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Abstract: We give a short proof of the following theorem due to Borodin [2]. Every planar graph with maximum degree $\Delta \geq 9$ is $(\Delta + 1)$-edge-choosable.

Key-words: edge-colouring, list colouring, List Colouring Conjecture, planar graphs

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Les graphes planaires de degré maximum $\Delta \geq 9$ sont $(\Delta + 1)$-arête-choisissables – une preuve courte

Résumé : Nous présentons une preuve courte d'un résultat de Borodin [2] : tout graphe planaire de degré maximum $\Delta \geq 9$ est $(\Delta + 1)$-arête-choisissable.

Mots-clés : arête-coloration, coloration par liste, Conjecture de la coloration par liste, graphes planaires

Planar graphs with maximum degree $\Delta \geq 9$ are $(\Delta + 1)$-edge-choosable – a short proof

Nathann Cohen*, Frédéric Havet

We give a short proof of the following theorem due to Borodin [2]. Every planar graph with maximum degree $\Delta \geq 9$ is $(\Delta + 1)$-edge-choosable.

1 Introduction

All graphs considered in this paper are simple and finite. An *edge-colouring* of a graph G is a mapping f from $E(G)$ into a set S of *colours* such that incident edges have different colours. If $|S| = k$ then f is a *k-edge-colouring*. A graph is *k-edge-colourable* if it has a k-edge-colouring. The *chromatic index* $\chi'(G)$ of a graph G is the least k such that G is k-edge-colourable.

Since edges sharing an end-vertex need different colours, $\chi'(G) \geq \Delta(G)$ where $\Delta(G)$ denotes the maximum degree of G. The celebrated Vizing's Theorem [13] (also shown independently by Gupta [5]) states that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

Theorem 1 (Vizing [13]) *Let G be a graph. Then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

An *edge-list-assignment* of a graph G is an application L which assigns to each edge $e \in E(G)$ a prescribed list of colours $L(e)$. An edge-list-assignment is a *k-edge-list-assignment* if each list is of size at least k. An *L-edge-colouring* of G is an edge-colouring such that $\forall v \in V(G), v \in L(v)$. A graph G is *L-edge-colourable* if there exists an edge-colouring of G. It is *k-edge-choosable* if it is L-colourable for every k-list-assignment L. The *choice index* or *list chromatic index* $ch'(G)$ is the least k such that G is k-edge-choosable.

One of the most celebrated conjecture on graph colouring is the List Colouring Conjecture asserting that the chromatic index is always equal to the list chromatic index.

Conjecture 2 (List Colouring Conjecture) *For every graph G, $\chi'(G) = ch'(G)$.*

Bollobás and Harris [1] proved that $ch'(G) < c\Delta(G)$ when $c > 11/6$ for sufficiently large Δ. Using probabilistic methods, Kahn [9] proved Conjecture 2 asymptotically: $ch'(G) \leq (1 + o(1))\Delta(G)$. The error term was sharpened by Häggkvist and Janssen [7]: $ch'(G) \leq \Delta(G) + O(\Delta(G)^{2/3}\sqrt{\log \Delta(G)}$ and further on by Molloy and Reed [10]: $ch'(G) \leq \Delta(G) +$

$O(\Delta(G)^{1/2}(\log \Delta(G))^4)$. Galvin [6] proved the List Colouring Conjecture for bipartite graphs. (See also Slivnik [12]).

The List-Colouring Conjecture and Vizing's Theorem imply the following conjecture :

Conjecture 3 *For any graph G, $ch'(G) \leq \Delta(G) + 1$.*

Borodin [2] settled this conjecture for planar graphs of maximum degree at least 9.

Theorem 4 (Borodin [2]) *Let $\Delta \geq 9$. Every planar graph of maximum degree at most Δ is $(\Delta + 1)$-edge-choosable.*

This theorem does not imply the List Colouring Conjecture for planar graphs of large maximum degree. Indeed, Sanders and Zhao [11] showed that planar graphs with maximum degree $\Delta \geq 7$ are Δ-edge-colourable. Vizing Edge-Colouring Conjecture [14] asserts that it remains true for $\Delta = 6$. This would be best possible as for any $\Delta \in \{2, 3, 4, 5\}$, there are some planar graphs with maximum degree Δ and chromatic index equal to $\Delta + 1$ [14].

Borodin, Kostochka and Woodall [3] showed that if G is planar and $\Delta(G) \geq 12$ then $ch'(G) \leq \Delta(G)$, thus proving the List Colouring Conjecture for such planar graphs of maximum degree at least 12. Another proof has been given by Cole, Kowalik and Škrekovski [4] which yields a linear time algorithm to L-edge-colour a planar graph G for any $\max\{\Delta(G), 12\}$-list edge-assignment. Conjecture 3 is still open for planar graphs of maximum degree between 5 and 8 and it is still unknown if planar graphs of maximum degree Δ are Δ-edge-choosable for $6 \leq \Delta \leq 11$.

In this paper, we give a short proof of Theorem 4.

2 Proof of Theorem 4

Our proof uses the discharging method.

A vertex of degree d (respectively at least d, respectively at most d) is said to be a *d-vertex* (respectively a *$(\geq d)$-vertex*, respectively a *$(\leq d)$-vertex*). The notion of a *d-face* (respectively a *$(\leq d)$-face*, respectively a *$(\geq d)$-face*) is defined analogously regarding the size of a face.

Consider a minimal counter-example G to the theorem. Let L be a $(\Delta + 1)$-list edge-assignment so that G is not L-edge-colourable. G has no edge uv such that $d(u) + d(v) \leq \Delta + 2$, otherwise any L-colouring of $G \setminus uv$ could be extended to one of G by giving to e a colour distinct from the ones of its Δ adjacent edges. In particular, $\delta(G) \geq 3$ and for any $i \geq 3$ the neighbours of a i-vertex have degree at least $\Delta + 3 - i$.

Let V_3 be the set of 3-vertices and V_Δ the set of vertices of degree Δ.

Claim 4.1 $|V_\Delta| > 2|V_3|$.

Proof. Let F the set of edges with an end-vertex of degree 3 (and so the other end-vertex of degree Δ) and H the bipartite subgraph $(V_3 \cup V_\Delta, F)$ of G.

Let us first show that H is a forest. Suppose by way of contradiction that H has cycle C. Then C is even because H is bipartite. By minimality of G, $G \setminus E(C)$ has an L-edge-colouring. Now every edge of C has at least two available colours since it is adjacent to $\Delta+1$ edges and $\Delta-1$ coloured ones. Since the even cycles are 2-edge-choosable, one can extend the L-edge-colouring to G, which is a contradiction. Then, as any $v \in V_3$ is of degree 3 in H (implying $|E(H)| = 3|V_3|$), we can write $|V_\Delta| + |V_3| > 3|V_3|$. □

Let us assign a charge of its degree to every vertex and face. It follows easily from Euler's Formula that $\Sigma = \sum_{v \in V(G)}(d(v)-4) + \sum_{f \in F(G)}(d(f)-4) = -8$. Let us now discharge along the following rules:

(R1) Every Δ-vertex gives $1/2$ to a common pot from which each 3-vertex receives 1;

(R2) Every (≥ 8)-vertex gives $1/2$ to each of its incidents 3-faces;

(R3) Every d-vertex with $d \in \{5,6,7\}$ gives $\frac{d-4}{d}$ to each of its incident 3-faces.

Let us show that after the final charge f of every vertex or face is non-negative as well as the charge of the common pot which contradicts $\Sigma < 0$.

- As $|V_\Delta| > 2|V_3|$ by Claim 4.1, the charge of the common pot is positive.
- Let x be a d-vertex.

If $d=3$ then x receives at least $1/3$ from each of its neighbours (they must have degree Δ), so $f(x) \geq 0$. If $d=4$, the charge of x does not change so $f(x) = d \geq 0$. If $d \in \{5,6,7\}$, then x sends at most $\frac{d-4}{d}$ to each of its incident face so $f(x) \geq d(1-\frac{d-4}{d})-4 \geq 0$. If $8 \leq d \leq \Delta-1$, then x sends at most $1/2$ to each of its incident faces so $f(x) \geq d-4-d/2 \geq 0$. If $d=\Delta$, then the most x can send is $d \times 1/2 + d/2 \times 1/3$ since a 3-face contains at most one 3-vertex. So $f(x) \geq d-4-d/2-d/6 \geq 4$ because $d \geq 12$.

- Let x be a d-face.

If $d \geq 4$ then its charge does not change so $f(x) = d(x)-4 \geq 0$. Suppose now that $d=3$. If x contains a (≤ 4)-vertex then the two other neighbours have degree at least $\Delta-1 \geq 8$ so it receives $1/2$ from each of those two. So $f(x) = 3-4+2\times 1/2 = 0$. If x contains a 5-vertex then its two other vertices have degree at least $\Delta-2 \geq 7$. So it receives at $\frac{1}{5}$ from its 5-vertex and at least $\frac{3}{7}$ from the other two vertices. So $f(x) \geq 3-4+1/5+2\times 3/7 > 0$. Otherwise, all the vertices incident to x are (≥ 6)-vertices. Hence $f(x) \geq 3-4+3\times 1/3 = 0$.

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