On the Degree of Standard Geometric Predicates for Line Transversals in 3D

Hazel Everett, Sylvain Lazard, Bill Lenhart, Linqiao Zhang

On the Degree of Standard Geometric Predicates for Line Transversals in 3D

Hazel Everett* Sylvain Lazard* William Lenhart† Linqiao Zhang‡

November 8, 2007

Abstract

In this paper we study various geometric predicates for determining the existence of and categorizing the configurations of lines in 3D that are transversal to lines or segments. We compute the degrees of standard procedures of evaluating these predicates. The degrees of some of these procedures are surprisingly high (up to 168), which may explain why computing line transversals with finite-precision floating-point arithmetic is prone to error. Our results suggest the need to explore alternatives to the standard methods of computing these quantities.

1 Introduction

Computing line transversals to lines or segments is an important operation in solving 3D visibility problems arising in computer graphics [2, 7, 8, 9, 10, 14]. In this paper, we study various predicates and their degrees concerning line transversals to lines and segments in 3D.

A predicate is a function that returns a value from a discrete set. Typically, geometric predicates answer questions of the type "Is a point inside, outside or on the boundary of a set?". We consider predicates that are evaluated by boolean functions of more elementary predicates, the latter being functions that return the sign ($-$, 0 or $+$) of a multivariate polynomial whose arguments are a subset of the input parameters of the problem instance (see, for instance [1]). By *degree* of a procedure for evaluating a predicate, we mean the maximum degree in the input parameters among all polynomials used in the evaluation of the predicate by the procedure. In what follows we casually refer to this measure as the degree of the predicate. We are interested in the degree because it provides a measure of the number of bits required for an exact evaluation of our predicates when the input parameters are integers or floating-point numbers; the number of bits required is then roughly the product of the degree with the number of bits used in representing each input value.

In this paper, we first study the degree of standard procedures for determining the number of line transversals to four lines or four segments in 3D; recall that four lines in $\mathbb{R}^3$ admit 0, 1, 2 or an infinite number of line transversals and that four segments admit up to 4 or an infinite number of line transversals [3]. We also consider the predicate for determining whether a minimal (i.e., locally shortest) segment transversal to four line segments is intersected by a triangle. These predicates are ubiquitous in 3D visibility problems. The latter predicate, for instance, can be used for determining whether two triangles see each other in a scene of triangles (that is, for determining whether there exists a segment joining the two triangles and that does not properly intersect any of the other triangles). Finally, we study the predicate for ordering planes through two fixed points, each plane containing a third rational point or a line transversal to four segments or lines. This predicate arises in the rotating plane-sweep algorithm that computes the minimal free segments tangent to four among k convex polyhedra in 3D [2].

Our study shows that standard procedures for solving these predicates have high degrees. We study, in particular, procedures that involve computing the Plücker coordinates of the line transversals involved in the predicates.

*LORIA (INRIA, CNRS, Nancy Université), Campus Scientifique, 54506 Nancy, France. Firstname.Name@loria.fr

†Computer Science Department, 47 Lab Campus Drive, Williams College, Williamstown, MA 01267, USA. lenhart@cs.williams.edu

‡School of Computer Science, McGill University, 3480 University Street, Montreal, Quebec H3A 2A7, Canada. lzhang15@cs.mcgill.ca.
This research was carried out while this author was visiting LORIA and partially funded by the INRIA associated team ISA-McGill.

Throughout the paper, the points defining input geometric primitives (which can be lines, segments, and triangles) are, by assumption, given by their Cartesian coordinates and the degrees of the procedures for evaluating predicates are expressed in these coordinates. We show that, for determining the number of transversals to four lines or four segments, such standard methods lead to procedures of degree 22 or 36, respectively. For determining whether a minimal segment transversal to four line segments is intersected by a triangle, we show that these methods lead to a procedure of degree 78. Also, for ordering, in a rotational sweep about a line, two planes, each defined by a line transversal to four lines, such methods lead to a procedure of degree 144. Furthermore, in some implementations, the Plücker coordinates of the relevant line transversals are computed in a way that the degrees of these procedures are even higher; for instance, the procedure for evaluating the latter predicate for ordering planes then become of degree 168 instead of 144. These very high degrees may help explain why using fixed-precision floating-point arithmetic in implementations for solving 3D visibility problems are prone to errors when given real-world data (see, for instance, [11]).

The degrees we present are tight, that is, they correspond to the maximum degree of the polynomials to be evaluated, in the worst case, in the procedures we consider. It should be stressed that these degrees refer to polynomials used in specific evaluation procedures and we make no claim on the optimality of these procedures.

In the next section we describe a standard method used for computing the line transversals to four lines, which is common to all our predicates. In Section 3 we describe the predicates and their degrees. Some experimental results are presented in Section 4.

2 Computing lines through four lines

We describe here a method for computing the line transversals to four lines in real projective space $\mathbb{P}^3(\mathbb{R})$. This method is a variant, suggested by Devillers and Hall-Holt [6] and also described in Redburn [15], of that by Hohmeyer and Teller [12]; note that, for evaluating predicates, the latter method is not appropriate because it uses singular value decomposition for which we only know of numerical methods and thus the line transversals cannot be computed exactly, when needed.

Each line can be described using Plücker coordinates (see [17], for example, for a review of Plücker coordinates). If a line ℓ in $\mathbb{R}^3$ is represented by a direction vector $\vec{u}$ and a point p in $\mathbb{R}^3$ then ℓ can be represented by the six-tuple $(\vec{u}, \vec{u} \times \overrightarrow{Op})$ in real projective space $\mathbb{P}^5(\mathbb{R})$, where O is any arbitrarily, fixed, origin and $\times$ denote the cross product. The side product $\odot$ of any two six-tuples $\ell = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $k = (x_1, x_2, x_3, x_4, x_5, x_6)$ is $\ell \odot k = a_4x_1 + a_5x_2 + a_6x_3 + a_1x_4 + a_2x_5 + a_3x_6$. The fundamental importance of the side product lies in the fact that a six-tuple $k \in \mathbb{P}^5(\mathbb{R})$ represents a line in 3D if and only if $k \odot k = 0$; this defines a quadric in $\mathbb{P}^5(\mathbb{R})$ called the Plücker quadric. More generally, recall that two lines intersect in real projective space $\mathbb{P}^3(\mathbb{R})$ if and only if the side product of their Plücker coordinates is zero. Notice that this implies that there is a predicate for determining whether two lines intersect in $\mathbb{P}^3(\mathbb{R})$ which is of degree two in the Plücker coordinates of the lines and, if the lines are each defined by two points, of degree three in the Cartesian coordinates of these points.

Oriented lines of $\mathbb{R}^3$, with direction vector $\vec{u}$ and through a point p, can be represented similarly by a six-tuple $(\vec{u}, \vec{u} \times \overrightarrow{Op})$ in real oriented projective space (*i.e.*, the quotient of $\mathbb{R}^6 \setminus \{0\}$ by the equivalence relation induced by positive scaling). The sign (positive or negative) of the side operator of two oriented lines ℓ and k then determines on which "side" of ℓ, k lies; for instance, if op and oq are two lines oriented from o to p and from o to q and ℓ is an arbitrarily oriented line such that ℓ, p, q, and o are not coplanar, then $(\ell \odot op)(\ell \odot oq) \leqslant 0$ if and only if ℓ intersects segment pq (see Figure 1(a)).

Given four lines $\ell_1, \ldots, \ell_4$, our problem here is to compute all lines $k = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{P}^5(\mathbb{R})$ such that $k \odot \ell_i = 0$, for $1 \leqslant i \leqslant 4$, which can be written in the following form:

$$\begin{pmatrix} a_4 & a_5 & a_6 & a_1 & a_2 & a_3 \\ b_4 & b_5 & b_6 & b_1 & b_2 & b_3 \\ c_4 & c_5 & c_6 & c_1 & c_2 & c_3 \\ d_4 & d_5 & d_6 & d_1 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

where the rows of the 4×6 matrix contain the Plücker coordinates of the four lines. This can be rewritten as

$$\begin{pmatrix} a_6 & a_1 & a_2 & a_3 \\ b_6 & b_1 & b_2 & b_3 \\ c_6 & c_1 & c_2 & c_3 \\ d_6 & d_1 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} + \begin{pmatrix} a_4x_1 + a_5x_2 \\ b_4x_1 + b_5x_2 \\ c_4x_1 + c_5x_2 \\ d_4x_1 + d_5x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

Let δ denote the determinant of the above 4×4 matrix. Assuming $\delta \neq 0$, we can solve the system for x_3, x_4, x_5, and x_6 in terms of x_1 and x_2. Applying Cramer's rule, we get

$$\left\{ \begin{array}{lcl} x_3 & = & -(\alpha_1 x_1 + \beta_1 x_2)/\delta \\ x_4 & = & -(\alpha_2 x_1 + \beta_2 x_2)/\delta \\ x_5 & = & -(\alpha_3 x_1 + \beta_3 x_2)/\delta \\ x_6 & = & -(\alpha_4 x_1 + \beta_4 x_2)/\delta \end{array} \right.$$

where α_i (respectively β_i) is the determinant δ with the i^{th} column replaced by $(a_4,b_4,c_4,d_4)^T$ (respectively $(a_5,b_5,c_5,d_5)^T$). We rewrite this system as

$$\left\{ \begin{array}{lcl} x_1 & = & -u\delta \\ x_2 & = & -v\delta \\ x_3 & = & \alpha_1 u + \beta_1 v \\ x_4 & = & \alpha_2 u + \beta_2 v \\ x_5 & = & \alpha_3 u + \beta_3 v \\ x_6 & = & \alpha_4 u + \beta_4 v \end{array} \right. \tag{3}$$

with $(u,v) \in \mathbb{P}^1(\mathbb{R})$. Since k is a line, we have $k \odot k = 0$, which implies

$$x_1x_4 + x_2x_5 + x_3x_6 = 0.$$

Substituting in the expressions for $x_1 \ldots x_6$, we get

$$Au^2 + Buv + Cv^2 = 0 \tag{4}$$

where

$$\begin{array}{lcl} A & = & \alpha_1\alpha_4 - \alpha_2\delta, \\ B & = & \alpha_1\beta_4 + \beta_1\alpha_4 - \beta_2\delta - \alpha_3\delta, \\ C & = & \beta_1\beta_4 - \beta_3\delta. \end{array}$$

Solving this degree-two equation in (u,v) and replacing in (3), we get (assuming that $A \neq 0$) that the Plücker coordinates of the transversal lines k are:

$$\left\{ \begin{array}{lcl} x_1 & = & B\delta \mp \delta\sqrt{B^2-4AC} \\ x_2 & = & -2A\delta \\ x_3 & = & -B\alpha_1 + 2A\beta_1 \pm \alpha_1\sqrt{B^2-4AC} \\ x_4 & = & -B\alpha_2 + 2A\beta_2 \pm \alpha_2\sqrt{B^2-4AC} \\ x_5 & = & -B\alpha_3 + 2A\beta_3 \pm \alpha_3\sqrt{B^2-4AC} \\ x_6 & = & -B\alpha_4 + 2A\beta_4 \pm \alpha_4\sqrt{B^2-4AC}. \end{array} \right. \tag{5}$$

Lemma 1. *Consider four lines, given by the Cartesian coordinates of pairs of points, that admit finitely many line transversals in $\mathbb{P}^3(\mathbb{R})$. If the four lines are not parallel to a common plane, the Plücker coordinates of their transversals in $\mathbb{P}^3(\mathbb{R})$ can be written as $\phi_i + \varphi_i\sqrt{\Delta}$, $i = 1,\ldots,6$, where ϕ_i, φ_i, and Δ are polynomials of degree at most* 17, 6, *and* 22, *respectively, in the coordinates of the input points. Otherwise, the Plücker coordinates of the transversals can be written as polynomials of degree at most* 19. *Moreover, these bounds are, in the worst case, reached for three of the coordinates.*

Proof. The assumption that the four lines admit finitely many transversals in $\mathbb{P}^3(\mathbb{R})$ ensures that the 4×6 matrix of Plücker coordinates (in (1)) has rank 4. Consider first the case where the four input lines are not all parallel to a common plane. Then, the 4×3 matrix of the direction vectors of the four lines has rank 3. By the basis extension theorem, this matrix can be complemented by one of the other columns of the matrix of Plücker coordinates (of (1)) in order to get a 4×4 matrix of rank 4. We can thus assume, without loss of generality, that the 4×4 matrix of (2) has rank 4.

Since, by assumption, the four lines admit finitely many transversals in $\mathbb{P}^3(\mathbb{R})$, A, B, and C in (4) are not all zero. We compute the degree, in the coordinates of the input points, of the various polynomial terms in (5). For each input line ℓ_i, the first three and last three coordinates of its Plücker representation have degree 1 and 2, respectively. Hence δ, α_1, and β_1 have degree 5 and α_i and β_i have degree 6 for $i = 2,3,4$. Hence, A, B, and C have degree 11 and the bounds on the degrees of ϕ_i, φ_i, and Δ follow. Note, in particular, that, if $A \neq 0$, these bounds are reached for $i = 4,5,6$.

Consider now the case where the four input lines are parallel to a common plane. Since the four lines admit finitely many transversals in $\mathbb{P}^3(\mathbb{R})$, they are not parallel. It follows that the 4×3 matrix of the direction vectors of the four lines has rank 2. Two vectors, say (a_i, b_i, c_i, d_i) for $i = 1,2$, are thus linearly independent and, by the basis extension theorem, the corresponding 4×2 matrix can be complemented by two other columns (say, (a_i, b_i, c_i, d_i) for $i = 4,5$) of the matrix of Plücker coordinates (of (1)) in order to define a 4×4 matrix of rank 4. As above, a straightforward computation gives the Plücker coordinates of the line transversal. We get

$$x_1 = \alpha_1 u, \quad x_2 = \alpha_2 u, \quad x_3 = -u\delta, \quad x_4 = \alpha_3 u + \beta_3 v, \quad x_5 = \alpha_4 u + \beta_4 v, \quad x_6 = -v\delta$$

where $(u, v) \in \mathbb{P}^1(\mathbb{R})$ is solution of the equation

$$A' u^2 + B' uv = 0 \quad \text{where} \quad A' = \alpha_1\alpha_3 + \alpha_2\alpha_4 \quad \text{and} \quad B' = \alpha_1\beta_3 + \alpha_2\beta_4 + \delta^2. \tag{6}$$

$\delta, \alpha_1, \alpha_2, \beta_3, \beta_4$ have degree 6 and α_3, α_4 have degree 7 (and $\beta_1 = \beta_2 = 0$) thus A' and B' have degree 13 and 12, respectively. Note that A' and B' are not both zero since there are finitely many transversals. The Plücker coordinates of the transversals can thus be written as polynomials of degree at most 19 and, for one of the transversals (the one not in the plane at infinity), this bound is reached for three coordinates (namely, x_4, x_5, x_6). $\square$

Lemma 2. *Consider four lines, given by the Cartesian coordinates of pairs of points, that admit finitely many line transversals in* $\mathbb{P}^3(\mathbb{R})$. *If the four lines are not parallel to a common plane, we can compute on each transversal two points whose homogeneous coordinates have the form* $\phi_i + \varphi_i\sqrt{\Delta}$, $i = 1,\ldots,4$, *where* ϕ_i, φ_i, *and* Δ *are polynomials of degree at most* 17, 6, *and* 22, *respectively, in the coordinates of the input points. Otherwise, we can compute on each transversal two points whose homogeneous coordinates are polynomials of degree at most* 19. *Moreover, these bounds are reached, in the worst case, for some coordinates.*

Proof. Denote by w_1 (resp. w_2) the vector of the first (resp. last) three coordinates of $(x_1, \ldots, x_6)$, the Plücker coordinates of a line k, and let n denote any vector of $\mathbb{R}^3$. Then, if the four-tuple $(w_2 \times n, w_1 \cdot n)$ is not equal to $(0,0,0,0)$, it is a point (in homogeneous coordinates) on the line k (by Lagrange's triple product expansion formula). By considering the axis unit vectors for n, we get that the four-tuples $(0, x_6, -x_5, x_1)$, $(-x_6, 0, x_4, x_2)$, $(x_5, -x_4, 0, x_3)$ that are non-zero are points on the transversal lines k. Either five of the six Plücker coordinates of k are zero or at least two of these four-tuples are non-zero and thus are points on k. In the latter case, the result follows from Lemma 1. In the former case, two points with coordinates 0 or 1 can easily be computed on line k since the line is then one of the axis or a line at infinity defined by the directions orthogonal to one of the axis. $\square$

Remark 3. *In some implementations (for instance, the one of [15]), the* 4×4 *submatrix of the matrix of Plücker coordinates (see* (1)*) used for computing the line transversals is chosen, by default, as the leftmost submatrix whose determinant has degree 7 in the coordinates of the input points. In this case, the Plücker coordinates of the line transversals are written as* $\phi_i + \varphi_i\sqrt{\Delta}$, $i = 1,\ldots,6$, *where* ϕ_i, φ_i, *and* Δ *are polynomials of degree at most* 20, 7, *and* 26, *respectively, in the coordinates of the input points (and these bounds are reached). Similarly for the homogeneous coordinates of two points on the transversals.*

3 Predicates

3.1 Preliminaries

We start by two straightforward lemmas on the degree of predicates for determining the sign of simple algebraic numbers. If x is a polynomial expression in some variables, we denote by $\deg(x)$ the degree of x in these variables. This first lemma is trivial and its proof is omitted.

Lemma 4. *If a, b, and c are polynomial expressions of (input) rational numbers, the sign of $a + b\sqrt{c}$ can be determined by a predicate of degree* $\max\{2\deg(a), 2\deg(b) + \deg(c)\}$.

Lemma 5. *If $\alpha_i, \beta_i, \delta, \mu$, $i = 1, 2$, are polynomial expressions of (input) rational numbers, the sign of $\alpha_1 + \beta_1\sqrt{\delta} + (\alpha_2 + \beta_2\sqrt{\delta})\sqrt{\mu}$ can be obtained by a predicate of degree*

$$\max\{4\deg(\alpha_1), \quad 4\deg(\beta_1) + 2\deg(\delta), \quad 4\deg(\alpha_2) + 2\deg(\mu), \quad 4\deg(\beta_2) + 2\deg(\delta) + 2\deg(\mu), \\ 2\deg(\alpha_1) + 2\deg(\beta_1) + \deg(\delta), \quad 2\deg(\alpha_2) + 2\deg(\beta_2) + 2\deg(\mu) + \deg(\delta)\}.$$

Proof. The predicate is to evaluate the sign of an expression of the form $a + b\sqrt{\mu}$, where $a = \alpha_1 + \beta_1\sqrt{\delta}$, $b = \alpha_2 + \beta_2\sqrt{\delta}$, and $\alpha_i, \beta_i, \mu, \delta$ are rational. This can be done by evaluating the signs of a, b, and $a^2 - b^2\mu$. The first two signs can be obtained by directly applying Lemma 4. On the other hand, $a^2 - b^2\mu$ is equal to $A + B\sqrt{\delta}$ with $A = \alpha_1^2 + \beta_1^2\delta - \alpha_2^2\mu - \beta_2^2\mu\delta$ and $B = 2\alpha_1\beta_1 - 2\alpha_2\beta_2\mu$. The sign of $A + B\sqrt{\delta}$ can be determined by another application of Lemma 4, which gives the result. $\square$

3.2 Transversals to four lines

We consider first the predicate of determining whether four lines admit 0, 1, 2, or infinitely many line transversals in $\mathbb{P}^3(\mathbb{R})$ (that is lines in $\mathbb{P}^3(\mathbb{R})$ that intersect, in $\mathbb{P}^3(\mathbb{R})$, the four input lines). An evaluation of this predicate directly follows from the algorithm described in Section 2 for computing the line transversals. Recall that, in the sequel, all input points are, by assumption, given by their Cartesian coordinates.

Theorem 6. *Given four lines defined by pairs of points, there is a predicate of degree* 22 *in the coordinates of these points to determine whether the four lines admit 0, 1, 2, or infinitely many line transversals in* $\mathbb{P}^3(\mathbb{R})$.

Proof. We consider three cases. First, if the four lines are parallel, which can easily be determined by a predicate of degree 3, then they admit infinitely many line transversals in $\mathbb{P}^3(\mathbb{R})$. Second, if the four lines are not parallel but parallel to a common plane, which can easily be determined by a predicate of degree 3, then the four lines admit infinitely many transversals if Equation (6) is identically zero and, otherwise, 2 line transversals in $\mathbb{P}^3(\mathbb{R})$; this can thus be determined with a predicate of degree 13 (see the proof of Lemma 1). Finally, if the four lines are not parallel to a common plane, they admit infinitely many transversals if Equation 4 is identically zero and, otherwise, 0, 1, or 2 transversals depending on the sign of Δ (in Lemma 1) which is of degree 22 in the coordinates of the points defining the lines. $\square$

Note that if the leftmost (instead of the rightmost) 4×4 submatrix of the matrix of Plücker coordinates (in (1)) is used for computing line transversals (see Remark 3) then the procedure described in the above proof has degree 26 instead of 22.

All line transversals are defined in $\mathbb{R}^3$ except in the case where the four input lines are parallel to a common plane, in which case the intersection of this plane with the plane at infinity is a line transversal at infinity. Note also that, determining whether a line transversal in $\mathbb{P}^3(\mathbb{R})$ is transversal in $\mathbb{R}^3$ amounts to determining whether the transversal is parallel to one of the four input lines ℓ_i, that is if their direction vectors are collinear. This can be done, by Lemmas 1 and 4, by a predicate of degree 36 in the Cartesian coordinates of the points defining the input lines.

Note, however, that if the points defining the ℓ_i have rational coordinates and if the transversal is parallel to one of the ℓ_i, the Plücker coordinates of the transversal are rational; indeed, the multiplicative factor of the direction vectors is rational (since one of the coordinates of the direction vector of the transversal is rational, *e.g.*, x_2 in (5)) and thus all the coordinates of this direction vector are rational, which implies that Δ is a square in (5). Hence, deciding whether

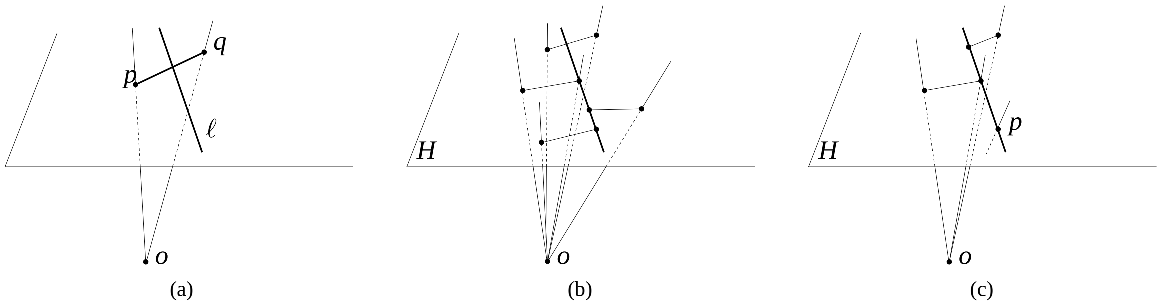

Figure 1. (a): Transversal ℓ intersects segment pq only if $(\ell \odot op)\,(\ell \odot oq) \leqslant 0$. (b-c): An illustration for the proof of Lemma 10.

a transversal is parallel to one of the input lines ℓ_i can be done by first determining whether Δ is a square and, if so, testing whether the direction vectors are collinear. It thus follows from Lemma 1 that determining whether a transversal is parallel to one of the input lines ℓ_i can be done with a fixed-precision floating-point arithmetic using a number of bits roughly equal to 22 times the number of bits used in representing each input value. This should be compared to the degree 36 of the above procedure. In this paper we have restricted our attention to evaluation procedures for predicates that consist entirely of determining the signs of polynomial expressions in the input parameters. We see here an example of a predicate which may be more effeciently evaluated by a procedure which permits other operations, in this case, dertermining whether a rational number is a square. This provides an interesting example of a geometric predicate whose algebraic degree does not seem to be an entirely adequate measure of the number of bits needed for the computation.

3.3 Transversals to four segments

We consider here the predicate of determining how many transversals four segments of $\mathbb{R}^3$ admit. Recall that four segments may admit up to 4 or infinitely many line transversals [3]. In this section, we prove the following theorem.

Theorem 7. *Given four line segments, there is a predicate of degree 36 in the coordinates of their endpoints to determine whether those segments admit 0, 1, 2, 3, 4, or infinitely many line transversals.*

Note that if, the leftmost (instead of the rightmost) 4×4 submatrix of the matrix of Plücker coordinates (in (1)) is used for computing line transversals (see Remark 3) then the procedure described below for the predicate of Theorem 7 has degree 42 instead of 36.

We consider, in the following, the supporting lines of the four segments, that is, the lines containing the segments; in the case where one (or several) segment is reduced to a point, we consider as supporting line, any line through this point and parallel to at least another supporting line. We first consider the case where the four supporting lines admit finitely many transversals in $\mathbb{P}^3(\mathbb{R})$; this can be determined by a predicate of degree 22, by Theorem 6.

Lemma 8. *Given four segments in $\mathbb{R}^3$ whose supporting lines admit finitely many line transversals in $\mathbb{P}^3(\mathbb{R})$, determining the number of transversals to the four segments can be done with a predicate of degree 36 in the coordinates of their endpoints.*

Proof. Let ℓ denote an (arbitrarily) oriented line, as well as its Plücker coordinates, that is transversals to the four lines; ℓ can be computed as described in Section 2. We consider the predicate of determining whether ℓ intersects each of the four segments, in turn. Let p and q denote the endpoints of one of these segments. For any two distinct points r and s, denote by rs the Plücker coordinates of the line rs oriented from r to s; depending on the context, rs also denotes the line through r and s or the segment from r to s.

If a point o does not lie in the plane containing line ℓ and segment pq (see Figure 1(a)), then line ℓ intersects segment pq if and only if the oriented line ℓ is on opposite sides of the two oriented lines from o to p and from o to q, that is if $(\ell \odot op)\,(\ell \odot oq) \leqslant 0$ (recall that $\odot$ denotes the side operator – see Section 2).

On the other hand, point o lies in a plane containing line ℓ and segment pq if and only if ℓ intersects (in $\mathbb{P}^3(\mathbb{R})$) both lines op and oq, that is both side operators $\ell \odot op$ and $\ell \odot oq$ are zero. By choosing point o to be for instance $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, or $(1,1,1)$, we ensure that one of these points will not be coplanar with ℓ and segment pq unless segment pq lies on ℓ.

Hence the predicate follows from the sign of side operators of the line transversal and of a line defined by two points, one of which with coordinates equal to 0 or 1. The degree of the Plücker coordinates of the line through these two points is thus 1 (in the coordinates of the input points). Hence, by Lemma 1, the predicate can be computed by determining the sign of polynomials of degree at most 20 if the input lines are parallel to a common plane and, otherwise, by determining the sign of expressions of the form $a+b\sqrt{c}$ where a, b and c have degree at most 18, 7, and 22, respectively; moreover, these bounds are reached. By Lemma 4, the predicate thus has degree 36, which concludes the proof. □

We now consider the case where the four lines admit infinitely many transversals. Recall that, in $\mathbb{P}^3(\mathbb{R})$, four lines or line segments admit infinitely many transversals only if [3]:

1. they lie in one ruling of a hyperbolic paraboloid or a hyperboloid of one sheet,
2. they are all concurrent, or
3. they all lie in a plane, with the possible exception of a group of one or more that all meet that plane at the same point.

We treat the cases independently.

Lemma 9. *Given four segments in $\mathbb{R}^3$ whose supporting lines are pairwise skew and admit infinitely many line transversals, determining the number of their line transversals can be done with a predicate of degree at most 36 in the coordinates of their endpoints.*

Proof. When four lines are pairwise skew, their common transversals can be parameterized by their points of intersection with one of the lines; moreover, the set of common transversals to the four segments corresponds (through this parameterization) to up to four intervals on that line and the transversals that correspond to the endpoints of these intervals contain (at least) one endpoint of the segments [3]. We can compute and order all these interval endpoints and determine whether there exists a transversal (to the four segments) through each midpoint of two consecutive distinct interval endpoints. By construction and by [3], the four segments admit such a transversal if and only if they admit infinitely many transversals.

The set of interval endpoints, on, say, segment s_1 is a subset of the endpoints of s_1 and of the intersection points of s_1 with the planes containing s_2 and an endpoint of s_3 or s_4 and of the intersection points of s_1 with the planes containing s_3 and an endpoint of s_2. The coordinates of these points can be trivially computed as rational expressions of degree 4 in the coordinates of the segment endpoints. The coordinates of the midpoints are thus rational expressions of degree at most 8.

The transversal to the four lines through (any) one of these midpoints intersects line ℓ_2 and lies in the plane containing line ℓ_3 and the considered midpoint; the coordinates of the intersection point between this plane and ℓ_2 are rational expressions of degree at most 19. Finally, determining whether a transversal (to the four lines) through two points whose coordinates are rational expressions of degree 8 and 19 is a transversal to each of the four segments can be done, as in the proof of Lemma 8, using side operators. Hence, we can decide whether the four segments admit infinitely many transversals with a predicate of degree at most 36 since the Plücker coordinates of the line transversal are of degree at most 35.

Now, if the four segments admit finitely many transversals, we can determine the number of transversals as follows. As mentioned above, the set of transversals can be parameterized by intervals on a line and the interval endpoints correspond to transversals that go through a segment endpoint. A transversal is isolated if and only if it corresponds to an interval that is reduced to a point. Thus, a transversal is isolated only if it goes through two distinct segment endpoints (the segments necessarily have distinct endpoints since, by assumption, their supporting lines are pairwise skew and thus no segment is reduced to a point). Determining whether the lines through two distinct endpoints intersect the other segments can easily be done, as described in the proof of Lemma 8, by computing the sign of side operators which are here of degree 3 in the coordinates of the segment endpoints. □

Lemma 10. *Given four segments in $\mathbb{R}^3$ whose supporting lines are not pairwise skew and admit infinitely many line transversals, determining the number of their line transversals can be done with a predicate of degree 7 in the coordinates of their endpoints.*

Proof. First, note that testing whether two segments intersect can be done using side operators with a predicate of degree 3. The four lines containing the segments are not pairwise skew and they admit infinitely many line transversals. Thus, they are all concurrent or they all lie in a plane H, with the possible exception of a group of one or more that all meet that plane at the same point [3]. Four cases may occur:

(i) all four lines lie in a plane H,
(ii) three lines lie in a plane H and the fourth line intersects H in exactly one point,
(iii) two lines lie in a plane H and two other lines intersect H in exactly one and the same point,
(iv) three lines are concurrent but not coplanar.

Differentiating between these cases can be done by determining whether sets of four segment endpoints are coplanar (which is a predicate of degree 3). We study each case in turn.

Case (i). The four segments are coplanar. Any component of transversals contains a line through two distinct segment endpoints. Hence the four segments have finitely many transversals if and only if any line through two distinct endpoints that is a transversal to the four segments is an isolated transversal. This only occurs[1] (see Figure 1(b)) when the transversal goes through the endpoints of three segments such that the segment, whose endpoint is in between the two others, lies (in H) on the opposite side of the transversal than the two other segments. This can be tested by computing the sign of scalar products and side operators between the transversal and the lines through a point o not in H and the segment endpoints (see Figure 1(b)). This leads to a predicate of degree 4.

Case (ii). Three lines lie in a plane H. Testing whether the fourth segment intersects the plane H can easily be done by computing the point of intersection between H and the line containing the fourth segment, leading to a predicate of degree 3. If the fourth segment does not intersect plane H, the four segments have no transversal unless the first three segments are concurrent in which case the four segments have one or infinitely many transversals depending on whether the four lines supporting the segments are concurrent. Otherwise, let p denote the point of intersection. We assume that the three segments in H are not concurrent; otherwise the four segments have infinitely many transversals. Thus, any component of transversals contains a line through p and through a segment endpoint. Hence the four segments have finitely many transversals if and only if any line through p and a segment endpoint that is a transversal to the four segments is an isolated transversal. Testing whether such a line is a transversal to all segments can be done, as in the proof of Lemma 8, by computing the sign of side operators of the line transversal and of lines through a segment endpoint and a point o not in H; the coordinates of point p are rational expressions of degree 4, thus the Plücker coordinates of the transversal have degree at most 6, which leads to a predicate of degree 7. Such a line transversal is isolated (see Figure 1(c)) if and only if[2] the transversal goes through two endpoints of two distinct segments that lie on the same side (in plane H) of the transversal or not depending whether p is in between the two endpoints or not. This test can be done by computing the sign of scalar products and side operators between the transversal and the lines through a point o not in H and the segment endpoints (see Figure 1(c)). This test also leads to a predicate of degree 7. We can thus determine the number of isolated transversals with a predicate of degree 7.

Case (iii). Two lines lie in a plane H and two other lines intersect H in exactly one and the same point. (Note that there may be two instances of plane H for a given configuration.) This case can be treated similarly as Case (ii).

Case (iv). Three lines are concurrent but not coplanar. If none of the three corresponding segments intersect, they have no common transversal. If only two segments intersect, the three segments have exactly one transversal; checking whether that transversal intersects the fourth segment can easily be done with a predicate of degree 3. Now, if the three segments intersect, then the four segments have infinitely many transversals if they are concurrent or if their

[1] For simplicity, we do not discuss here the case where the line transversal contains one of the four segments.
[2] We assume here for simplicity that the line transversal contains no segment.

supporting lines are not concurrent. Otherwise, if the four segments are not concurrent but their supporting lines are, the four segments then have a unique transversal. This can also be checked with a predicate of degree 3. □

We can now conclude the proof of Theorem 7. By Theorem 6, we can determine with a predicate of degree 22 whether the four lines containing the four segments admit finitely many transversals in $\mathbb{P}^3(\mathbb{R})$. If the four lines admit finitely many transversals, then, by Lemma 8, determining the number of transversals to the four segments can be done with a predicate of degree 36. Assume now that the four lines admit infinitely many transversals. Note that determining whether the input lines are pairwise skew can easily be done with a predicate of degree 3. Thus, by Lemmas 9 and 10, determining whether the four segments admit 0, 1, 2, 3, 4, or infinitely many line transversals can be done by a predicate of degree at most 36. Hence, we can determine the number of transversals to four segments with a predicate of degree 36. □

3.4 Transversals to four segments and a triangle

We consider here the predicate of determining whether a minimal segment transversal to four line segments is intersected by a triangle. Given a line transversal ℓ to a set S of segments, a triangle T *occludes* ℓ if ℓ intersects T and if there exist two segments in S whose intersections with ℓ lie on opposite sides of T. We describe a method for evaluating the predicate for determining whether a triangle occludes a transversal to a given set of line segments and establish its degree.

Theorem 11. *Let ℓ be a line transversal to four line segments that admit finitely many transversals and let T be a triangle. There is a predicate of degree 78 in the coordinates of the points defining the segments and the triangle to determine whether T occludes ℓ.*

Proof. Let ℓ denote an oriented line transversal to segments $s_1,\ldots,s_4$, each defined by two points e_i and f_i, $i=1,\ldots,4$, and let T be a triangle defined by three points p,q, and r. The Plücker coordinates of ℓ can be computed as described in Section 2. We only consider the case where the four lines containing segments s_i have finitely many transversals because, otherwise, since the four segments admit finitely many transversals, each transversal goes through at least one endpoint of the four segments and it is straightforward that the degree of the predicate is then much smaller.

We first determine whether ℓ intersects T by taking the side product of ℓ with each supporting line of T (oriented consistently); ℓ intersects T if and only if no two side products have opposite signs (*i.e.*, ± 1). Similarly as in the proof of Lemma 8, there is a predicate of degree 38 for determining the sign of these side operators.

Assuming that ℓ intersects T, we next find the point of intersection. By Lemma 2, ℓ can be represented parametrically in the form $\pi+\rho t$. We determine the value of t for which the determinant of $p,q,r,\pi+\rho t$ is equal to zero; denote this value of t by t_0. This determinant has the form $a_0+b_0t_0$, where, by Lemma 2, a_0 and b_0 are polynomials of degree 22 if $s_1,\ldots,s_4$ are parallel to a common plane or, otherwise, have the form $\phi+\varphi\sqrt{\Delta}$ where ϕ,φ, and Δ have degree 20, 9, and 22, respectively, in the coordinates of p,q,r,e_i,f_i.

Now, for each segment s_i, we compute the point of intersection of s_i with ℓ in terms of the parameter t using the method similar to that of the previous section: choose a point o_i not in the plane determined by s_i and ℓ and compute the value t for which the determinant of $e_i, f_i, o_i, \pi+\rho t$ equals 0. Denote this value by t_i. Since o_i can be chosen with all coordinates equal to 0 or 1, we get, similarly as in the previous paragraph, that each of these determinants has the form $a_i+b_it_i$ where a_i and b_i are polynomials of degree 21 if $s_1,\ldots,s_4$ are parallel to a common plane or, otherwise, have the form $\phi+\varphi\sqrt{\Delta}$ where ϕ,φ, and Δ have degree 19, 8, and 22, respectively.

Determining whether T occludes ℓ is now only a matter of determining whether t_0 lies between two of the values $t_i, i=1,\ldots,4$, which requires only that we be able to compare t-values, that is, compute $\operatorname{sign}(t_i-t_j)$. Observe that $t_i-t_j=\frac{a_jb_i-a_ib_j}{b_ib_j}<0$, so $\operatorname{sign}(t_i-t_j)=\operatorname{sign}(a_jb_i-a_ib_j)\operatorname{sign}(b_i)\operatorname{sign}(b_j)$. It follows from the above characterization of the a_i and b_i that a product a_ib_j is either a polynomial of degree 43 if $s_1,\ldots,s_4$ are parallel to a common plane or, otherwise, has the form $\phi+\varphi\sqrt{\Delta}$ where ϕ,φ, and Δ have degree at most 39, 28, and 22, respectively (and these bounds are reached in the worst case). Applying Lemma 4 yields a predicate of degree 78, which concludes the proof. □

Note that, if the leftmost (instead of the rightmost) 4×4 submatrix of the matrix of Plücker coordinates (in (1)) is used for computing line transversals (see Remark 3) then the procedure described above for the predicate of Theorem 11 has degree 90 instead of 78.

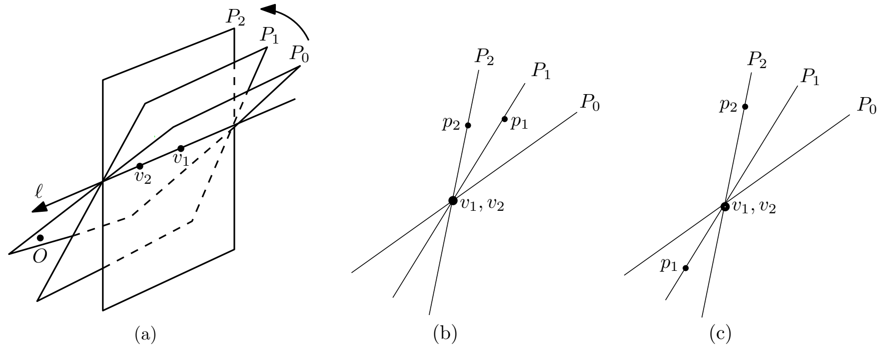

Figure 2. Planes P_1 and P_2 such that $P_1 < P_2$

3.5 Ordering planes through two fixed points, each containing a third (rational) point or a line transversal

Let ℓ be a line defined by two points v_1 and v_2, and $\vec{\ell}$ be the line ℓ oriented in the direction $\overrightarrow{v_1v_2}$.

We define an ordering of all the planes containing ℓ with respect to the oriented line $\vec{\ell}$ and a reference point O (not on ℓ). Let P_0 be the plane containing O and ℓ, and let P_1 and P_2 be two planes containing ℓ. We say that $P_1 < P_2$ if and only if P_1 is encountered strictly before P_2 when rotating counterclockwise about $\vec{\ell}$ a plane from P_0 (see Figure 2a).

Let p_i be any point on plane P_i but not on ℓ, for $i = 1, 2$, and let $D(p,q)$ denote the determinant of the four points (v_1, v_2, p, q) given in homogeneous coordinates.

Lemma 12. *With $\chi = D(O, p_1) \cdot D(O, p_2) \cdot D(p_1, p_2)$, we have:*

(a) If $\chi > 0$ then $P_1 > P_2$.
(b) If $\chi < 0$ then $P_1 < P_2$.
(c) If $\chi = 0$ then
  (i) if $D(p_1, p_2) = 0$, then $P_1 = P_2$,
  (ii) else if $D(O, p_1) = 0$, then $P_1 < P_2$,
  (iii) else $P_1 > P_2$.

Proof. Assume first that $D(O, p_1) \cdot D(O, p_2) > 0$, that is, that p_1 and p_2 lie strictly on the same side of the plane P_0 (see Figure 2b). Then the order of P_1 and P_2 is determined by the orientation of the four points (v_1, v_2, p_1, p_2), that is by the sign of $D(p_1, p_2)$. It is then straightforward to notice that $P_1 > P_2$ if and only if $D(p_1, p_2) > 0$. Hence, if $\chi > 0$, then $P_1 > P_2$ and, if $\chi < 0$, then $P_1 < P_2$.

Suppose now that $D(O, p_1) \cdot D(O, p_2) < 0$, that is, that p_1 and p_2 lie strictly on opposite sides of the plane P_0 (see Figure 2c). The order of P_1 and P_2 is then still determined by the sign of $D(p_1, p_2)$. However, $P_1 > P_2$ if and only if $D(p_1, p_2) < 0$. Hence, we have in all cases that, if $\chi > 0$, then $P_1 > P_2$ and, if $\chi < 0$, then $P_1 < P_2$.

Suppose finally that $\chi = 0$. If $D(p_1, p_2) = 0$, then p_1 and p_2 are coplanar, and $P_1 = P_2$. Otherwise, if $D(O, p_1) = 0$, then $P_0 = P_1$ thus P_1 is smaller to all other planes (containing $\vec{\ell}$), and in particular $P_1 \leqslant P_2$. Furthermore, since $D(p_1, p_2) \neq 0$, $P_1 \neq P_2$ and thus $P_1 < P_2$. Otherwise, $D(O, p_2) = 0$ and we get similarly that $P_2 < P_1$. $\square$

Computing a point on a plane defined by ℓ and a line transversal. We want to order planes P_i that are defined by line ℓ and either a rational point not on ℓ, or by a line transversal to ℓ and three other lines. In the latter case, we consider a point on the line transversal (which is non-rational, in general; see Lemma 2). The following lemma tells us that, in general, such a plane P_i contains no rational points outside of ℓ, and that in the cases where it does contain such a rational point, the line transversal is then rational. Hence, if the points computed on the line transversal, as described in Lemma 2, are not rational, there is no need to search for simpler points on the plane (but not on ℓ).

Lemma 13. *The plane P containing a rational line ℓ and a line transversal to ℓ and three other segments, each determined by two rational points, contains in general no rational points except on ℓ. Furthermore, if plane P contains a rational point not on ℓ then the line transversal is rational.*

Proof. Suppose that the plane P contains a rational point p not on ℓ. Then the plane contains three (non-collinear) rational points, p and two points on ℓ, and thus P is a rational plane. This plane intersects the three other segments in three points, all of which are rational and lie on the transversal. So the transversal is a rational line which implies that the discriminant $B^2 - 4AC$ in Equation (5) is a square, which is not the case in general. $\square$

Comparing two planes. We want to order planes P_i that are defined by either line ℓ and another (input rational) point not on ℓ, or by line ℓ and a line transversal to ℓ and three other lines.

By Lemma 12, ordering such planes about ℓ amounts to computing the sign of determinants of four points (in homogeneous coordinates). Two of these points are input (affine rational) points on ℓ (v_1 and v_2) and each of the two other points is either an input (affine rational) point r_i, $i = 1,2$, or is, by Lemma 2 (and Lemma 13), a point u_i whose homogeneous coordinates have degree at most 19 (in the coordinates of the input points) or a point of the form $p_i + q_i\sqrt{\Delta_i}$, $i = 1,2$, where the Δ_i have degree 22 and where the p_i and q_i are points with homogeneous coordinates of degree at most 17 and 6, respectively. If the four points are all input points, then the determinant of the four points has degree 3 in their coordinates.

If only three of the four points are input points, then the determinant of the four points is either a polynomial of degree 22 or it has the form $D(p_1,r_1) + D(q_1,r_1)\sqrt{\Delta_1}$ where the degrees of the $D()$ are 20 and 9, respectively, in the coordinates of the input points. Hence, by Lemma 4, the sign of this expression can be determined with a predicate of degree 40.

Finally, if only two of the four points are input points, then the determinant has one of the following forms (depending on whether the quadruples of lines defining the transversals are parallel to a common plane); the degrees are given in terms of the coordinates of the input points:

(i) $D(u_1,u_2)$ which is of degree 40.

(ii) $D(u_1,p_1) + D(u_1,q_2)\sqrt{\Delta_1}$ where the $D()$ have degree 38 and 27, respectively.

(iii) $D(p_1,p_2) + D(q_1,p_2)\sqrt{\Delta_1} + (D(p_1,q_2) + D(q_1,q_2)\sqrt{\Delta_1})\sqrt{\Delta_2}$ where the $D()$ have degree 36, 25, 25, and 14, respectively.

Hence, by Lemma 5, the sign of these expressions can be determined with a predicate of degree at most 144 (and the bound is reached in the worst case). We thus get the following result.

Theorem 14. *Let ℓ be an oriented line defined by two points, let p_0 be a point not on ℓ, and let P_0 be the plane determined by ℓ and p_0. Given two planes P_1, P_2 containing ℓ there is a predicate which determines the relative order of P_1 and P_2 about ℓ with respect to P_0 having the following degree in the coordinates of the input points:*

(i) degree 3 if $P_i, i = 1,2$ are each specified by a (input) point p_i;

(ii) degree 40 if P_1 is specified by a point p_1 and P_2 is determined by a line transversal to ℓ and three other lines ℓ_1, ℓ_2, ℓ_3, each specified by two (input) points;

(iii) degree 144 if $P_i, i = 1,2$ are each determined by a line transversal to ℓ and three other lines $\ell_{i,1}, \ell_{i,2}, \ell_{i,3}$, each specified by two (input) points.

Remark 15. *Similarly as before, note that, if the leftmost (instead of the rightmost) 4×4 submatrix of the matrix of Plücker coordinates (in (1)) is used for computing line transversals (see Remark 3) then the predicates of Theorem 14 have degree 3, 46, and 168.*

4 Experiments

In this section, we report the results of experiments that analyze the behavior of the predicate for ordering, in a rotational sweep about a line, two planes each defined by a line transversal to four lines, that is the predicate related to Theorem 14(iii). The degree of the procedure we use for evaluating this predicate is 168 because we use for computing line transversals to four lines the code of Redburn [15], which, as noted in Remarks 3 and 15, leads to degree 168 instead of 144 as in Theorem 14(iii).

predicates \ ε	10^{-12}	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}
degree 168	99.6%	50.4%	7.6%	0.8%	0.08%	0.008%
degree 3	99.5%	8.2%	0.08%	0.001%		

Table 1. Percentages of failure of the degree 168 and degree 3 predicates using double-precision floating-point interval-arithmetic, for ε varying from 10^{-12} to 10^{-2}.

The standard approach to comparing two such planes is to first evaluate the predicate using fixed-precision interval-arithmetic. This is very efficient but may fail when the sign of an expression cannot be successfully determined because the result of the evaluation of the expression is an interval that contains zero. If this happens, the answer to the predicate is then obtained by either evaluating exactly the expression (and thus its sign) using exact arithmetic or by increasing the precision of the interval arithmetic until either the result of the evaluation of the expression is an interval that does not contain zero or the separation bound is attained (see for instance [4, 13, 16, 18]); in both approaches the computation is much slower than when using fixed-precision interval-arithmetic. We are thus interested in determining how often the fixed-precision interval-arithmetic evaluation of our predicate fails.

To test our predicate, we generate pairs of planes, each defined by two lines, one chosen at random and common to the two planes, and the other defined as a transversal to the common line and to three other random lines. We are interested in evaluating our predicate in the case where the two planes are very close together, that is, when there is significant risk of producing an error when using finite-precision floating-point arithmetic.

We generate two sets of four lines. Each line of the first set is determined by two points, all of whose coordinates are double-precision floating-point numbers chosen uniformly at random from the interval $[-5000, 5000]$. The second set of lines is obtained by perturbing the points defining three of the lines of the first set; the fourth line is not perturbed and is thus common to the two sets. To perturb a point p, we translate it to a point chosen uniformly at random in a sphere centered at p, with radius ε.

We compute, for each of these two sets of four lines, a line transversal. If either set of four lines does not admit a transversal (which happens roughly 24% of the time), we throw out that data and start again. Otherwise, we choose a transversal in a consistent way for the two sets of four lines, that is, such that one transversal converges to the other when ε tends to zero. Each transversal, together with the common line, defines a plane.

For various values of ε, varying from 10^{-2} to 10^{-10}, we evaluate the predicate using double-precision floating-point interval arithmetic until we obtain 1000 pairs of planes for which the computation of the predicate fails. We measure the percentage of time that the computation fails. The results of these experiments are shown in Table 1.

We observe, as expected, that when ε is sufficiently small (10^{-10}), that is, when the two planes are often close enough to each other, the fixed-precision interval-arithmetic predicate fails with high probability and that this probability decreases as ε increases. When $\varepsilon = 10^{-2}$, the probability of failure is close to zero. Finally, we have also observed that the predicate fails when the angle between the two planes is less than roughly 10^{-8} radians, which is, of course, independent of ε.

Note finally that the percentage of failure of the degree 168 predicate using fixed-precision interval-arithmetic is, as expected, high compared to lower-degree predicates. Table 1 also shows the failure rate for the degree 3 predicate related to Theorem 14(i). We use the same experimental scheme as above, that is, we chose at random three points that define a plane and perturb one of these points by at most ε.

All the experiments were made on a i686 machine with AMD Athlon 1.73 GHz CPU and 1 GB of main memory using the CGAL interval number type with double-precision floating-point numbers [5].

Acknowledgment

The authors gratefully acknowledge fruitful discussions on this topic with Olivier Devillers and Marc Glisse. We also especially thank Hervé Brönnimann for his comments on the paper and Günter Rote for pointing out an improvement, in Lemma 1, to the degrees appearing in the Plücker coordinates of line transversals.

References

[1] J.-D. Boissonnat and F. Preparata. Robust plane sweep for intersecting segments. *SIAM Journal on Computing*, 29(5):1401–1421, 2000.

[2] H. Bronnimann, O. Devillers, V. Dujmovic, H. Everett, M. Glisse, X. Goaoc, S. Lazard, H.-S. Na, and S. Whitesides. Lines and free line segments tangent to arbitrary three-dimensional convex polyhedra. *SIAM Journal on Computing*, 37(2):522–551, 2007.

[3] H. Brönnimann, H. Everett, S. Lazard, F. Sottile, and S. Whitesides. Transversals to line segments in three-dimensional space. *Discrete and Computational Geometry*, 34(3):381–390, 2005.

[4] C. Burnikel, S. Funke, K. Mehlhorn, S. Schirra, and S. Schmitt. A separation bound for real algebraic expressions. In Proceedings of the *9th Annual European Symposium on Algorithms*, volume 2161 of *Lecture Notes in Computer Science*, pages 254–265. Springer-Verlag, 2001.

[5] CGAL: Computational Geometry Algorithms Library. `http://www.cgal.org`.

[6] O. Devillers and O. Hall-Holt. Private communication, 2000.

[7] F. Duguet and G. Drettakis. Robust epsilon visibility. In J. Hughes, editor, *Proceedings of ACM SIGGRAPH 2002*, pages 567–575. ACM Press / ACM SIGGRAPH, July 2002.

[8] F. Durand, G. Drettakis, and C. Puech. The visibility skeleton: a powerful and efficient multi-purpose global visibility tool. *Computer Graphics Proceedings, Annual Conference Series*, 31:89–100, 1997. Proceedings of Siggraph'97.

[9] F. Durand, G. Drettakis, and C. Puech. The 3D visibility complex. *ACM Transactions on Graphics*, 21(2):176–206, 2002.

[10] A. Efrat, L. Guibas, O. Hall-Holt, and L. Zhang. On incremental rendering of silhouette maps of a polyhedral scene. In *Proceedings of the 11th ACM-SIAM Symp. on Discrete Algorithms*, pages 910–917, 2000.

[11] L. Glaves. An exploration of the 3D visibility complex. Master's thesis, Polytechnic University, Brooklyn, NY., 2007.

[12] M. Hohmeyer and S. Teller. Determining the lines through four lines. *Journal of Graphics Tools*, 4(3):11–22, 1999.

[13] M. Mignotte. Identification of algebraic numbers. *CJournal of Algorithms*, 3(3):197–204, 1982.

[14] H. Plantinga and C. Dyer. Visibility, occlusion, and the aspect graph. *International Journal of Computer Vision*, 5(2):137–160, 1990.

[15] J. Redburn. Robust computation of the non-obstructed line segments tangent to four amongst *n* triangles. Master's thesis, B.A. Thesis, Williams College, Massachusetts, May 2003., 2003.

[16] E. R. Scheinerman. When close enough is close enough. *American Mathematical Monthly*, 107:489–499, 2000.

[17] K. Shoemake. Plücker coordinate tutorial. *Ray Tracing News*, 11(1), 1998.

[18] C. K. Yap. Towards exact geometric computation. *Computational Geometry: Theory and Applications*, 7(1-2):3–23, 1997.