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ELIMINATION AND NONLINEAR EQUATIONS OF REES ALGEBRA

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with an appendix by Joseph Oesterlé

Abstract. A new approach is established to computing the image of a rational map, whereby the use of approximation complexes is complemented with a detailed analysis of the torsion of the symmetric algebra in certain degrees. In the case the map is everywhere defined this analysis provides free resolutions of graded parts of the Rees algebra of the base ideal in degrees where it does not coincide with the corresponding symmetric algebra. A surprising fact is that the torsion in those degrees only contributes to the first free module in the resolution of the symmetric algebra modulo torsion. An additional point is that this contribution – which of course corresponds to non linear equations of the Rees algebra – can be described in these degrees in terms of non Koszul syzygies via certain upgrading maps in the vein of the ones introduced earlier by J. Herzog, the third named author and W. Vasconcelos. As a measure of the reach of this torsion analysis we could say that, in the case of a general everywhere defined map, half of the degrees where the torsion does not vanish are understood.

1. Introduction

Let k stand for an arbitrary field, possibly assumed to be of characteristic zero in some parts of this work. Let $R := k[X_1, \ldots, X_n]\,(n \geq 2)$ denote a standard graded polynomial ring over the field k and let $I \subset R$ denote an ideal generated by k-linearly independent forms $\mathbf{f} = \{f_0, \ldots, f_n\}$ of the same degree $d \geq 1$. Set $\mathfrak{m} := (X_1, \ldots, X_n)$.

Throughout the paper I will be assumed to be of codimension at least $n-1$, i.e., that $\dim R/I \leq 1$. In the terminology of rational maps, we are assuming that the base locus of the rational map defined by $\mathbf{f}$ consists of a finite (possibly, empty) set of points. Furthermore, for the purpose of elimination theory we will always assume that $\dim k[\mathbf{f}] = \dim R$, i.e., that the image of the rational map is a hypersurface.

The background for the contents revolves around the use of the so-called approximation complex $\mathcal{Z}$ ([HSV83]) associated to I in order to extract free complexes over a polynomial ring that yield the equation of the eliminated hypersurface, at least in principle. This idea was originated in [BJ03] to which subsequent additions were made in [BC05] and [BCJ09].

We note that the complex $\mathcal{Z} = \mathcal{Z}(I)$ is in the present case an acyclic complex of bigraded modules over the standard bigraded polynomial ring $S := R[T_0, \ldots, T_n]$. The gist of the idea has been to look at the the one-side $\mathbb{N}$-grading of S given by $S = \oplus_{\mu \geq 0} S_\mu$, where $S_\mu := R_\mu \otimes_k k[T_0, \ldots, T_n]$ is naturally a free $k[T_0, \ldots, T_n]$-module. When "restricted" to this $\mathbb{N}$-grading, $\mathcal{Z}(I)$ gives a hold of the corresponding graded

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pieces of the symmetric algebra $\mathrm{Sym}_R(I)$ of I. In order to set up the next stage one has to assume some *threshold degree* beyond which the annihilator of the graded piece of the symmetric algebra stabilizes. The final step is to read the eliminated equation off a matrix of the presentation map of such a graded piece.

One basic question is to express this sort of threshold degree in terms of the numerical invariants stemming from the data, i.e., from I. In [BC05] one such invariant was introduced which involved solely the integers n, d and the initial degree of the $\mathfrak{m}$-saturation of I.

In the present incursion into the question we take a slight diversion by bringing up the symmetric algebra of I modulo its $\mathfrak{m}$-torsion. In a precise way, we shift the focus to the R-algebra $\mathcal{S}_I^* := \mathrm{Sym}_R(I)/H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))$. This algebra is an intermediate homomorphic image of $\mathrm{Sym}_R(I)$ in the way to get the Rees algebra $\mathrm{Rees}_R(I)$ of I. In fact, when I is $\mathfrak{m}$-primary – so to say, half of the cases we have in mind – one has $\mathcal{S}_I^* = \mathrm{Rees}_R(I)$.

Correspondingly, we introduce yet another threshold degree $\mu_0(I)$ involving, besides the basic integers n, d, also numerical data of the Koszul homology of I. All results will of this paper will deal with integers (degrees) μ satisfying $\mu \geq \mu_0(I)$ – any such integer will be named a *threshold integer*. Moreover, a dimension theoretic restriction will be assumed, namely that $\dim \mathrm{Sym}_R(I) = \dim \mathrm{Rees}_R(I)$. By [HSV83, Proposition 8.1], this is equivalent to requiring a typical bound on the local number of minimal generators of I, to wit

$$\nu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + 1, \quad \text{for every prime ideal} \quad \mathfrak{p} \supset I.$$

In the present context, this is no requirement whatsoever if I happens to be $\mathfrak{m}$-primary, and in the codimension $n-1$ case it is imposing that I be generically generated by $n = \dim R$ elements – i.e., a drop by one from the global number of equations. The need for this assumption stems clear from mimicking an almost complete intersection of codimension n.

Section 2 contains the main structural result related to the threshold degree (Theorem 1). Firstly, we prove the vanishing of the graded components, beyond the threshold degree, of all local cohomology modules (of order $i \geq 1$) of $\mathrm{Sym}_R(I)$ with support on $\mathfrak{m}$; and secondly, we prove the freeness as $k[T_0, \ldots, T_n]$-modules, of the graded components, beyond the threshold degree, of $J\langle \ell \rangle / J\langle \ell - 1 \rangle$, for $\ell \geq 2$, alongside with the values of their ranks. Here, $J := \ker(S \twoheadrightarrow \mathcal{S}_I^*)$ with $T_j \mapsto f_j$ and $J\langle \ell \rangle$ denotes its degree ℓ homogeneous part in the standard grading of $R[T_0, \ldots, T_n]$.

As it turns the free $k[T_0, \ldots, T_n]$-modules $(J\langle \ell \rangle / J\langle \ell - 1 \rangle)_\mu$ are crucial in writing a free $k[T_0, \ldots, T_n]$-resolution of the module $(\mathcal{S}_I^*)_\mu$ (μ a threshold integer). This resolution is given in Corollary 1.

Section 3 deals with the $\mathfrak{m}$-primary case. The one main result is a sharp lower bound for the threshold degree in terms of n, d. This bound is actually attained in characteristic zero provided the forms $\mathbf{f}$ are general. The proof depends on the form of the Hilbert series of a well-known $\mathfrak{m}$-primary almost complete intersection – the guessed form of the series is actually not entirely obvious. There are at least two ways of getting it, one of which a Lefschetz type of argument. We added an appendix with a more elementary proof due to Oesterlé.

This bound in turn allows, by tuning up a threshold integer μ, to bound the degrees of the syzygies of I that may appear in the presentation matrix of $(\mathcal{S}_I^*)_\mu$ in the aforementioned free $k[T_0, \ldots, T_n]$-resolution. As a consequence, the form of the resolution for such a choice of a threshold integer becomes more explicit (see Corollary 3).

Another piece of interest in this section is that, in the way of proving Theorem 1, we obtain in the $\mathfrak{m}$-primary case an isomorphism of $k[T_0,\dots,T_n]$-modules

$$(J\langle\ell\rangle/J\langle\ell-1\rangle)_\mu \simeq (H_1)_{\mu+\ell d}\otimes_k k[T_0,\dots,T_n](-\ell),$$

for μ a threshold integer, $\ell\geq 2$. We show that this isomorphism is really the expression of a so-called *downgrading map* (see Proposition 4). Versions of such maps have been considered in [HSV09] and even earlier in a slightly different form ([HSV83]).

In Section 4 we try to replay the results of the previous section when $\dim R/I = 1$. We still obtain a good lower bound for the threshold number in terms of n, d. The argument is different since there is no obvious model to compare the respective Hilbert functions as in the $\mathfrak{m}$-primary case.

Finally, Section 5 is devoted to a few examples of application in implicitization to illustrate how the present theory works in practice.

2. The main theorem

Let $R := k[X_1,\dots,X_n]$, with $n\geq 2$, stand for the standard graded polynomial ring over a field k and let $I\subset R$ denote an ideal generated by k-linearly independent forms $\mathbf{f}=\{f_0,\dots,f_n\}$ of the same degree $d\geq 1$. Set $\mathfrak{m} := (X_1,\dots,X_n)$.

Throughout it will be assumed that $\dim R/I\leq 1$. In addition, for the purpose of implicitization, we assume that $\dim k[\mathbf{f}] = \dim R$, i.e., that the image of the rational map is a hypersurface.

Let $K_i := K_i(f_0,\dots,f_n;R)$ denote the term of degree i of the Koszul complex associated to $\mathbf{f}$, with $Z_i, B_i, H_i = Z_i/B_i$ standing for the module of cycles, the module of borders and the homology module in degree i, respectively. Since the ideal I is homogeneous, these modules inherit a natural structure of graded R-modules.

Letting $T_0,\dots,T_n$ denote new variables over k, set $R' := k[T_0,\dots,T_n]$ and $S := R\otimes_k R' \simeq R[T_0,\dots,T_n]$. Let J stand for the kernel of the following graded R-algebra homomorphism

$$\begin{array}{ccl} S & \to & \mathcal{S}_I^* := \mathrm{Sym}_R(I)/H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I)) \\ T_i & \mapsto & f_i \end{array}$$

The ideal $H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))$ – which could be called the $\mathfrak{m}$-*torsion* of $\mathrm{Sym}_R(I)$ – is contained in the full R-torsion of $\mathrm{Sym}_R(I)$. Therefore, there is a surjective a graded R-homomorphism onto the Rees algebra of I

$$\mathcal{S}_I^* \twoheadrightarrow \mathrm{Rees}_R(I)$$

which is injective if and only if $\nu(I_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$ for every prime $\mathfrak{p}\supset I$ such that $\mathfrak{p}\neq\mathfrak{m}$, where $\nu(\_)$ denotes minimal number of generators. In particular, if I has codimension n (i.e., if I is $\mathfrak{m}$-primary) then $\mathcal{S}_I^*$ is the Rees algebra $\mathrm{Rees}_R(I)$.

Given an integer $\ell\geq 0$ we consider the ideal $J\langle\ell\rangle\subset J$ generated by elements in J whose degree in the T_i's is at most ℓ. Thus $J\langle 0\rangle = 0$ and $J\langle 1\rangle\simeq \mathrm{Syz}_R(f_0,\dots,f_n)S$ via the identification of a syzygy $(a_0,\dots,a_n)$ with the linear form $a_0T_0+\ldots+a_nT_n$, and $\mathrm{Sym}_R(I)\simeq S/J\langle 1\rangle$.

We will denote by $\mathfrak{KS}$ (for $\mathfrak{K}$oszul $\mathfrak{S}$yzygies) the S-ideal generated by the elements $f_iT_j - f_jT_i$ $(0\leq i,j\leq n)$ and set $\overline{S} := S/\mathfrak{KS}$. Notice that $\mathfrak{KS}\subset J\langle 1\rangle$ where the inclusion is strict (for the sequence $\mathbf{f}$ cannot be R-regular). Observe also that the module $J\langle \ell \rangle / J\langle \ell - 1\rangle$ is generated exactly in degree ℓ.

Finally, for any $\mathbb{N}$-graded module M, we will denote

$$\mathrm{indeg}(M) := \inf\{\mu \mid M_\mu \neq 0\},$$

with the convention that $\mathrm{indeg}(0) = +\infty$, and

$$\mathrm{end}(M) := \sup\{\mu \mid M_\mu \neq 0\},$$

with the convention $\mathrm{end}(0) = -\infty$.

We next introduce the basic numerical invariant of this work and give it a name for the sake of easy reference throughout the text.

Definition 1. The *threshold degree* of the ideal $I = (\mathbf{f})$ is the integer

$$\mu_0(I) := (n-1)(d-1) - \min\{\mathrm{indeg}(H_1(\mathbf{f};R)), \mathrm{indeg}(H^0_{\mathfrak{m}}(H_1(\mathbf{f};R))) - d\}.$$

Any integer μ such that $\mu \geq \mu_0(I)$ will be likewise referred to as a *threshold integer*.

Note that the threshold degree does not depend on the choice of a minimal set of generators.

If $\dim(R/I) \leq 1$ then

$$\mu_0(I) = \max\{\mathrm{end}(H^0_{\mathfrak{m}}(R/I)) - d, \mathrm{end}(H^1_{\mathfrak{m}}(H_1(\mathbf{f};R))) - 2d\} + 1$$

by Koszul duality.

The threshold degree will play a key role throughout this paper and it will soon become clear why it is called this way. Notice that whenever I is $\mathfrak{m}$-primary, then $\mu_0(I) = \mathrm{reg}(I) - d$, where

$$\mathrm{reg}(I) = \min\{\nu \text{ such that } H^i_{\mathfrak{m}}(I)_{>\nu - i} = 0\}$$

stands for the Castelnuovo-Mumford regularity of I.

Also, in the $\mathfrak{m}$-primary case, the threshold degree is related to the numerical invariant $r(I)$ introduced in [HSV09, Theorem 2.14], namely, one has $r(I) + d = \mu_0(I)$.

The more detailed nature of $\mu_0(I)$ will be discussed in Sections 3.1 and 4.1.

We will hereafter consider the one-side $\mathbb{N}$-grading of S given by $S = \oplus_{\mu \geq 0} S_\mu$, where $S_\mu := R_\mu \otimes_k R'$. Likewise, if M is a bigraded S-module, then M_μ stands for the homogeneous component of degree μ of M as an $\mathbb{N}$-graded module over S endowed with the one-side grading. Note that M_μ is an R'-module.

Recall that our standing setup has $\dim(R/I) \leq 1$, i.e., either I is $\mathfrak{m}$-primary or has codimension one less. Most of the subsequent results will deal with integers (degrees) μ satisfying $\mu \geq \mu_0(I)$ – recall that any such integer is being named a threshold integer. Moreover, it will be assumed throughout that $\dim \mathrm{Sym}_R(I) = \dim \mathrm{Rees}_R(I)$, which by [HSV83, Proposition 8.1] is tantamount to requiring the well-known bounds

$$\nu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + 1, \quad \text{for every prime ideal} \quad \mathfrak{p} \supset I.$$

The following basic preliminary seems to have gone unnoticed.

Lemma 1. *Let* $\dim(R/I) \leq 1$. *If* $\nu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + 1$ *for every prime ideal* $\mathfrak{p} \supset I$, *then* $\mathrm{indeg}(H_2) \geq \mathrm{indeg}(H_1) + d$.

Proof. First, if $\dim(R/I) = 0$ then $H_2 = 0$ and the claimed inequality holds by convention. Consequently, from now on we assume that $\dim(R/I) = 1$. We may assume that k is an infinite field. Let $\mathbf{g} := \{g_1, \ldots, g_n\}$ be general k-linear combinations of the f_i's and set $J := (\mathbf{g})$. Since $\nu(I_\mathfrak{p}) \leq \dim R_\mathfrak{p} + 1$ for every non maximal prime ideal $\mathfrak{p} \supset I$, J and I have the same saturation with respect to $\mathfrak{m}$. Also, since the f_i's are minimal generators of I, they form a k-basis of the vector space I_d. Therefore, $I = J + (f)$ for some $f \in I_d$, hence $H_i \simeq H_i(\mathbf{g}, f; R)$ for every integer i. Moreover, one has the exact sequence of complexes

$$0 \to K_\bullet(\mathbf{g}; R) \xrightarrow{\iota} K_\bullet(\mathbf{g}, f; R) \xrightarrow{\pi} K_{\bullet-1}(\mathbf{g}; R)[-d] \to 0$$

where ι is the canonical inclusion and π the canonical projection. It yields the long exact sequence

$$\begin{aligned}(1) \qquad \ldots \to H_2(\mathbf{g}; R) \xrightarrow{\overline{\iota_2}} H_2 \xrightarrow{\overline{\pi_2}} H_1(\mathbf{g}; R)[-d] \xrightarrow{\cdot(-f)} H_1(\mathbf{g}; R)& \\ \xrightarrow{\overline{\iota_1}} H_1 \xrightarrow{\overline{\pi_1}} H_0(\mathbf{g}; R)[-d] \xrightarrow{\cdot f} H_0(\mathbf{g}; R) \xrightarrow{\overline{\iota_0}} H_0 \to 0&\end{aligned}$$

Now, since $\mathbf{g}$ is an almost complete intersection $H_1(\mathbf{g}; R)$ is isomorphic to the canonical module $\omega_{A/J}$ (see [BC05, Proof of Lemma 2]). But the latter is annihilated by J, hence the map

$$H_1(\mathbf{g}; R)[-d] \xrightarrow{\cdot(-f)} H_1(\mathbf{g}; R)$$

is the null map. By inspecting (1), it follows that $H_2 \simeq H_1(\mathbf{g}; R)[-d]$ and that $H_1(\mathbf{g}; R)$ is a submodule of H_1. Consequently, for every ν such that $(H_1)_\nu = 0$, we have $(H_2)_{\nu+d} = 0$. $\square$

We now prove a vanishing result for the local cohomology modules of the modules of cycles of $\mathbf{f}$ in terms of the threshold degree $\mu_0(I)$.

Proposition 1. *Let* $\dim(R/I) \leq 1$. *If* $\nu(I_\mathfrak{p}) \leq \dim R_\mathfrak{p} + 1$ *for every prime ideal* $\mathfrak{p} \supset I$, *then for every threshold integer* μ *one has*

$$(2) \qquad H^q_\mathfrak{m}(Z_p)_{\mu+pd} = 0 \quad \textit{for} \quad p \neq q$$

Proof. Consider the approximation complex of cycles associated to the ideal $I = (f_0, \ldots, f_n)$. We denote it by $\mathcal{Z}$. By definition, we have $\mathcal{Z}_i = Z_i[id] \otimes_R S\{-i\}$ so that $(\mathcal{Z}_i)_\mu = (Z_i)_{\mu+id} \otimes_k R'\{-i\}$. As proved in [BC05, Theorem 4], the complex $\mathcal{Z}$ is acyclic under our assumptions. To take advantage of the spectral sequences associated to the double complex $\mathcal{C}^p_\mathfrak{m}(\mathcal{Z}_q)$, the knowledge of the local cohomology of the cycles of the Koszul complex associated to $f_0, \ldots, f_n$ is helpful. One has the following graded (degree zero) isomorphisms of R-modules [BC05, Lemma 1]:

$$(3) \qquad H^q_\mathfrak{m}(Z_p) \simeq \begin{cases} 0 & \text{if } q < 2 \text{ or } q > n \\ H^0_\mathfrak{m}(H_{q-p})^*[n-(n+1)d] & \text{if } q = 2 \\ {H_{q-p}}^*[n-(n+1)d] & \text{if } 2 < q \leq n-1 \\ {Z_{n-p}}^*[n-(n+1)d] & \text{if } q = n \end{cases}$$

where $—^* := \mathrm{Homgr}_R(—, k)$, and also [BC05, Proof of Lemma 1]

$$(4) \qquad H^0_\mathfrak{m}(H_0)^*[n-(n+1)d] \simeq H^2_\mathfrak{m}(Z_2) \simeq H^0_\mathfrak{m}(H_1)$$

We now consider various cases.

The vanishing is obvious if $q < 2$ and $q > n$.

If $q = 2$ then there is only one non-trivial module

$$H^2_{\mathfrak{m}}(Z_1) \simeq H^0_{\mathfrak{m}}(H_1)^*[n-(n+1)d]$$

(observe that $Z_0 = S$ so that $H^2_{\mathfrak{m}}(Z_0)_\mu = 0$ for $\mu \geq -1$). This isomorphism shows that $H^2_{\mathfrak{m}}(Z_1)_\mu = 0$ for every

$$\mu \geq (n+1)d - n + 1 - \operatorname{indeg}(H^0_{\mathfrak{m}}(H_1)) = (n-1)(d-1) + 2d - \operatorname{indeg}(H^0_{\mathfrak{m}}(H_1))$$

so that $H^2_{\mathfrak{m}}(Z_1)_{\mu+d} = 0$ for every $\mu \geq (n-1)(d-1) - (\operatorname{indeg}(H^0_{\mathfrak{m}}(H_1)) - d)$.

Now, assume that $3 \leq q \leq n-1$. From (3) one only has to consider the modules H_1^* and H_2^* because $H_i = 0$ if $i > 2$. By definition, $H_1^*[n-(n+1)d]_\mu = 0$ for every $\mu \geq (n-1)(d-1) + 2d - \operatorname{indeg}(H_1)$ and similarly $H_2^*[n-(n+1)d]_\mu = 0$ for every $\mu \geq (n-1)(d-1) + 2d - \operatorname{indeg}(H_2)$. It follows that $H^q_{\mathfrak{m}}(Z_p)_{\mu+pd} = 0$, $p \neq q$, for every

$$\mu \geq (n-1)(d-1) - \min(\operatorname{indeg}(H_1), \operatorname{indeg}(H_2) - d) \geq (n-1)(d-1) - \operatorname{indeg}(H_1)$$

where the last inequality holds by Lemma 1.

Finally, if $q = n$ then for every $\mu \in \mathbb{Z}$ and $p < n$ we have

$$H^n_{\mathfrak{m}}(Z_p)_{\mu+pd} \simeq ({Z_{n-p}}^*)_{\mu+n-(n-p+1)d}$$

Moreover, since $\operatorname{indeg}(B_{n-p}) = (n-p+1)d$ the exact sequence

$$0 \to B_{n-p} \to Z_{n-p} \to H_{n-p} \to 0$$

shows that

$$({Z_{n-p}}^*)_{\mu+n-(n-p+1)d} \simeq ({H_{n-p}}^*)_{\mu+n-(n-p+1)d}$$

for every μ such that $\mu + n - (n-p+1)d > -(n-p+1)d$, that is to say for every $\mu > -n$. Therefore, we get that for every $\mu > -n$ and $p < n$

$$H^n_{\mathfrak{m}}(Z_p)_{\mu+pd} \simeq ({H_{n-p}}^*)_{\mu+n-(n-p+1)d}$$

It follows, as in the previous case, that $H^n_{\mathfrak{m}}(Z_p)_{\mu+pd} = 0$ for $\mu > \mu_0(I)$ and $p < n$. □

Theorem 1. *Let* $\dim(R/I) \leq 1$. *If* $\nu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + 1$ *for every prime ideal* $\mathfrak{p} \supset I$ *then for every threshold integer* μ, *one has:*

(i) $H^i_{\mathfrak{m}}(\operatorname{Sym}_R(I))_\mu = 0$ *for* $i > 0$ *and*
(ii) *for every* $\ell \geq 2$ *the* R'*-module* $(J\langle \ell \rangle / J\langle \ell - 1 \rangle)_\mu$ *is free of rank*

$$\begin{cases} \dim_k(H^0_{\mathfrak{m}}(H_1))_{\mu+2d} & \text{if} \quad \ell = 2 \\ \dim_k(H^0_{\mathfrak{m}}(H_1))_{\mu+\ell d} + \dim_k(R/I^{sat})_{(n+1-\ell)d-n-\mu} & \text{otherwise.} \end{cases}$$

Proof. By definition of the $\mathcal{Z}$-complex, $H^q_{\mathfrak{m}}(Z_p)_{\mu+pd} = H^q_{\mathfrak{m}}(\mathcal{Z}_p)_\mu$ for any μ. Fix an integer $\mu \geq \mu_0(I)$. By (2), the spectral sequence $H^q_{\mathfrak{m}}(\mathcal{Z}_p)_\mu \Rightarrow H^{q-p}_{\mathfrak{m}}(\operatorname{Sym}_R(I))_\mu$ implies that $H^i_{\mathfrak{m}}(\operatorname{Sym}_R(I))_\mu = 0$ for $i > 0$, which proves (i), also providing a filtration of the R'-module $H^0_{\mathfrak{m}}(\operatorname{Sym}_R(I))_\mu$

$$0 = F_0 = F_1 \subset F_2 \subset \cdots \subset F_t = H^0_{\mathfrak{m}}(\operatorname{Sym}_R(I)_\mu) = (J/J\langle 1 \rangle)_\mu \tag{5}$$

such that, by (4),

$$\begin{aligned} F_2 &\simeq H^2_{\mathfrak{m}}(\mathcal{Z}_2)_\mu \simeq H^0_{\mathfrak{m}}(H_0)^*[n-(n+1)d]_{\mu+2d} \otimes_k R'\{-2\} \\ &\simeq H^0_{\mathfrak{m}}(H_1)_{\mu+2d} \otimes_k R'\{-2\}. \end{aligned} \tag{6}$$

Clearly, (6) shows that F_2 is a finite free R'-module of rank $\dim_k(H^0_{\mathfrak{m}}(H_1))_{\mu+2d}$.

By a similar token, for every $\ell \geq 3$,

$$F_\ell/F_{\ell-1} \simeq H^\ell_{\mathfrak{m}}(\mathcal{Z}_\ell)_\mu \simeq {H_0}^*[n-(n+1)d]_{\mu+\ell d} \otimes_k R'\{-\ell\} \tag{7}$$

In particular, $F_\ell/F_{\ell-1}$ is a free R'-module which is generated in degree ℓ. Now, as $(J\langle\ell\rangle/J\langle\ell-1\rangle)_\mu$ is also generated in degree ℓ, an easy recursive argument on $\ell \geq 1$ yields $F_\ell = (J\langle\ell\rangle/J\langle 1\rangle)_\mu$.

Therefore $(J\langle 2\rangle/J\langle 1\rangle)_\mu$ is a free R'-module of rank $\dim_k(H^0_{\mathfrak{m}}(H_1))_{\mu+2d}$, which shows the case $\ell = 2$ of item (ii).

To get the case $\ell \geq 3$, recall that $H^1_{\mathfrak{m}}(Z_2) = 0$. According to [BC05, Equation (3) in Proof 3], there is an exact sequence of graded R-modules

$$0 \to H^1_{\mathfrak{m}}(H_2) \to {H_0}^*[n-(n+1)d] \to H^2_{\mathfrak{m}}(Z_2) \to 0 \tag{8}$$

Moreover, $H^2_{\mathfrak{m}}(Z_2) \simeq H^0_{\mathfrak{m}}(H_1)$ by (4) and by local duality

$$H^1_{\mathfrak{m}}(H_2) \simeq (R/I^{\mathrm{sat}})^*[n-(n+1)d].$$

Therefore, we deduce that if $\ell \geq 3$ then $F_\ell/F_{\ell-1} \simeq (J\langle\ell\rangle/J\langle\ell-1\rangle)_\mu$ is a finite free R'-module of rank

$$\dim_k((H_0)^*[n-(n+1)d])_{\mu+\ell d} = \dim_k(H^0_{\mathfrak{m}}(H_1))_{\mu+\ell d} + \dim_k(R/I^{\mathrm{sat}})_{(n+1-\ell)d-n-\mu}.$$

This finishes the proof. ☐

Corollary 1. *Let* $\dim(R/I) \leq 1$. *If* $\nu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + 1$ *for every prime ideal* $\mathfrak{p} \supset I$ *then, for every threshold integer* μ*, the* R'*-module* $(\mathcal{S}^*_I)_\mu$ *admits a minimal graded free* R'*-resolution of the form*

$$\cdots \to (\mathcal{Z}_i)_\mu \to \cdots \to (\mathcal{Z}_2)_\mu \to (\mathcal{Z}_1)_\mu \oplus^n_{\ell=2} (J\langle\ell\rangle/J\langle\ell-1\rangle)_\mu \to (\mathcal{Z}_0)_\mu = R_\mu \otimes_k R'. \tag{9}$$

Proof. By Theorem 1, $(J\langle\ell\rangle/J\langle\ell-1\rangle)_\mu$ is a free R'-module for $\ell \geq 2$. Therefore the split exact sequence

$$0 \to J\langle\ell-1\rangle_\mu \to J\langle\ell\rangle_\mu \to (J\langle\ell\rangle/J\langle\ell-1\rangle)_\mu \to 0$$

gives rise to an exact sequence

$$0 \to J\langle\ell-1\rangle_\mu \otimes_{R'} k \to J\langle\ell\rangle_\mu \otimes_{R'} k \to (J\langle\ell\rangle/J\langle\ell-1\rangle)_\mu \otimes_{R'} k \to 0,$$

which shows that the minimal generators of $J\langle\ell\rangle_\mu$ are the minimal generators of $J\langle\ell-1\rangle_\mu$ plus the generators of $(J\langle\ell\rangle/J\langle\ell-1\rangle)_\mu$. Moreover, we also deduce the isomorphisms $\mathrm{Tor}^{R'}_i(J\langle\ell\rangle_\mu, k) \simeq \mathrm{Tor}^{R'}_i(J\langle\ell-1\rangle_\mu, k)$ for $i > 0$ that imply by induction that $\mathrm{Tor}^{R'}_i(J_\mu, k) \simeq \mathrm{Tor}^{R'}_i(J\langle 1\rangle_\mu, k)$ for $i > 0$. It follows that the complex (9), which is built by adding to the complex $(\mathcal{Z}_\bullet)_\mu$ the canonical map

$$\oplus^n_{\ell=2}(J\langle\ell\rangle/J\langle\ell-1\rangle)_\mu \to R_\mu \otimes_k R'$$

is acyclic. ☐

Corollary 2. *Let* $\dim(R/I) \leq 1$. *If* $\nu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}}$ *for every non maximal prime ideal* $\mathfrak{p} \supset I$ *then, for every threshold integer* μ*, the* R'*-module* $\mathrm{Rees}_R(I)_\mu$ *admits* (9) *as a minimal graded free* R'*-resolution*

Proof. This follows from the well known property that $\mathcal{S}^*_I = \mathrm{Rees}_R(I)$ if and only if $\nu(I_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$ for every non maximal prime ideal $\mathfrak{p} \supset I$. ☐

3. The $\mathfrak{m}$-primary case

In this section, we will concentrate on the case where the ideal I is $\mathfrak{m}$-primary. As we will see, in such a situation it is remarkable that all the syzygies of I used in the matrix-based representation of a hypersurface $\mathcal{H}$ obtained by this method can be recovered from the linear syzygies via downgrading maps; this is described in Section 3.3.

Recall that in the $\mathfrak{m}$-primary case, $H^1_{\mathfrak{m}}(H_1) = 0$, $H^0_{\mathfrak{m}}(H_1) = H_1$ and we have

$$\mu_0 = \mathrm{reg}(I) - d = \mathrm{end}(H^0_{\mathfrak{m}}(R/I)) - d + 1 = n(d-1) - \mathrm{indeg}(H_1) + 1$$

3.1. Bounds for the threshold degree.

It is clear that $\mu_0(I) \leq (n-1)(d-1)$. In this section, we will give a sharp lower bound for $\mu_0(I)$. We begin with a preliminary result.

Proposition 2. *We have*

$$\mathrm{end}(R/I) \geq \left\lfloor \frac{(n+1)(d-1)}{2} \right\rfloor$$

and equality holds if the forms $f_0, \ldots, f_n$ are sufficiently general, and k has characteristic 0.

Proof. [Iarrobino-Stanley; see also the Appendix at the end of this paper] By the Künneth formula, $X := (\mathbf{P}^{d-1})^n$ has de Rham cohomology ring isomorphic to

$$C = \mathbf{Z}[\omega_1, \ldots, \omega_n]/(\omega_1^d, \ldots, \omega_n^d),$$

where ω_i is a Kähler form on the i-th factor $\mathbf{P}^{d-1}$ and the cup product in the cohomology ring corresponds to the usual product in C. The form $\omega := \omega_1 + \cdots + \omega_n$ is a Kähler form on X and the Hard Lefschetz Theorem applied to ω shows that for $0 \leq \mu < n(d-1)/2$, the multiplication by ω in C_μ is injective and the multiplication by $\omega^{n(d-1)-2\mu}$ induces the Poincaré duality from $C_\mu \otimes_{\mathbf{Z}} \mathbf{Q}$ to $C_{n(d-1)-\mu} \otimes_{\mathbf{Z}} \mathbf{Q}$. It follows that for $\mu \leq \nu$, multiplication by $\omega^{\nu-\mu}$ in $C \otimes_{\mathbf{Z}} \mathbf{Q}$ is injective if $\dim C_\mu \leq \dim C_\nu$ and onto if $\dim C_\mu \geq \dim C_\nu$.

This shows that the Hilbert function of $B := \mathbf{Q}[\omega_1, \ldots, \omega_n]/(\omega_1^d, \ldots, \omega_n^d, \omega^d)$ is $h(i) := \max\{0, a_i\}$, where a_i is given by

$$\frac{(1-t^d)^{n+1}}{(1-t)^n} = \sum_{i=0}^{(n+1)d-n} a_i t^i.$$

As $a_i = -a_{(n+1)d-n-i}$, $a_i > 0$ if and only if $0 < i \leq \left\lfloor \frac{(n+1)(d-1)}{2} \right\rfloor$, hence $\mathrm{end}(B) = \left\lfloor \frac{(n+1)(d-1)}{2} \right\rfloor$. Now notice that if the f_i's are general forms of degree d, then n of them, say $f_1, \ldots, f_n$, form a regular sequence. The quotient A by this regular sequence has the same Hilbert function as C and therefore the Hilbert function of R/I is bounded below by $h(i) := \max\{0, \dim_k A_i - \dim_k A_{i-d}\}$. As this lower bound is reached by B, it is reached on a non empty Zariski open subset of the coefficients of $n+1$ forms of degree d, if the field contains $\mathbf{Q}$. □

Corollary 3. *If I is $\mathfrak{m}$-primary, then the threshold degree satisfies the inequalities*

$$\left\lfloor \frac{(n-1)(d-1)}{2} \right\rfloor \leq \mu_0(I) \leq (n-1)(d-1)$$

and equality on the left holds if the forms $f_0, \dots, f_n$ are sufficiently general, and k has characteristic 0.

Proof. If the forms $f_0, \dots, f_n$ are sufficiently general, then the ideal I is $\mathfrak{m}$-primary and it follows that $H^0_{\mathfrak{m}}(H_i) = H_i$, $i = 0, 1$. Therefore

$$\mu_0(I) = (n-1)(d-1) - \operatorname{indeg}(H_1) + d = \\ (n-1)(d-1) - (n+1)d + n + \left\lfloor \frac{(n+1)(d-1)}{2} \right\rfloor + d$$

where the second equality follows by Koszul duality, and the stated formula follows from a straightforward computation. □

3.2. **Free resolutions.** Let M_μ denote the matrix of the presentation map in Corollary 1

$$(\mathcal{Z}_1)_\mu \oplus_{\ell=2}^n (J\langle \ell \rangle / J\langle \ell - 1 \rangle)_\mu \to (\mathcal{Z}_0)_\mu = R_\mu \otimes_k R,$$

with respect to a fixed, but otherwise arbitrary, basis.

The following result gives the degree of the syzygies that appears in this matrix. This degree depends on the choice of the integer μ.

Proposition 3. *Assume that I is $\mathfrak{m}$-primary and let μ denote a threshold integer. Suppose that ℓ is an integer such that some syzygy of the power I^ℓ appears in the matrix M_μ as defined above. Then*

$$\ell \leq \left\lceil \frac{\operatorname{indeg}(H_1)}{d} \right\rceil \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. By Theorem 1, if $\mu + (\ell+1)d > \operatorname{end}(H_1)$, then M_μ involves syzygies of I^j with $1 \leq j \leq \ell$ for $\mu \geq \mu_0$. Now $\operatorname{end}(H_1) = (n+1)d - n$ and the equation

$$(n-1)(d-1) - \operatorname{indeg}(H_1) + d + (\ell+1)d \geq (n+1)d - n + 1$$

can be rewritten $\ell d \geq \operatorname{indeg}(H_1)$. By Corollary 3 it obtains

$$\operatorname{indeg}(H_1) \leq (n-1)(d-1) + d - \left\lfloor \frac{(n-1)(d-1)}{2} \right\rfloor < \frac{(n+1)d}{2},$$

which shows the second estimate. □

The above proposition also shows that one can tune a threshold integer μ in order to bound the degree of the syzygies of I that may appear in the matrix M_μ. More precisely, choose an integer $l \in \{1, \dots, \lceil \frac{n+1}{2} \rceil\}$. Then, M_μ involves only syzygies of I of degree at most l for every

$$\mu \geq \max\{(n-1)(d-1) - (l-1)d, \mu_0(I)\} = \max\{(n-l)(d-1) - (l-1), \mu_0(I)\}$$

For instance, M_μ involves only linear syzygies of I (i.e. $l = 1$) for every $\mu \geq (n-1)(d-1)$, and it involves only linear and quadratic syzygies of I for every $\mu \geq \max\{(n-2)(d-1) - 1, \mu_0(I)\}$, and so forth.

Corollary 4. *If the forms $f_0, \dots, f_n$ define an $\mathfrak{m}$-primary ideal then for any threshold integer μ the graded R'-module $(\mathcal{S}_I^*)_\mu$ has a minimal graded free R'-resolution of the form*

$$\cdots \to R'\{-i\}^{b_i} \to \cdots \to R'\{-2\}^{b_2} \to R'\{-1\}^{b_1} \oplus_{\ell=1}^{\lceil \frac{n+1}{2} \rceil} R'\{-\ell\}^{\beta_\ell} \to R'^{\binom{\mu+n-1}{n-1}} \tag{10}$$

with $\beta_\ell := \dim_k(H_0)_{(n+1-\ell)d-n-\mu}$ *and*

$$b_i := \dim_k(B_i)_{\mu+id} = \sum_{k=1}^{\min\{n-i,\lfloor \frac{\mu}{d} \rfloor\}} (-1)^{k+1} \binom{\mu - kd + n - 1}{n-1}^{\binom{n}{i+k}}$$

Proof. $\mathfrak{K}\mathfrak{S}$ is resolved by the back part of the $\mathcal{Z}$-complex and, as $H_i = 0$ for $i \geq 2$, this coincides with the corresponding part of the $\mathcal{B}$-complex. □

It is interesting to describe explicitly the maps involved in the resolution (10).

- The map $R'\{-i\}^{b_i} \to R'\{-i+1\}^{b_{i-1}}$ is the degree μ part of the i-th map in the $\mathcal{B}$-complex. It is given by $(B_i)_{\mu+id} \otimes_k R'\{-i\} \xrightarrow{d_i^T} (B_{i-1})_{\mu+(i-1)d} \otimes_k R'\{-i+1\}$ (composed with the inclusion $(B_0)_\mu \otimes_k R' = I_\mu \otimes_k R' \subset R_\mu \otimes_k R'$ for $i = 1$).
- The matrix of the map $R'\{-1\}^{\beta_1} \to R'^{\binom{\mu+n-1}{n-1}} = R_\mu \otimes_k R'$ is given by pre-images in $S_{\mu,1}$ of a basis over k of $(H_1)_{\mu+d}$ (the syzygies $\sum_i a_i T_i$ with a_i of degree μ, modulo the Koszul syzygies).
- The matrix of the map $R'\{-\ell\}^{\beta_\ell} \to R'^{\binom{\mu+n-1}{n-1}} = R_\mu \otimes_k R'$ for $\ell \geq 2$ is given by pre-images in $S_{\mu,\ell}$ of a basis of $(J\langle \ell \rangle / J\langle \ell - 1 \rangle)_{\mu,\ell}$ over k

3.3. **Downgrading maps.** Assuming that I is $\mathfrak{m}$-primary (notice that in this case $J = (J\langle 1 \rangle : \mathfrak{m}^\infty)$), the homology modules H_0 and H_1 are supported on $V(I) = V(\mathfrak{m})$ so that (8) shows that

$$H_0{}^*[n-(n+1)d] \simeq H^0_{\mathfrak{m}}(H_0)^*[n-(n+1)d] \simeq H^2_{\mathfrak{m}}(Z_2) \simeq H^0_{\mathfrak{m}}(H_1) = H_1$$

Therefore, (6) and (7) imply that for every $\ell \geq 2$ and every threshold integer μ we have a graded isomorphism of R'-modules

$$(J\langle \ell \rangle / J\langle \ell - 1 \rangle)_\mu \simeq (H_1)_{\mu+\ell d} \otimes_k R'\{-\ell\} \tag{11}$$

The purpose of this section is to show that (11) is realized by a *downgrading map* – versions of such maps have been considered in [HSV09] and even earlier in a slightly different form ([HSV83]).

Namely, define the map $\delta : S/\mathfrak{K}\mathfrak{S} \to (S/\mathfrak{K}\mathfrak{S})[d]\{-1\}$ by

$$\begin{array}{rcl} \delta_p : (S/\mathfrak{K}\mathfrak{S})_p & \to & (S/\mathfrak{K}\mathfrak{S})_{p-1}[d] \\ \displaystyle\sum_{0 \leq i_1, \ldots, i_p \leq n} c_{i_1,\ldots,i_p} T_{i_1} \ldots T_{i_p} & \mapsto & \displaystyle\sum_{0 \leq i_1, \ldots, i_p \leq n} c_{i_1,\ldots,i_p} f_{i_1} T_{i_2} \ldots T_{i_p} \end{array}$$

Note that this map is well defined. In addition, it induces for any integer $\mu \geq 0$ a (well-defined) homogeneous map of graded R'-modules

$$\lambda_2^\mu : (J\langle 2 \rangle / J\langle 1 \rangle)_\mu \to (J\langle 1 \rangle / \mathfrak{K}\mathfrak{S})\{-1\}_{\mu+d},$$

and, for any integers $\mu \geq 0$ and $p \geq 3$, a (well-defined) homogeneous map of graded R'-modules

$$\lambda_p^\mu : (J\langle p \rangle / J\langle p-1 \rangle)_\mu \to (J\langle p-1 \rangle / J\langle p-2 \rangle)\{-1\}_{\mu+d}.$$

Lemma 2. *The map* λ_2^μ *is injective for every* $\mu \geq 0$ *and is surjective for every threshold integer* μ.

Proof. By definition, $(J\langle 1\rangle/\mathfrak{K}\mathfrak{S})_\nu \simeq (H_1)_{\nu+d} \otimes_k R'\{-1\}$ for every integer $\nu \geq 0$. Therefore, since λ_2^μ is a graded map and since (11) is a graded isomorphism, it suffices to show that λ_2^μ is injective for every $\mu \geq 0$.

Let $\alpha := \sum_{0\leq i,j\leq n} c_{i,j}T_iT_j \in \ker(\lambda_2^\mu)$. By a standard property of Koszul syzygies, there exists a skew-symmetric matrix $(a_{i,j})_{0\leq i,j\leq n}$ with entries in R_μ such that

$$\sum_{0\leq i,j\leq n} c_{i,j} f_i T_j = \sum_{0\leq i,j\leq n} a_{i,j} f_i T_j$$

in $R_{\mu+d} \otimes_k R'$. It follows that, for every $j = 0,\ldots,n$, $\sum_{0\leq i\leq n}(c_{i,j} - a_{i,j})f_i = 0$, i.e., $\sum_{0\leq i\leq n}(c_{i,j} - a_{i,j})T_i \in J\langle 1\rangle$, $\quad j = 0,\ldots,n$. Thus

$$\sum_{0\leq i,j\leq n} (c_{i,j} - a_{i,j})T_iT_j \in J\langle 1\rangle, \quad j = 0,\ldots,n. \tag{12}$$

Therefore, since $\sum_{0\leq i,j\leq n} a_{i,j}T_iT_j \in J\langle 1\rangle$, one has $\alpha \in J\langle 1\rangle$, as was to be shown. $\square$

Remark 1. The above result can also be deduced from [HSV09, Lemma 2.11] and [HSV09, Theorem 2.14].

Proposition 4. *For every integer $p \geq 2$ and every threshold integer μ the map λ_p^μ is an isomorphism.*

Proof. By the same token, according to (11), it suffices to show that λ_p^μ is injective for every $p \geq 2$ and every $\mu \geq \mu_0(I)$. By Lemma 2, the claim holds for $p = 2$.

Now assume that $p \geq 3$. If $p = 3$ pick an element

$$\alpha = \sum_{i=0}^n B_iT_i \in (J\langle 3\rangle)_{\mu,3}$$

such that $\lambda_3^\mu(\overline{\alpha}) = 0$. It follows that

$$\delta_3^\mu(\alpha) = \sum_{i=0}^n \delta_2^\mu(B_i)T_i \in (J\langle 1\rangle)_{\mu+d,2}$$

By a similar argument to the one employed in the proof of Lemma 2, it follows that $\delta_2^\mu(B_i) \in (J\langle 1\rangle/\mathfrak{K}\mathfrak{S})_{\mu+d,1}$ for every $i = 0,\ldots,n$. Therefore, since λ_2^μ is an isomorphism, it follows that $B_i \in (J\langle 2\rangle/J\langle 1\rangle)_{\mu,2}$ and hence that $\overline{\alpha} = 0$ in $(J\langle 3\rangle/J\langle 2\rangle)_\mu$.

Now, we proceed by induction on the integer $p \geq 2$ and assume that $p > 3$. Pick an element

$$\alpha = \sum_{i=0}^n B_iT_i \in (J\langle p\rangle)_{\mu,p}$$

such that $\lambda_p^\mu(\overline{\alpha}) = 0$, it follows that

$$\delta_p^\mu(\alpha) = \sum_i \delta_{p-1}^\mu(B_i)T_i \in (J\langle p-2\rangle)_{\mu+d,p-1}$$

By Theorem 1, $(J\langle p-2\rangle/J\langle p-3\rangle)_{\mu+d}$ is a free graded R'-module which is generated in degree $p-2$. Therefore, we deduce that $\delta_{p-1}^\mu(B_i) \in (J\langle p-2\rangle/J\langle p-3\rangle)_{\mu+d,p-2}$ for every $i = 0,\ldots,n$. Since λ_{p-1}^μ is an isomorphism by our inductive hypothesis, it follows that $B_i \in (J\langle p-1\rangle/J\langle p-2\rangle)_{\mu,p-1}$ and hence that $\alpha = 0$ in $(J\langle p\rangle/J\langle p-1\rangle)_\mu$. $\square$

4. The one-dimensional case

In this section, we go back to the general situation where $\dim(R/I) \leq 1$ and will no longer assume that the ideal I is $\mathfrak{m}$-primary. This more general class of ideals have interesting applications to the implicitization of surfaces in a projective space defined by a parametrization whose base locus is a finite set of points.

4.1. **Bounds for the threshold degree.** Recall the definition of the threshold degree:

$$\mu_0(I) := (n-1)(d-1) - \min\{\mathrm{indeg}(H_1), \mathrm{indeg}(H^0_{\mathfrak{m}}(H_1)) - d\}.$$

Following [BC05], one sets

$$\nu_0(I) := (n-1)(d-1) - \mathrm{indeg}(I^{\mathrm{sat}})$$

Proposition 5. *Let* $\dim(R/I) \leq 1$. *If* $\nu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + 1$ *for every prime ideal* $\mathfrak{p} \supset I$ *then*

$$\mathrm{reg}(R/I) - d \leq \mu_0(I) \leq \nu_0(I) \leq (n-1)(d-1)$$

for $n \geq 3$ *and* $d \geq 2$.

Proof. First, notice that since $0 \leq \mathrm{indeg}(I^{\mathrm{sat}}) \leq d$, it is clear that $0 \leq \nu_0(I) \leq (n-1)(d-1)$.

Now, since $\mathrm{indeg}(H_1) \geq d$ and $\mathrm{indeg}(I^{\mathrm{sat}}) \leq d$, if one proves that

$$\mathrm{indeg}(H^0_{\mathfrak{m}}(H_1)) \geq d + \mathrm{indeg}(I^{\mathrm{sat}}) \tag{13}$$

then the inequality $\mu_0(I) \leq \nu_0(I)$ will follow. We may assume that k is an infinite field. Let $g_1, \ldots, g_n$ be general k-linear combinations of the f_i's and set $J := (g_1, \ldots, g_n)$. The ideals I and J have the same saturation with respect to $\mathfrak{m}$ and hence $\mathrm{indeg}(I^{sat}) = \mathrm{indeg}(J^{sat})$. Moreover, since I is minimally generated we necessarily have $J \subsetneq J^{\mathrm{sat}}$ (for $I^{\mathrm{sat}} = J^{sat} = J \subset I \subset I^{\mathrm{sat}}$ implies that $I = J$). As shown in the proof of Lemma 1, one has an exact sequence

$$0 \to H_1(g_1, \ldots, g_n; R) \to H_1 \to 0 :_{R/J} (f_i)[-d] \to 0.$$

Therefore, since $H^0_{\mathfrak{m}}(H_1(g_1, \ldots, g_n; R)) \simeq H^0_{\mathfrak{m}}(H_2) = 0$, e.g., by (3), it obtains

$$\begin{aligned}
\mathrm{indeg}(H^0_{\mathfrak{m}}(H_1)) &\geq \mathrm{indeg}(H^0_{\mathfrak{m}}(0 :_{R/J} (f_i))[-d]) \\
&= d + \mathrm{indeg}(0 :_{H^0_{\mathfrak{m}}(R/J)} (f_i)) \\
&\geq d + \mathrm{indeg}(H^0_{\mathfrak{m}}(R/J)) \\
&= d + \mathrm{indeg}(J^{\mathrm{sat}}/J) \\
&\geq d + \mathrm{indeg}(J^{\mathrm{sat}}) = d + \mathrm{indeg}(I^{\mathrm{sat}})
\end{aligned}$$

By Theorem 1, $H^i_{\mathfrak{m}}(\mathrm{Sym}^1_R(I))_\mu = H^i_{\mathfrak{m}}(I)_{\mu+d} \simeq H^{i-1}_{\mathfrak{m}}(R/I)_{\mu+d} = 0$ for $\mu \geq \mu_0$ and $i > 0$. This proves that $\mathrm{reg}(R/I) \leq \mu_0(I) + d$, as $H^i_{\mathfrak{m}}(R/I) = 0$ for $i > 1$. $\square$

In addition to the above result, it is also possible to provide a lower bound for the threshold degree solely in terms of n and d. For this purpose, we begin with a technical result.

Lemma 3. *Let* $n \geq 2$ *and* $d \geq 2$ *be two integers and consider the polynomial*

$$\frac{(1-t^d)^n}{(1-t)^{n-1}} = \sum_{i=0}^{n(d-1)+1} c_i t^i \in \mathbb{Z}[t]$$

Then, we have

- $c_i > 0$ *for every* $0 \leq i \leq \lfloor \frac{n(d-1)}{2} \rfloor$,
- $c_i < 0$ *for every* $\lceil \frac{n(d-1)}{2} + 1 \rceil \leq i \leq n(d-1)+1$,
- *if* $n(d-1)+1$ *is even then* $c_{\frac{n(d-1)+1}{2}} = 0$.

Proof. First, observe that

$$\frac{(1-t^d)^n}{(1-t)^{n-1}} = (1-t)\left(\frac{1-t^d}{1-t}\right)^n$$

Now, the coefficients of the polynomial

$$\left(\frac{1-t^d}{1-t}\right)^n = \sum_{i=0}^{n(d-1)} d_i t^i$$

rank along a symmetric sequence that increases up to index $\frac{n(d-1)}{2}$, which corresponds to two indexes if $n(d-1)$ is odd) and then decreases [RRR91, Theorem 1]. Multiplying out by $1-t$ leads to the sequence of coefficients $(d_i - d_{i-1})_i$ from which the result follows easily. $\square$

Proposition 6. *Let* $\dim(R/I) \leq 1$. *If* $\nu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + 1$ *for every prime ideal* $\mathfrak{p} \supset I$ *then*

$$\mu_0(I) \geq \left\lfloor \frac{(n-2)(d-1)-1}{2} \right\rfloor$$

for $n \geq 3$ *and* $d \geq 2$.

Proof. We assume that $H_2 \neq 0$ for otherwise Corollary 3 applies. Recalling that K_i denotes the ith Koszul homology module of I, we have the following well-known formula in $\mathbb{Z}[t]$:

$$\begin{aligned} \frac{(1-t^d)^{n+1}}{(1-t)^n} &= \sum_{\mu=0}^{n(d-1)+d} \left(\sum_{i\geq 0} (-1)^i \dim_k(K_i)_\mu \right) t^\mu \\ &= \sum_{\mu=0}^{(n+1)(d-1)} \left(\sum_{i\geq 0} (-1)^i \dim_k(H_i)_\mu \right) t^\mu \end{aligned}$$

But since $H_i = 0$ for $i \geq 3$, the above simplifies to

$$\sum_{\mu=0}^{n(d-1)+d} (h_0(\mu) - h_1(\mu) + h_2(\mu))\, t^\mu = \frac{(1-t^d)^{n+1}}{(1-t)^n} \in \mathbb{Z}[t],$$

where we have set $h_i(\mu) := \dim_k(H_i)_\mu$, for $i = 0,1,2$.

Next consider the difference operator Δh_i acting by $\Delta h_i(\mu) := h_i(\mu) - h_i(\mu-1)$, for $i = 0,1,2$. It follows that

$$\begin{aligned} \sum_{\mu=0}^{n(d-1)+d+1} (\Delta h_0(\mu) - \Delta h_1(\mu) + \Delta h_2(\mu))\, t^\mu &= (1-t)\frac{(1-t^d)^{n+1}}{(1-t)^n} \\ &= (1-t^d)\frac{(1-t^d)^n}{(1-t)^{n-1}}. \end{aligned}$$

Applying Lemma 3, we find that

$$\Delta h_0(\mu) - \Delta h_1(\mu) + \Delta h_2(\mu) = c_\mu - c_{\mu-d}$$

is non positive (< 0) for every

$$\left\lceil \frac{n(d-1)+1}{2} \right\rceil \leq \mu \leq \left\lfloor \frac{n(d-1)+1}{2} \right\rfloor + d. \tag{14}$$

Note that $\Delta h_2(\nu) \in \mathbb{N}$ for all ν since $H^0_{\mathfrak{m}}(H_2) = 0$. Therefore, for every integer μ satisfying (14) and such that $\Delta h_0(\mu) \geq 0$ we have $\Delta h_1(\mu) > 0$ and hence $h_1(\mu) \neq 0$.

The condition $\Delta h_0(\mu) \geq 0$ is fulfilled when $H^0_{\mathfrak{m}}(H_0)_{\mu-1} = 0$, that is to say, when $\operatorname{end}(H^0_{\mathfrak{m}}(H_0)) \leq \mu - 2$ or, still equivalently, when

$$\operatorname{indeg}(H^0_{\mathfrak{m}}(H_1)) \geq (n+1)d - n - \mu + 2.$$

These considerations, applied to the lowest possible value of μ satisfying (14), so we claim now, imply that

$$\min\{\operatorname{indeg}(H_1), \operatorname{indeg}(H^0_{\mathfrak{m}}(H_1)) - d\} \leq \left\lceil \frac{n(d-1)+1}{2} \right\rceil, \tag{15}$$

from which the required lower bound follows by the definition of $\mu_0(I)$.

To see why (15) holds, note that we proved the inequality $\operatorname{indeg}(H_1) \leq \left\lceil \frac{n(d-1)+1}{2} \right\rceil$ provided

$$\operatorname{indeg}(H^0_{\mathfrak{m}}(H_1)) \geq (n+1)d - n - \left\lceil \frac{n(d-1)+1}{2} \right\rceil + 2 = \left\lfloor \frac{n(d-1)+1}{2} \right\rfloor + d + 1.$$

Thus, negating the latter inequality yields

$$\operatorname{indeg}(H^0_{\mathfrak{m}}(H_1)) \leq \left\lfloor \frac{n(d-1)+1}{2} \right\rfloor + d$$

and (15) follows. □

Example 1. Let $n = 3$. Let I denote the ideal generated by the 3-minors of a matrix of general forms $\oplus_{i=1}^3 R(-e_i) \to R^4$ where $\sum_{i=1}^3 e_i = d$. As is well-known, I is a codimension 2 saturated ideal, hence $H^0_{\mathfrak{m}}(H_1) = 0$ and I is not $\mathfrak{m}$-primary. By [AH80, §1], the module H_1 is generated in degree $d + \min_i\{e_i\}$. We deduce that

$$\left\lceil \frac{2d}{3} \right\rceil - 2 \leq \mu_0(I) = d - 2 - \min_i\{e_i\} \leq d - 3$$

Also, we obtain that $\operatorname{reg}(R/I) - d = \min_i\{e_i\} - 2$ and $\nu_0(I) = d - 2$, which is coherent with Proposition 5. Notice also that if the lower bound given in Proposition 6 is satisfied, the one given in Corollary 3 is not. This shows that the assumption that I be $\mathfrak{m}$-primary in Corollary 3 is not superfluous.

5. Application to the hypersurface implicitization problem

Given a parametrization

$$\begin{array}{ccc} \mathbb{P}^{n-1} & \to & \mathbb{P}^n \\ (X_1 : \cdots : X_n) & \mapsto & (f_0 : \cdots : f_n)(X_1 : \cdots : X_n) \end{array}$$

of a rational hypersurface $\mathcal{H}$, the approximation complex of cycles associated to the ideal $I = (f_0, \ldots, f_n)$ has been used (see e.g. [BJ03, BC05]) to derive a matrix-based representation of $\mathcal{H}$. Such a representation only uses the linear syzygies of the ideal

I. The results obtained in the previous sections allow to produce new matrix-based representations of $\mathcal{H}$ that involve not only the linear syzygies but also higher order syzygies of the ideal I. Indeed, the following proposition shows that the divisor associated to $(\mathcal{S}_I^*)_\mu$ has the expected property for every threshold integer μ.

Proposition 7. *Let* $\dim(R/I) \leq 1$. *If* $\nu(I_\mathfrak{p}) \leq \dim R_\mathfrak{p} + 1$ *for every prime ideal* $\mathfrak{p} \supset I$ *then for every* $\mu \geq 0$, *one has* $\mathrm{ann}_{R'}((\mathcal{S}_I^*)_\mu) = \mathrm{ann}_{R'}((\mathcal{S}_I^*)_0) = H^0_\mathfrak{m}(\mathrm{Sym}_R(I))_0$ *while for every threshold integer* μ *it obtains*

$$\mathrm{Div}((\mathcal{S}_I^*)_{\mu+1}) = \mathrm{Div}((\mathcal{S}_I^*)_\mu).$$

Proof. Let $\ell \in R_1$ not in any minimal prime of I. Then ℓ is a nonzero divisor on $\mathcal{S}_I^*$. Hence for $\mu \geq 0$ the canonical inclusion $R_1\mathrm{ann}_{R'}((\mathcal{S}_I^*)_\mu) \subseteq \mathrm{ann}_{R'}((\mathcal{S}_I^*)_{\mu+1})$ is an equality. Furthermore, the exact sequence

$$0 \to (\mathcal{S}_I^*)_\mu \xrightarrow{\times \ell} (\mathcal{S}_I^*)_{\mu+1} \to (\mathcal{S}_I^*/(\ell)\mathcal{S}_I^*)_{\mu+1} \to 0$$

shows that

$$\mathrm{Div}((\mathcal{S}_I^*)_{\mu+1}) = \mathrm{Div}((\mathcal{S}_I^*)_\mu) + \mathrm{Div}((\mathcal{S}_I^*/(\ell)\mathcal{S}_I^*)_{\mu+1}).$$

Let $\overline{R} := R/(\ell)$, $\overline{I} := I/(\ell) \subset \overline{R}$, $H := \mathrm{Proj}(\overline{R}) \subset \mathrm{Proj}(R) = \mathbf{P}^{n-1}$, notice that $\mathrm{Proj}(\mathcal{S}_I^*/(\ell)\mathcal{S}_I^*) \subset H \times \mathbf{A}^{n+1}$ coincides with $\mathrm{Proj}(\mathrm{Sym}_{\overline{R}}(\overline{I})) = \mathrm{Proj}(\mathrm{Rees}_{\overline{R}}(\overline{I}))$ and that $\mathrm{ann}_{R'}(\mathrm{Rees}_{\overline{R}}(\overline{I}))_\mu)$ is a prime ideal of height two that does not depend on μ for any $\mu \geq 0$.

It follows that

$$\mathrm{Div}((\mathcal{S}_I^*/(\ell)\mathcal{S}_I^*)_{\mu+1}) = \mathrm{Div}(\mathrm{Rees}_{\overline{R}}(\overline{I})_{\mu+1}) = 0$$

if $H^0_\mathfrak{m}((\mathcal{S}_I^*/(\ell)\mathcal{S}_I^*))_{\mu+1} = 0$, which in turns hold if $H^1_\mathfrak{m}(\mathcal{S}_I^*)_\mu = 0$. The conclusion then follows from Theorem 1. □

As a consequence of Proposition 7 and Corollary 1, the matrix of the first map of the resolution of $\mathcal{S}_I^*$, in any basis with respect to the chosen degree, provides a matrix-based representation of the hypersurface $\mathcal{H}$ if the base points, if any, are locally complete intersection. Otherwise, if the base points are almost complete intersections, then some known extraneous factors appear; we refer the interested reader to [BCJ09] for more details. We end this paper by summing up the consequence of the results presented in this paper for the purpose of matrix-based representation of parameterized hypersurfaces.

5.1. **The $\mathfrak{m}$-primary case.** This case is particularly comfortable because all the non-linear syzygies that appear in these matrix representations can be computed by downgrading some linear syzygies of higher degree. This is a consequence of the isomorphisms given in Section 3.3.

Recall that, as we explained just after Proposition 3, it is possible to tune the integer ν so that there is only linear and quadratic syzygies in the matrix-based representation. Such a framework has been intensively studied by the community of Computer Aided Geometric Design under the name "moving surfaces method" (see [CGZ00, BCD03] and the references therein).

In the particular case $n = 3$, we see that only linear and quadratic syzygies appear in the family of matrices M_μ with $\mu \geq \mu_0$. If the f_i's are in generic position, then $\mu_0 = d - 1$ and the matrix M_{μ_0} is a square matrix (all the b_i's are equal to zero). In the paper [CGZ00], a condition on the rank of the moving planes matrix is used. It is interesting to notice that it implies that $\mu_0 = d - 1$ and hence that the matrix M_{μ_0} is square. Indeed, with the notation of this paper, the condition in [CGZ00] is $\dim(Z_1)_{2d-1} = d$. From the exact sequence

$$0 \to Z_1 \to R(-d)^4 \to R \to H_0 \to 0$$

we get $\dim(Z_1)_{2d-1} = d + \dim(H_0)_{2d-1}$. Therefore, the condition in [CGZ00] implies that $\dim(H_0)_{2d-1} = 0$. Moreover, since $(B_1)_{2d-1} = 0$ we have $\dim(H_1)_{2d-1} = d$ and the isomorphism $H_1 \simeq {H_0}^*[3-4d]$ shows that $\dim(H_0)_{2d-2} = d \neq 0$. Therefore, $\mathrm{end}(H_0) = 2d-2$ so that $\mu_0 = d-1$ (as if the f_i's were in generic position).

5.2. **In the presence of base points.** In this case, the downgrading maps are no longer available. So that the higher order syzygies have to be computed as linear syzygies of a suitable power of the ideal I.

Notice that similarly to the $\mathfrak{m}$-primary case, it is also possible to tune the integer ν in order to bound the order of the syzygies appearing in the matrix representation. Mention also that if the ideal I is saturated, so that $H^0_{\mathfrak{m}}(H_1) = 0$, it is remarkable that one never gets quadratic syzygies in the first map of the complex. This is a direct consequence of Theorem 1.

Example 2. Take again Example 1 and assume that $d = 3$, that is to say that we start with a matrix of general linear forms ($e_i = 1$ for every $i = 1, 2, 3$). In this case, $\mu_0(I) = 0$ and $\nu_0(I) = 1$. The implicit equation, which is a cubic form, is then directly obtained in the case by taking $\mu = 0$ and is represented by a matrix of linear syzygies when $\mu \geq 1$. According to our previous observation, whatever $\mu \geq 0$ is, there is no quadratic syzygies involved in the associated complex.

Example 3. We treat in detail the following example taken from [BCD03, Example 3.2]. All the computations have been done with the software Macaulay2 [GS].

$$f_0 = X_0X_2^2,\ f_1 = X_1^2(X_0 + X_2),\ f_2 = X_0X_1(X_0 + X_2),\ f_3 = X_1X_2(X_0 + X_2)$$

and $d = 3$. The ideal $I = (f_0, f_1, f_2, f_3)$ defines 6 base points: $(0:0:1)$, $(1:0:0)$ with multiplicity 2 and $(0:1:0)$ with multiplicity 3. Its saturation I^{sat} is the complete intersection $(X_0X_1 + X_1X_2, X_0X_2^2)$, so that $\mathrm{indeg}(I^{sat}) = 2$. The method developed in [BC05] shows that for every $\mu \geq 2 \times (3-1) - 2 = 2$ one can obtain a matrix, filled exclusively with linear syzygies, representing our parameterized surface. For instance, such a matrix for $\mu = 2$ is given by

$$\begin{pmatrix} T_1 & 0 & 0 & 0 & 0 & T_3 & 0 & 0 & 0 \\ -T_2 & T_1 & 0 & 0 & 0 & 0 & T_3 & 0 & T_0 \\ 0 & 0 & T_1 & 0 & 0 & -T_2 & 0 & T_3 & 0 \\ 0 & -T_2 & 0 & T_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -T_2 & -T_1 & T_3 & 0 & -T_2 & 0 & T_0 \\ 0 & 0 & 0 & 0 & -T_1 & 0 & 0 & -T_2 & -T_2 \end{pmatrix}$$

Although this is something that one does not want to do from a computational point of view, one can extract from the above matrix the implicit equation of our surface which is $T_1T_2T_3 + T_1T_2T_4 - T_3T_4^2 = 0$.

Now, we have $\mu_0(I) = 2 \times 2 - 4 = 0$ for $\mathrm{indeg}(H_1) = \mathrm{indeg}(H^0_{\mathfrak{m}}(H_1)) - d = 4$. Moreover, since $H^0_{\mathfrak{m}}(H_1)$ is concentrated in degree 7, Theorem 1 shows that in the case $\mu = 0$ the matrix representing our surface is simply a 1×1-matrix whose entry is an implicit equation of the surface. However, the case $\mu = 1$ is much more interesting because in this case the matrix representing the surface is filled with

$\dim(Z_1)_{1+d} = 3$ linear syzygies and $\dim H^0_{\mathfrak{m}}(H_1)_{1+3d} = 1$ quadratic syzygies since $\dim(R/I^{sat})_{-1} = 0$. Here is this matrix

$$\begin{pmatrix} T_2 & 0 & T_4 & -T_4^2 \\ -T_3 & T_4 & 0 & T_1T_3 + T_1T_4 \\ 0 & -T_2 & -T_4 & 0 \end{pmatrix}$$

It gives a representation of our parameterized surface. Notice that, as observed in [BCD03, Example 3.2], there does not exists a *square* matrix of linear and/or quadratic syzygies whose determinant is an implicit equation of the surface.

Appendix A. An argument of Joseph Oesterlé

A.1. **Un théorème à la Lefschetz.** Soient n et m deux entiers naturels. Considérons l'anneau gradué

$$\mathrm{R} = \mathbf{Q}[x_1, \dots, x_n]/(x_1^m, \dots, x_n^m),$$

où les x_i sont des indéterminées. L'ensemble R_k de ses éléments homogènes de degré k est un espace vectoriel de dimension finie sur $\mathbf{Q}$ pour tout $k \in \mathbf{Z}$; il est non nul si et seulement si $0 \leq k \leq d$, où $d = n(m-1)$.

L'espace vectoriel R_d est de dimension 1 sur $\mathbf{Q}$, et $(x_1 \dots x_n)^{m-1}$ en est une base. Pour tout entier k tel que $0 \leq k \leq d$, l'application bilinéaire $\mathrm{R}_k \times \mathrm{R}_{d-k} \to \mathrm{R}_d$ induite par la multiplication de R est inversible ; le rang de R_k est donc égal à celui de R_{d-k}.

Posons $\omega = x_1 + \dots + x_n$. Nous allons démontrer le résultat suivant:

Théorème. *Soient $k \in \mathbf{Z}$ et $t \in \mathbf{N}$. L'application $\mathbf{Q}$-linéaire de R_k dans R_{k+t} induite par la multiplication par ω^t est injective si $2k + t \leq d$, et surjective si $2k + t \geq d$.*

Il nous suffira de démontrer la première assertion, car la seconde s'en déduit par dualité. Nous le ferons par récurrence sur n, en nous servant du lemme suivant, que nous démontrerons au n°A.2:

Lemme. *Soient A une $\mathbf{Q}$-algèbre, a un élément de A, m et t des entiers naturels, et x une indéterminée. Pour qu'un élément de $\mathrm{A}[x]/x^m\mathrm{A}[x]$ soit annulé par $(x+a)^t$, il faut et il suffit que ce soit la classe d'un polynôme $\mathrm{P}(x) \in \mathrm{A}[x]$ de la forme $\sum_{j=1}^{\inf(m,t)} b_j \mathrm{P}_j(x)$, où $\mathrm{P}_j(x) = \sum_{i=0}^{m-j} \frac{(m+t-2j-i)!(j+i-1)!}{(m-j-i)!(j-1)!}(-a)^i x^{m-j-i}$ et où $b_j \in \mathrm{A}$ est annulé par $a^{m+t+1-2j}$.*

Remarque. Comme $\mathrm{P}_j(x)$ est de degré $m-j$ et que son coefficient dominant est inversible, les b_j dont il est question dans le lemme sont uniques.

Le théorème étant clair pour $n = 0$, nous supposerons $n \geq 1$. Nous appliquerons le lemme en prenant pour A l'anneau $\mathbf{Q}[x_1, \dots, x_{n-1}]/(x_1^m, \dots, x_{n-1}^m)$, $a = x_1 + \dots + x_{n-1}$ et $x = x_n$, de sorte que $\mathrm{A}[x]/x^m\mathrm{A}[x]$ s'identifie à R et que $\omega = x + a$.

Soit k un entier relatif tel que $2k + t \leq d = n(m-1)$. Si un élément de R_k est annulé par ω^t, il est la classe d'un polynôme P de la forme $\sum_{j=1}^{\inf(m,t)} b_j \mathrm{P}_j$, où les P_j sont comme dans le lemme et où $b_j \in \mathrm{A}$ est annulé par $a^{m+t+1-2j}$. Lorsqu'on munit $\mathrm{A}[x]$ de la graduation déduite de celle de $\mathbf{Q}[x_1, \dots, x_n]$, P_j est homogène de degré

$m-j$. Vu l'assertion d'unicité de la remarque ci-dessus, les b_j sont homogènes de degré $k-m+j$. Comme

$$2(k-m+j)+(m+t+1-2j)=2k+t-m+1\leq d-(m-1)=(n-1)(m-1),$$

l'hypothèse de récurrence implique que les b_j sont tous nuls et donc que $\mathrm{P}=0$. Cela démontre le théorème.

A.2. **Démonstration du lemme.** Nous adoptons les notations du lemme : A est une $\mathbf{Q}$-algèbre, a est un élément de A, m et t sont des entiers naturels et x est une indéterminée.

Notons B l'anneau $\mathrm{A}((x^{-1}))$ des séries de Laurent en x^{-1}. Remarquons que $x+a=x(1+ax^{-1})$ est un élément inversible de B, dont l'inverse est $\sum_{i=0}^{\infty}(-a)^i x^{-i-1}$. Considérons le sous-A-module

$$\mathrm{E}=\mathrm{A}x^m+\mathrm{A}x^{m+1}+\ldots+\mathrm{A}x^{m+t-1}$$

de B; il est libre de rang t. Notons F l'ensemble des $f\in\mathrm{B}$ tels que $(x+a)^t f\in\mathrm{E}$. C'est un sous-A-module de B libre de rang t, puisque l'application $f\mapsto(x+a)^t f$ définit un isomorphisme de F sur E. Considérons les éléments $f_1,f_2,\ldots,f_t$ de B définis par :

$$f_j=\begin{cases}(\frac{d}{dx})^{t-j}(\frac{x^{m+t-j}}{(x+a)^j}) & \text{si } 1\leq j\leq \inf(m,t)\\ \frac{x^m}{(x+a)^j} & \text{si } m+1\leq j\leq t.\end{cases}$$

Il est clair que $(x+a)^t f_j$ est un polynôme en x, que ce polynôme appartient à $x^m\mathrm{A}[x]$, que son degré est $m+t-j$ et que son coefficient dominant est un entier naturel non nul (à savoir $\frac{(m+t-2j)!}{(m-j)!}$ si $1\leq j\leq\inf(m,t)$ et 1 si $m+1\leq j\leq t$). Il s'en suit que les $(x+a)^t f_j$ forment une base du A-module E et que les f_j forment une base du A-module F.

Pour qu'un élément de $\mathrm{A}[x]/x^m\mathrm{A}[x]$ soit annulé par $(x+a)^t$, il faut et il suffit que le polynôme $\mathrm{P}(x)\in\mathrm{A}[x]$ de degré $\leq m-1$ qui le représente appartienne à F. Examinons donc à quelle condition une série de Laurent de la forme $b_1f_1+\ldots+b_tf_t$, où les b_j appartiennent à A, est un polynôme.

Si $1\leq j\leq\inf(m,t)$, la série de Laurent f_j s'écrit $\mathrm{P}_j+a^{m+t-2j+1}g_j$, où $\mathrm{P}_j\in\mathrm{A}[x]$ est le polynôme en x figurant dans le lemme et g_j est une série formelle en x^{-1} sans terme constant dont le terme de plus bas degré en x^{-1} est le produit d'un entier relatif non nul par $x^{-(t-j+1)}$. Si $m+1\leq j\leq t$, on pose $g_j=f_j$: dans ce cas g_j est une série formelle en x^{-1} sans terme constant dont le terme de plus bas degré est $x^{-(j-m)}$. De ces propriétés, on déduit que les séries formelles $g_1,\ldots,g_t$ sont linéairement indépendantes sur A. Il s'en suit que $b_1f_1+\ldots+b_tf_t$ est un polynôme si et seulement si on a $b_ja^{m+t-2j+1}=0$ pour $1\leq j\leq\inf(m,t)$ et $b_j=0$ pour $m+1\leq j\leq t$. Et ce polynôme est alors égal à $\sum_{1\leq j\leq\inf(m,t)}b_j\mathrm{P}_j$, d'où le lemme.

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