The Hilbert scheme of points and its link with border basis

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THE HILBERT SCHEME OF POINTS AND ITS LINK WITH BORDER BASIS

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ABSTRACT. This paper examines the effective representation of the punctual Hilbert scheme. We give new equations, which are simpler than Bayer and Iarrobino-Kanev equations. These new Plücker-like equations define the Hilbert scheme as a subscheme of a single Grassmannian and are of degree two in the Plücker coordinates. This explicit complete set of defining equations for $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$ are deduced from the commutation relations characterising border bases and from generating equations. We also prove that the punctual Hilbert functor $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ can be represented by the scheme $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$ defined by these relations and the well-known Plücker relations on the Grassmanian. A new description of the tangent space at a point of the Hilbert scheme, seen as a subvariety of the Grassmannian, is also given in terms of projections with respect to the underlying border basis.

1. Introduction

A natural question when studying systems of polynomial equations is how to characterize the family of ideals which defines a fixed number μ of points counted with multiplicities. It is motivated by practical issues related to the solution of polynomial systems, given with approximate coefficients. Understanding the allowed deformations of a zero-dimensional algebra, which keep the number of solutions constant, is an actual challenge, in the quest for efficient and stable numerical polynomial solvers. From a theoretical point of view, this question is related to the study of the Hilbert Scheme of μ points, which is an active area of investigation in Algebraic Geometry.

The notion of Hilbert Scheme was introduced by [9]: it is defined as a scheme representing a contravariant functor from the category of schemes to the one of sets. This functor associates to any scheme S the set of flat families $\chi \subset \mathbb{P}^r \times S$ of closed subschemes of $\mathbb{P}^r$ parametrized by S, whose fibers have Hilbert polynomial μ.

Many works were developed to analyze its geometric properties (see eg. [17]), which are still not completely understood. Among them, it is known to be reducible for $n > 2$ [16], but the components are not known for $\mu \geq 8$ [3]. Its connectivity firstly proved by Hartshorne (1965), is studied in [22] with a more constructive approach.

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Though the Hilbert functor $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ is known to be representable [25], its effective representation is under investigation. Using the persistence theorem of [6], a global explicit description of the Hilbert scheme as a subscheme of a product of two Grassmannians is given in [15], and in [11] for a multi-graded extension. Equations defining $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$ in a single Grassmannian are also given in [15]. These equations, obtained from rank conditions in the vector space of polynomials in successive "degrees", have a high degree in the Plücker coordinnates, namely the number of monomials of degree μ in $n+1$ variables minus μ.

In [1], a different set of equations of degree n in the Plücker coordinates is proposed. It is conjectured that these equations define the Hilbert scheme, which is proved in [11]. Nevertheless, these equations are not optimal, as noticed in an example in dimension 3 in [11][p. 756]: they are of degree 3 whereas the corresponding Hilbert scheme can be defined in this case by quadratic equations.

The problem of representation is also studied through subfunctor constructions and open covering of charts of the Hilbert scheme. Covering charts corresponding to subsets of ideals with a fixed initial ideal for a given term ordering are analysed in several works, starting with [5], and including more recent one like [19]. These open subsets can be embedded into affine open subsets of the Hilbert scheme, corresponding to ideals associated to quotient algebras with a given monomial basis. Explicit equations of these affine varieties are developped in [10] for the planar case, [13], [14], using syzygies or in [24].

In this paper, we concentrate on the Hilbert scheme of μ points in the projective space $\mathbb{P}^n_{\mathbb{K}}$ and on its effective representation. We give new equations for the punctual Hilbert scheme, which are simpler than Bayer and Iarrobino-Kanev equations. They are quadratic in the Plücker coordinnates, and define the Hilbert scheme as a subscheme of a single Grassmannian. We give a new proof that Hilbert functor can be represented by this scheme $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$ given by this explicit quadratic equations. Reformulating a result in [20], we recall how the open chart corresponding to quotient algebras with fixed (monomial) basis connected to 1 can simply be defined by the commutation relations characterising border basis (see also [23], [18]). We show how these commutation relations can be further exploited to provide an explicit complete set of defining equations for $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$ as a projective variety. Following a dual point of view, this approach yields new Plücker-like equations of degree two in the coordinates on the Grassmannian, which are explicit and of smaller degree than those in [1], [15]. We show moreover that the scheme $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$ defined by these relations and the Plücker relations on the Grassmanian represents the punctual Hilbert functor $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$. It has a natural structure of projective variety, as a subvariety of the Grassmannian. Finally, we give a new description of the tangent space to this variety in terms of projections with respect to the underlying border basis.

After setting our notations, we analyse in section 2 the Hilbert functor, starting with a local description based on commutation relations, followed by and subfunctor constructions. In Section 3, we describe the new quadratic equations in the Plücker coordinnates, related them with the commutation and generating relation for border basis, prove that they characterise completely elements of the punctual Hilbert scheme and deduce an explicit representation of the Hilbert functor. Finally in Section 4, we show how the tangent space to the Hilbert scheme at a given point can be defined in terms border basis computation. Standard results on functors and on genericity are collected in an appendix for the seek of self-contain.

1.1. **Notations.** Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and $R = \mathbb{K}[x_1, \dots, x_n] = \mathbb{K}[\underline{\mathbf{x}}]$ be the set of polynomials in the variables $x_1, \dots, x_n$ and coefficients in $\mathbb{K}$. We also denote by $S = \mathbb{K}[x_0, \dots, x_n] = \mathbb{K}[\mathbf{x}]$ the polynomial ring in $x_0, \dots, x_n$ for a new variable x_0 "of homogenization". For any $\alpha \in \mathbb{N}^{n+1}$ (resp. $\mathbb{N}^n$), let $\mathbf{x}^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ (resp. $\underline{\mathbf{x}}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$). The canonical basis of $\mathbb{N}^{n+1}$ is denoted by $(e_i)_{i=0,\dots,n}$, so that $\mathbf{x}^{\alpha+e_i} = \mathbf{x}^\alpha x_i$ ($\alpha \in \mathbb{N}^{n+1}, i = 0, \dots, n$). For $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, we denote by $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

For a given set $B = \{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_D}\}$ of monomials in $x_0, \dots, x_n$, we will identify B with its set of exponents $\{\alpha_1, \dots, \alpha_D\}$. For a set of exponents $E = \{\alpha_1, \dots, \alpha_D\} \subset \mathbb{N}^{n+1}$, we denote by $\mathbf{x}^E$ the corresponding set of monomials with exponents in E: $\{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_D}\}$.

For $B \subset \mathbb{N}^n$, we say that B is connected to 1 if $\mathbf{0} \in B$ (i.e $1 \in \underline{\mathbf{x}}^B$) and for all $\beta \in B \setminus \{\mathbf{0}\}$, there exist $\beta' \in B$ and $i \in 1 \dots n$ such that $\beta = \beta' + e_i$ (i.e $\underline{\mathbf{x}}^\beta = \underline{\mathbf{x}}^{\beta'} x_i$).

Given an ideal I of S, we denote by I_d the vector space of homogeneous polynomials of degree d that belong to I. We also denote s_d the dimension of the vector space S_d of polynomials $\in S$ of degree d.

For $B \subset \mathbb{N}^n$, we denote by $B^+ = e_1 + B \cup \cdots \cup e_n + B \cup B$ and $\partial B = B^+ - B$.

A *rewriting family* associated to a set $B \subset R$ of monomials is a set of polynomials of the form $(h_\alpha)_{\alpha \in \partial B}$ with:

$$h_\alpha(\underline{\mathbf{x}}) = \underline{\mathbf{x}}^\alpha - \sum_{\beta \in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta$$

with $z_{\alpha,\beta} \in \mathbb{K}$ for all $\alpha \in \partial B,\ \beta \in B$. We call it a *border basis* of B if moreover B is a basis of $\mathcal{A} = R/(h_\alpha(\underline{\mathbf{x}}))$.

If $B = (\beta_1, \dots, \beta_m)$ is a sequence of elements of $\mathbb{N}^n$ and $\beta \in \mathbb{N}^n$, $B^{\beta_i|\beta}$ is the sequence $(\beta_1, \dots, \beta_{i-1},\ \beta,\ \beta_{i+1}, \dots, \beta_m)$ obtained from B, by replacing β_i by β. Finally we denote by $\langle B \rangle$ the vector space generated by B.

We study the set of $\mathbb{K}$-algebras $\mathcal{A}$ generated by $x_1, \dots, x_n$, that admit B as a monomial basis. For any $a \in \mathcal{A}$, we consider the operator M_a of multiplication by a in $\mathcal{A}$:

$$\begin{aligned} M_a \ :\ & \mathcal{A} \to \mathcal{A} \\ & b \mapsto ab \end{aligned}$$

As $\mathcal{A}$ is a commutative algebra, the multiplication operators by the variables x_i commute. Thus for any $p \in R$, we can define the operator $p(M_{x_1}, \dots, M_{x_n})$ obtained by substitution of the variable x_i by M_{x_i} $(i = 1, \dots, n)$.

We define $\mathfrak{I}(\mathcal{A}) := \{p \in R; p(M_{x_1}, \dots., M_{x_n}) = 0\}$ and call it the ideal associated to $\mathcal{A}$.

The "dehomogenization by x_0" is the application from S to R that maps a polynomial $p \in S$ to $p(1, x_1, \dots, x_n) \in R$. For any subset $I \subset S$, we denote $\underline{I}$ its image by the dehomogenization.

Let A be a ring and I an ideal of $A[x_0, \dots, x_n]$, we will say that a polynomial P is not a zero divisor of I if $I : P = I$.

2. Hilbert functor representation

In this section we give a new proof of the existence of the Hilbert scheme $\mathrm{Hilb}^\mu(\mathbb{P}^n)$ using border basis relations. We will focus on open subfunctors of the Hilbert functor $\mathbf{Hilb}^\mu_{\mathbb{P}^n}$ that are represented by affine schemes and consist of a covering of $\mathrm{Hilb}^\mu(\mathbb{P}^n)$. We will use border bases and commutation relations to define these open affine subschemes of $\mathrm{Hilb}^\mu(\mathbb{P}^n)$.

2.1. **Border basis representation.** Let A be a local noehterian ring with maximal ideal m and residue field $k := A/m$. Suppose that $\mathcal{A}$ is a quotient algebra of $A[x_1, \dots, x_n]$ that is a free A-module. Assume that $\mathcal{A}$ has a monomial basis B of size μ, connected to 1. Then for any $\alpha \in \partial B$, the monomial $\underline{\mathbf{x}}^\alpha$ is a linear combination in $\mathcal{A}$ of the monomials of B: For any $\alpha \in \partial B$, there exists $z_{\alpha,\beta} \in \mathbb{K}$ $(\beta \in B)$ such that $h^{\mathbf{z}}_\alpha(\underline{\mathbf{x}}) := \underline{\mathbf{x}}^\alpha - \sum_{\beta \in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta \equiv 0$ in $\mathcal{A}$. The equations $h^{\mathbf{z}}_\alpha(\underline{\mathbf{x}})$ will be called, hereafter, the *border relations* of $\mathcal{A}$ in B.

Given these border relations, we define a projection $N^{\mathbf{z}} : \langle B^+\rangle \to \langle B^+\rangle$ by:

- $N^{\mathbf{z}}(\underline{\mathbf{x}}^\beta) = \underline{\mathbf{x}}^\beta$ if $\beta \in B$,
- $N^{\mathbf{z}}(\underline{\mathbf{x}}^\alpha) = \underline{\mathbf{x}}^\alpha - h^{\mathbf{z}}_\alpha(\underline{\mathbf{x}}) = \sum_{\beta \in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta$ if $\alpha \in \partial B$.

This construction is extended by linearity to $\langle B^+\rangle$.

Similarly, the tables of multiplication $M^{\mathbf{z}}_{x_i} : \langle B\rangle \to \langle B\rangle$ are constructed using $M^{\mathbf{z}}_{x_i}(\underline{\mathbf{x}}^\beta) = N^{\mathbf{z}}(x_i \underline{\mathbf{x}}^\beta)$ for $\beta \in B$. Notice that the coefficients of the matrix of $M^{\mathbf{z}}_{x_i}$ in the basis B are linear in the coefficients $\mathbf{z}$.

More generally, a monomial m can be reduced modulo the polynomials $(h^{\mathbf{z}}_\alpha(\underline{\mathbf{x}}))_{\alpha \in \partial B}$ to a linear combination of monomials in B, as follows: decompose $m = x_{i_1} \cdots x_{i_l}$ and compute $N^{\mathbf{z}}(m) = M^{\mathbf{z}}_{i_1} \circ \cdots \circ M^{\mathbf{z}}_{i_l}(1)$. We easily check that $m - N^{\mathbf{z}}(m) \in (h^{\mathbf{z}}_\alpha(\underline{\mathbf{x}}))_{\alpha \in \partial B}$.

Given a free quotient algebra of $A[x_1, \dots, x_n]$ with basis B connected to 1 of size μ, we have seen that there exist coefficients $(z_{\alpha,\beta} \in \mathbb{K})_{\alpha \in \partial B, \beta \in B}$ which describe completely an ideal defining μ points with multiplicity. Conversely, we are interested in characterizing the coefficients $\mathbf{z} := (z_{\alpha,\beta})_{\alpha \in \partial B, \beta \in B}$ such that the polynomials $(h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}}))_{\alpha\in B}$ are the border relations of some quotient algebra $\mathcal{A}^{\mathbf{z}}$ in the basis B.

The following result [20], also used in [23, 18] for special cases of base B and adapted to local rings, answers the question:

Theorem 2.1. *Let B be a set of μ monomials connected to 1. The polynomials $h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}})$ are the border relations of some free quotient algebra $\mathcal{A}^{\mathbf{z}}$ of $A[x_1,\ldots,x_n]$ of basis B iff*

$$(1)\qquad M_{x_i}^{\mathbf{z}}\circ M_{x_j}^{\mathbf{z}} - M_{x_j}^{\mathbf{z}}\circ M_{x_i}^{\mathbf{z}} = 0 \quad \text{for} \quad 1\leqslant i<j\leqslant n.$$

Proof. See [20]. □

In the next propositions, we consider $\boldsymbol{z}=(z_{\alpha,\beta})_{\alpha\in\partial B,\beta\in B}$ as variables. Then, note that the relations (1) induce polynomial equations of degree $\leqslant 2$ in $\boldsymbol{z}$ that we will denote:

$$(2)\qquad M_{x_i}(\boldsymbol{z})\circ M_{x_j}(\boldsymbol{z}) - M_{x_j}(\boldsymbol{z})\circ M_{x_i}(\boldsymbol{z}) = 0 \quad \text{for} \quad 1\leqslant i<j\leqslant n.$$

Proposition 2.2. *Let $B\subset A[x_1,\ldots,x_n]$ be a set of μ monomials connected to 1. Let N be the size of ∂B, then*

$$\{I\subset A[x_1,\ldots,x_n]\mid \mathcal{A}=R/\underline{I}\ is\ free\ with\ basis\ B\}$$

is a variety of $\mathbb{K}^{\mu\times N}$ in the variables $\boldsymbol{z}\in\mathbb{A}_{\mathbb{K}}^{\mu\times N}$ defined by $\mathfrak{H}_B:=\{\boldsymbol{z}\in\mathbb{K}^{\mu\times N}; M_{x_i}(\boldsymbol{z})\circ M_{x_j}(\boldsymbol{z}) - M_{x_j}(\boldsymbol{z})\circ M_{x_i}(\boldsymbol{z})=0, 1\leqslant i<j\leqslant n\}$. We call it the variety of free quotient algebras with basis B.

These varieties depending on monomial sets B are used in [14] to define the global punctual Hilbert scheme, via a glueing construction.
Hereafter, we will give a direct and explicite construction of the Hilbert scheme, based on these relations.

Example 2.3. *To illustrate the construction, we consider the very simple case where $B=(1,x)$ connected to 1 in $\mathbb{K}[x,y]$. Then we have $\partial B=(y,xy,x^2)$ and the formal border relations are:*

$$\begin{aligned} f_y &= y - z_{y,1} + z_{y,x}\,x \\ f_{xy} &= xy - z_{xy,1} + z_{xy,x}\,x \\ f_{x^2} &= x^2 - z_{x^2,1} + z_{x^2,x}\,x \end{aligned}$$

where $z_{y,1}, z_{y,x}, z_{xy,1}, z_{xy,x}, z_{x^2,1}, z_{x^2,x}$ are the 6 variables of the border relations. The multiplication matrices are:

$$M_x=\begin{pmatrix} 0 & z_{x^2,1} \\ 1 & z_{x^2,x}\end{pmatrix},\ M_y=\begin{pmatrix} z_{y,1} & z_{xy,1} \\ z_{y,x} & z_{xy,x}\end{pmatrix}.$$

The equations of $\mathfrak{H}_B$ are then given by $M_x\, M_y - M_y\, M_x = 0$. This yields the following equations of degree 2 in the 6 variables $(z_{\alpha,\beta})$:

$$\begin{cases} z_{xy,1} - z_{x^2,1}\, z_{y,x} = 0, \\ z_{xy,x} - z_{y,1} - z_{y,x}\, z_{x^2,x} = 0, \\ z_{x^2,1}\, z_{y,1} + z_{x^2,x}\, z_{xy,1} - z_{xy,x}\, z_{x^2,1} = 0, \\ z_{x^2,1}\, z_{y,x} - z_{xy,1} = 0 \end{cases}$$

defining the ideal generated by the two polynomials $z_{xy,1} - z_{x^2,1}\, z_{y,x}$, $z_{xy,x} - z_{y,1} - z_{y,x}\, z_{x^2,x}$, which define a (parameterized) variety of dimension 4.

2.2. **The Hilbert functor.** Let $\mu \in \mathbb{N}$ and I be and ideal of S. The *Hilbert function* of I associates to $k \in \mathbb{N}$ the dimension of S_k/I_k. It coincides with a polynomial called the *Hilbert polynomial* of I, for k large enough.
We consider the category $\mathcal{C}$ of noetherian schemes over $\mathbb{K}$. Let $\mathbb{P}^n$ be the projective scheme $\mathbf{Proj}(S)$. Let A be a commutative ring and p be a prime of A. We will denote by A_p the localization of A by p. Let m_p be its maximal ideal. We will denote by $k(p)$ the residue field $A_p/A_p m_p$.

Definition 2.4. *Let I be a graded ideal of S. I is said to be saturated if for all integers k and d such that $k \leq d$, $I_d : S_k = I_{d-k}$.*

Definition 2.5. *Let X and Y be schemes and $f : X \to Y$ be a morphism of schemes. X is said to be flat over Y if $\mathcal{O}_X$ is f-flat over Y i.e for every $x \in X$, $O_{X,x}$ is a $O_{Y,f(x)}$ flat module (see [12][Chap.III, p.254]).*

Definition 2.6. *The Hilbert functor of $\mathbb{P}^n$ relative to μ denoted $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ is the contravariant functor from the category $\mathcal{C}$ to the category of Sets which maps an object X of $\mathcal{C}$ to the set of flat families $Z \subset X \times \mathbb{P}^n$ of closed subschemes of $\mathbb{P}^n$ parametrized by X with fibers having Hilbert polynomial μ (flat families $Z \subset X \times \mathbb{P}^n$ means that Z is flat over X).*

Example 2.7. If $X = \mathbf{Spec}(A)$, where A is a noetherian $\mathbb{K}$-algebra, $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is given by the set of saturated homogeneous ideals I of $A[x_0, \ldots, x_n]$ such that $\mathbf{Proj}(A[x_0, \ldots, x_n]/I)$ is flat over $\mathbf{Spec}(A)$ and for every prime ideal $p \subset A$, the Hilbert polynomial of the $k(p)$-graded algebra $(A[x_0, \ldots, x_n]/I) \otimes_A k(p)$ is equal to μ where $k(p)$ is the residue field A_p/pA_p.

Definition 2.8. *Let A be a noetherian $\mathbb{K}$-algebra. Let $p \in \mathbf{Spec}(A)$ be a prime of A with residue field $k(p) := A_p/pA_p$. Let I be a homogeneous ideal of $A[x_0, \ldots, x_n]$. Consider the following exact sequence:*

$$0 \longrightarrow I \longrightarrow A[x_0, \ldots, x_n] \longrightarrow A[x_0, \ldots, x_n]/I \longrightarrow 0$$

Tensoring by $k(p)$ we get the exact sequence

$$I \otimes k(p) \longrightarrow k(p)[x_0, \ldots, x_n] \longrightarrow A[x_0, \ldots, x_n]/I \otimes k(p) \longrightarrow 0$$

Then, we will denote by $I(p)$ the homogeneous ideal of $k(p)[x_0, \ldots, x_n]$ which consists of the image of $I \otimes k(p)$ in $k(p)[x_0, \ldots, x_n]$. Thus we have

$$A[x_0, \ldots, x_n]/I \otimes k(p) \sim k(p)[x_0, \ldots, x_n]/I(p).$$

Remark 2.9. *Note that in general $I(p)$ is not isomorphic to $I \otimes k(p)$ because tensoring by $k(p)$ is not a left exact functor i.e the morphism:*

$$I \otimes k(p) \longrightarrow k(p)[x_0, \dots, x_n]$$

is surjective but not injective.

In our analysis, we will use the affine setting described in Section 2.1. In order to identify the good affinizations which lead to this setting, we introduce the following definition and characterization:

Definition 2.10. *Given an homogeneous ideal J in $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(\overline{k})$, one has from the Nullstellensatz theorem that J has the following primary decomposition:*

$$J = \bigcap_i q_i$$

with q_i homogeneous $m_{\overline{k},P_i}$-primary ideal, for some points P_i in the projective space $\mathbb{P}^n_{\overline{k}}$. The set $\{P_i\}$ will be called the set of points defined by J in $\mathbb{P}^n_{\overline{k}}$.

More generaly, let J be an homogeneous ideal (not necessarily saturated) of S with Hilbert polynomial equal to the constant μ. The set of points defined by J in $\mathbb{P}^n_{\overline{k}}$ is the set of points defined below by its saturation (denoted $Sat(J)$):

$$Sat(J) := \bigcup_{j \in \mathbb{N}} J : (m_{\overline{k}})^j$$

in $\mathbb{P}^n_{\overline{k}}$.

Proposition 2.11. *Let $X = \mathbf{spec}(A)$ be a scheme in $\mathcal{C}$ and $Z = \mathbf{Proj}(A[x_0, \dots, x_n]/I)$ be an element of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$. Let u be a linear form in $\mathbb{K}[x_0, \dots, x_n]$ and Z_u be the open set associated to u considered as an element of $H^0(Z, O_Z(1))$ (see [7][(0.5.5.2), p.53]). Let π be the natural morphism from Z to X. Let $p \in \mathbf{Spec}(A)$ be a prime of A and $k(p) := A_p/pA_p$ its residue field. Then, $\pi_*(O_{Z_u})_p$ is a free $\mathcal{O}_{X,p}$-module of rank μ if and only if u does not vanish at any of the points defined by $\overline{I(p)} = I(p) \otimes_{k(p)} \overline{k(p)}$ in $\mathbb{P}^n_{\overline{k(p)}}$ (see definition 2.10 and 2.8), with $\overline{k(p)}$ the algebraic closure of $k(p)$.*

Proof. By a change of variables in $\mathbb{K}[x_0, \dots, x_n]$ we can assume that $\mathbf{u} = x_0$. Moreover, without loss of generality, we can assume that A is a local ring with maximal ideal p.
Let $\underline{I} \subset A[x_1, \dots, x_n]$ be the affinization of I by x_0 (set $x_0 = 1$). One has that $\underline{I}(p) = \underline{I(p)}$ and that

$$(A[x_0, \dots, x_n]/I) \otimes_A k(p) = k(p)[x_0, \dots, x_n]/I(p)$$

and

$$(A[x_1, \dots, x_n]/\underline{I}) \otimes_A k(p) = k(p)[x_1, \dots, x_n]/\underline{I}(p).$$

We also know that $Z_{x_0} = D_+(x_0)$ (see [8][Prop (2.6.3), p.37]) and that $Z_{|D_+(x_0)} = \mathbf{Spec}(A[x_1, \dots, x_n]/\underline{I})$. Thus $\pi_*(\mathcal{O}_{Z\ |D_+(x_0)})$ is the sheaf of $\mathcal{O}_X$-module associated to the A-module $A[x_1, \dots, x_n]/\underline{I}$ on $\mathbf{Spec}(A)$. Finally,

$\pi_*(O_{Z_{x_0}})_p$ is a free $\mathcal{O}_{X,p}$-module of rank μ if and only if $A[x_1,\ldots,x_n]/\underline{I}$ is free of rank μ.

First let us prove that there exists an integer $N > 0$ such that for all $d \geq N$, the multiplication by x_0:

$$k(p)[x_0,\ldots,x_n]_d/I(p)_d \xrightarrow{*x_0} k(p)[x_0,\ldots,x_n]_{d+1}/I(p)_{d+1} \tag{3}$$

is injective if and only if x_0 does not vanish at any of the points defined by $\overline{I(p)}$ in $\mathbb{P}^n_{\overline{k(p)}}$.
As a matter of fact, $I(p)$ defines μ points in $k(p)[x_0,\ldots,x_n]$, there exists and integer $N \geq \mu$ such that the dimension of $k(p)[x_0,\ldots,x_n]_d/I(p)_d$ is equal to μ for all degree $d \geq N$. One has from [15][C.28] that $I(p)_{d+1} : S_1 = I(p)_d$ for all $d \geq N$ i.e:

$$Sat(I(p)) = \sum_{1\leq i\leq d} I(p)_d : S_i + (I(p)_d)\ \forall d \geq N.$$

with $Sat(I(p)) := \bigcup_{d\in\mathbb{N}} I(p) : (S_d)$ (i.e $I(p)$ is saturated in degree greater than N).
Thus the multiplication by x_0 is injective (and by dimension bijective) for all $d \geq N$ if and only if x_0 is not a zero divisor of the saturation $Sat(I(p))$ of $I(p)$. By proposition D.4 and definition 2.10, this is equivalent to x_0 does not vanish at any of the points defined by $\overline{I(p)} = \overline{k(p)} \otimes I(p)$ in $\mathbb{P}^n_{\overline{k(p)}}$.

Consider now the following commutative diagram for all $d \geq N$:

$$\begin{tikzcd}
(A[x_0,\ldots,x_n]_d/I_d)\otimes k(p) \arrow[r, "\sim"] \arrow[d, "\phi"] & k(p)[x_0,\ldots,x_n]_d/I(p)_d \arrow[d, "\delta"] \\
(A[x_1,\ldots,x_n]_{\leq d}/\underline{I}_{\leq d})\otimes k(p) \arrow[r, two heads, "\psi"] \arrow[d, "j"] & k(p)[x_1,\ldots,x_n]_{\leq d}/\underline{I}(p)_{\leq d} \arrow[d, "i"] \\
(A[x_1,\ldots,x_n]/\underline{I})\otimes k(p) \arrow[r, "="] & k(p)[x_1,\ldots,x_n]/\underline{I}(p)
\end{tikzcd} \tag{4}$$

First, as $k(p)[x_1,\ldots,x_n]/\underline{I}(p)$ is a $k(p)$-algebra of dimension less or equal to μ, one has that

$$k(p)[x_1,\ldots,x_n]_{\leq d}/\underline{I}(p)_{\leq d} \xrightarrow{i} k(p)[x_1,\ldots,x_n]/\underline{I}(p) \tag{5}$$

is an isomorphism for all $d \geq N \geq \mu$ (this comes from the fact that the Hilbert function of $k(p)[x_1,\ldots,x_n]/\underline{I}(p)$ is strictly increasing until it is the constant function equal to the dimension of $k(p)[x_1,\ldots,x_n]/\underline{I}(p) \leq \mu$). Thus, for all $d \geq N$, j is surjective.

Let us prove the equivalence between $A[x_1,\ldots,x_n]/\underline{I}$ is a free A-module of rank μ and x_0 does not vanish at any point defined by $\overline{I(p)}$ in $\mathbb{P}^n_{\overline{k(p)}}$.

First, assume x_0 does not vanish at any point defined by $\overline{I(p)}$ in $\mathbb{P}^n_{\overline{k(p)}}$. Thus, the multiplication by x_0

$$k(p)[x_0,\ldots,x_n]_d/I(p)_d \xrightarrow{x_0} k(p)[x_0,\ldots,x_n]_{d+1}/I(p)_{d+1} \tag{6}$$

is a bijection for all $d \geq N$ i.e the morphism δ in diagram (4) is an isomorphism. Thus all the morphisms in diagram (4) are isomorphisms for all $d \geq N \geq \mu$. Consequently, using the multiplication (6) from degree d to $d+1$, one has that the natural morphism

$$(A[x_1,\ldots,x_n]_{\leq d}/\underline{I}_{\leq d}) \otimes k(p) \longrightarrow (A[x_1,\ldots,x_n]_{\leq d+1}/\underline{I}_{\leq d+1}) \otimes k(p) \tag{7}$$

is an isomorphism for all $d \geq N$. By Nakayama Lemma, the natural inclusion:

$$A[x_1,\ldots,x_n]_{\leq d}/\underline{I}_{\leq d} \longrightarrow A[x_1,\ldots,x_n]_{\leq d+1}/\underline{I}_{\leq d+1} \tag{8}$$

is an isomorphism and

$$A[x_1,\ldots,x_n]_{\leq d}/\underline{I}_{\leq d} = A[x_1,\ldots,x_n]_{\leq d+1}/\underline{I}_{\leq d+1} = \ldots = A[x_1,\ldots,x_n]/\underline{I}. \tag{9}$$

Finally, $A[x_1,\ldots,x_n]/\underline{I}$ is a flat A-module of finite type such that $A[x_1,\ldots,x_n]/\underline{I} \otimes k(p)$ is of dimension μ. Using [26][lem.7.51, p.55], we deduce that $A[x_1,\ldots,x_n]/\underline{I}$ is a free A-module of rank μ.

Reciprocally, assume that $A[x_1,\ldots,x_n]/\underline{I}$ is a free A-module of rank μ. Then the dimension of $A[x_1,\ldots,x_n]/\underline{I} \otimes k(p)$ is equal to μ . Thus all the morphisms in diagram (4) (in particular δ) are isomorphisms for all $d \geq N$. Consequently x_0 is not a zero divisor of $Sat(I(p))$ and thus does not vanish at any point defined by $\overline{I(p)}$ in $\mathbb{P}^n_{\overline{k(p)}}$. $\square$

Example 2.12. *Let $A = \mathbb{K}[s,t]/(s\,t)$ and $I = (s\,x_0^2 + x_0x_1 + t\,x_1^2) \subset A[x_0,x_1]$. For each prime ideal p of A, the quotient $k(p)[x_0,x_1]/I$ is of dimension 2. Thus, $I \in \mathbf{Hilb}^2_{\mathbb{P}^1}(A)$. If we take the prime ideal $p = (s)$ of A, then $A_p = k(p) = \mathbb{K}(t)$ and the roots of $I(p)$ in $\overline{\mathbb{K}(t)}$ are $(1:0), (-t:1) \in \mathbb{P}^1(\overline{\mathbb{K}(t)})$. If we take $u = x_0 + x_1$, it does not vanish at the roots of $I(p)$. By a change of variables, $X_0 = x_0 + x_1, X_1 = x_1$ and taking $X_0 = 1$, we obtain $\underline{I} = (s + (1+2\,s)\,X_1 + (1+s+t)X_1^2)$. Thus*

$$\pi_*(O_{Z_{X_0}})_p = A_p[X_1]/\underline{I}_p = A_p[X_1]/(X_1 + (1+t)X_1^2)$$

is a free A_p module of rank 2 generated by $\{1, X_1\}$, since $(1+t)$ is invertible in A_p.

If we take $p = (s,t)$, then $k(p) = \mathbb{K}$ and the roots of $I(p)$ are $(1:0), (0:1) \in \mathbb{P}^1(\overline{\mathbb{K}})$. By the same change of variables, we obtain

$$\pi_*(O_{Z_{X_0}})_p = A_p[X_1]/\underline{I}_p = A_p[X_1]/(s + (1+2\,s)\,X_1 + (1+s+t)X_1^2)$$

which is also a free A_p module of rank 2 generated by $\{1, X_1\}$, since $(1+s+t)$ is invertible in A_p.

Notice that the localisation is needed: $A[X_1]/(X_1+(1+t)X_1^2)$ or $A[X_1]/(s+(1+2\,s)\,X_1+(1+s+t)X_1^2)$ are not free A-modules nor of finite type.

Corollary 2.13. *Let A be a local ring with maximal ideal m and a noetherian $\mathbb{K}$-algebra. Then*

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(A) =$$

$$\{\text{Saturated ideal } I \subset A[x_0,\ldots,x_n] \mid A[x_0,\ldots,x_n]_d/I_d \text{ is free of rank } \mu \text{ for } d \geq \mu\}.$$

Proof. First, let $I \subset A[x_0,\ldots,x_n]$ be a saturated homogeneous ideal such that $A[x_0,\ldots,x_n]_d/I_d$ is free of rank μ for all degree $d \geq \mu$, then I belongs to $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(A)$.

We deduce $\underline{I}(p)$ defines an affine zero dimensional algebra of multiplicity μ in $k(p)[x_1,\ldots,x_n]$. Thus, using homogenization by x_0 on $\underline{I}(p)$ we obtain an homogeneous ideal in $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(k(p))$. Then, by the Gotzmann's persistence theorem [15][C.17, p.297] we deduce that for all degree $d \geq \mu$ the natural inclusion

$$i : k(p)[x_1,\ldots,x_n]_{\leq d}/\underline{I}(p)_{\leq d} \longrightarrow k(p)[x_1,\ldots,x_n]/\underline{I}(p)$$

is an isomorphism (by dimension). Thus, the morphism

$$j : (A[x_1,\ldots,x_n]_{\leq d}/\underline{I}_{\leq d}) \otimes k(p) \longrightarrow (A[x_1,\ldots,x_n]/\underline{I}) \otimes k(p)$$

is surjective for all $d \geq \mu$. Then, by Nakayama's Lemma, we get that

$$A[x_1,\ldots,x_n]_{\leq\mu}/\underline{I}_{\leq\mu} = A[x_1,\ldots,x_n]_{\leq\mu+1}/\underline{I}_{\leq\mu+1} = \cdots = A[x_1,\ldots,x_n]/\underline{I}.$$

Finally, as x_0 is not a zero divisor of I, we get that $A[x_1,\ldots,x_n]_{\leq d}/\underline{I}_{\leq d} \simeq A[x_0,\ldots,x_n]_d/I_d$ is a free A-module of rank μ for all degree $d \geq \mu$. □

Corollary 2.14. *Let A be a local ring with maximal ideal m and a noethrian $\mathbb{K}$-algebra. Let $I \subset A[x_0,\ldots,x_n]$ be a homogeneous ideal in $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(A)$. Then I is generated in degree μ:*

$$I_{\mu+k} = A[x_0,\ldots,x_n]_k\, I_\mu$$

for all $k \geq 0$.

Proof. As in the proof of corollary 2.13, we can assume x_0 is not a zero divisor of I and does not vanish at any point defined by $\overline{I(m)}$ (see definitions 2.8 and 2.10). Then, from proposition 2.11, $A[x_1,\ldots x_n]/\underline{I}$ is a free A-module of rank μ and we get the following diargam:

$$\begin{tikzcd}
(A[x_1,\ldots,x_n]_{\leq d}/\underline{I}_{\leq d}) \otimes k(p) \arrow[r, two heads, "\psi"] \arrow[d, "j"] & k(p)[x_1,\ldots,x_n]_{\leq d}/\underline{I}(p)_{\leq d} \arrow[d, "i"] \\
(A[x_1,\ldots,x_n]/\underline{I}) \otimes k(p) \arrow[r, "\sim"] & k(p)[x_1,\ldots,x_n]/\underline{I}(p)
\end{tikzcd}$$

for which we proved in corollary 2.13 that i, j (and thus ψ) are isomorphism for all $d \geq \mu$. As $k(p)[x_1,\ldots,x_n]/\underline{I}(p)$ is of dimension μ, we can find a basis B with polynomials of degree less or equal to $\mu - 1$ (take for instance B connected to 1). Thus, B is a basis of $k(p)[x_1, \ldots, x_n]_{\leq d}/\underline{I}(p)_{\leq d} \sim (A[x_1, \ldots, x_n]_{\leq d}/\underline{I}_{\leq d}) \otimes k(p)$ for all $d \geq \mu$. By Nakayama's lemma, we deduce that B is a basis of the free A-module $A[x_1, \ldots, x_n]_{\leq d}/\underline{I}_{\leq d}$ for all $d \geq \mu$. As the degree of all the polynomials in B is less or equal to $\mu - 1 < \mu \leq d$, we can define operators of multiplication by the variables $(x_i)_{1 \leq i \leq n}$ in $A[x_1, \ldots, x_n]_{\leq d}/\underline{I}_{\leq d}$ for all $d \geq \mu$. Then, using these operators of multiplication, we can easily prove that $\underline{I}_{\leq d+1} = A[x_1, \ldots, x_n]_{\leq 1}.\underline{I}_{\leq d}$ for all $d \geq \mu$. As x_0 is not a zero divisor of I, we deduce

$$I_{d+1} = S_1 \, I_d$$

for all $d \geq \mu$. $\square$

Remark 2.15. *Let A be a noetherian $\mathbb{K}$-algebra. Let $X = \mathbf{Spec}(A)$ and Z be a closed subscheme of $X \times \mathbb{P}^n$ and let $\mathcal{I} \subset \mathcal{O}_{X \times \mathbb{P}^n}$ be the sheaf of ideals that defines Z. Corollary 2.14 means that if Z belongs to $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$, then the natural map*

$$H^0(X \times \mathbb{P}^n, \mathcal{I}(d)) \otimes_{\mathbb{K}} \mathcal{O}_{X \times \mathbb{P}^n}(1) \longrightarrow H^0(X \times \mathbb{P}^n, \mathcal{I}(d+1))$$

is surjective.

2.3. **Open covering of the Hilbert functor.** Let A be a ring and M is an A-module. We denote by $\widetilde{M}$ the quasi-coherent sheaf of modules associated to M in $\mathbf{Spec}(A)$. We will say that M is locally free on $\Omega \subset \mathbf{Spec}(A)$ if for all $p \in \Omega$, M_p is a free A_p-module. We will say that M is locally free if it is locally free on $\mathbf{Spec}(A)$. Thus, M is locally free if and only if $\widetilde{M}$ is locally free on $\mathbf{Spec}(A)$ as a sheaf of modules.
We recall some definitions about functors (see eg. [25][Appendix E]):

Definition 2.16. *A contravariant functor F from the category $\mathcal{C}$ to the category of Sets is representable if there exists an object Y in $\mathcal{C}$ such that the functor $\mathbf{Hom}(-, Y)$ is isomorphic to the functor F. In particular for every X in $\mathcal{C}$,*

$$F(X) \simeq Hom(X, Y).$$

Definition 2.17. *A contravariant functor F from the category $\mathcal{C}$ to the category of Sets is called a sheaf if for every scheme X in $\mathcal{C}$, the presheaf of sets on the topological space associated to X given by:*

$$U \to F(U)$$

is a sheaf. Namely, if for all schemes X in $\mathcal{C}$ and for every open covering $\{U_i\}$ of X, the following is an exact sequences of sets:

$$0 \to F(X) \to \prod_i F(U_i) \to \prod_{i,j} F(U_i \cap U_j)$$

Notice that by construction, representable functors are sheaves.

Definition 2.18. *Let F be a contravariant functor from $\mathcal{C}$ to the category of Sets. A subfunctor G of F is said to be an open subfunctor if for every scheme X in $\mathcal{C}$ and for every morphism of functors*

$$\mathbf{Hom}(-,X) \to F$$

the fiber product $\mathbf{Hom}(-,X) \times_F G$ (which is a subfunctor of $\mathbf{Hom}(-,X)$) is represented by an open subscheme of X.
A family of open subfunctors $\{G_i\}$ of F is a covering if for every scheme X in $\mathcal{C}$, the family of subschemes that represent the subfunctors $\{\mathbf{Hom}(-,X) \times_F G_i\}$ is an open covering of X.

Definition 2.19. *Let u be a linear form in $\mathbb{K}[x_0,\dots,x_n]$. Let $\mathbf{H}_u$ be the subfunctor of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ which associates to X in $\mathcal{C}$ the set $\mathbf{H}_u(X)$ of flat families $Z \subset X \times \mathbb{P}^n$ of closed subschemes of Y parametrized by X with fibers having Hilbert polynomial μ and such that $\pi_*(O_{Z_u})$ is locally free sheaf of rank μ of X, where π is the natural morphism from Z to X and Z_u is the open set associated to u considered as an element of $H^0(Z, O_Z(1))$ (see [7][(0.5.5.2) p.53]).*

Proposition 2.20. *The family of subfunctors $(\mathbf{H}_u)_{u\in\mathbb{K}[x_0,\dots,x_n]_1}$ consists of an open covering of subfunctors of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$.*

Proof. From proposition A.2, it is enough to consider affine schemes $X = \mathbf{Spec}(A)$ (with A a noetherian $\mathbb{K}$-algebra) and to prove that, given a morphism of functors from $\mathbf{Hom}(-,X)$ to $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ (i.e given an element $Z \in \mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$) the functor

$$G := \mathbf{Hom}(-,X) \times_{\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}} \mathbf{H}_u$$

restricted to the category of affine noetherian schemes over $\mathbb{K}$ is represented by an open subscheme of X.
Let $X' = \mathbf{Spec}(A')$ be an affine noetherian scheme over $\mathbb{K}$. Let f be a morphism of $\mathbb{K}$-algebras from A to A'. Let ϕ be the morphism of schemes from $X' = \mathbf{Spec}(A')$ to $\mathbf{Spec}(A)$ associated to f. Let Z be an element of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ and I be its associated saturated homogeneous ideal of $A[x_0,\dots,x_n]$. Let Z' be the element of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X')$ given by $(\phi \times Id_{\mathbb{P}^n})^*(Z)$. Let I' be the homogeneous ideal of $A'[x_0,\dots,x_n]$ associated to the quotient algebra $(A[x_0,\dots,x_n]/I) \otimes_A A'$.
By a change of variables in $\mathbb{K}[x_0,\dots,x_n]$, we can assume $u = x_0$. Then, $Z'_{x_0} = (\phi \times Id_{\mathbb{P}^n})^*(Z_{x_0})$ is equal to $\mathbf{Spec}(A'[x_1,\dots,x_n]/\underline{I}')$ (where $\underline{I}'$ denote the affinization of I').
Then, we need to prove that for all A' and f, the sheaf of module of $\mathbf{Spec}(A')$ associated to the A'-module $A'[x_1,\dots,x_n]/\underline{I}'$ is locally free of rank μ if and only if the morphism

$$\phi : \mathbf{Spec}(A') \to \mathbf{Spec}(A)$$

factors through an open subscheme Ω_{x_0} of $\mathbf{Spec}(A)$.

Let q be a prime of A' and p be its image in $\mathbf{Spec}(A)$. From proposition 2.11, $A'_q[x_1,\dots,x_n]/\underline{I'_q}$ is free of rank μ if and only if x_0 does not vanish at any point defined by $\overline{I'(q)} = I'(q) \otimes_{k'(q)} \overline{k'(q)}$ in $\mathbb{P}^n_{\overline{k'(q)}}$ (see Definition 2.8). Note that we have:

$$I'(q) = I(p) \otimes_{k(p)} k'(q).$$

Thus, by Proposition D.5, the points defined by $\overline{I'(q)}$ are the same as those defined by $\overline{I(p)} = I(p) \otimes_{k(p)} \overline{k(p)}$ in $\mathbb{P}^n_{\overline{k(p)}}$ using the natural field inclusions

$$\begin{array}{ccc} k(p) & \xrightarrow{i} & k'(q) \\ \downarrow & & \downarrow \\ \overline{k(p)} & \xrightarrow{\overline{i}} & \overline{k'(q)} \end{array}$$

Thus, x_0 does not vanish at the points defined by $\overline{I'(q)}$ if and only if x_0 does not vanish at the points defined by $\overline{I(p)}$. Equivalently: $A'_q[x_1,\dots,x_n]/\underline{I'_q}$ is free of rank μ if and only if $A_p[x_1,\dots,x_n]/\underline{I_p}$ is free of rank μ (with $p = \phi(q)$). Thus, $A'[x_1,\dots,x_n]/\underline{I'}$ is locally free of rank μ if and only if ϕ factors through the subset $\Omega_{x_0} \subset \mathbf{Spec}(A)$ on which $A[x_1,\dots,x_n]/\underline{I}$ is locally free of rank μ.
Let $p \in \mathbf{Spec}(A)$ be a prime of A. Then, using diagram 4, one has the following equivalence:
(i) $A[x_1,\dots,x_n]/\underline{I} \otimes_A A_p = A_p[x_1,\dots,x_n]/\underline{I_p}$ is an A_p free module,
(ii) $A[x_1,\dots,x_n]/\underline{I} \otimes_A A_p = A_p[x_1,\dots,x_n]/\underline{I_p}$ is an A_p-module of finite type,
(iii) $A_p[x_1,\dots,x_n]_{\leq d}/\underline{I_p}_{\leq d} = A_p[x_1,\dots,x_n]_{\leq d+1}/\underline{I_p}_{\leq d+1}$ for all $d \geq \mu$,
(iv) there exists an integer $d \geq \mu$ such that

$$A_p[x_1,\dots,x_n]_{\leq d}/\underline{I_p}_{\leq d} = A_p[x_1,\dots,x_n]_{\leq d+1}/\underline{I_p}_{\leq d+1}.$$

Thus, the set $\Omega \subset \mathbf{Spec}(A)$ on which $A[x_1,\dots,x_n]/\underline{I}$ is locally free is equal to the subset of $\mathbf{Spec}(A)$ on which the morphism of inclusion

$$i : A[x_1,\dots,x_n]_{\leq \mu}/\underline{I}_{\leq \mu} \longrightarrow A[x_1,\dots,x_n]_{\leq \mu+1}/\underline{I}_{\leq \mu+1}$$

is surjective. Let M be the cokernel of i, then Ω is equal to the subset of $\mathbf{Spec}(A)$ on which $\widetilde{M}$ is equal to zero. As M is of finite type (i.e $\widetilde{M}$ is a coherent scheaf of module), Ω is an open subset of $\mathbf{Spec}(A)$.
Finally, Ω_{x_0} is an open subset of Ω and $A'[x_1,\dots,x_n]/\underline{I'}$ is locally free of rank μ if and only if ϕ factors through the open subscheme associated to Ω_{x_0}. Consequently, $\mathbf{H}_{x_0}$ is an open subfunctor of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$.

Let us prove that $(\mathbf{H}_u)_{u \in \mathbb{K}[x_0,\dots,x_n]_1}$ consists of a covering of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$. By definition 2.18, we need to prove that $(\Omega_u)_{u \in S_1}$ consists of a covering of $\mathbf{Spec}(A)$. Let p be a point of $\mathbf{Spec}(A)$. Consider the points of $\mathbb{P}^n_{k(p)}$ defined by $I(p)$. One can find a linear form $u \in S_1 = \mathbb{K}[x_0,\dots,x_n]_1$ that does not

vanish at any of these points. By proposition 2.11, p belongs to Ω_u.
Thus the family $(\Omega_u)_{u\in\mathbb{K}[x_0,\dots,x_n]_1}$ consists of a covering of $\mathbf{Spec}(A)$ and the family of open subfunctors $\mathbf{H}_u$ is a covering of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$. □

2.4. **Representation of the Hilbert functor.** We are now going to prove that the Hilbert functor is representable.

Definition 2.21. *Let B be a family of μ monomials of degree d in $\mathbb{K}[x_0,\dots,x_n]$. Let $\mathbf{H}^B_{x_0}$ be the subfunctor of $\mathbf{H}_{x_0}$ which associates to X in $\mathcal{C}$ the set $\mathbf{H}^B_{x_0}(X)$ of flat families $Z\subset X\times\mathbb{P}^n$ of closed subschemes of Y parametrized by X with fibers having Hilbert polynomial μ such that $\pi_*(O_{Z_{x_0}})$ is a locally free sheaf of rank μ of X with basis $\underline{B}:=B/x_0^d$ considered as elements of $H^0(Z,O_{Z_{x_0}})$.*

Lemma 2.22. *Let $d\geq\mu$ be an integer. Let $\mathcal{B}_d$ be the set of families B of μ monomials of degree d in $\mathbb{K}[x_0,\dots,x_n]$ such that the affinization $\underline{B}$ is connected to one in $\mathbb{K}[x_1,\dots,x_n]$. Then, the family of contravariant functors $(\mathbf{H}^B_{x_0})_{B\in\mathcal{B}_d}$ consists of open covering of representable subfunctors of $\mathbf{H}_{x_0}$.*

Proof. First, let us prove that $\mathbf{H}^B_{x_0}$ is an open subfunctor. By proposition A.2, we can reduce to the case of affine schemes. Let A and A' be noetherian $\mathbb{K}$-algebras. Let f be any morphism of $\mathbb{K}$-algebras from A to A' and ϕ its corresponding morphism from $\mathbf{Spec}(A')$ to $\mathbf{Spec}(A)$.
Let Z be an element of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ and I be its associated saturated homogeneous ideal of $A[x_0,\dots,x_n]$. Let Z' be the element of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X')$ given by $(\phi\times Id_{\mathbb{P}^n})^*(Z)$. Let I' be the homogeneous ideal of $A'[x_0,\dots,x_n]$ associated to the quotient algebra $(A[x_0,\dots,x_n]/I)\otimes_A A'$.
Then, $Z'_{x_0}=(\phi\times Id_{\mathbb{P}^n})^*(Z_{x_0})$ is equal to $\mathbf{Spec}(A'[x_1,\dots,x_n]/\underline{I}')$ (where $\underline{I}'$ denote the affinization of I'). Thus, we need to prove that for all noetherian $\mathbb{K}$-algebras A' and for all morphisms of $\mathbb{K}$-algebras f from A to A', the sheaf of modules of $\mathbf{Spec}(A')$ associated to the A'-module $A'[x_1,\dots,x_n]/\underline{I}'$ is locally free of rank μ with basis $\underline{B}$ if and only if the morphism

$$\phi:\mathbf{Spec}(A')\to\mathbf{Spec}(A)$$

factors through an open subscheme Γ_B of $\mathbf{Spec}(A)$.
From proposition 2.20, one has that Γ_B exists and is equal to the open subscheme of Ω_{x_0} associated to the open subset on which the sheaf of module given by the A-module $A[x_1,\dots,x_n]/\underline{I}$ is locally free of rank μ with basis $\underline{B}$. Thus, $\mathbf{H}^B_u$ is an open subfunctor of $\mathbf{H}_{x_0}$.
To prove that the family $(\mathbf{H}^B_u)_{B\in\mathcal{B}_d}$ is a covering, we need to prove that the family $(\Gamma_B)_{B\in\mathcal{B}_d}$ is a covering of $\mathbf{Spec}(A)$. This is a straightforward consequence of the fact that any zero dimensional k-algebra $k[x_1,\dots,x_n]/J$ (where k is a field and J an ideal of $k[x_1,\dots,x_n]$) has a basis of monomials connected to one (take for instance the complement of the initial ideal of J for a monomial ordering).

Finally we need to prove that $\mathbf{H}^B_{x_0}$ is representable. By proposition A.3, we can reduce to affine schemes.

Let A be a noetherian $\mathbb{K}$-algebra and $X = \mathbf{Spec}(A)$. Recall that $\mathbf{H}^B_{x_0}(X)$ is the set of saturated homogeneous ideals I of $A[x_0, \ldots, x_n]$ such that for all $d \geq \mu$ $(A[x_0, \ldots, x_n]/I)_d$ is a flat A module, for every prime ideal $p \subset A$, the Hilbert polynomial of the $k(p)$-graded algebra $(A[x_0, \ldots, x_n]/I) \otimes_A k(p)$ is equal to μ and $A_p[x_1, \ldots, x_n]/\underline{I_p}$ is a free A_p-module of basis $\underline{B}$.

Let F be the contravariant functor from the category of affine schemes to the category of Sets which associates to X the set $F_B(X)$ of ideals J of $A[x_1, \ldots, x_n]$ such that $A[x_1, \ldots, x_n]/J$ is a free A module of basis $\underline{B}$.

Let ψ be the morphism of functors from $\mathbf{H}^B_{x_0}$ to F given by

$$\begin{array}{rcl} \psi : \mathbf{H}^B_{x_0}(X) & \longrightarrow & F_B(X) \\ I & \longmapsto & \underline{I} \end{array}$$

The map ψ is a bijection whose inverse consists of the homogenization. Then the functors F_B and $\mathbf{H}^B_{x_0}$ are isomorphic.

By proposition 2.2, F_B is represented by $\mathbf{Spec}(\mathbb{K}[(z_{\alpha,\beta})_{\alpha \in \delta B, \beta \in B}]/\mathcal{R})$, where $\mathcal{R}$ is the ideal generated by the commutation relations (2). Thus $\mathbf{H}^B_{x_0}$ is representable in the category of affine schemes. By proposition A.3, $\mathbf{H}^B_{x_0}$ is representable in $\mathcal{C}$. □

Theorem 2.23. *The contravariant functor* $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ *from* $\mathcal{C}$ *to the category of Sets is representable.*

Proof. From lemma 2.22 and proposition A.1, $\mathbf{H}_{x_0}$ is a representable functor. More generally, by a change of variables in $\mathbb{K}[x_0, \ldots, x_n]$, $\mathbf{H}_u$ is a representable functor for all $u \in \mathbb{K}[x_0, \ldots, x_n]_1$.
Thus, from propositions 2.20 and A.1, $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ is a representable contravariant functor.

□

Remark 2.24. *Note that the* $\mathbb{K}$*-rational points of* $\mathbf{Hilb}^{\mu}(\mathbb{P}^n)$ *are by definition in bijection with the set of homogeneous saturated ideal* I *of* $\mathbb{K}[x_0, \ldots, x_n]$ *such that the quotient algebra* S/I *has Hilbert polynomial equal to* μ.

3. Global equations of the Hilbert Scheme

Let A be $\mathbb{K}$-algebra and a noetherian local ring of maximal ideal m and residue field $k := A/m$. Recall that if $a \notin m$ then a is invertible in A. Let $X := \mathbf{Spec}(A)$ be the affine scheme associated to A and μ be an integer. Let $T := A[x_0, \ldots, x_n]$ be the polynomial over A in $n+1$ variables and $V := A[x_1, \ldots, x_n]$ the polynomial ring over A in n variables.

3.1. **Gotzmann's persistence and regularity theorems.** Recall that from corollary 2.13, $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is equal to set of homogeneous ideals I of T such that T_d/I_d is a free A-module of rank μ for all $d \geq \mu$.
Finally, recall the definition of the Grasmmannian functor (see eg. [25][Chap.4.3.3, p.209]):

Definition 3.1. *Let V be a $\mathbb{K}$-vector space of finite dimension N and $n \leq N$ an integer. Let X be a noetherian scheme over $\mathbb{K}$. The n Grassmannian functor of V is the contravariant functor from the category $\mathcal{C}$ to the category of Sets which associates to X the set $\mathbf{Gr}_V^n(X)$ of locally free sheaves ϵ of rank n such that ϵ is a quotient of $V^* \otimes_{\mathbb{K}} \mathcal{O}_X$ on X.*

The following theorems come from [6]:

Theorem 3.2 (Persistence theorem)**.** *Let d, μ be integers such that $d \geq \mu$. Let B be any noetherian ring and $F = B[x_0, \ldots, x_n]$. Let I be an homogeneous ideal of F generated by I_d and let $M = F/I$. If M_i is a flat B-module of rank μ for $i = d, d+1$, then M_i is so for all $i \geq d$.*

Theorem 3.3 (Gotzmann's regularity theorem)**.** *Let I be an homogeneous ideal of S with Hilbert polynomial μ. Then I is μ regular:*

$$H^i(\mathbb{P}^n, \widetilde{I}(\mu - i)) = 0$$

for $i > 0$, where $\widetilde{I}$ is the quasi-coherent sheaf associated to I.

For definition of $H^i(\mathbb{P}^n, -)$, see [12][Chap.III, §8].

Using Persistence theorem 3.2 and corollary 2.14, we deduce the following propositions:

Proposition 3.4. *Given an integer d such that $d \geq \mu$, $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is in bijection with the subset W of $\mathbf{Gr}^{\mu}_{S_d^*}(X) \times \mathbf{Gr}^{\mu}_{S_{d+1}^*}(X)$ defined by*

$$W = \{(T_d/I_d, T_{d+1}/I_{d+1}) \in \mathbf{Gr}^{\mu}_{S_d^*}(X) \times \mathbf{Gr}^{\mu}_{S_{d+1}^*}(X) \mid T_1.I_d = I_{d+1}\}.$$

Proposition 3.5 ([15])**.** *Given an integer d such that $d \geq \mu$, $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is in bijection with the subset G of $\mathbf{Gr}^{\mu}_{S_d^*}(X)$ given by*

$$G = \{T_d/I_d \in \mathbf{Gr}^{\mu}_{S_d^*}(X) \mid T_{d+1}/(T_1.I_d) \text{ is a free } A\text{-module of rank } \mu\}.$$

Actually, proposition 3.4 can be reformulated this way:

Proposition 3.6. *Given an integer d such that $d \geq \mu$, $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is in bijection with the subset W of $\mathbf{Gr}^{\mu}_{S_d^*}(X) \times \mathbf{Gr}^{\mu}_{S_{d+1}^*}(X)$ defined by*

$$W = \{(T_d/I_d, T_{d+1}/I_{d+1}) \in \mathbf{Gr}^{\mu}_{S_d^*}(X) \times \mathbf{Gr}^{\mu}_{S_{d+1}^*}(X) \mid T_1.I_d \subset I_{d+1}\}.$$

Proof. Thanks to proposition 3.4, we only need to prove that if (I_d, I_{d+1}), with $I_d.S_1 \subset I_{d+1}$, belongs to $\mathbf{Gr}^{\mu}_{S_d^*}(X) \times \mathbf{Gr}^{\mu}_{S_{d+1}^*}(X)$ then $I_d.S_1 = I_{d+1}$. Consider $I_d(m)$ and $I_{d+1}(m)$ introduced in definition 2.8. Then one has

$$(A[x_0, \ldots, x_n]_j/I_j) \otimes k \sim k[x_0, \ldots, x_n]/I_j(m)$$

for $j = d, d+1$. Thus, one has that $S_1.I_d(m) \subset I_{d+1}(m)$ and $\dim_k k[x_0, \ldots, x_n]/I_d(m) = \dim_k k[x_0, \ldots, x_n]/I_{d+1}(m) = \mu$ with $d \geq \mu$. Then, by minimal growth of an ideal in $k[x_0, \ldots, x_n]$ (see [15][Cor.C.4, p.291]), we deduce $S_1.I_d(m) = I_{d+1}(m)$. Let $I(m)$ (resp. $\mathcal{M}$) be the ideal generated by $I_d(m)$ (resp. S_1)

in $k[x_0,\ldots,x_n]$. Thus, by the persistence theorem 3.2 and minimal growth [15][Cor.C.4, p.291], we get that

$$\mathrm{Sat}(I(m)) := \bigcup_{j\in\mathbb{N}} I(m) : (\mathcal{M})^j = \overline{I_d(m)}$$

with $\overline{I_d(m)}$ equal to $I(m)+(I_d(m):S_1)+(I_d(m):S_2)+\ldots+(I_d(m):S_{d-1})$, and that $\overline{I_d(m)}$ belongs to $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(k)$. Then, as $\overline{I_d(m)}$ is saturated, we can assume by a change of coordinates in $\mathbb{K}[x_0,\ldots,x_n]$ that x_0 is not a zero divisor of $\overline{I_d(m)}$. It implies that the multiplication by x_0:

$$*x_0 : k[x_0,\ldots,x_n]/I_d(m) \longrightarrow k[x_0,\ldots,x_n]/I_{d+1}(m)$$

is injective (and by dimension is an isomorphism).
Thus one has the following diagram:

$$\begin{tikzcd}
(A[x_0,\ldots,x_n]_d/I_d)\otimes k \arrow[r, "\sim"] \arrow[d, "*x_0"] & k(p)[x_0,\ldots,x_n]_d/I_d(m) \arrow[d, "*x_0"] \\
(A[x_0,\ldots,x_n]_{d+1}/I_{d+1})\otimes k \arrow[r, "\sim"] & k(p)[x_0,\ldots,x_n]_{d+1}/I_{d+1}(m)
\end{tikzcd}$$

in which the morphisms of multiplication by x_0 are isomorphisms. Thus, as $A[x_0,\ldots,x_n]_j/I_j$ are free A-module for $j=d,d+1$, the morphism of multiplication by x_0:

$$A[x_0,\ldots,x_n]_d/I_d \xrightarrow{*x_0} A[x_0,\ldots,x_n]_{d+1}/I_{d+1}$$

is an isomorphism.
Then, we proceed by dehomogenization by x_0. One has that the natural inclusion

$$A[x_1,\ldots,x_n]_{\le d}/\underline{I_d} \longrightarrow A[x_1,\ldots,x_n]_{\le d+1}/\underline{I_{d+1}}$$

is an isomorphism. As $d\ge\mu=\mathrm{rank}_A A[x_1,\ldots,x_n]_{\le d}/\underline{I_d}$, we can find a basis B of $A[x_1,\ldots,x_n]_{\le d}/\underline{I_d}$ of polynomials of degree stricly less than d and define operators of multiplications by the variables $(x_i)_{1\le i\le n}$ in $A[x_1,\ldots,x_n]_{\le d}/\underline{I_d}$. Finally, we conclude as in proof of corollary 2.14 that $S_1\, I_d = I_{d+1}$. □

Remark 3.7. *Note that the bijection introduced in proposition 3.6 is the following:*

$$\{(I_d,I_{d+1})|T_1.I_d\subset I_{d+1} \textit{ and } T_k/I_k \textit{ is free of rank } \mu \textit{ for } k=d,d+1\}\to \mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$$

$$(I_d,I_{d+1})\mapsto \overline{I_d}$$

where $\overline{I_d}=(I_d)+(I_d:T_1)+(I_d:T_2)+\ldots+(I_d:T_{d-1})$.

The same way, the bijection introduced in proposition 3.5 is the following:

(10) $\{I_d\subset T_d|\ T_d/I_d$ *is a free* A*-module of rank* $\mu\}\to \mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$

$$(I_d)\mapsto \overline{I_d}$$

Remark 3.8. *Note that the previous bijections induce the following commutative diagram:*

$$
\begin{tikzcd}
\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & W \subset \mathbf{Gr}^{\mu}_{S^*_d}(X) \times \mathbf{Gr}^{\mu}_{S^*_{d+1}}(X) \arrow[dl, "\pi"] \\
& G \subset \mathbf{Gr}^{\mu}_{S^*_d}(X) &
\end{tikzcd}
\tag{11}
$$

where ϕ and ψ are bijections and π is the natural projection on the first Grassmannian $\mathbf{Gr}^{\mu}_{S^*_d}(X)$. Thus, we deduce π is also a bijection and G' is exactly the projection of W' on $\mathbf{Gr}^{\mu}_{S^*_d}(X)$.

3.2. **Global description.** From the previous section, we can consider $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ as a subset W of the product of the two Grassmannians: $\mathbf{Gr}^{\mu}_{S^*_d}(X) \times \mathbf{Gr}^{\mu}_{S^*_{d+1}}(X)$ or as a subset G of the single Grassmannian: $\mathbf{Gr}^{\mu}_{S^*_d}(X)$ for $d \geq \mu$. In this section we will prove that $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ can actually be considered as an algebraic subvariety of this product of two Grassmannians or as an algebraic subvariety of this single Grassmannian. We will get the global equations of these subvarieties in the Plücker coordinates and will connect it to the border basis description of Section 2.1.

We consider the well known embedding of $\mathbf{Gr}^{\mu}_{S^*_d}(X)$ into the projective space $\mathbb{P}(\wedge^{\mu} T^*_d)$ given as follow: let $\Delta := T_d/I_d$ be an element of $\mathbf{Gr}^{\mu}_{S^*_d}(X)$ and $(e_1, \ldots, e_{\mu})$ be any basis of the free A-module Δ. For any ordered family $(\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_{\mu}})$ of μ monomials of degree d (for some monomial ordering $<$) write:

$$\mathbf{x}^{\alpha_1} \wedge \cdots \wedge \mathbf{x}^{\alpha_{\mu}} = \Delta_{\alpha_1, \ldots, \alpha_{\mu}} e_1 \wedge \cdots \wedge e_{\mu}$$

in $\wedge^{\mu}(T_d/I_d)$ which is free of rank 1 and has basis $e_1 \wedge \cdots \wedge e_{\mu}$ with $\Delta_{\alpha_1, \ldots, \alpha_{\mu}}$ in A. Finally, let us associate to $\Delta \in \mathbf{Gr}^{\mu}_{S^*_d}(X)$ the point in $\mathbb{P}(\wedge^{\mu} T^*_d)$ corresponding to the family $(\Delta_{\alpha_1, \ldots, \alpha_{\mu}})_{\alpha_1 < \ldots < \alpha_{\mu}}$ (note that this construction does not depend on the choice of the basis $(e_1, \ldots, e_{\mu})$ of Δ).
The $(\Delta_{\alpha_1, \ldots, \alpha_{\mu}})$ will be called the Plücker coordinates. They satisfy the well known Plücker relations.

Let Δ be an element of $\mathbf{Gr}^{\mu}_{S^*_d}(X)$. Let $(\delta_1, \ldots, \delta_{\mu})$ in T^*_d be the dual basis of $(e_1, \ldots, e_{\mu})$ in the dual space $\Delta^* = \mathrm{Hom}_A(\Delta, A)$ which is also a free A-module of rank μ. Then the Plücker coordinates of Δ as an element of $\mathbb{P}(\wedge^{\mu} T^*_d)$ are given by:

$$\Delta_{\beta_1, \ldots, \beta_{\mu}} = \begin{vmatrix} \delta_1(\mathbf{x}^{\beta_1}) & \cdots & \delta_1(\mathbf{x}^{\beta_{\mu}}) \\ \vdots & & \vdots \\ \delta_{\mu}(\mathbf{x}^{\beta_1}) & \cdots & \delta_{\mu}(\mathbf{x}^{\beta_{\mu}}) \end{vmatrix}$$

for $\beta_i \in \mathbb{N}^{n+1},\ |\beta_i| = d$ and $\beta_1 < \cdots < \beta_\mu$.
More generally, consider the following determinant:

$$\begin{vmatrix} \delta_1(p_1) & \cdots & \delta_1(p_\mu) \\ \vdots & & \vdots \\ \delta_\mu(p_1) & \cdots & \delta_\mu(p_\mu) \end{vmatrix}$$

for $(p_1, \ldots, p_\mu)$ any family of polynomials in S_d (not necessarily monomials and not necessarily ordered). Using the multilinearity properties of the determinant and the equality above, it is easy to prove that this determinant can be written as a linear form in the Plücker coordinates $(\Delta_{\beta_1,\ldots,\beta_\mu})_{|\beta_i|=d,\beta_1<\cdots<\beta_\mu}$. We will denote it $\Delta_{p_1,\ldots,p_\mu}$ or Δ_E where $E = (p_1, \ldots, p_\mu)$ is a family of polynomials in T.
For example, let $F = (\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_\mu})$ be a family of μ monomials in T_d (not necessarily ordered). Then $\Delta_F = \epsilon.\Delta_{F'}$ where $F^{'}$ is the family of monomials associated to F ordered by the monomial ordering $<$, and $\epsilon \in \{\pm 1\}$ is the signature of the permutation which transforms F into $F^{'}$.

From now on, if $\Delta \in \mathbf{Gr}^{\mu}_{S_d^*}(X)$, we will denote by $\ker(\Delta)$ the A-submodule I_d of T_d such that $\Delta = T_d/I_d$.

We are going to describe the border relations with respect to a basis $B \subset S_d$ of $\Delta := T_d/I_d$ in terms of these Plücker coordinates.

Lemma 3.9. *Let $\Delta := T_d/I_d$ be an element of $\mathbf{Gr}^{\mu}_{S_d^*}(X)$. Let $B = (b_1, \ldots, b_\mu)$ be a family of polynomials of degree d. Then we have the following relation:*

$$\Delta_B\, a - \sum_{i=1}^{\mu} \Delta_{B^{[b_i|a]}}\, b_i = 0 \text{ in } \Delta, \text{for } a \in T_d$$

where $B^{[b_i|a]} = (b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_\mu)$.

Proof. The previous relation is a straightforward consequence of basic properties of determinants. Consider the following matrix

$$M := \begin{bmatrix} \delta_1(a) & \delta_1(b_1) & \cdots & \delta_1(b_\mu) \\ \vdots & & & \vdots \\ \delta_\mu(a) & \delta_\mu(b_1) & \cdots & \delta_\mu(b_\mu) \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

and develop its determinant along the last row of M. Then one has the following relation

$$M \begin{bmatrix} \Delta_B \\ \Delta_{B^{[b_1|a]}} \\ \vdots \\ \Delta_{B^{[b_\mu|a]}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \det(M) \end{bmatrix}.$$

From this relation, we conclude that $\Delta_B\, a - \sum_{i=1}^{\mu} \Delta_{B^{[b_i|a]}}\, b_i = 0$ in Δ. $\square$

Theorem 3.10. *Let $d \geq \mu$ be an integer. $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is the projection on $\mathbf{Gr}^{\mu}_{S^*_d}(X)$ of the variety of $\mathbf{Gr}^{\mu}_{S^*_d}(X) \times \mathbf{Gr}^{\mu}_{S^*_{d+1}}(X)$ defined by the equations*

$$\Delta_B \, \Delta^{'}_{B',x_k a} - \sum_{b \in B} \Delta_{B[b|a]} \, \Delta^{'}_{B',x_k b} = 0, \tag{12}$$

for all families B (resp. B') of μ (resp. $\mu - 1$) monomials of degree d (resp. $d+1$), all monomial $a \in T_d$ and for every k (where $B^{'}, x_k a$ is the family $(b^{'}_1, \ldots, b^{'}_{\mu-1}, x_k a)$).

Proof. From remark 3.8 about the commutative diagram (11), G is equal to the projection of W onto $\mathbf{Gr}^{\mu}_{S^*_d}(X)$. Thus it is enough to prove that $W \subset \mathbf{Gr}^{\mu}_{S^*_d}(X) \times \mathbf{Gr}^{\mu}_{S^*_{d+1}}(X)$ is the subvariety defined by the equations (12). By proposition 3.6, W is defined by the single condition: $T_1 . \ker \Delta \subset \ker \Delta^{'}$. Thus we need to prove (12) is equivalent to $T_1 . \ker \Delta \subset \ker \Delta^{'}$.
First of all, let $(\Delta, \Delta^{'})$ be an element of $\mathbf{Gr}^{\mu}_{S^*_d}(X) \times \mathbf{Gr}^{\mu}_{S^*_{d+1}}(X)$ satisfying the equations (12). We need to prove that $T_1 \cdot \ker \Delta \subset \ker \Delta^{'}$. Let B be a basis of Δ (so that $\Delta_B \notin m$ is invertible), and let p be an element of $\ker \Delta$. By linearity, equations (12) imply that $\Delta^{'}_{B',x_k p} = 0$ for all $k = 1, \ldots, n$ and all subset B' of $\mu - 1$ monomials of degree $d+1$ (because $\Delta_{B[b|p]} = 0$). Thus, by lemma 3.9, $x_k \cdot p$ belongs to $\ker \Delta^{'}$ for all $k = 1, .., n$ and $S_1 \cdot \ker \Delta \subset \ker \Delta^{'}$.

Conversely, let $(\Delta, \Delta^{'})$ satisfy $T_1 . \ker \Delta \subset \ker \Delta^{'}$. Thus by proposition 3.6, $(\Delta, \Delta^{'})$ is in $W^{'}$ and corresponds to a homogeneous saturated ideal I in $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ so that $I_d = \ker \Delta$ and $I_{d+1} = \ker \Delta^{'}$. We are going to prove that the equations (12) are satisfied for $(\Delta, \Delta^{'})$.
This is a straightforward consequence to lemma 3.9: for any family (b_i) of μ polynomials in T_d and any polynomial $a \in T_d$, one has the following formula:

$$a \, \Delta_B = \sum_i \Delta_{B[b_i|a]} \, b_i$$

in Δ. As $T_1 . \ker \Delta \subset \ker \Delta^{'}$, one also has

$$a \, x_k \, \Delta_B = \sum_i \Delta_{B[b_i|a]} \, b_i \, x_k$$

in $\Delta^{'}$ for any $0 \leq k \leq n$.
Then by linearity, for any family B' of $\mu - 1$ monomials,

$$\Delta_B \, \Delta^{'}_{B',x_k a} = \Delta^{'}_{B',x_k \, \Delta_B a} = \sum_{b \in B} \Delta_{B[b|a]} \, \Delta^{'}_{B',x_k b}$$

which are precisely equations (12). □

Hereafter, we describe the equations of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ as a variety of the single Grassmannian $\mathbf{Gr}^{\mu}_{S^*_d}(X)$ (i.e the equations of G introduced in proposition 3.5). This is also equal to the projection on $\mathbf{Gr}^{\mu}_{S^*_d}(X)$ of the variety defined by the equations (12) in $\mathbf{Gr}^{\mu}_{S_d^*}(X) \times \mathbf{Gr}^{\mu}_{S_{d+1}^*}(X)$. Let us introduce a generic linear form $\mathbf{u} = u_0 x_0 + \cdots + u_n x_n$ where u_i are parameters in A.

For any family $B = (b_1, \ldots, b_\mu)$ in S_{d-1}, we define

$$\Delta_{\mathbf{u}\cdot B} = \det(\delta_i(\mathbf{u} \cdot b_j)) = \sum_{\mathcal{I} \in \{0,\ldots,n\}^\mu} \mathbf{u}^{(\mathcal{I})} \Delta_{\mathcal{I}\cdot B},$$

where $\Delta_{\mathcal{I}\cdot B} = \Delta_{x_{\mathcal{I}_1} b_1, \ldots, x_{\mathcal{I}_\mu} b_\mu}$ and $(\mathcal{I})$ is the element of $\mathbb{N}^{n+1}$ such that $\mathbf{u}^{(\mathcal{I})} = u_{\mathcal{I}_1} \cdots u_{\mathcal{I}_\mu}$ for all $\mathcal{I} \in \{0, \ldots, n\}^\mu$. In this context, $\mathcal{I}_i$ will also be denoted $\mathcal{I}_{b_i}$ and more generaly $\mathcal{I}_b$ for $b \in B$.

For two families of monomials $B, B' \subset T_{d-1}$, we have

$$\Delta_{\mathbf{u}\cdot B}\, \Delta_{\mathbf{u}\cdot B'} = \sum_{\mathcal{K} \in \mathbb{N}^{n+1}, |\mathcal{K}| = 2\mu} \mathbf{u}^{\mathcal{K}} \sum_{\mathcal{I},\mathcal{J} \in \{0,\ldots,n\}^\mu,\ (\mathcal{I})+(\mathcal{J}) = \mathcal{K}} \Delta_{\mathcal{I}\cdot B}\, \Delta_{\mathcal{J}\cdot B}.$$

Proposition 3.11. *Let $d \geq \mu$ be an integer $\Delta \in \mathbf{Gr}^{\mu}_{S_d^*}(X)$ be an element of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$. Then for all families B of μ monomials of degree $d-1$, for all $\mathcal{K} \in \mathbb{N}^{n+1}$ with $|\mathcal{K}| = 2\mu$, for all $b, b'' \in B$ and all $1 \leq i < j \leq n$, we have*

(13)
$$\sum_{(\mathcal{I})+(\mathcal{J})=\mathcal{K}} \sum_{b' \in B} (\Delta_{\mathcal{I}\cdot B^{[x_{\mathcal{I}_{b'}} b' | x_i b]}}\, \Delta_{\mathcal{J}\cdot B^{[x_{\mathcal{J}_{b''}} b'' | x_j b']}} - \Delta_{\mathcal{I}\cdot B^{[x_{\mathcal{I}_{b'}} b' | x_j b]}}\, \Delta_{\mathcal{J}\cdot B^{[x_{\mathcal{J}_{b''}} b'' | x_i b']}}) = 0.$$

Proof. Note that the relations (13) are obtained (using previous notations) as the coefficients in $\mathbf{u}$ of the relations:

$$\sum_{b' \in B} (\Delta_{\mathbf{u}\cdot B^{[\mathbf{u}b' | x_i b]}}\, \Delta_{\mathbf{u}\cdot B^{[\mathbf{u}b'' | x_j b']}} - \Delta_{\mathbf{u}\cdot B^{[\mathbf{u}b' | x_j b]}}\, \Delta_{\mathbf{u}\cdot B^{[\mathbf{u}b'' | x_i b']}}) = 0. \tag{14}$$

By proposition C.1, proving (13) is thus equivalent to proving (14) for generic values of $\mathbf{u}$ in $S_1 = \mathbb{K}[x_0, \ldots, x_n]$.

Let I be the homogeneous ideal in $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ associated to $\Delta \in \mathbf{Gr}^{\mu}_{S_d^*}(X)$. Let $\Delta' = T_{d+1}/I_{d+1}$. As $d \geq \mu$, we know by propositions 3.5 and 3.6 that Δ and Δ' are free A modules of rank μ. Let $F := (e_1, \ldots, e_\mu)$ be a basis of Δ. Let $P_1 \in A[\mathbf{u}]$ be the polynomial in $\mathbf{u}$ given by:

$$P_1(\mathbf{u}) := \Delta'_{\mathbf{u}\cdot F}.$$

As I is saturated, one can find $\mathbf{v} \in T_1$ such that $\mathbf{v}$ is not a zero divisor of I and thus $P_1(\mathbf{v}) \notin m$. Thus the polynonial $\overline{P_1}$ in $T \otimes (A/m) = k[x_0, \ldots, x_n]$ is not equal to zero. Consequently, $\Delta'_{\mathbf{u}.F} \notin m$ for generic values of $\mathbf{u}$ in S_1. Let us choose a generic $\mathbf{u}$ such that $\Delta'_{\mathbf{u}.F} \notin m$. Finally by a change of variables in $\mathbb{K}[x_0, \ldots, x_n]$, we can assume that $u = x_0$.

Then $F := (e_1, \ldots, e_\mu)$ and $x_0 \cdot F := (x_0\, e_1, \ldots, x_0\, e_\mu)$ are basis of respectively Δ and Δ'. Thus, for any family B of μ monomials of degree $d-1$ one has:

$$\frac{\Delta_{x_0\cdot B}}{\Delta_F} = \frac{\Delta'_{x_0^2\cdot B}}{\Delta'_{x_0\cdot F}}$$

because for any homogeneous polynomial a of degree d the following decomposition in Δ:

$$a = \sum_i z_i\, e_i$$

($z_i \in A$) induces this decomposition in Δ' (since $x_0 \mathrm{Ker}\Delta \subset \ker\Delta'$):

$$x_0\, a = \sum_i z_i\, x_0\, e_i$$

For, any family B of μ monomials of degree $d-1$ and any $b', b, b'' \in B$, we have

$$\sum_{b' \in B} \Delta_{x_0 \cdot B\,[x_0 b' | x_i b]}\, \Delta_{x_0 \cdot B\,[x_0 b'' | x_j b']} = \frac{\Delta_F}{\Delta'_{x_0 F}} \sum_{b' \in B} \Delta_{x_0 \cdot B\,[x_0 b' | x_i b]}\, \Delta'_{x_0^2 \cdot B\,[x_0^2 b'' | x_0 x_j b']}$$

But by lemma 3.9 we have

$$\sum_{b' \in B} x_0\, x_j b'\, \Delta_{x_0 \cdot B\,[x_0 b' | x_i b]} = x_j\, x_i b\, \Delta_{x_0 \cdot B}.$$

in Δ'.
Finally, by linearity we get

$$\sum_{b' \in B} \Delta_{x_0 \cdot B\,[x_0 b' | x_i b]}\, \Delta_{x_0 \cdot B\,[x_0 b'' | x_j b']} = \frac{\Delta_F}{\Delta'_{x_0 \cdot F}}\, \Delta_{x_0 \cdot B}\, \Delta'_{x_0^2 \cdot B\,[x_0^2 b'' | x_i x_j b]}$$

As this expression is symmetric in i and j, we get

$$\sum_{b' \in B} \Delta_{\mathbf{u} \cdot B\,[\mathbf{u} b' | x_i b]}\, \Delta_{\mathbf{u} \cdot B\,[\mathbf{u} b'' | x_j b']} = \sum_{b' \in B} \Delta_{\mathbf{u} \cdot B\,[\mathbf{u} b' | x_j b]}\, \Delta_{\mathbf{u} \cdot B\,[\mathbf{u} b'' | x_i b']},$$

which proves the relations (14). □

Remark 3.12. Note that if B is a basis of Δ and $\mathbf{u} = x_0$ is not a zero divisor of I, then $\underline{B}$ is a basis of the quotient:

$$\mathcal{A} := A[x_1, \ldots, x_n]/\underline{I}$$

and equations (14) are equivalent to the commutation relations between the operators of multiplication by the variable (x_i) in the quotient algebra $\mathcal{A}$.

Proposition 3.13. *Let $d \geq \mu$ be an integer and $\Delta \in \mathbf{Gr}^{\mu}_{S_d^*}(X)$ be an element of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$. Then for all families B of μ monomials of degree $d-1$, for all $\mathcal{K} \in \mathbb{N}^{n+1}$ with $|\mathcal{K}| = 2\,\mu$, for all monomial $a \in S_{d-1}$, for all $b \in B$ and for all $k = 0, \ldots, n$, we have*
(15)

$$\sum_{(\mathcal{I})+(\mathcal{J})=\mathcal{K}} \left(\Delta_{\mathcal{I} \cdot B^{[x_{\mathcal{I}_b} b | x_k a]}}\, \Delta_{\mathcal{J} \cdot B} - \sum_{b' \in B} \Delta_{\mathcal{I} \cdot B^{[x_{\mathcal{I}_b} b | x_k b']}}\, \Delta_{\mathcal{J} \cdot B^{[x_{\mathcal{J}_{b'}} b' | x_{\mathcal{J}_{b'}} a]}} \right) = 0.$$

Proof. Here also the relations (15) are obtained as the coefficients in $\mathbf{u}$ of the relations

$$\Delta_{\mathbf{u}\cdot B^{[\mathbf{u}b|x_k a]}}\,\Delta_{\mathbf{u}\cdot B} - \sum_{b'\in B}\Delta_{\mathbf{u}\cdot B^{[\mathbf{u}b|x_k b']}}\,\Delta_{\mathbf{u}\cdot B^{[\mathbf{u}b'|\mathbf{u}a]}} = 0. \tag{16}$$

By proposition C.1, proving (15) is equivalent to proving (16) for generic values of $\mathbf{u}\in S_1$.

Let I be the homogeneous ideal in $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ associated to $\Delta\in\mathbf{Gr}^{\mu}_{S_d^*}(X)$. Let $\Delta' = T_{d+1}/I_{d+1}$. As $d\geq\mu$, we know by propositions 3.5 and 3.6 that Δ and Δ' are free A modules of rank μ. Let $F := (e_1,\dots,e_\mu)$ be a basis of Δ''.

As we did in the proof of proposition 3.11, we can assume $\mathbf{u} = x_0$ and $F := (e_1,\dots,e_\mu)$ and $x_0.F := (x_0\,e_1,\dots,x_0\,e_\mu)$ are basis of respectively Δ and Δ'. Then, for any family B of μ monomials of degree $d-1$ one has:

$$\frac{\Delta_{x_0\cdot B}}{\Delta_F} = \frac{\Delta'_{x_0^2\cdot B}}{\Delta'_{x_0\cdot F}}$$

Thus for any family B of μ monomials of degree $d-1$, any monomials a of degree $d-1$ and any $k\in\mathbb{N}$:

$$\Delta_{x_0\cdot B^{[x_0\cdot b|x_k b']}} = \frac{\Delta_F}{\Delta'_{x_0\cdot F}}\,\Delta'_{x_0^2\cdot B^{[x_0^2 b|x_k x_0 b']}}.$$

By lemma 3.9 we also have:

$$\sum_{b'\in B}\Delta_{x_0\cdot B^{[x_0 b'|x_0 a]}}\,x_0\,b'\,x_k = \Delta_{x_0\cdot B}\,x_k\,a$$

in Δ'.

Finally, by linearity we have:

$$\sum_{b'\in B}\Delta_{x_0\cdot B^{[x_0 b|x_k b']}}\,\Delta_{x_0\cdot B^{[x_0 b'|x_0 a]}} = \frac{\Delta_F}{\Delta'_{x_0\cdot F}}\,\Delta_{x_0 B}\,\Delta'_{x_0^2\cdot B^{[x_0^2 b|x_k x_0 a]}} = \Delta_{x_0\cdot B^{[x_0 b|x_k a]}}\,\Delta_{x_0\cdot B}$$

□

Remark 3.14. Note that if B is a basis of Δ and $\mathbf{u} = x_0$ is not a zero divisor of I, then $\underline{B}$ is a basis of the affinization of T/I:

$$\mathcal{A} := A[x_1,\dots,x_n]/\underline{I}$$

and equations (16) are equivalent to the fact that the decomposition of $x_k a$ on $\underline{B}$ in $\mathcal{A}$ is obtained by computing the decomposition of a on $\underline{B}$ and applying to it the operator of multiplication by x_k.

Example 3.15. *Here is an example of equations that are obtained from propositions 3.11 and 3.13 in the case $n=2$, $S=\mathbb{K}[x,y,z]$ and $\mu=2$.*
We will give the equations given by (14) *and* (16) *in the case $\mathbf{u}=x$. There are three families of two monomials of degree $\mu-1=1$:*

$$B_1 = (x,y), B_2 = (x,z), B_3 = (y,z).$$

Thus,

$$x \cdot B_1 = (x^2, xy), x \cdot B_2 = (x^2, xz), x \cdot B_3 = (xy, xz).$$

The variables of multiplications are y and z. Let us focus on the case $B = B_1$. By homogeneisation of the equations of example 2.3:

$$\Delta_{x\cdot B_1} M_y = \begin{pmatrix} 0 & \Delta_{xy,y^2} \\ \Delta_{x^2,xy} & \Delta_{x^2,y^2} \end{pmatrix}, \ \Delta_{x\cdot B_1} M_z = \begin{pmatrix} \Delta_{xy,xz} & \Delta_{xy,yz} \\ \Delta_{x^2,xz} & \Delta_{x^2,yz} \end{pmatrix}.$$

we obtain a first set of equations of type (14) *given by the commutation property:*

$$\begin{cases} \Delta_{y^2,xy}\Delta_{x^2,xz} - \Delta_{xy,yz}\Delta_{x^2,xy} = 0, \\ \Delta_{y^2,xy}\Delta_{x^2,yz} - \Delta_{xy,xz}\Delta_{xy,y^2} - \Delta_{xy,yz}\Delta_{x^2,y^2} = 0, \end{cases}$$

Notice that these equations are not enough to define the Hilbert scheme, since it is irredicble of dimension 4 *but these equations vanish when $\Delta_{y^2,xy} = 0, \Delta_{x^2,xy} = 0, \Delta_{x^2,y^2} = 0$. The second set of equations of type* (16) *in the case $\mathbf{u} = x$ and $B = B_1$, is given by the decomposition in $x \cdot B_1$ of the monomials $v\, a$ and by the operator of multiplication by v applied to $x\, a$ with $a = x, y, z$ and $v = y, z$. For instance, for $v = y$ and $a = z$ we have:*

$$\Delta^2_{x\cdot B_1}\, yz \equiv \begin{pmatrix} \Delta_{x^2,xy}\Delta_{xy,yz} \\ \Delta_{x^2,xy}\Delta_{x^2,yz} \end{pmatrix}$$

and

$$\Delta_{x\cdot B_1} xz \equiv \begin{pmatrix} \Delta_{xy,xz} \\ \Delta_{x^2,xz} \end{pmatrix}.$$

Then, we write

$$\Delta^2_{x\cdot B_1}\, yz - \Delta^2_{x\cdot B_1} M_y(xz) \equiv 0$$

and we get

$$\begin{cases} \Delta_{x^2,xy}\Delta_{xy,yz} - \Delta_{x^2,xz}\Delta_{y^2,xy} = 0, \\ \Delta_{x^2,xy}\Delta_{x^2,yz} - \Delta_{xy,xz}\Delta_{x^2,xy} - \Delta_{x^2,xz}\Delta_{x^2,y^2} = 0. \end{cases}$$

We do the same for $v = z$ and obtain:

$$\begin{cases} \Delta_{x^2,xy}\Delta_{z^2,xy} - \Delta_{xy,xz}\Delta_{xy,xz} - \Delta_{x^2,xz}\Delta_{xy,yz} = 0, \\ \Delta_{x^2,xy}\Delta_{x^2,z^2} - \Delta_{xy,xz}\Delta_{x^2,xz} - \Delta_{x^2,xz}\Delta_{x^2,yz} = 0. \end{cases}$$

If we do the same for $a = x, y$ we do not get any new equations. Consequently, the equations obtained from relations (14) *and* (16) *in the case $\mathbf{u} = x$ and $B = B_1 = (x, y)$ are:*

$$\begin{cases} \Delta_{y^2,xy}\Delta_{x^2,xz} - \Delta_{xy,yz}\Delta_{xy,x^2} = 0, \\ \Delta_{y^2,xy}\Delta_{x^2,yz} - \Delta_{xy,xz}\Delta_{xy,y^2} - \Delta_{xy,yz}\Delta_{x^2,y^2} = 0, \\ \Delta_{x^2,xy}\Delta_{xy,xz} + \Delta_{x^2,y^2}\Delta_{x^2,xz} - \Delta_{x^2,yz}\Delta_{x^2,xy} = 0, \\ \Delta_{x^2,xy}\Delta_{xy,z^2} - \Delta^2_{xy,xz} - \Delta_{x^2,xz}\Delta_{xy,yz} = 0, \\ \Delta_{x^2,xy}\Delta_{x^2,z^2} - \Delta_{xy,xz}\Delta_{x^2,xz} - \Delta_{x^2,xz}\Delta_{x^2,yz} = 0. \end{cases}$$

In the case $B = B_2$, we get the equations by permutation of y and z. Finally, in the case $B = B_3$, we get:

$$\begin{cases} \Delta_{y^2,xz}\Delta_{yz,xz} + \Delta_{yz,xz}\Delta_{xy,yz} - \Delta_{yz,xz}\Delta_{y^2,xz} - \Delta_{z^2,xy}\Delta_{xy,y^2} = 0, \\ \Delta_{xy,y^2}\Delta_{yz,xz} + \Delta^2_{xy,yz} - \Delta_{xy,yz}\Delta_{y^2,xz} - \Delta_{xy,z^2}\Delta_{xy,y^2} = 0, \\ \Delta_{y^2,xz}\Delta_{xy,z^2} + \Delta_{xz,yz}\Delta_{xy,z^2} - \Delta^2_{xz,yz} - \Delta_{xy,z^2}\Delta_{xy,yz} = 0, \\ \Delta_{xy,y^2}\Delta_{z^2,xy} + \Delta_{xy,yz}\Delta_{xy,z^2} - \Delta_{xy,yz}\Delta_{yz,xz} - \Delta_{xy,z^2}\Delta_{xy,yz} = 0. \end{cases}$$

from relation (14), *and*

$$\begin{cases} \Delta^2_{xy,xz} - \Delta_{y^2,xz}\Delta_{x^2,xz} - \Delta_{yz,xz}\Delta_{xy,x^2} = 0, \\ \Delta_{xy,y^2}\Delta_{x^2,xz} + \Delta_{xy,yz}\Delta_{xy,x^2} = 0, \\ \Delta_{yz,xz}\Delta_{x^2,xz} + \Delta_{z^2,xz}\Delta_{xy,x^2} = 0, \\ \Delta^2_{xy,xz} - \Delta_{xy,yz}\Delta_{x^2,xz} - \Delta_{xy,z^2}\Delta_{xy,x^2} = 0. \end{cases}$$

from relation (16).
Finally, the equations that are obtained from (14) *and* (16) *in the case* $\mathbf{u} = x$ *are:*

$$\begin{cases} \Delta_{y^2,xy}\Delta_{x^2,xz} - \Delta_{xy,yz}\Delta_{x^2,xy} = 0, \\ \Delta_{y^2,xy}\Delta_{x^2,yz} - \Delta_{xy,xz}\Delta_{y^2,xy} - \Delta_{xy,yz}\Delta_{x^2,y^2} = 0, \\ \Delta_{x^2,xy}\Delta_{xy,xz} + \Delta_{x^2,y^2}\Delta_{x^2,xz} - \Delta_{x^2,yz}\Delta_{x^2,xy} = 0, \\ \Delta_{x^2,xy}\Delta_{z^2,xy} - \Delta^2_{xy,xz} - \Delta_{x^2,xz}\Delta_{xy,yz} = 0, \\ \Delta_{x^2,xy}\Delta_{x^2,z^2} - \Delta_{xy,xz}\Delta_{x^2,xz} - \Delta_{x^2,xz}\Delta_{x^2,yz} = 0, \\ \Delta_{z^2,xz}\Delta_{x^2,xy} - \Delta_{xz,yz}\Delta_{x^2,xz} = 0, \\ \Delta_{z^2,xz}\Delta_{x^2,zy} - \Delta_{xy,xz}\Delta_{z^2,xz} - \Delta_{zy,xz}\Delta_{x^2,z^2} = 0, \\ \Delta_{x^2,xz}\Delta_{xy,xz} + \Delta_{x^2,z^2}\Delta_{x^2,xy} - \Delta_{x^2,zy}\Delta_{x^2,xz} = 0, \\ \Delta_{x^2,xz}\Delta_{y^2,xz} - \Delta^2_{xy,xz} - \Delta_{x^2,xy}\Delta_{xz,yz} = 0, \\ \Delta_{x^2,xz}\Delta_{x^2,y^2} - \Delta_{xy,xz}\Delta_{x^2,xy} - \Delta_{x^2,xy}\Delta_{x^2,zy} = 0, \\ \Delta_{y^2,xz}\Delta_{yz,xz} + \Delta_{yz,xz}\Delta_{xy,yz} - \Delta_{yz,xz}\Delta_{y^2,xz} - \Delta_{z^2,xy}\Delta_{xy,y^2} = 0, \\ \Delta_{xy,y^2}\Delta_{yz,xz} + \Delta_{xy,yz}\Delta_{xy,yz} - \Delta_{xy,yz}\Delta_{y^2,xz} - \Delta_{xy,z^2}\Delta_{xy,y^2} = 0, \\ \Delta_{y^2,xz}\Delta_{z^2,xy} + \Delta_{yz,xz}\Delta_{xy,z^2} - \Delta_{yz,xz}\Delta_{yz,xz} - \Delta_{z^2,xy}\Delta_{xy,yz} = 0, \\ \Delta_{xy,y^2}\Delta_{z^2,xy} + \Delta_{xy,yz}\Delta_{xy,z^2} - \Delta_{xy,yz}\Delta_{yz,xz} - \Delta_{xy,z^2}\Delta_{xy,yz} = 0, \\ \Delta^2_{xy,xz} - \Delta_{xz,y^2}\Delta_{x^2,xz} - \Delta_{xz,yz}\Delta_{x^2,xy} = 0, \\ \Delta_{xz,yz}\Delta_{xz,x^2} + \Delta_{xz,z^2}\Delta_{xy,x^2} = 0, \\ \Delta^2_{xy,xz} - \Delta_{xy,yz}\Delta_{x^2,xz} - \Delta_{xy,z^2}\Delta_{xy,x^2} = 0. \end{cases}$$

A complete set of equations of $\mathbf{Hilb}^2_{\mathbb{P}^2}$ *is obtained by permutation of* x, y *and* z *in these equations. Notice that such quadratic equations have also been computed by Gröbner basis techniques for* $\mathbf{Hilb}^2_{\mathbb{P}^2}$ *in* [2, p. 3].

Theorem 3.16. *Let* $d \geq \mu$. *An element* $\Delta \in \mathbf{Gr}^{\mu}_{S^*_d}(X)$ *is in* $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ *iff it satisfies the relations* (13) *and* (15).

Proof. First, let us prove that there exists a family B of μ monomials of degree $d-1$ and a linear form $\mathbf{u}$ such that $\mathbf{u} \cdot B$ is a basis of Δ.

Let $I_d = \ker \Delta \subset T_d$. Tensoring by the residue field k, we can reduce to the case $A = k$ and thus $T = S = k[x_0, \ldots, x_n]$. To prove that there exists a family B of μ monomials of degree $d-1$ and a linear form $\mathbf{u}$ such that $\mathbf{u} \cdot B$ is a basis of Δ, it is enough to prove this result for $\Delta_{in} := S_d/J_d$ with J equal to the initial ideal of (I_d) (for the degree reverse lexicographic ordering $<$ such that $x_i > x_{i+1}$, $i = 0, \ldots, n-1$). From [4][Thm. 15.20, p. 351], by a generic change of variables, we can assume that J is Borel-fix i.e if $x_i x^{\alpha} \in J$ then $x_j x^{\alpha} \in J$ for all $j > i$.

Let us prove now that there exists a family B of μ monomials of degree $d-1$ such that $x_0 \cdot B$ is a basis of S_d/J_d. It is enough to prove $J_d + x_0\, S_{d-1} = S_d$. Consider $J'_d := (J_d + x_0\, S_{d-1})/x_0\, S_{d-1}$ as a subvector space of $S'_d := S_d/x_0\, S_{d-1}$ (which is isomorphic to $k[x_1, \ldots, x_n]_d$). We need to prove that $S'_d = J'_d$. Let $L \subset S_{d-1}$ be the following set

$$L := \{x \in S_{d-1} |\ x_0 x \in J_d\} = (J_d : x_0).$$

One has the exact sequence

$$0 \longrightarrow S_{d-1}/L \xrightarrow{*x_0} S_d/J_d \longrightarrow S'_d/J'_d \longrightarrow 0.$$

Assume that $\dim(S'_d/J'_d) > 0$, then $\dim(L) > s_{d-1} - \mu$. Thus, as $d \geq \mu$, by [6][(2.10), p.66], $\dim(S_1 \cdot L) > s_d - \mu$. As J is Borel-fix, $S_1 \cdot L \subset J_d$. Thus, $\dim(J_d) \geq \dim(S_1 \cdot L) > s_d - \mu$. By assumption, this is impossible, thus $J'_d = S'_d$. we deduce that there exists a family B of μ monomials of degree $d-1$ such that $x_0 \cdot B$ is a basis of S_d/J_d.

Let B be a family of μ monomials of degree $d-1$ such that $x_0 \cdot B$ is a basis of $\Delta = T_d/I_d$. Let $\underline{I}_d \subset V_{\leq d} = A[x_1, \ldots, x_n]_{\leq d}$ be the dehomogenization of I_d (the free A submodule of rank μ of $V_{\leq d}$ defined by putting $x_0 = 1$ in I_d). Let $\pi : T_d/I_d \to \langle B \rangle$ be the natural isomorphism of A modules between T_d/I_d and $\langle B \rangle$ (the free A-module of basis B). The dehomogenization is also an isomorphism of A-modules between $V_{\leq d}/\underline{I}_d$ and T_d/I_d. Thus $V_{\leq d}/\underline{I}_d$ is naturally isomorphic to $\langle B \rangle$. Let ψ be this isomorphism, we have the following commutative diagram:

$$\begin{tikzcd}
V_{\leq d}/\underline{I}_d \arrow[rr, "\sim"] \arrow[dr, "\psi"'] & & T_d/I_d \arrow[dl, "\pi"] \\
& \langle B \rangle &
\end{tikzcd}$$

We introduce the linear operators $(M_i)_{i=1,\ldots,n}$ operating on $\langle B \rangle$. From remarks 3.12 and 3.14, relations (14) and (16) for $\mathbf{u} = x_0$ give us that the operators $(M_i)_{i=1,\ldots,n}$ commute and that, for every $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| = d-1$ and every $i = 1, \ldots, n$, $\pi(\mathbf{x}^\alpha\, x_i) = M_i(\pi(\mathbf{x}^\alpha\, x_0))$ (i.e $\psi(\underline{\mathbf{x}}^\alpha\, x_i) = M_i(\psi(\underline{\mathbf{x}}^\alpha)))$. Note that, by definition in section Notations 1.1, for all $\alpha \in \mathbb{N}^{n+1}$ such that $|\alpha| = d$ we have

$$\pi(\mathbf{x}^\alpha) = \psi(\underline{\mathbf{x}}^\alpha).$$

We define σ, an application from $V = A[x_1, \ldots, x_n]$ to $\langle B \rangle$, as follows: $\forall p \in V, \sigma(p) = p(M)(\pi(x_0^d))$ (or $p(M)(\psi(1))$) where $\underline{\mathbf{x}}^\alpha(M) = M_1^{\alpha_1} \cdots M_n^{\alpha_n}$. As the operators $(M_i)_{i=1,\ldots,n}$ commute, $p(M)$ is well defined and $\mathfrak{J} = \ker \sigma$ is an ideal of V. Let us prove by induction on the degree k of p that $\sigma(p) = \psi(p)$ for polynomials p in $V_{\leq d}$. For $k = 0$, $\sigma(1) = \psi(1)$ by definition. From k to $k+1$, we assume that p is a monomial of degree $k+1$ i.e p is of the form:

$$p = x_i\, \underline{\mathbf{x}}^\alpha$$

with $i \in 1 \ldots n$ and $\alpha \in \mathbb{N}^{n+1}$ such that $|\alpha| = d$ and degree of $\underline{\mathbf{x}}^\alpha$ is equal k. Then, $\sigma(x_i\, \underline{\mathbf{x}}^\alpha) = M_i(\sigma(\underline{\mathbf{x}}^\alpha))$. By induction on k, $\sigma(\underline{\mathbf{x}}^\alpha) = \psi(\underline{\mathbf{x}}^\alpha)$. Thus, we have $\sigma(x_i\, \underline{\mathbf{x}}^\alpha) = M_i(\psi(\underline{\mathbf{x}}^\alpha)) = M_i(\pi(\mathbf{x}^\alpha))$. From the end of the previous paragraph, this is equal to $\psi(x^\alpha\, x_i) = \psi(p)$.

Then it follows that σ is surjective because for all $\mathbf{x}^\alpha \in B, \sigma(\underline{\mathbf{x}}^\alpha) = \psi(\underline{\mathbf{x}}^\alpha) = \pi(\mathbf{x}^\alpha) = \mathbf{x}^\alpha$. Thus we get that $V/\mathfrak{J}$ is a free A-module of rank $\mu = |B|$. Moreover, since σ and ψ coincide on $V_{\leq d}$, $\mathfrak{J}_{\leq d} = \underline{I_d}$. Let J be the homogenization of $\mathfrak{J}$, then by proposition 3.6 J belongs to $\mathbf{Hilb}^\mu_{\mathbb{P}^n}(X)$ and $J_d = I_d$ (because $\mathfrak{J}_{\leq d} = \underline{I_d}$).

□

Now, using theorem 3.16, we can finally give equations for the Hilbert scheme $\mathrm{Hilb}^\mu(\mathbb{P}^n)$.

By definition $\mathrm{Hilb}^\mu(\mathbb{P}^n)$ represents the Hilbert functor $\mathbf{Hilb}^\mu_{\mathbb{P}^n}$. We can reduce to the case of affine schemes $X = \mathbf{Spec}(A)$, with A a noetherian $\mathbb{K}$-algebra. In the following, we will say that an A module M is locally free of rank r if the quasi coherent sheaf of modules denoted $\widetilde{M}$ on $X = \mathbf{Spec}(A)$ is locally free of rank r.

Recall that the Hilbert functor associates to X in the category $\mathcal{C}$ of noetherian schemes over $\mathbb{K}$, the set $\mathbf{Hilb}^\mu_{\mathbb{P}^n}(X)$ of saturated homogeneous ideals I of $A[x_0,\dots,x_n]$ such that $(A[x_0,\dots,x_n]/I)_d$ is a flat A-module for every $d \geq \mu$ and for every prime $p \subset A$,the Hilbert polynomial of the $k(p)$-graded algebra $(A[x_0,\dots,x_n]/I) \otimes_A k(p)$ is equal to μ.

By [26][Lem.7.51, p.55] and the bijection (10) introduced in subsection 3.1, for $d \geq \mu$, $\mathbf{Hilb}^\mu_{\mathbb{P}^n}(X)$ is the set of A-submodules I_d of $A[x_0,\dots,x_n]_d$ such that $A[x_0,\dots,x_n]_d/I_d$ and $A[x_0,\dots,x_n]_{d+1}/I_{d+1}$ are locally free of rank μ, where $I_{d+1} = A[x_0,\dots,x_n]_1.I_d$.

We can rewrite it as the set of ϵ locally free sheaves of modules of rank μ on X along with $g := \mathbb{K}[x_0,\dots,x_n]_d \otimes_{\mathbb{K}} \mathcal{O}_X \to \epsilon \to 0$ such that $A[x_0,\dots,x_n]_{d+1}/I_{d+1}$ is locally free of rank μ, where $I_{d+1} = A[x_0,\dots,x_n]_1.\mathrm{Ker}(g)$.

Thus, we get a morphism of functors for $d \geq \mu$

$$\Phi := \mathbf{Hilb}^\mu_{\mathbb{P}^n} \longrightarrow \mathbf{Gr}^\mu_{S_d^*}$$

and a morphism of schemes

$$\phi := \mathrm{Hilb}^\mu(\mathbb{P}^n) \longrightarrow \mathrm{Gr}^\mu(S_d^*).$$

Theorem 3.17. *The morphism ϕ is a closed immersion whose equations are* (13) *and* (15)*. Equivalently, we have the following commutative diagram:*

$$\begin{tikzcd}
\mathrm{Hilb}^\mu(\mathbb{P}^n) \arrow[rr, "\phi"] \arrow[dr, "\sim"'] & & \mathrm{Gr}^\mu(S_d^*) = \mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#)) \\
& \mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#, (13), (15))) \arrow[ur, "\iota"'] &
\end{tikzcd}$$

where ι is the natural closed immersion from $\mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#, (13), (15)))$ *to* $\mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#))$.

Proof. Let us prove that $\mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#, (13), (15)))$ represents the Hilbert functor. Again we can reduce to the case of affine schemes $X = \mathbf{Spec}(A)$ with A a noetherian $\mathbb{K}$ algebra.

Consider an element (ϵ, g) of $\mathbf{Hilb}^\mu_{\mathbb{P}^n}(X)$ with ϵ locally free sheaf of rank μ on X and

$$g : \mathbb{K}[x_0,\dots,x_n]_d \otimes_{\mathbb{K}} \mathcal{O}_X \to \epsilon \to 0.$$

Let I_d be the A submodule of $A[x_0,\dots,x_n]_d$ such that $\widetilde{I_d} = \mathrm{Ker}(g)$ (thus ϵ is the quasi coherent sheaf associated to $A[x_0,\dots,x_n]_d/I_d$). Let I_{d+1} be the submodule of $A[x_0,\dots,x_n]_{d+1}$ equal to $A[x_0,\dots,x_n]_1.I_d$. By By proposition 3.4, $A[x_0,\dots,x_n]_{d+1}/I_{d+1}$ is locally free of rank μ. As we did in the case of the Grassmannian functor, let $s_\alpha \in \epsilon(X)$ for $\alpha \in \mathbb{N}^n$ such that $|\alpha| = d$, be the image of the monomial $x^\alpha \in S_d$ by g. For $B = (x^{\alpha_1},\dots,x^{\alpha_\mu})$ a family of μ monomials of degree d in S_d, we will denote by $p_B \in \wedge^\mu \epsilon(X)$ the global section $p_{\alpha_1,\dots,\alpha\mu} = s_{\alpha_1} \wedge \cdots \wedge s_{\alpha_\mu}$ (see lemma B.2).

We already know from theorem B.1 that (p_B), for families B of μ ordered monomials of degree d in $A[x_0,\dots,x_n]_d$ (for some monomial ordering $<$), satisfy the Plücker relations (#).

For all primes p of A, let $(I_p)_d = I_d \otimes A_p$ and $(I_p)_{d+1} = I_{d+1} \otimes A_p$. Then, $A_p[x_0,\dots,x_n]_d/(I_p)_d$ is the fiber of ϵ at $p \in \mathbf{Spec}(A)$. Thus it a free A_p-module of rank μ. By definition, $A_p[x_0,\dots,x_n]_{d+1}/(I_p)_{d+1}$ is also a free A_p-module of rank μ.

One can easily check that the image of the global sections (p_B), in the stalk $\wedge^\mu \epsilon(X)_p = A_p$ of $\wedge^\mu \epsilon(X)$ at $p \in \mathbf{Spec}(A)$, are exactly the coordinates (Δ_B) of $\Delta = T_d/(I_p)_d$ introduced in subsection 3.2. Thus, using one side of the equivalence of theorem 3.16, (p_B) satisfy equations (13) and (15) at any $p \in \mathbf{Spec}(A)$. Thus, (p_B) satisfies globally equations (13) and (15) and we get a morphism

$$X \longrightarrow \mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#,(13),(15))).$$

We deduce a morphism of contravariant functors

$$\Psi := \mathbf{Hilb}^\mu_{\mathbb{P}^n} \longrightarrow \mathbf{Hom}(-,\mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#,(13),(15)))).$$

$\Psi(X)$ is injective because, by definition, $\Phi(X)$ is injective and the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{Hilb}^\mu_{\mathbb{P}^n}(X) \arrow[rr, "\Phi(X)"] \arrow[dr, "\Psi(X)"'] & & \mathbf{Gr}^\mu_{S_d^*}(X) \\
& \mathbf{Hom}(X,\mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#,(13),(15)))) \arrow[ur, "\iota"'] &
\end{tikzcd}.$$

Let us prove now that $\Psi(X)$ is surjective.
Given an element of $\mathbf{Hom}(X,\mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#,(13),(15))))$, we get (through its image in $\mathbf{Gr}^\mu_{S_d^*}(X)$) an element (ϵ, g) of $\mathbf{Hilb}^\mu_{\mathbb{P}^n}(X)$ with ϵ locally free sheaf of rank μ on X and

$$g : \mathbb{K}[x_0,\dots,x_n]_d \otimes_{\mathbb{K}} \mathcal{O}_X \to \epsilon \to 0.$$

Using the same notations as above, for all primes $p \in \mathbf{Spec}(A)$, as (p_B) satisfy (13) and (15), (Δ_B) also satisfy (13) and (15). Using the other side of the equivalence of theorem 3.16, we get that $A_p[x_0,\dots,x_n]_{d+1}/(I_p)_{d+1}$ is locally free of rank μ for all primes $p \in \mathbf{Spec}(A)$. Thus, $A[x_0,\dots,x_n]_{d+1}/I_{d+1}$ is locally free of rank μ and (ϵ, g) belongs to $\mathbf{Hilb}^\mu_{\mathbb{P}^n}(X)$.

Finally $\Psi(X)$ is a bijection and Ψ is an isomorphism of contravariant functors. We conclude $\mathbf{Proj}(\mathbb{K}[\wedge^\mu S_d^*]/(\#,(13),(15)))$ represents the Hilbert functor. $\square$

4. Tangent space

Our objective in this section is to determine the tangent space at a $\mathbb{K}$-rational point I_0 of the Hilbert scheme $\mathrm{Hilb}^\mu(\mathbb{P}^n)$. From Theorem 3.17, $\mathrm{Hilb}^\mu(\mathbb{P}^n)$ is a projective scheme whose equations are the equations (13), (15) and the Plücker relations (#).

For $\mathbf{u}$ in S_1, let $H_\mathbf{u}$ be the open subscheme of $\mathrm{Hilb}^\mu(\mathbb{P}^n)$ associated to the open subfunctor $\mathbf{H_u}$ introduced in definition 2.19. Let $\mathbf{v} \in S_1$ be a linear form such that the open subscheme $H_\mathbf{v}$ contains I_0 (ie. $\mathbf{v}$ is not a zero divisor of I_0). After a change of coordinates ($\mathbf{v} = x_0$) we can assume this open subset is of the form $H_{\mathbf{x}_0}$. Let $B \in \mathcal{B}_d$ (see lemma 2.22) be a family of μ monomials in S_d (for $d \geq \mu$) connected to 1. Let $H_{x_0}^B$ be the open affine subscheme associated to the open subfunctor $\mathbf{H}_{x_0}^B$ of the Hilbert scheme $\mathrm{Hilb}^\mu(\mathbb{P}^n)$ (see definition 2.21). Using proposition 2.2 and lemma 2.22, $H_{\mathbf{x}_0}^B$ is the affine scheme associated to the affine variety

$$\mathfrak{H}_{\underline{B}} := \{\boldsymbol{z} \in \mathbb{K}^{\mu\times N}; M_{x_i}(\boldsymbol{z}) \circ M_{x_j}(\boldsymbol{z}) - M_{x_j}(\boldsymbol{z}) \circ M_{x_i}(\boldsymbol{z}) = 0, 1 \leqslant i < j \leqslant n\}.$$

The system of coordinates of this variety is the set of parameters $\boldsymbol{z} = (z_{\alpha,\beta})_{\alpha\in\partial B,\beta\in B}$ such that

$$h_\alpha^0(\underline{\mathbf{x}}) = \underline{\mathbf{x}}^\alpha - \sum_{\beta\in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta$$

is a border basis of $\underline{I_0}$ for B. Then, the equations of $\mathrm{Hilb}^\mu(\mathbb{P}^n)$ in this system of coordinates reduce to the commutation relations:

$$M_i^0(\boldsymbol{z})\, M_j^0(\boldsymbol{z}) - M_j^0(\boldsymbol{z})\, M_i^0(\boldsymbol{z}) = 0,$$

where $M_i^0(\boldsymbol{z})$ is the operator of multiplication by x_i in the basis B modulo the affine ideal $\underline{I_0}$. We will compute the tangent space of the Hilbert scheme using the previous system of coordinates.

By definition, the tangent space of $\mathrm{Hilb}^\mu(\mathbb{P}^n)$ at I_0 is the set of vectors $\mathbf{h}^1 = (h_{\alpha,\beta}^1)$ such that line $h_\alpha^\varepsilon(\mathbf{x}) = h_\alpha^0(\mathbf{x}) + \varepsilon h_\alpha^1(\mathbf{x})$ intersects $\mathrm{Hilb}^\mu(\mathbb{P}^n)$ with multiplicity ≥ 2, where $h_\alpha^1(\underline{\mathbf{x}}) := \sum_{\beta\in B} h_{\alpha,\beta}^1 \underline{\mathbf{x}}^\beta$.

Substituting in the commutation relations, we obtain

$$\begin{aligned}
&M_{x_i}^\varepsilon \circ M_{x_j}^\varepsilon - M_{x_j}^\varepsilon \circ M_{x_i}^\varepsilon \\
&\quad = \quad (M_{x_i}^0 \circ M_{x_j}^0 - M_{x_j}^0 \circ M_{x_i}^0) \\
&\qquad\quad + \varepsilon(M_{x_i}^1 \circ M_{x_j}^0 + M_{x_i}^0 \circ M_{x_j}^1 - M_{x_j}^1 \circ M_{x_i}^0 - M_{x_j}^0 \circ M_{x_i}^1) + \mathcal{O}(\varepsilon^2) \\
&\quad = \quad \varepsilon(M_{x_i}^1 \circ M_{x_j}^0 + M_{x_i}^0 \circ M_{x_j}^1 - M_{x_j}^1 \circ M_{x_i}^0 - M_{x_j}^0 \circ M_{x_i}^1) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

where $M_{x_i}^\varepsilon = M_{x_i}^0 + \varepsilon M_{x_i}^1$, $M_{x_i}^0$ is the operator of multiplication by x_i in $\mathcal{A}^0$ and $M_{x_i}^1$ is linear in $\boldsymbol{h}^1$. We deduce the linear equations in $\boldsymbol{h}^1$ defining the tangent space of $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$ at I_0:

(17) $M^1_{x_i}\circ M^0_{x_j}+M^0_{x_i}\circ M^1_{x_j}-M^1_{x_j}\circ M^0_{x_i}-M^0_{x_j}\circ M^1_{x_i}=0\ (1\leqslant i<j\leqslant n).$

Definition 4.1. *Let $\boldsymbol{h}^0 := (h^0_\alpha)_{\alpha\in\partial B}$ be a border basis for B. We denote by $T_{\boldsymbol{h}_0}$, the set of $\boldsymbol{h}^1 = (h^1_\alpha)_{\alpha\in\partial B}$ with $h^1_\alpha \in \langle B\rangle$, which satisfies the linear equations (17).*

In the following, we will also denote by $H^0 : \langle B^+\rangle \to \langle B^+\rangle$ the linear map such that for $\beta\in B$, $H^0(\underline{\mathbf{x}}^\beta)=0$ and for $\alpha\in\partial B$, $H^0(\underline{\mathbf{x}}^\alpha)=h^0_\alpha$. We also denote by $N_0 : \langle B^+\rangle \to \langle B\rangle$ the normal form modulo $\mathfrak{I}^0$, that is the projection of $\langle B^+\rangle$ on $\langle B\rangle$ along $\langle h^0_\alpha\rangle$. By construction, for any $p=\sum_{\alpha\in B^+}\lambda_\alpha\underline{\mathbf{x}}^\alpha \in \langle B^+\rangle$, $H^0(p)=\sum_{\alpha\in\partial B}\lambda_\alpha h^0_\alpha$ and we have $N^0+H^0=\mathrm{Id}_{\langle B^+\rangle}$. Similarly, we also denote by $H^1 : \langle B^+\rangle \to \langle B\rangle$ the map defined by $H^1(\underline{\mathbf{x}}^\beta)=0$ if $\underline{\mathbf{x}}^\beta\in B$, $H^1(\underline{\mathbf{x}}^\alpha)=h^1_\alpha$ if $\alpha\in\partial B$. By construction, for all $m\in B$, $M^1_i(m)=H^1(x_i m)$.

Theorem 4.2. *Let $I_0\in H^B_{x_0}$ be an ideal, with the border relations $\boldsymbol{h}^0 := (h^0_\alpha)_{\alpha\in\partial B}$ for the basis B of $\mathcal{A}^0 = R/\mathfrak{I}_0$, where $\mathfrak{I}_0=\underline{I_0}$. Then*

$$\begin{array}{rcl}\phi : T_{\boldsymbol{h}^0} & \to & \mathrm{Hom}_R(\mathfrak{I}_0, R/\mathfrak{I}_0)\\ \boldsymbol{h}^1 & \to & \phi(\boldsymbol{h}^1) : h^0_\alpha \mapsto h^1_\alpha\end{array}$$

is an isomorphism of $\mathbb{K}$-vector spaces.

Proof. We first prove that $\phi(\boldsymbol{h}^1)$ is well-defined, ie. if $g=\sum_i u_\alpha h^0_\alpha = \sum_\alpha u'_\alpha h^0_\alpha\in\mathfrak{I}_0$ with $u_\alpha, u'_\alpha\in R$, then $\sum_\alpha u_\alpha h^1_\alpha=\sum_\alpha u'_\alpha h^1_\alpha$ in $R/\mathfrak{I}_0$. In other words, if $\sum_\alpha v_\alpha h^0_\alpha=0$ in R, then $\sum_\alpha v_\alpha h^1_\alpha\equiv 0$ in $R/\mathfrak{I}_0$. By [21][Theorem 4.3], the syzygies of the border basis elements $\boldsymbol{h}^0 := (h^0_\alpha)$ are generated by the commutation polynomials:

$$x_iH^0(x_{i'}m)-x_{i'}H^0(x_im)+H^0(x_iN^0(x_{i'}m))-H^0(x_{i'}N^0(x_im)),$$

for all $m\in B$, $1\leqslant i<i'\leqslant n$. Let us prove that these relations are also satisfied modulo $\mathfrak{I}_0$, if we replace H^0 by H^1. As $\boldsymbol{h}^1=(h^1_\alpha)\in T_{\boldsymbol{h}_0}$, we have

$$\begin{array}{rcl}0 & = & M^0_{x_i}\circ M^1_{x_{i'}}(m)-M^0_{x_{i'}}\circ M^1_{x_i}(m)+M^1_{x_i}\circ M^0_{x_{i'}}(m)-M^1_{x_{i'}}\circ M^0_{x_i}(m)\\ & = & N^0(x_iH^1(x_{i'}m))-N^0(x_{i'}H^1(x_im))+H^1(x_iN^0(x_{i'}m))-H^1(x_{i'}N^0(x_im))\\ & = & x_iH^1(x_{i'}m)-H^0(x_iH^1(x_{i'}m))\\ & & -x_{i'}H^1(x_im)+H^0(x_{i'}H^1(x_im))\\ & & +H^1(x_iN^0(x_{i'}m))-H^1(x_{i'}N^0(x_im))\\ & \equiv & x_iH^1(x_{i'}m)-x_{i'}H^1(x_im)+H^1(x_iN^0(x_{i'}m))-H^1(x_{i'}N^0(x_im))\ \mathrm{modulo}\ \mathfrak{I}_0.\end{array}$$

This proves that the generating syzygies are mapped by $\phi(\boldsymbol{h}^1)$ to 0 in $R/\mathfrak{I}_0$ and thus the image by $\phi(\boldsymbol{h}^1)$ of any syzygy of $\boldsymbol{h}^0$ is 0, that is, $\phi(\boldsymbol{h}^1)$ is a well-defined element of $\mathrm{Hom}_R(\mathfrak{I}_0, R/\mathfrak{I}_0)$.

Conversely, let us prove that if $\psi_0\in\mathrm{Hom}_R(\mathfrak{I}_0, R/\mathfrak{I}_0)$, then $\boldsymbol{h}^1 := (\psi_0(h^0_\alpha))\in T_{\boldsymbol{h}_0}$. As $\psi_0\in\mathrm{Hom}_R(\mathfrak{I}_0, R/\mathfrak{I}_0)$, the syzygies of $\boldsymbol{h}^0$ are mapped by ψ_0 to 0. Thus, for all $m\in B$, $1\leqslant i<i'\leqslant n$,

$$\begin{array}{rcl} 0 & \equiv & \psi_0(x_i H^0(x_{i'}m) - x_{i'}H^0(x_i m) + H^0(x_i N^0(x_{i'}m)) - H^0(x_{i'}N^0(x_i m))) \\ & \equiv & x_i H^1(x_{i'}m) - x_{i'}H^1(x_i m) + H^1(x_i N^0(x_{i'}m)) - H^1(x_{i'}N^0(x_i m)) \,\text{modulo}\, \mathfrak{I}_0. \end{array}$$

As $H^1(p) \in \langle B \rangle$ and $N^0(H^1(p)) = H^1(p)$ for all $p \in \langle B^+ \rangle$, we have

$$\begin{array}{rcl} 0 & = & N^0(x_i H^1(x_{i'}m)) - N^0(x_{i'}H^1(x_i m)) + H^1(x_i N^0(x_{i'}m)) - H^1(x_{i'}N^0(x_i m)) \\ & = & M_i^0 \circ M_{i'}^1(m) - M_{i'}^0 \circ M_i^1(m) + M_i^1 \circ M_{i'}^0(m) - M_{i'}^1 \circ M_i^0(m), \end{array}$$

which proves that $\boldsymbol{h}^1 \in T_{\boldsymbol{h}^0}$. □

We can notice that the tangent space of the variety $\mathrm{Hilb}^\mu(\mathbb{P}^n)$ locally defined by the equations (17) is also isomorphic to $\mathrm{Hom}_R(\mathfrak{I}_0/\mathfrak{I}_0^2, R/\mathfrak{I}_0)$. Our construction gives a new (simple) proof of this well known result [25][p. 217].

The results in the following appendix parts can be considered as "classical", though not necessarily explicit in the literature. They are recalled here for the sake of completeness.

Appendix A. Representable functors

We consider the category $\mathcal{C}$ of noetherian schemes over $\mathbb{K}$. Let $\mathcal{C}^a$ be the category of affine noetherian schemes over $\mathbb{K}$. Let $\mathbb{P}^n$ be the projective scheme $\mathbf{Proj}(S)$. For the notions of presheaf, sheaf and scheme, see [12][Chap. II]. The objective of this section is to find conditions to the representation of contravariant functors from the category of schemes to the category of sets. Most of the material used in this section comes from appendix E of [25].

Proposition A.1. *Let F be a contravariant functor from the category $\mathcal{C}$ to the category of Sets. Suppose that:*
- F is a sheaf
- F admits an open covering of representable functors,
then F is also representable.

Proof. See appendix E in [25][Prop.E.10, p.318] □

Proposition A.2. *Let F be a contravariant functor from $\mathcal{C}$ to the category of Sets and G a subfunctor of F. Assume that for every affine scheme X in $\mathcal{C}$ and every morphism of functors:*

$$\mathbf{Hom}(-, X) \to F$$

the functor $H := \mathbf{Hom}(-, X) \times_F G$ restricted to the category of affine noetherian schemes over $\mathbb{K}$ is represented by an open subscheme of X. Then G is an open subfunctor of F (in $\mathcal{C}$).

Proof. Let X and Y be objects of $\mathcal{C}$. Let $(U_i)_{i \in I}$ be any affine covering of X. Consider the morphism of functors from $\mathbf{Hom}(-, X)$ to F given by an element $\lambda \in F(X)$ (see appendix E in [25][Lem.E.1, p.313]). Then, the contravariant functor $H := \mathbf{Hom}(-, X) \times_F G$ is given by:

$$H(Y) = \{\phi \in \mathbf{Hom}(Y, X) | F(\phi)(\lambda) \in G(Y) \subset F(Y)\}$$

Let $(V_{i,j})_{j\in J}$ be an affine covering of $\phi^{-1}(U_i) \subset Y$ for all i. Let $\phi_{i,j}$ be the restriction of ϕ to $V_{i,j}$:

$$\phi_{i,j} : V_{i,j} \to U_i.$$

As F and G are sheaves, $F(\phi)(\lambda) \in G(Y)$ if and only if $F(\phi_{i,j})(\lambda_i) \in G(V_{i,j})$ (where $\lambda_i \in F(U_i)$ is the restriction of $\lambda \in F(X)$ to $F(U_i)$).
As G is an open subfunctor of F in the category of affine schemes, there exists an open subscheme Ω_i of the affine scheme U_i such that:

$$F(\phi)(\lambda) \in G(Y) \Leftrightarrow F(\phi_{i,j})(\lambda_i) \in G(V_{i,j}) \Leftrightarrow \phi_{i,j} \text{ factors through } \Omega_i : \tag{18}$$

$$\begin{tikzcd} V_{i,j} \arrow[rr, "\phi_{i,j}"] \arrow[dr] & & U_i \\ & \Omega_i \arrow[ur] & \end{tikzcd}$$

If $\phi \in \mathbf{Hom}(Y, X)$ belongs to $H(Y)$, $F(\phi)(\lambda) \in G(Y)$ by definition. Thus, from the previous equivalence (18), ϕ factors through $\Omega := \bigcup_i \Omega_i \subset X$:

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr] & & X \\ & \Omega = \bigcup \Omega_i \arrow[ur] & \end{tikzcd}$$

Reciprocally, if ϕ factors through Ω, consider $(W_{i,j})_{j\in J}$ an open affine covering of $\phi^{-1}(\Omega_i) \subset Y$ for all $i \in I$. We denote now by $\phi_{i,j}$ the restriction of ϕ to $W_{i,j}$:

$$\begin{tikzcd} W_{i,j} \arrow[rr, "\phi_{i,j}"] \arrow[dr] & & U_i \\ & \Omega_i \arrow[ur] & \end{tikzcd}$$

This commutative diagram and (18) imply that $F(\phi_{i,j})(\lambda_i) \in F(W_{i,j})$ belongs to $G(W_{i,j})$ for all i, j. It implies (as F and G are sheaves) that $F(\phi)(\lambda) \in F(Y)$ belongs to $G(Y)$ and thus ϕ belongs to $H(Y)$.

Thus, the functor H is isomorphic $\mathbf{Hom}(-, \Omega)$, i.e H is represented by the open subscheme Ω of X. We conclude that G is an open subfunctor of F (in $\mathcal{C}$). □

Proposition A.3. *Let F be a contravariant functor from $\mathcal{C}$ to the category of Sets. F is represented by the scheme X if and only if the functors F and* $\mathbf{Hom}(-, X)$ *are isomorphic in the category of affine schemes over $\mathbb{K}$.*

Proof. This is a straightforward consequence of the fact that every scheme has a topological basis which consists of open affine subschemes. □

Appendix B. The Grassmannian

The objective of this section is to present a construction of the Grassmannian as a scheme representing a contravariant functor. Most of the material used for this construction comes from [25][Chap.4.3.3, p.209].

Theorem B.1. *For all $\mathbb{K}$-vector spaces V of finite dimension N and integers $n \leq N$, the Grassmannian functor n of V is representable. It is represented by a projective scheme we will denote $\mathrm{Gr}^n(V)$:*

$$\mathrm{Gr}^n(V) \sim \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#)).$$

where $(\#)$ is the ideal generated by the Plücker relations.

Proof. By definition, the n Grassmannian functor of V is:

$$X \longrightarrow \{V^* \otimes_{\mathbb{K}} \mathcal{O}_X \to \epsilon \to 0 \mid \epsilon \text{ is a locally free sheaf of rank } n \text{ of } \mathcal{O}_X\}.$$

Let

$$g := V^* \otimes_{\mathbb{K}} \mathcal{O}_X \to \epsilon \to 0$$

be an element of $\mathbf{Gr}^n_V(X)$. Let $\wedge$ denote the exterior product. Then we have

$$\wedge^n g := \wedge^n V^* \otimes_{\mathbb{K}} \mathcal{O}_X \to \wedge^n \epsilon = \mathcal{L} \to 0$$

where $\mathcal{L}$ is an invertible sheaf. Let $(e_0, \dots, e_N)$ be a basis of V. Let $s_i \in \epsilon(X)$ be the image of e_i by g and $p_{i_1,\dots,i_n} := s_{i_1} \wedge \dots \wedge s_{i_n} \in \mathcal{L}(X)$. $(p_{i_1,\dots,i_n})$ satisfies the well-known Plücker relations

$$(\#) \qquad \sum_{\lambda=1,\dots,n+1} p_{i_1,\dots,i_{n-1},j_\lambda} \otimes p_{i_1,\dots,\hat{j_\lambda},\dots,i_n} = 0.$$

Thus, by [8][Prop.4.2.3, p.73] we have a morphism:

$$X \longrightarrow \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#))$$

Thus we constructed a morphism of functors from the Grassmannian functor $\mathbf{Gr}^n_V$ to the functor $\mathbf{Hom}(-, \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#)))$:

$$\Phi := \mathbf{Gr}^n_V \longrightarrow \mathbf{Hom}(-, \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#)))$$

Lemma B.2. *The morphism Φ is an isomorphism of functors. Thus the Grassmannian functor is represented by $\mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#))$.*

Proof. From proposition A.3, we can reduce to affine schemes $X = \mathbf{Spec}(A)$. Let us prove that

$$\Phi(X) := \mathbf{Gr}^n_V(X) \to \mathbf{Hom}(X, \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#)))$$

is a bijection.

Let locally free sheaves of rank μ ϵ and ϵ' together with surjective morphisms

$$g = V^* \otimes \mathcal{O}_X \to \epsilon \to 0$$

and

$$g' = V^* \otimes \mathcal{O}_X \to \epsilon' \to 0$$

be two elements of $\mathbf{Gr}_V^n(X)$. Let (s_i) and $(p_{i_1,\dots,i_n})$ (resp. (s'_i) and $(p'_{i_1,\dots,i_n})$) be the global sections of ϵ and $\mathcal{L} = \wedge^n \epsilon$ (resp. ϵ' and $\mathcal{L}' = \wedge^n \epsilon'$) introduced before. Assume $\Phi(X)(\epsilon, g) \in \mathbf{Hom}(X, \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#)))$ and $\Phi(X)(\epsilon', g') \in \mathbf{Hom}(X, \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#)))$ are equal. Then there exists an isomorphism ϕ between $\mathcal{L}$ and $\mathcal{L}'$ such that the following diagram is commutative:

$$\begin{tikzcd}
\wedge^n V^* \otimes \mathcal{O}_X \arrow[rr] \arrow[dr] & & \mathcal{L} \arrow[dl, "\phi"] \\
& \mathcal{L}' &
\end{tikzcd}$$

and such that $\phi(p_{i_1,\dots,i_n}) = p'_{i_1,\dots,i_n}$.
Consider the open subset $X_{p_{i_1,\dots,i_n}}$ where $p_{i_1,\dots,i_n} \neq 0$ (which is equal to $X_{p'_{i_1,\dots,i_n}}$ because ϕ is an isomorphism that sends $p_{i_1,\dots,i_n}$ on $p'_{i_1,\dots,i_n}$). On $X_{p_{i_1,\dots,i_n}}$ (resp. $X_{p'_{i_1,\dots,i_n}}$), $(s_{i_1}, \dots, s_{i_n})$ (resp. $(s'_{i_1}, \dots, s'_{i_n})$) is a basis of ϵ (resp. ϵ ') and we have

$$s_j = \sum_{k=1,\dots,n} a_{j,k}.s_{i_k} \ (resp.\ s'_j = \sum_{k=1,\dots,n} a'_{j,k}.s'_{i_k})$$

with

$$a_{jk} = (-1)^{n-k} \frac{p_{i_1,\dots,\hat{i_k},\dots,i_n,j}}{p_{i_1,\dots,i_n}} = a'_{jk}. \tag{19}$$

Thus, on $X_{p_{i_1,\dots,i_n}}$, we have the natural isomorphism $f_{i_1,\dots,i_n}$ from ϵ to ϵ' that sends s_{i_k} to s'_{i_k} for all $k \le n$. Then, by equations (19), the morphisms $(f_{i_1,\dots,i_n})$ patch together to form an isomorphism f from ϵ to ϵ' such that:

$$\begin{tikzcd}
V^* \otimes \mathcal{O}_X \arrow[rr, "g"] \arrow[dr, "g'"'] & & \epsilon \arrow[dl, "f"] \\
& \epsilon' &
\end{tikzcd}$$

is commutative. Thus

$$g = V^* \otimes \mathcal{O}_X \to \epsilon \to 0$$

and

$$g' = V^* \otimes \mathcal{O}_X \to \epsilon' \to 0$$

are equal as elements of $\mathbf{Hom}(X, \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#)))$. Thus $\Phi(X)$ is injective.

To prove $\Phi(X)$ is surjective, let

$$\phi := \wedge^n V^* \otimes \mathcal{O}_X \to \mathcal{L} \to 0$$

be an element of $\mathbf{Hom}(X, \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#)))$ where $\mathcal{L}$ is an invertible sheaf on X and such that $p_{i_1,\ldots,i_n} := \phi(e_{i_1,\ldots,i_n})$ satisfy the relations $(\#)$. On $X_{p_{i_1,\ldots,i_n}}$ let $\epsilon_{i_1,\ldots,i_n}$ be a free sheaf of rank n with basis $(e_{i_1}, \ldots, e_{i_n})$. Using relations (19) and $(\#)$, we can glue the sheaves $(\epsilon_{i_1,\ldots,i_n})$ to form a locally free sheaf ϵ of rank n together with a surjective morphism

$$g := V^* \otimes \mathcal{O}_X \to \epsilon \to 0$$

which satisfies

$$\Phi(X)(\epsilon, g) = (\mathcal{L}, \phi).$$

Thus $\Phi(X)$ is surjective and Φ is an isomorphism of functors. □

From lemma B.2, the Grassmannian functor is represented by

$$\mathrm{Gr}^n(V) \sim \mathbf{Proj}(\mathbb{K}[\wedge^n V]/(\#)).$$

□

Appendix C. Generic linear forms

The objective of this section is the to extend the notion of generecity in the case of polynomial rings over a field $\mathbb{K}$ to the case of polynomial rings over a $\mathbb{K}$-algebra.
In the following, $\mathbb{K}$ will denote a field of characteristic zero.

Proposition C.1. *Let A be a $\mathbb{K}$-algebra and n be an integer. Let P be a polynomial in $A[x_1, \ldots, x_n]$ such that P vanishes on generic values of $\mathbb{K}^n$, then $P = 0$.*

Proof. Let $\mathbf{P_d}$ be the following proposition: for all $m \in \mathbb{N}$ and all polynomial $P \in A[x_1, \ldots, x_m]$ of degree less or equal to d, if P vanishes on generic bvalues of $\mathbb{K}^m$ then $P = 0$. We will prove by induction that $\mathbf{P_d}$ is true for all $d \geq 0$.
For $d = 0$, $\mathbf{P_0}$ is obviously true.
Assume $\mathbf{P_k}$ is true for all $k \leq d$, let us prove that $\mathbf{P_{d+1}}$ is true. Let m be an integer and P be a polynomial in $A[x_1, \ldots, x_m]$ of degree less or equal to $d+1$ such that P vanishes on generic values of $\mathbb{K}^m$. Let $U \subset \mathbb{K}^m$ be the zero set of P. Let $Q_i \in A[x_1, \ldots, x_m, y_1, \ldots, y_m]$ be the polynomial given by

$$Q_i(\mathbf{x}, \mathbf{y}) = \frac{P(\mathbf{x}) - P(\mathbf{y})}{x_i - y_i}.$$

As $P = (x_i - y_i).Q_i$, Q_i vanishes on $V := U \times U \setminus \{(\mathbf{x}, \mathbf{y}) | \; x_i - y_i = 0\} \subset \mathbb{K}^{2m}$. Thus Q_i vanishes on generic values of $\mathbb{K}^{2m}$ and is of degree less or equal to d. By $\mathbf{P_d}$, Q_i is equal to 0 for all $1 \leq i \leq m$. Thus all the partial derivatives $\partial_i P = Q_i(\mathbf{x}, \mathbf{x})$ of P are equal to 0. As $\mathbb{K}$ is of charasteristic zero, we conclude P is equal to zero.
Thus $\mathbf{P_d}$ is true for all d and m in $\mathbb{N}$. This proves the proposition C.1. □

Appendix D. Zero dimensional algebra

In this section, we recall why an ideal I remains in $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ by coefficient field extension and give a characterization of non-zero divisibility for linear forms.

Let k be a field of characteristic zero and $\overline{k}$ its algebraic closure. Let $S = k[x_0, \ldots, x_n]$ (resp. $\overline{S} := S \otimes_k \overline{k}$) be the polynomial ring in $n+1$ variables over k (resp. $\overline{k}$). Recall that $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(\mathbf{Spec}(k))$ (or simply $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(k)$) is equal to the set of homogeneous saturated ideal of S such S/I has Hilbert polynomial equal to the constant μ.
Given a point P in the projective space $\mathbb{P}^n_k$ we denote by $m_{k,P}$ the homogeneous ideal of $k[x_0, \ldots, x_n]$ generated by

$$\{Q \text{ homogeneous polynomial in } k[x_0, \ldots, x_n] |\ Q(P_i) = 0\}.$$

Finally, denote by m_k the homogeneous ideal of $k[x_0, \ldots, x_n]$ generated by

$$\{P \text{ homogeneous polynomial in } k[x_0, \ldots, x_n] \text{ of degree} \geq 1\}.$$

Definition D.1. *Let I be a homogeneous ideal of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(k)$. Then, $\overline{I}$ is the ideal of $\overline{S}$ given by*

$$\overline{I} := I \otimes_k \overline{k}.$$

Proposition D.2. *Let $I \subset S$ be a homogeneous ideal of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(k)$. Then, one has*

$$\overline{I} \cap S = I$$

Proof. First, one has that $I \subset \overline{I} \cap S$. Then, tensoring by $\overline{k}$, one has that $\overline{I} \subset (\overline{I} \cap S) \otimes \overline{k} \subset \overline{I}$. Thus $\overline{I} = (\overline{I} \cap S) \otimes \overline{k}$. Then, looking at the dimensions for all degree $d \leq 1$, one has that

$$\dim_k(S_d/(\overline{I} \cap S)_d) = \dim_{\overline{k}}(S_d/(\overline{I} \cap S)_d \otimes \overline{k}) = \dim_{\overline{k}}(S_d/I_d \otimes \overline{k}) = \dim_k(S_d/I_d).$$

We conclude that

$$I = \overline{I} \cap S.$$

$\square$

Corollary D.3. *Let $I \subset S$ be a homogeneous ideal of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(k)$. Then $\overline{I}$ belongs to $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(\overline{k})$.*

Proof. We just need to prove that $\overline{I}$ is saturated (i.e $\overline{I} : m_{\overline{k}} = \overline{I}$). One has that $\overline{I}$ is saturated if and only if $m_{\overline{k}}$ is not a prime associated to $\overline{I}$. Assume $\overline{I}$ is not saturated. From the Nullstellensatz theorem, $\overline{I}$ has a reduce primary decomposition of the form

$$\overline{I} = \bigcap_i q_i \cap q$$

with q_i homogeneous $m_{\overline{k},P_i}$-primary ideal with P_i a point in the projective space $\mathbb{P}^n_{\overline{k}}$, and q a homogeneous $m_{\overline{k}}$-primary ideal. From proposition D.2, we have

$$I = \overline{I} \cap S.$$

Thus, one has

$$I = \bigcap_i q_i \cap S \bigcap q \cap S$$

with $q_i \cap S$ homogeneous $m_{\overline{k},P_i} \cap S$-primary ideal and $q \cap S$ homogeneous $m_{\overline{k}} \cap S = m_k$-primary ideal. As I is saturated this is impossible. Thus $\overline{I}$ is saturated. □

Proposition D.4. *Let I be a homogeneous ideal in $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(k)$ and u a linear form in S_1. Then $(I : u) = I$ if and only if u does not vanish at any point defined by $\overline{I}$ in $\mathbb{P}^n_{\overline{k}}$.*

Proof. We already know from proposition D.3 that $\overline{I}$ has a primary decomposition of the form:

$$\overline{I} = \bigcap_{i \in E} q_i$$

with q_i homogeneous $m_{\overline{k},P_i}$-primary ideal and $\{P_i |\ i \in E\}$ is the set of points defined by $\overline{I}$ in $\mathbb{P}^n_{\overline{k}}$. From proposition D.2, we have that

$$I = \bigcap_{i \in E} q_i \cap S. \tag{20}$$

One has that $q_i \cap S$ is a homogeneous $m_{\overline{k},P_i} \cap S$-primary ideal. Thus the primary decomposition of I is deduced from (20) by discarding those $q_i \cap S$ that contain $\bigcap_{j \neq i} q_j \cap S$ and intersecting those $q_i \cap S$ that are $m_{\overline{k},P_{i_0}} \cap S$-primary for the same P_{i_0}. Firstly, if $\bigcap_{j \neq i} q_j \cap S \subset q_i \cap S$, then there exists a $j_0 \neq i$ such that $m_{\overline{k},P_{j_0}} \cap S = m_{\overline{k},P_i} \cap S$, i.e P_{j_0} and P_i are conjugate. Secondly, $q_i \cap S$ and $q_j \cap S$ are both $m_{\overline{k},P_{i_0}} \cap S$-primary if and only if P_i, P_j and P_{i_0} are conjugate. Thus I has a primary decomposition of the form

$$I = \bigcap_{i \in F} q_i \cap S$$

with $F \subset E$ satisfying that for all $i \in E$ there exists a unique $j \in F$ such that P_i and P_j are conjugate. Thus, $(I : u) = I$ if and only if u does not vanish at any point P_j for all $j \in F$, i.e u does not vanish at any point P_i for all $i \in E$. □

Proposition D.5. *Consider a field extension $k \subset L$ of k. Let $\overline{L}$ be the algebraic closure of L. Let I be a homogeneous ideal in $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(k)$ and let $I_L = I \otimes_k L$. Then, the points defined by $\overline{I_L}$ in $\mathbb{P}^n_{\overline{L}}$ are exactly the image by the field extension:*

$$\overline{k} \subset \overline{L}$$

of the points defined by $\overline{I}$ in $\mathbb{P}^n_{\overline{k}}$.

Proof. The points defined by $\overline{I_L}$ in $\mathbb{P}^n_{\overline{L}}$ are given by the primary decomposition of $\overline{I_L} = I_L \otimes_L \overline{L}$ in $S \otimes \overline{L}$. We just need to prove that these points are the same as those obtained by the primary decomposition of $\overline{I}$ in $\overline{S}$. In fact, one has that

$$I_L \otimes_L \overline{L} = \overline{I} \otimes_{\overline{k}} \overline{L}.$$

Thus, as

$$\overline{I} = \bigcap_{i \in E} q_i$$

with q_i homogeneous $m_{\overline{k},P_i}$-primary ideal and $\{P_i | \, i \in E\}$ is the set of points defined by $\overline{I}$ in $\mathbb{P}^n_{\overline{k}}$; we deduce that $I_L \otimes_L \overline{L}$ can be written

$$I_L \otimes_L \overline{L} = \bigcap_{i \in E} q_i \otimes \overline{L}$$

with $q_i \otimes \overline{L}$ homogeneous $m_{\overline{k},P_i} \otimes \overline{L}$-primary ideal. But one has that $m_{\overline{k},P_i} \otimes \overline{L} = m_{\overline{L},P_i}$ (P_i considered as a point of $\mathbb{P}^n_{\overline{L}}$ via the field inclusion $\overline{k} \subset \overline{L}$). Thus, the points defined by $\overline{I_L}$ in $\mathbb{P}^n_{\overline{L}}$ are exactly the image by the field extension:

$$\overline{k} \subset \overline{L}$$

of $\{P_i \in \mathbb{P}^n_{\overline{k}} | \, i \in E\}$. □

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