_{1,i_2}, L_2[i_2, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$ .
- (7)  $e_1 = (a_{1,i_1}, a_{2,i_1})$ ,  $e_2 = (a_{2,i_1}, a_{3,i_1})$  and  $e_3 = (a_{2,i_3}, a_{2,i_3+1})$  with  $i_1 < i_3$ .  $W_1 = (L_1[1, i_1], L_2[i_1, i_3], a_{1,i_3}, L_3[i_3, l+1])$ ,  $W_2 = (L_2[1, i_1], a_{3,i_1}, L_1[i_1, l+1])$ , and  $W_3 = (L_3[1, i_3], L_2[i_3, l+1])$ .



- (8)  $e_1 = (a_{1,i_1}, a_{2,i_1})$ ,  $e_2 = (a_{2,i_1}, a_{3,i_1})$  and  $e_3 = (a_{3,i_3}, a_{3,i_3+1})$ .  $W_1 = (L_1[1, i_1], L_2[i_1, l+1])$ ,  $W_2 = (L_2[1, i_1], a_{3,i_1}, L_1[i_1, l+1])$ , and  $W_3 = (L_3[1, l+1])$ .
- (9)  $e_1 = (a_{1,i_1}, a_{1,i_1+1})$ ,  $e_2 \in \{(a_{1,i_2}, a_{2,i_2}), (a_{2,i_2}, a_{3,i_2}), (a_{3,i_2}, a_{1,i_2})\}$  and  $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$  with  $i_2 < i_3$ .  $W_1 = (L_1[1, l+1])$ ,  $W_2 = (L_2[1, i_2], a_{3,i_2}, a_{1,i_2}, L_2[i_2, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$ .
- (10)  $e_1 = (a_{1,i_1}, a_{1,i_1+1})$ ,  $e_2 = (a_{2,i_2}, a_{2,i_2+1})$  and  $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$ .  $W_1 = (L_1[1, l+1])$ ,  $W_2 = (L_2[1, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$ . ■

**Proof of Theorem 51** By Theorem 29,  $R(n, 3, 0) \geq \frac{3n}{2} + \frac{\sqrt{n}}{2} - 5$ .

Let us now define the  $(n, 3, 0)$ -network  $G_3(n)$  inductively as follows:

For  $l \geq 2$ , let  $n_l = 36l^2 - 78l + 51$ . If  $n = n_l$  for some integer  $l$ , then  $G_3(n) = H_l$ . Otherwise let  $n_{l-1} < n < n_l$ . If  $n$  is odd then  $G_3(n)$  is obtained from  $G_3(n+2)$  by removing a switch  $s_{e,f}$  and its two adjacent outputs, and deleting the two switches  $s(e) \in P(e)$  and  $s(f) \in P(f)$  adjacent to  $s_{e,f}$  and their adjacent inputs, and adding an edge joining the two neighbours of  $s(e)$  (resp.  $s(f)$ ) in the input path  $P(e)$  (resp.  $P(f)$ ).

If  $n$  is even then  $G_3(n)$  is obtained from  $G_3(n+1)$  by an L-contraction of a linking edge  $\{s(e), s_{e,f}\}$ .

Since  $H_l$  is a repartitor, then if  $n$  is odd  $G_3(n)$  is obviously an  $(n, 3, 0)$ -repartitor and if  $n$  is even, by Lemma 44,  $G_3(n)$  is an  $(n, 3, 0)$ -repartitor.  $G_3(n)$  has  $N = N_l - \frac{3}{2}(n - n_l) + \varepsilon/2 = \frac{3n}{2} + \frac{\sqrt{n_l}}{2} - 5/4 + \varepsilon/2$  switches where  $\varepsilon = 1$  if  $n$  is even. But as  $n > n_{l-1}$ ,  $\sqrt{n_l} - \sqrt{n} < \sqrt{n_l} - \sqrt{n_{l-1}} \leq 6$ . So  $N \leq \frac{3n}{2} + \frac{\sqrt{n}}{2} + \frac{9}{4}$ . ■

### Theorem 55

$$R(n, 2, 1) = \frac{3n}{2} + \frac{\sqrt{n+1}}{2} + c,$$

where  $-3 \leq c \leq \frac{23}{4}$ .

**Proof.** From Lemma 42, we obtain  $R(n, 2, 1) \leq R(n+1, 3, 0) + R(2, 2, 1)$ . By Theorem 51,  $R(n+1, 3, 0) \leq \frac{3n}{2} + \frac{\sqrt{n+1}}{2} + \frac{15}{4}$ . And by Proposition 3,  $R(2, 2, 1) = R(2, 0, 1)$  and  $R(2, 0, 1) = 2$  (see [4]). Hence  $R(n, 2, 1) \leq \frac{3n}{2} + \frac{\sqrt{n+1}}{2} + \frac{23}{4}$ . Theorem 29 gives the other inequality. ■

## 4.6 $(n, 4, 0)$ -networks

The aim of this subsection is to prove the following theorem :

### Theorem 56

$$R(n, 4, 0) = \frac{7n}{4} + c$$

where  $-7 \leq c \leq \frac{89}{4}$ .

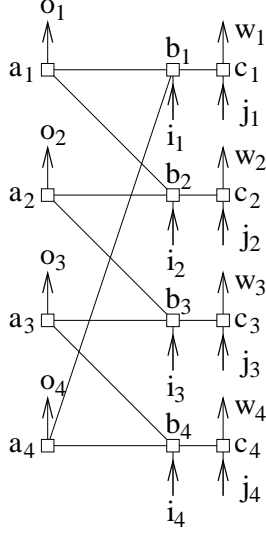


Figure 11: A brick.

We will construct a network  $H(m)$ , which is a  $(16m + 4, 4, 0)$ -repartitor.

We start with a basis brick  $B$  (see Figure 11). It contains 12 switches  $a_l, b_l, c_l$  for  $1 \leq l \leq 4$ . Switch  $a_l$  is connected to  $b_l$  and  $b_{l+1}$  (the indices being taken modulo 4), and  $b_l$  is connected to  $c_l$ . There are 8 inputs: for  $1 \leq l \leq 4$ ,  $i_l$  is connected to  $b_l$ , and  $j_l$  connected to  $c_l$ . There are 8 outputs: for  $1 \leq l \leq 4$ ,  $o_l$  connected to  $a_l$  and  $w_l$  connected to  $c_l$ .

Let  $G(m)$  be the network obtained by concatenating  $m$  bricks  $B^k$ ,  $1 \leq k \leq m$ , that is, by joining  $c_l^k$  to  $a_l^{k+1}$  for  $1 \leq k \leq m - 1$  and  $1 \leq l \leq 4$ .

Let  $F(m)$  be the network obtained by taking two copies of  $G(m)$ ,  $G(m)$  and  $\overline{G}_m$ , deleting the  $8m$  outputs  $w_l^k$  and  $\overline{w}_l^k$ , adding  $4m$  switches  $s_l^k$  connected to  $c_l^k, \overline{c}_l^k$  and two new outputs  $r_l^k$  and  $\overline{r}_l^k$ .

Finally let  $IS$  be an *input selector* with 4 inputs  $p_l$ ,  $1 \leq l \leq 4$ , and 8 outputs  $\alpha_l$  and  $\overline{\alpha}_l$ ,  $1 \leq l \leq 4$ , such that for any set of four outputs, there are four edge-disjoint paths joining the inputs to these outputs.

Let  $OS$  be an *output selector* with 8 inputs  $\gamma_l$  and  $\overline{\gamma}_l$ ,  $1 \leq l \leq 4$ , and 4 outputs  $t_l$ ,  $1 \leq l \leq 4$ , such that for any set of four inputs of the form  $C_1 \cup C_2$  and any partition  $T_1 \cup T_2$  of  $T = \{t_1, t_2, t_3, t_4\}$  with  $|C_1| = |T_1|$ , there are four edge-disjoint paths,  $|C_1|$  of them joining  $C_1$  to  $T_1$  and  $|C_2|$  of them joining  $C_2$  to  $T_2$ .

Then  $H(m)$  is obtained by concatenating  $IS$ ,  $F(m)$  and  $OS$ , that is, by identifying  $\alpha_l$  (resp.  $\overline{\alpha}_l$ ) with  $a_l^1$  (resp.  $\overline{a}_l^1$ ) and  $c_l^m$  (resp.  $\overline{c}_l^m$ ) with  $\gamma_l$  (resp.  $\overline{\gamma}_l$ ). The network  $H(m)$  is depicted in Figure 12.

$H(m)$  has 4 priorities  $p_l$ ,  $1 \leq l \leq 4$ ,  $16m$  ordinary inputs  $i_l^k, \overline{i}_l^k, j_l^k, \overline{j}_l^k$  for  $1 \leq k \leq m$  and  $1 \leq l \leq 4$  and  $n = 16m + 4$  outputs,  $o_l^k, \overline{o}_l^k, r_l^k, \overline{r}_l^k$  for  $1 \leq k \leq m$  and  $1 \leq l \leq 4$  and  $t_l$ ,  $1 \leq l \leq 4$ . It has  $28m + N(IS) + N(OS)$  switches where  $N(IS)$  (resp.  $N(OS)$ ) is the minimum number of switches of an input (resp. output) selector. An input selector is nothing more than a  $(4, 0, 4)$ -repartitor and it is proved in [6] that  $R(4, 0, 4) = 5$ , so  $N(IS) = 5$ . An optimum

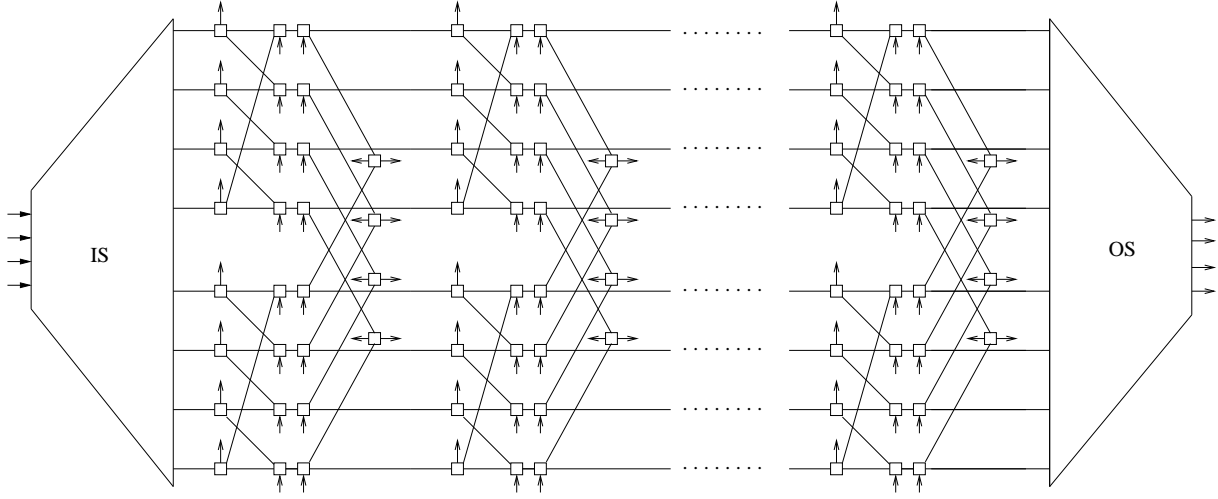


Figure 12: The network  $H(m)$ .

input selector is drawn in Figure 13 left. The network depicted in Figure 13 right is an output selector. Indeed, it is constructed from a Waksman network (see [2, 3]), which is able to realize any permutation of the 8 inputs into 8 outputs, by removing the useless outputs, switches and links (dotted on the figure). Hence  $N(OS) \leq 13$ .

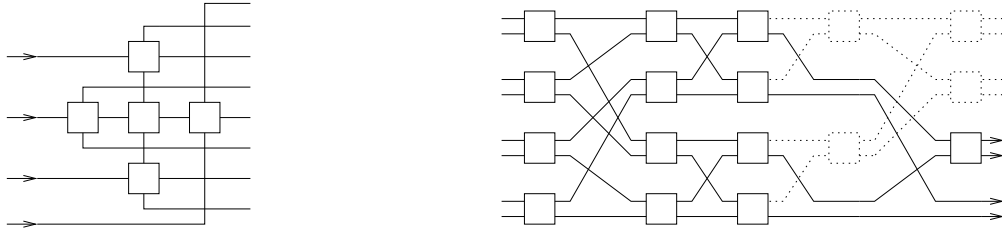


Figure 13: An optimum input selector and an output selector.

**Lemma 57**  $H(m)$  is a  $(16m + 4, 4, 0)$ -repartitor.

**Proof.** In what follows we will denote  $O = \bigcup_{k,l} o_l^k$ ,  $\bar{O} = \bigcup_{k,l} \bar{o}_l^k$ ,  $R = \bigcup_{k,l} r_l^k$ ,  $\bar{R} = \bigcup_{k,l} \bar{r}_l^k$ ,  $W = \bigcup_{k,l} w_l^k$ ,  $\bar{W} = \bigcup_{k,l} \bar{w}_l^k$ .

Let  $Q$  be any set of four outputs of  $H(m)$ . Let  $Q_1 = Q \cap (O \cup \bar{O} \cup R \cup \bar{R})$  and  $Q_2 = Q \cap T$  and  $q_i = |Q_i|$ ,  $1 \leq i \leq 2$ .

To prove that  $H(m)$  is a repartitor it suffices to find in  $F(m)$  for any set  $Q = Q_1 \cup Q_2$  a set of  $16m + 4$  edge-disjoint paths such that:

$q_1$  of them join a set  $A_1$  of  $q_1$  elements of  $\bigcup_l \{a_l^1, \bar{a}_l^1\}$  to  $Q_1$ ,

$q_2$  of them join a set  $A_2$  of  $q_2$  elements of  $\bigcup_l \{a_l^1, \bar{a}_l^1\} \setminus A_1$  to a set  $C_2$  of  $q_2$  elements of  $\bigcup_l \{c_l^m, \bar{c}_l^m\}$ ,

and the  $16m$  other paths join the ordinary inputs to the set of outputs  $(O \cup \bar{O} \cup R \cup \bar{R}) \setminus Q_1$  and a set  $C_1$  of  $q_1$  elements of  $\bigcup_l \{c_l^m, \bar{c}_l^m\} \setminus C_2$ .

Indeed  $IS$  being an input selector, we can connect the 4 priorities of  $H(m)$  to the 4 outputs assigned to  $A_1 \cup A_2$  with edge-disjoint paths; then one can link  $A_1$  to  $Q_1$  and  $A_2$  to  $C_2$ , and the ordinary inputs either to outputs of  $(O \cup \bar{O} \cup R \cup \bar{R}) \setminus Q_1$  or vertices of  $C_1$ ; at last,  $OS$  being an output selector, we can find edge-disjoint paths from  $C_2$  to  $Q_2$ , and from  $C_1$  to  $T \setminus Q_2$ . Hence we have the  $16m + 4$  required edge-disjoint paths of  $H(m)$ , 4 of them joining the  $p_j$  to  $Q$  and the remaining  $16m$  joining the ordinary inputs to the outputs not in  $Q$ .

In fact, we can reduce the problem of finding the required set of paths in  $F(m)$  to that of finding in  $G(m)$  a  $Q_0$ -good path set defined as follows :

Let  $Q_0$  be any set of at most 4 outputs of  $G(m)$ ,  $Q_0 \subseteq O \cup W$  and  $|Q_0| = q_0 \leq 4$ . A  $Q_0$ -good path set consists of  $8m + 4$  edge-disjoint paths such that

- 4 of them link the vertices of  $\bigcup_{1 \leq l \leq 4} a_l^1$  to  $Q_0$  and a set  $D_1$  of  $4 - q_0$  vertices of  $\bigcup_{1 \leq l \leq 4} c_l^m$ , and
- the remaining  $8m$  link the  $8m$  ordinary inputs of  $G(m)$  to  $(O \cup W) \setminus Q_0$  and  $D_2 = \bigcup_{1 \leq l \leq 4} c_l^m \setminus D_1$ .

Let us define an *assignment* as a one-to-one mapping from  $R \cup \bar{R}$  to  $W \cup \bar{W}$  by associating to  $r_l^k$  one of  $w_l^k$  and  $\bar{w}_l^k$ , and to  $\bar{r}_l^k$  the other one.

Let  $Q_r = Q \cap (R \cup \bar{R})$ . For any assignment  $\phi$ , let  $Q_\phi = (Q \cap O) \cup (\phi(Q_r) \cap W)$  and  $\bar{Q}_\phi = (Q \cap \bar{O}) \cup (\phi(Q_r) \cap \bar{W})$ .

If  $G(m)$  contains a  $Q_\phi$ -good path set and  $\bar{G}(m)$  a  $\bar{Q}_\phi$ -good path set, then we obtain the required set of edge-disjoint paths of  $F(m)$  by taking the paths in  $G(m)$  and  $\bar{G}(m)$  and replacing for  $1 \leq k \leq m$  and  $1 \leq l \leq 4$ , the last edge  $(c_l^k, w_l^k)$  (resp.  $(\bar{c}_l^k, \bar{w}_l^k)$ ) of the path ending in  $w_l^k$  (resp.  $\bar{w}_l^k$ ) by the subpath  $(c_l^k, s_l^k, \phi^{-1}(w_l^k))$  (resp.  $(\bar{c}_l^k, s_l^k, \phi^{-1}(\bar{w}_l^k))$ ). Note that we actually get more paths than required; some of them joining some  $a_l^1$  or  $\bar{a}_l^1$  to some  $c_l^m$  or  $\bar{c}_l^m$  are not needed.

Finally note that we have many choices of possible assignments and we can always choose an assignment  $\phi$  such that  $Q_\phi$  (resp.  $\bar{Q}_\phi$ ) contains:

- either 2 outputs of  $W$  (resp.  $\bar{W}$ ) and none of  $O$  (resp.  $\bar{O}$ ),
- or 1 output of  $W$  (resp.  $\bar{W}$ ) and at most 2 of  $O$  (resp.  $\bar{O}$ ),
- or no outputs of  $W$  (resp.  $\bar{W}$ ).

Indeed choose  $\phi$  such that:

- if  $|Q_r| = 4$ , then  $|\phi(Q_r) \cap W| = |\phi(Q_r) \cap \bar{W}| = 2$ ;
- if  $|Q_r| = 3$  and  $|Q \cap O| = 1$ , then  $|\phi(Q_r) \cap W| = 1$  and  $|\phi(Q_r) \cap \bar{W}| = 2$ ;
- if  $|Q_r| = 3$  and  $|Q \cap \bar{O}| = 1$ , then  $|\phi(Q_r) \cap W| = 2$  and  $|\phi(Q_r) \cap \bar{W}| = 1$ ;
- if  $|Q_r| = 2$ , then  $|\phi(Q_r) \cap W| = |\phi(Q_r) \cap \bar{W}| = 1$ ;
- if  $|Q_r| = 1$  and  $|Q \cap O| > |Q \cap \bar{O}|$ , then  $|\phi(Q_r) \cap W| = 0$  and  $|\phi(Q_r) \cap \bar{W}| = 1$ ;

- if  $|Q_r| = 1$  and  $|Q \cap O| < |Q \cap \bar{O}|$ , then  $|\phi(Q_r) \cap W| = 1$  and  $|\phi(Q_r) \cap \bar{W}| = 0$ .

In summary, to prove that  $H(m)$  is a repartitor it suffices to prove the following lemma.

**Lemma 58** *For any set  $Q_0$  of outputs of  $G(m)$  such that either  $|Q_0 \cap W| = 2$  and  $|Q_0 \cap O| = 0$ , or  $|Q_0 \cap W| = 1$  and  $|Q_0 \cap O| \leq 2$ , or  $|Q_0 \cap W| = 0$ , there exists a  $Q_0$ -good path set in  $G(m)$ .*

**Proof.** We call a *default path* for  $w_l^k$ , (resp.  $o_l^k$ ) the path  $(j_l^k, c_l^k, w_l^k)$  (resp.  $(i_{l+1}^k, b_{l+1}^k, a_l^k, o_l^k)$ ). Let  $L_l$  be the path  $(a_l^1, b_l^1, c_l^1, a_l^2, b_l^2, c_l^2, \dots, a_l^m, b_l^m, c_l^m)$  and let us denote by  $L_l[x, y]$  the subpath of  $L_l$  with end vertices  $x$  and  $y$ . Let  $O_l = \{o_l^k | 1 \leq k \leq m\}$  and  $W_l = \{w_l^k | 1 \leq k \leq m\}$ .

By symmetry of  $G(m)$ , we only need to prove the lemma for the following cases:

- (i)  $|Q_0 \cap (O_l \cup W_l)| \leq 1$  for  $1 \leq l \leq 4$ ;
- (ii)  $Q_0 \cap (O \cup W) = \{w_1^{k_1}, w_1^{k_2}\}$  with  $k_1 < k_2$ ;
- (iii)  $Q_0 \cap W = \{w_1^{k_1}\}$  and  $Q_0 \cap O = \{o_1^{k_2}\}$ ;
- (iv)  $Q_0 \cap W = \{w_1^{k_0}\}$  and  $Q_0 \cap O = \{o_{l_1}^{k_1}, o_{l_2}^{k_2}\}$  with  $l_1 = 1$  or  $l_1 = l_2$ ;
- (v)  $Q_0 \cap (W \cup O) = \{o_1^{k_1}, o_1^{k_2}, o_{l_3}^{k_3}, o_{l_4}^{k_4}\}$ .

We only give here the proof in cases (i), (ii), (iii) and (v) with  $l_3 = l_4 = 1$ , the other cases being very similar.

- (i) - if  $|Q_0 \cap (O_l \cup W_l)| = 0$ , take the path  $L_l$ ;
- if  $Q_0 \cap (O_l \cup W_l) = w_l^{k_0}$ , take the paths  $(L_l[a_l^1, c_l^{k_0}], w_l^{k_0})$  and  $(j_l^{k_0}, L[c_l^{k_0}, c_l^m])$ .
- if  $Q_0 \cap (O_l \cup W_l) = o_l^{k_0}$ , take the paths  $(L_l[a_l^1, a_l^{k_0}], o_l^{k_0})$  and  $(i_{l+1}^{k_0}, b_{l+1}^{k_0}, L[a_l^{k_0}, c_l^m])$ .

These paths plus the default paths associated with the outputs not in  $Q_0$  form a  $Q_0$ -good path set.

- (ii) Consider the paths  $(L_1[a_1^1, c_1^{k_1}], w_1^{k_1})$ ,  $(j_1^{k_1}, L_1[c_1^{k_1}, a_1^{k_2}], o_1^{k_2})$ ,  $(L_2[a_2^1, b_2^{k_2}], L_1[a_1^{k_2}, c_1^{k_2}], w_1^{k_2})$ ,  $(i_2^{k_2}, L_2[b_2^{k_2}, c_2^m])$ ,  $(j_1^{k_2}, L_1[c_1^{k_2}, c_1^m])$ ,  $L_3$  and  $L_4$ . These paths plus the default paths associated with the outputs not in  $Q_0 \cup \{o_1^{k_2}\}$  form a  $Q_0$ -good path set.
- (iii) Consider the paths  $(L_1[a_1^1, c_1^{k_1}], w_1^{k_1})$ ,  $(j_1^{k_1}, L_1[c_1^{k_1}, c_1^m])$ ,  $(L_2[a_2^1, b_2^{k_2}], a_1^{k_2}, o_1^{k_2})$ ,  $(i_2^{k_2}, L_2[b_2^{k_2}, c_2^m])$ ,  $L_3$  and  $L_4$ . These paths plus the default paths associated with the outputs not in  $Q_0$  form a  $Q_0$ -good path set.
- (v) Suppose that  $l_3 = l_4 = 1$ . Without loss of generality, we may assume that  $k_1 < k_2 < k_3 < k_4$ . Consider the paths  $(L_1[a_1^1, a_1^{k_1}], o_1^{k_1})$ ,  $(L_2[a_2^1, b_2^{k_2}], a_1^{k_2}, o_1^{k_2})$ ,  $(L_3[a_3^1, b_3^{k_3}], a_2^{k_3}, b_2^{k_3}, a_1^{k_3}, o_1^{k_3})$ ,  $(L_4[a_4^1, b_4^{k_4}], a_3^{k_4}, b_3^{k_4}, a_2^{k_4}, b_2^{k_4}, a_1^{k_4}, o_1^{k_4})$ ,  $(i_2^{k_1}, b_2^{k_1}, L_1[a_1^{k_1}, c_1^m])$ ,  $(i_2^{k_2}, L_2[b_2^{k_2}, a_2^{k_3}], o_2^{k_3})$ ,  $(i_2^{k_3}, L_2[b_2^{k_3}, a_2^{k_4}], o_2^{k_4})$ ,  $(i_3^{k_3}, L_3[b_3^{k_3}, a_3^{k_4}], o_3^{k_4})$ ,  $(i_2^{k_4}, L_2[b_2^{k_4}, c_2^m])$ ,  $(i_3^{k_4}, L_3[b_3^{k_4}, c_3^m])$ , and  $(i_4^{k_4}, L_4[b_4^{k_4}, c_4^m])$ . See Figure 14.

These paths plus the default paths associated with the outputs not in  $Q_0 \cup \{o_2^{k_3}, o_2^{k_4}, o_3^{k_4}\}$  form a  $Q_0$ -good path set.

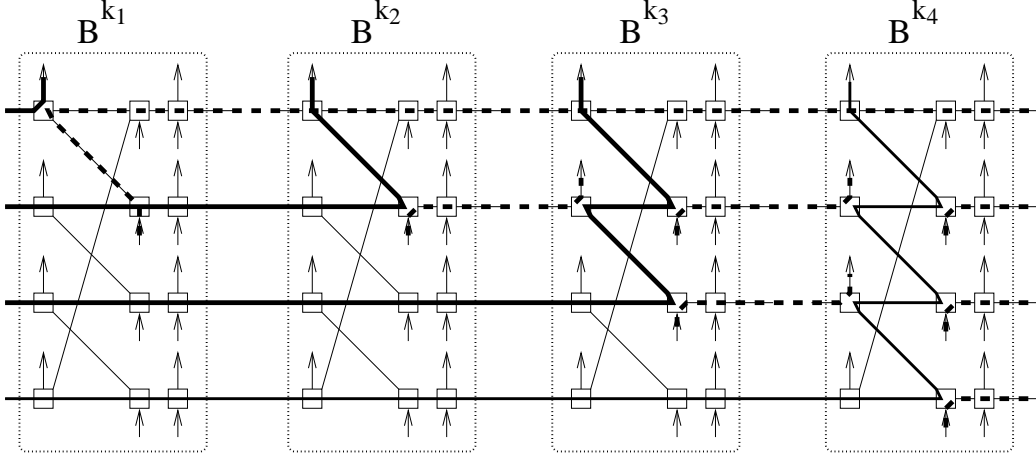


Figure 14: The non-default paths when  $Q_0 \cap (W \cup O) = \{o_1^{k_1}, o_1^{k_2}, o_1^{k_3}, o_1^{k_4}\}$ .

**Proof of Theorem 56.** By Theorem 33,  $R(n, 4, 0) \geq \frac{7n}{4} - 7$ . For  $n = 16m + 4$ , we have constructed an  $(n, 4, 0)$ -repartitor  $H(m)$  with  $28m + 18 = \frac{7n}{4} + 11$  switches. If  $n$  is not of the form  $16m + 4$ , let  $n = 16m + 4 - h$  with  $1 \leq h \leq 15$ . Let  $K(n)$  be the network obtained from  $H(m)$  by the L-contraction of  $h$  linking edges. That is always possible as the edges  $(b_l^k, a_{l+1}^k)$  or  $(c_l^k, s_l^k)$  are linking. By Lemma 44,  $K(n)$  is an  $(n, 4, 0)$ -repartitor. And  $K(n)$  has  $28m + 18 - h = \frac{7n}{4} + 11 + \frac{3h}{4} \leq \frac{7n}{4} + \frac{89}{4}$  switches. ■

**Theorem 59**

$$R(n, 3, 1) = \frac{7n}{4} + c,$$

where  $-\frac{19}{4} \leq c \leq 27$ .

**Proof.** From Lemma 42, we obtain  $R(n, 3, 1) \leq R(n + 1, 4, 0) + R(3, 3, 1)$ . By Theorem 56,  $R(n + 1, 4, 0) \leq \frac{7n}{4} + 24$ . And  $R(3, 3, 1) = R(3, 0, 1) = 3$ . Hence  $R(n, 3, 1) \leq \frac{7n}{4} + 27$ . Theorem 33 gives the other inequality. ■

## 5 Conclusion

In this paper, we studied the minimum size  $R(n, p, f)$  of an  $(n, p, f)$ -repartitor with  $n$  inputs,  $p$  of which are priorities, and  $n + f$  outputs. We have obtained general upper bounds, lower bounds, and tight bounds for small values of  $p$  and  $f$ . It would be interesting to see if our lower bound for  $p = 6$  and  $f = 0$  is attained and to study the cases of  $p = 1, 2, 3, 4$  and arbitrary  $f$ .

It would also be interesting to further explore the behaviour of the function  $R(n, p, f)$ . For example, we expect that  $R(n, p, f)$  is monotone in  $p$ .

**Conjecture 60** *If  $p \leq p' \leq \frac{n}{2}$  then  $R(n, p, f) \leq R(n, p', f)$ .*

Our results confirm that this holds for small values of  $n$ ,  $p$  and  $f$ . In view of this conjecture, the case  $p = n/2$  seems particularly interesting. Very recently, Havet [8] discovered an asymptotically better upper bound than our Theorem 8: He proved that  $R(n, p, f) \leq \frac{69}{2}n + \frac{35}{2}f - 33p + O(\log(n + f))$  if  $p \leq \frac{n-f}{2}$ ,  $R(n, p, f) \leq 18n + 34f + O(\log(n + f))$ , if  $\frac{n-f}{2} \leq p \leq \frac{n+f}{2}$ , and  $R(n, p, f) \leq \frac{3}{2}n + \frac{35}{2}f + 33p + O(\log(n + f))$  if  $p \geq \frac{n+f}{2}$ .

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