There are arbitrary large minimal 2-pinning configurations

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Let ℓ be a line and $\mathcal{C}$ a collection of disjoint convex sets in $\mathbb{R}^d$. We say that ℓ is a *transversal* to $\mathcal{C}$ if it intersects each of its members and that $\mathcal{C}$ *pins* ℓ if, in addition, no arbitrarily small perturbation of ℓ is a transversal to $\mathcal{C}$. We say that $\mathcal{C}$ k*-pins* ℓ if for any k-flat Π containing ℓ, $\Pi \cap \mathcal{C}$ pins ℓ in Π (so pinning and d-pinning are the same in $\mathbb{R}^d$). A *minimal k-pinning configuration* is a pair $(\ell, \mathcal{C})$ where $\mathcal{C}$ k-pins ℓ but no proper subset of $\mathcal{C}$ does.

When $\mathcal{C}$ consists of disjoint balls, all its transversal that meet the balls in the same order make up a connected set in line space [1]. So, by continuity, $\mathcal{C}$ pins ℓ if and only if no other transversal to $\mathcal{C}$ realizes the same order as ℓ. Since two lines span a space of dimension at most 3, this implies that $\mathcal{C}$ pins ℓ if and only if it 3-pins it. As a consequence, all minimal 3-pinning configurations of disjoint balls in $\mathbb{R}^d$ have size at most $2d - 1$ [1, 2]. Here, we prove that the situation is different for 2-pinning:

Theorem 1. *A minimal 2-pinning configuration of size $n \geq 6$ in $\mathbb{R}^3$ exists if and only if n is even.*

For simplicity we assume that the objects are smooth, e.g. balls. Our proof is essentially combinatorial:

1. Sign sequence. Let ℓ be an oriented line in $\mathbb{R}^2$ tangent to $O_1, \ldots, O_n$ in that order. Its *sign sequence* is a word in $\{-, +\}^n$ where the i^{th} letter is $+$ if and only if O_i is on the right of ℓ.

2. Encoding. We now return to 3-space, assume that $\mathcal{C}$ 2-pins ℓ and choose some arbitrary orientation on ℓ. The tangent planes to the objects of $\mathcal{C}$ at their contact point with ℓ partition the space into n sectors. For all planes containing ℓ lying in the same sector, the sign sequence of the associated planar configuration is the same (up to exchanging $-$ and $+$). We denote by $\sigma_0, \ldots, \sigma_{n-1}$ the sign sequences obtained successively as we go through all sectors around ℓ.

3. Translating geometric properties. The family $(\sigma_0, \ldots, \sigma_{n-1})$ corresponds to a geometric configuration $(\mathcal{C}, \ell)$ if and only if (1) there exists a permutation π of $\{1, \ldots, n\}$ such that σ_i differs from $\sigma_{i-1[n]}$ from the inversion of its $\pi(i)^{th}$ letter. Then, observe that $\mathcal{C}$ 2-pins ℓ if and only if (2) every σ_i contains an alternating triple. Last, notice that no proper subset of $\mathcal{C}$ 2-pins ℓ if and only if (3) for any $1 \leq i \leq n$ there is some t such that σ_t loses all alternating triples if its i^{th} letter is deleted.

4. Wrapping up. For odd n, rules (1)–(3) are incompatible. For even n, the sequence defined by

$$\sigma_0 = + - +^{n-2} \quad \text{and} \quad \pi = (4, 1, 6, 3, 8, 5, \ldots, n, n-3, 2, n-1)$$

satisfies rules (1)–(3) and thus corresponds to a minimal 2-pinning configuration.

References

[1] C. Borcea, X. Goaoc, and S. Petitjean. Line transversals to disjoint balls. *Discrete & Computational Geometry*, 2008. To appear.

[2] O. Cheong, X. Goaoc, A. Holmsen, and S. Petitjean. Hadwiger and Helly-type theorems for disjoint unit spheres. In J. E. Goodman, J. Pach, and R. Pollack, editors, *Computational Geometry - Twenty Years Later*. AMS. To appear.

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