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Geometric Tomography With Topological Guarantees

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Abstract: We consider the problem of reconstructing a compact 3-manifold (with boundary) embedded in $\mathbb{R}^3$ from its cross-sections with a given set of cutting planes having arbitrary orientations. Under appropriate sampling conditions that are satisfied when the set of cutting planes is dense enough, we prove that the algorithm presented by Liu et al. in [LBD$^+$08] preserves the homotopy type of the object. Using the homotopy equivalence, we also show that the reconstructed object is homeomorphic (and isotopic) to the original object. This is the first time that 3D shape reconstruction from cross-sections comes with such theoretical guarantees.

Key-words: Shape Reconstruction, Cross-Sections, Sampling Conditions.

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Tomographie géométrique avec garanties topologiques

Résumé : Nous considérons le problème de la reconstruction d'une 3-variété à bord plongée dans $\mathbb{R}^3$, à partir de ses intersections avec un ensemble de plans en position arbitraire, appellées coupes. Nous prouvons que si certaines conditions d'échantillonnage sont vérifiées, l'algorithme proposé par Liu et al. dans [LBD$^+$08] permet de reconstruire la topologie de l'objet à partir des coupes données. Nous prouvons également que l'objet reconstruit est homéomorphe (et isotope) à l'objet. C'est pour la première fois que ce problème est étudié en toute généralité, dans le but de fournir des garanties théoriques satisfaisantes sur le résultat de la reconstruction.

Mots-clés : Reconstruction à partir de coupes, Conditions d'échantillonnage.

1 Introduction

Overview. This paper deals with the reconstruction of 3-dimensional geometric shapes from unorganized planar cross-sections. The need for such reconstructions is a result of the advances in medical imaging technology, specially in ultrasound tomography. In this context, the purpose is to construct a 3D model of an organ from a collection of ultrasonic images. When the images are provided by free-hand ultrasound devices, the cross-sections of the organ belong to planes that are not necessarily parallel. However, it is only very recently that reconstruction from unorganized cross-sections has been considered: A very first work [PT94] by Payne and Toga was restricted to easy cases of reconstruction that do not require branching between sections. In [BG93], Boissonnat and Geiger proposed a Delaunay-based algorithm for the case of serial planes, that has been generalized to arbitrarily oriented planes in [DP97] and [BM07]. Some more recent work ([JWC$^+$05] and [LBD$^+$08]) can handle the case of multilabel sections (multiple materials). Barequet and Vaxman's work [BV09], that makes use of straight skeletons of polyhedra, can be cited as the most recent work that appeared in this area.
Most of previous work has been restricted to the case of parallel cross-sections and is mostly based on the simple idea of connecting two sections if their orthogonal projections overlap. This paper, analyzes a very natural generalization of this idea for the case of non parallel sections, that has been proposed in [LBD$^+$08]. We prove that under appropriate sampling conditions, the connection between the sections provided by this generalization is coherent with the connectivity structure of the object and the proposed reconstructed object is homeomorphic to the object. To the best of our knowledge, this work is the first to provide such a topological study in shape modeling from planar cross-sectional data. The only existing results studying the topology of the reconstructed object are restricted to the 2D variant of the problem ([SBG06] and [MB08]).

Reconstruction Problem. Let $\mathcal{O} \subset \mathbb{R}^3$ be a compact 3-manifold with boundary (denoted by $\partial\mathcal{O}$) of class $C^{1,1}$. The manifold $\mathcal{O}$ is cut by a set $\mathcal{P}$ of so-called cutting planes that are supposed to be in general position in the sense that none of these cutting planes are tangent to $\partial\mathcal{O}$. For any cutting plane $P \in \mathcal{P}$, we are given the intersection $\mathcal{O} \cap P$. There is no assumption on the geometry or the topology of these intersections. The goal is to reconstruct $\mathcal{O}$ from the given intersections. The problem being ill-posed, we are interested in finding an approximation $\mathcal{R}$ of $\mathcal{O}$ with the same intersection with all the cutting planes in $\mathcal{P}$.

Arrangement of the Cutting Planes: We can decompose the problem into several subproblems as follows. Consider the arrangement of the cutting planes, i.e., the subdivision of $\mathbb{R}^3$ into convex polyhedral cells induced by the cutting planes. Without loss of generality, we can restrict our attention to a cell of this arrangement and reduce the reconstruction of $\mathcal{O}$ to the reconstruction of $\mathcal{O} \cap \mathcal{C}$ for all cells $\mathcal{C}$ of the arrangement. Since the various reconstructed pieces will conform to the given sections, it will be easy to glue them together in the end to get the overall reconstructed object $\mathcal{R}$.

Sections: Input of the Reconstruction Algorithm. We now focus on a cell $\mathcal{C}$ of the arrangement and describe how the reconstructed object $\mathcal{R}_\mathcal{C}$ is defined in $\mathcal{C}$. On each face f of $\mathcal{C}$, the intersection of the object $\mathcal{O}$ with f is given and consists of a set of connected regions called *sections*. By definition, the sections of a face of $\mathcal{C}$ are disjoint. However, two sections (on two neighbor faces of $\mathcal{C}$) may intersect along the intersection between their two corresponding faces. The boundary of a section A is denoted by ∂A and is a set of closed curves, called *section-contours*, that may be nested.

Let us write $\partial\mathcal{C}$ for the boundary of $\mathcal{C}$, and $\mathcal{F}_\mathcal{C}$ for the set of faces of $\mathcal{C}$. In the sequel, $\mathcal{S}_\mathcal{C}$ denotes the union of sections of all the faces of $\mathcal{C}$, and a point of $\mathcal{S}_\mathcal{C}$ is called a *section-point*. So $\mathcal{S}_\mathcal{C}$ is the only information we have about $\mathcal{O}_\mathcal{C} = \mathcal{O} \cap \mathcal{C}$ and it constitutes the input of the reconstruction algorithm.

Methodology We know that a point on the boundary of $\mathcal{C}$ is in $\mathcal{O}$ if it lies in $\mathcal{S}_\mathcal{C}$. The goal is now to determine whether a point x inside $\mathcal{C}$ belongs to $\mathcal{O}$ or not. The reconstruction method that we will present here is based on the notion of *distance from* $\partial\mathcal{C}$:

A point $x \in \mathcal{C}$ is in the reconstructed object if one of its **nearest points** in $\partial\mathcal{C}$ is in $\mathcal{S}_\mathcal{C}$.

Different distance function (from the boundary of $\mathcal{C}$) may be used in order to satisfy properties of interest for different applications. For example, to promote the connection between sections in the case of sparse data, or to impose a favorite direction to connect the sections, etc. A natural idea is to use the Euclidean distance as the distance function from $\partial\mathcal{C}$. In this case, the reconstructed object coincides with the method introduced by Liu et al in [LBD+08]. In this paper, we analyze this method and present appropriate sampling conditions providing topological guarantees for the resulting reconstructed object.

Organization of the paper. After this brief introduction, in Section 2 we provide the definition of the reconstructed object $\mathcal{R}$. The rest of the paper will be then devoted to prove that under two appropriate sampling conditions, $\mathcal{R}$ and $\mathcal{O}$ are homotopy equivalent, and are in addition homeomorphic. To make the connection between the upcoming sections more clear, in Section 3 we shortly outline the general strategy employed in proving the homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$. In Section 4, we present the first sampling condition, called the *Separation Condition*, that ensures good connectivity between the sections, but does not necessarily imply the homotopy equivalence. As we will see, in order to ensure the homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$, a second so-called *Intersection Condition* is required, c.f., Section 7. In Section 8, we provide a set of properties on the sampling of cutting planes to ensure the Separation and the Intersection Conditions. Finally, in Section 9 we show that the two shapes $\mathcal{O}$ and $\mathcal{R}$ are indeed homeomorphic (and are isotopic). We note that some preliminary notions concerning the homotopy theory are provided in Appendix A.

2 Definition of the Reconstructed Object

The definition of the nearest point in $\partial\mathcal{C}$ is related to the Voronoi diagram of the faces of $\mathcal{C}$ defined as follows.

Voronoi Diagram of the Faces of a Cell. For a face f in $\mathcal{F}_\mathcal{C}$, the Voronoi cell of f, denoted by $\mathrm{Vor}_\mathcal{C}(f)$, is defined as the set of all points in $\mathcal{C}$ that have f as the nearest face in $\mathcal{F}_\mathcal{C}$, i.e.,

$$\mathrm{Vor}_\mathcal{C}(f) = \{\ x \in \mathcal{C} \mid d(x, f) \leq d(x, f'),\ \forall f' \in \mathcal{F}_\mathcal{C}\ \}.$$

Where $d(.,.)$ is the Euclidean distance. The collection of all $\mathrm{Vor}_\mathcal{C}(f)$, $f \in \mathcal{F}_\mathcal{C}$ forms a tiling of $\mathcal{C}$.

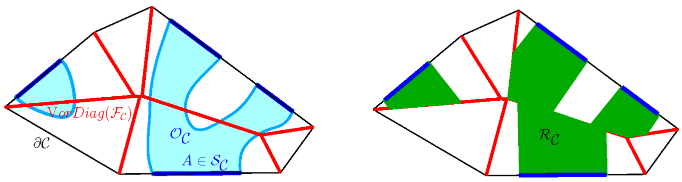

Figure 1: Definitions: The original shape $\mathcal{O}_\mathcal{C}$ and its sections are colored in blue. The Voronoi diagram $\mathrm{VorDiag}(\mathcal{F}_\mathcal{C})$ is colored in red. The reconstructed object $\mathcal{R}_\mathcal{C}$ is colored in green.

We write $\partial\mathrm{Vor}_\mathcal{C}(f)$ for the boundary of $\mathrm{Vor}_\mathcal{C}(f)$. The union of $\partial\mathrm{Vor}_\mathcal{C}(f)$ for all $f \in \mathcal{F}_\mathcal{C}$ is called the *Voronoi diagram of* $\mathcal{F}_\mathcal{C}$, and is denoted by $\mathrm{VorDiag}(\mathcal{F}_\mathcal{C})$. $\mathrm{VorDiag}(\mathcal{F}_\mathcal{C})$ is also called *the medial axis of the cell*, and is the locus of points in $\mathcal{C}$ that are at the same distance from at least two faces of $\mathcal{C}$. To simplify notation, when the cell $\mathcal{C}$ is understood from the context, we simply remove the index $\mathcal{C}$ and write $\mathrm{Vor}(f)$, $\partial\mathrm{Vor}(f)$, etc.

Definition 1 (Nearest Point) For any point x in $\mathcal{C}$, the *nearest point* in $\partial\mathcal{C}$ to x is the orthogonal projection of x onto the nearest face f of $\mathcal{C}$. This projection is denoted by $\mathrm{np}_f(x)$. The set of all nearest points to x in $\partial\mathcal{C}$ is denoted by $\mathrm{Np}_\mathcal{C}(x)$. Note that for any $x \notin \mathrm{VorDiag}(\mathcal{F}_\mathcal{C})$, $\mathrm{Np}_\mathcal{C}(x)$ is reduced to a single point. Based on this, and to simplify the presentation, sometimes we drop the index f, and by $\mathrm{np}(x)$ we denote a point of $\mathrm{Np}_\mathcal{C}(x)$.

We can now define the reconstructed object in a given cell $\mathcal{C}$. We first give the formal definition, and then present a more detailed geometric characterization of the reconstructed object using the lifting procedure described below.

Definition 2 (Reconstructed Object $\mathcal{R}_\mathcal{C}$ in a cell $\mathcal{C}$) The *reconstructed object* $\mathcal{R}_\mathcal{C}$ is the set of all points x in $\mathcal{C}$ such that a nearest point $\mathrm{np}(x)$ lies in $\mathcal{S}_\mathcal{C}$, i.e., $\mathrm{Np}_\mathcal{C}(x) \cap \mathcal{S}_\mathcal{C} \neq \emptyset$. Note that in the case where $\mathcal{S}_\mathcal{C}$ is empty, $\mathcal{R}_\mathcal{C}$ will be the empty set as well.

Definition 3 (Lift Function) Let $x \in \mathcal{C}$ be a point in the Voronoi cell of a face f of $\mathcal{C}$. The *lift of x in $\mathcal{C}$*, denoted by $\mathrm{lift}_\mathcal{C}(x)$ (or simply $\mathrm{lift}(x)$ if $\mathcal{C}$ is trivially implied), is defined to be the unique point of $\partial\mathrm{Vor}_\mathcal{C}(f)$ such that the line defined by the segment $[x, \mathrm{lift}(x)]$ is orthogonal to f. In other words, $\mathrm{lift}(x)$ is the unique point in $\partial\mathrm{Vor}_\mathcal{C}(f)$ that orthogonally projects to $\mathrm{np}(x)$ on f.
The *lift of a set of points* $X \subseteq \mathcal{C}$, denoted by $\mathrm{lift}(X)$, is the set of all the points $\mathrm{lift}(x)$ for $x \in X$, i.e., $\mathrm{lift}(X) := \{\,\mathrm{lift}(x) \mid x \in X\,\}$.

The function $\mathcal{L} : \mathcal{C} \to \mathrm{VorDiag}(\mathcal{F}_\mathcal{C})$ that maps each point $x \in \mathcal{C}$ to its lift in $\mathrm{VorDiag}(\mathcal{F}_\mathcal{C})$ will be called the *lift function* in the sequel. For any $Y \subset \mathrm{VorDiag}(\mathcal{F}_\mathcal{C})$, $\mathcal{L}^{-1}(Y)$ denotes the set of points $x \in \mathcal{C}$ such that $\mathrm{lift}(x) = y$ for some $y \in Y$.

Characterization of the Reconstructed Object $\mathcal{R}_\mathcal{C}$. If $\mathcal{S}_\mathcal{C} = \emptyset$, then as we said before, for any point $x \in \mathcal{C}$, $\mathrm{np}(x) \notin \mathcal{S}_\mathcal{C}$, and so $\mathcal{R}_\mathcal{C}$ is empty. Otherwise, let $A \in \mathcal{S}_\mathcal{C}$ be a section lying on a face of $\mathcal{C}$. For each point $a \in A$, the locus of all the points $x \in \mathcal{C}$ which have a as their nearest point in $\partial\mathcal{C}$ is the line segment $[a, \mathrm{lift}(a)]$ joining a to its lift. Therefore, the reconstructed object $\mathcal{R}_\mathcal{C}$ is the union of all the line-segments $[a, \mathrm{lift}(a)]$ for a point a in a section $A \in \mathcal{S}_\mathcal{C}$, i.e.,

$$\mathcal{R}_\mathcal{C} = \bigcup_{A \in \mathcal{S}_\mathcal{C}} \bigcup_{a \in A} [a, \mathrm{lift}(a)] = \mathcal{L}^{-1}(\mathrm{lift}(\mathcal{S}_\mathcal{C})).$$

Note that according to this characterization, if the lifts of two sections intersect in $\mathrm{VorDiag}(\mathcal{F}_\mathcal{C})$, then these two sections are connected in $\mathcal{R}_\mathcal{C}$. This is the generalization of the classical overlapping criterion for the case of parallel cutting planes. The union of all the pieces $\mathcal{R}_\mathcal{C}$ over all cells $\mathcal{C}$ will be the overall reconstructed object $\mathcal{R}$.

The rest of the paper is a devoted to prove that under two appropriate sampling conditions, $\mathcal{R}$ and $\mathcal{O}$ are homotopy equivalent, and are in addition homeomorphic. To make the connection between the upcoming sections more clear, we shortly outline in the next section the general strategy employed in proving the homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$.

3 Proof Outline of the Homotopy Equivalence Between $\mathcal{R}$ and $\mathcal{O}$

We will provide a homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ in each cell of the arrangement. (And then glue these homotopy equivalences together to form a global homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$.) To this aim, we first show in Section 4 that under the first sampling condition called *Separation Condition* the connection between the sections in the reconstructed object $\mathcal{R}_\mathcal{C}$ is the same as in $\mathcal{O}_\mathcal{C}$, in the sense that there is a bijection between the connected components of $\mathcal{R}_\mathcal{C}$ and the connected components of $\mathcal{O}_\mathcal{C}$. This implies that for proving the homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$, it will be enough to show that the corresponding connected components have the same homotopy type. In order to extend these homotopy equivalences to a homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$, we will have to glue together the homotopy equivalences we obtain in the cells of the arrangement. This needs some care since the restriction to a section S of the two homotopy equivalences defined in the two adjacent cells of S may be different. To overcome this problem, we need to define an intermediate shape $\mathcal{M}_\mathcal{C}$ in each cell $\mathcal{C}$, called the *medial shape*. The medial shape has the following three properties:

(i) The medial shape contains the sections of $\mathcal{C}$, i.e., $\mathcal{S}_\mathcal{C} \subseteq \mathcal{M}_\mathcal{C}$.

(*ii*) There is a (strong) deformation retract r from $\mathcal{O}_\mathcal{C}$ to $\mathcal{M}_\mathcal{C}$. In particular, this map is a homotopy equivalence between $\mathcal{O}_\mathcal{C}$ and $\mathcal{M}_\mathcal{C}$. And its restriction to $\mathcal{S}_\mathcal{C}$ is the identity map.

(*iii*) Under the first sampling condition (Separation Condition), $\mathcal{M}_\mathcal{C} \subseteq \mathcal{R}_\mathcal{C}$.

The first two properties will be crucial to guarantee that the homotopy equivalences conform on each section under the Separation Condition. Indeed, the map $\mathcal{O}_\mathcal{C} \to \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$, obtained by composing the deformation retract and the inclusion, restricts to the identity map on each section of $\mathcal{S}_\mathcal{C}$. Thus, we can glue all these maps to obtain a global map from $\mathcal{O}$ to $\mathcal{R}$.

Using a generalized version of the nerve theorem (see Section 7.1) and property (*ii*) above, we can then reduce the problem to prove that the inclusion $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ forms a homotopy equivalence in each cell. Using Whitehead's theorem, it will be enough to show that the inclusion i induces isomorphisms between the corresponding homotopy groups. Under the Separation Condition, we prove that i induces an injective map on the first homotopy groups, and that all higher homotopy groups of $\mathcal{M}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are trivial. Unfortunately, the Separation Condition does not ensure in general the surjectivity of i on the first homotopy groups. To overcome this problem, we need to impose a second condition called *Intersection Condition*. Under the Intersection Condition, the map i will be surjective on the first homotopy groups, leading to a homotopy equivalence between $\mathcal{O}$ and $\mathcal{R}$.

4 First Sampling Condition: Separation Condition

In this section, we provide the first sampling condition, under which the connection between the sections in the reconstructed object $\mathcal{R}$ are the same as in the original object $\mathcal{O}$. Our discussion will be essentially based on the study of the *medial axis*, that we define now.

Definition 4 (Medial Axis of $\partial\mathcal{O}$, Internal and External Retracts)

- Consider $\partial\mathcal{O}$ as a 2-manifold without boundary embedded in $\mathbb{R}^3$. The medial axis of $\partial\mathcal{O}$, denoted by $\mathrm{MA}(\partial\mathcal{O})$, contains two different parts: the so-called *internal* part, denoted by $\mathrm{MA}_i(\partial\mathcal{O})$, which lies in $\mathcal{O}$ and the so-called *external* part, denoted by $\mathrm{MA}_e(\partial\mathcal{O})$, which lies in $\mathbb{R}^3 \setminus \mathcal{O}$.

- The *internal retract* $m_i : \partial\mathcal{O} \to \mathrm{MA}_i(\partial\mathcal{O})$ is defined as follows: for a point $x \in \partial\mathcal{O}$, $m_i(x)$ is the center of the maximum ball entirely included in $\mathcal{O}$ which passes through x. For any $x \in \partial\mathcal{O}$, $m_i(x)$ is unique. Symmetrically, we define *the external retract* $m_e : \partial\mathcal{O} \to \mathrm{MA}_e(\partial\mathcal{O})$: for a point $x \in \partial\mathcal{O}$, $m_e(x)$ is the center of the maximum ball entirely included in $\mathbb{R}^3 \setminus \mathcal{O}$ which passes through x. For any $x \in \partial\mathcal{O}$, $m_e(x)$ is unique but may be at infinity. In the sequel, we may write $m(a)$ for a point in $\{m_i(a), m_e(a)\}$.

The interesting point is that as discussed below if the sample of cutting planes is sufficiently dense, then the internal part of $\mathrm{MA}(\partial\mathcal{O})$ lies inside the defined reconstructed object and the external part of this medial axis lies outside the reconstructed object.

Definition 5 (Separation Condition) We say that the set of cutting planes verifies the Separation Condition if:

$$\forall m \in \mathrm{MA}(\partial\mathcal{O}), \quad d(m, \mathrm{np}(m)) < d(m, \partial\mathcal{O}).$$

In other words, we impose that the medial ball of radius $d(m, \partial\mathcal{O})$ centered at m contains $\mathrm{np}(m)$. As a consequence, if $m \in \mathrm{MA}_i(\partial\mathcal{O})$, then $\mathrm{np}(m)$ belongs to $\mathcal{O}$ since this medial ball lies entirely inside $\mathcal{O}$. Thus, according to the definition of $\mathcal{R}$, we have $m \in \mathcal{R}$. Symmetrically, if $m \in \mathrm{MA}_e(\partial\mathcal{O})$, the medial ball lies entirely outside $\mathcal{O}$, and $\mathrm{np}(m)$ belongs to $\mathbb{R}^3 \setminus \mathcal{O}$. Hence, according to the definition of $\mathcal{R}$, we have $m \in \mathbb{R}^3 \setminus \mathcal{R}$. Therefore, the Separation Condition implies that $\partial\mathcal{R}$ *separates the internal and the external parts of the medial axis of* $\partial\mathcal{O}$. (That is where the name comes from.) Indeed, we will show that in each cell $\mathcal{C}$, the Separation Condition implies that $\partial\mathcal{R}_\mathcal{C}$ separates the internal and the external parts of the medial axis of $\partial\mathcal{O}_\mathcal{C}$.

Definition 6 (Medial Axis of $\partial\mathcal{O}_\mathcal{C}$, Internal & External Retracts in $\mathcal{C}$) In order to study the Separation Condition in a cell $\mathcal{C}$, we will need to consider the medial axis of $\mathcal{O}_\mathcal{C}$, denoted by $\mathrm{MA}(\partial\mathcal{O}_\mathcal{C})$, defined as the set of points in $\mathcal{C}$ with at least two closest points in $\partial\mathcal{O}_\mathcal{C}$. By $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$ (resp. $\mathrm{MA}_e(\partial\mathcal{O}_\mathcal{C})$) we denote the part of $\mathrm{MA}(\partial\mathcal{O}_\mathcal{C})$ that lies inside (resp. outside) $\mathcal{O}_\mathcal{C}$. Note that the two sets $\mathrm{MA}(\partial\mathcal{O}_\mathcal{C})$ and $\mathrm{MA}(\partial\mathcal{O}) \cap \mathcal{C}$ may be different.
We also consider the *internal retract* $m_{i,\mathcal{C}} : \partial\mathcal{O}_\mathcal{C} \to \mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$ defined as follows: for a point $x \in \partial\mathcal{O}_\mathcal{C}$, $m_{i,\mathcal{C}}(x)$ is the center of the maximum ball entirely included in $\mathcal{O}_\mathcal{C}$ which passes through x. Symmetrically, *the external retract* $m_{e,\mathcal{C}} : \partial\mathcal{O}_\mathcal{C} \to \mathrm{MA}_e(\partial\mathcal{O}_\mathcal{C})$ is defined as follows: for a point $x \in \partial\mathcal{O}_\mathcal{C}$, $m_{e,\mathcal{C}}(x)$ is the center of the maximum ball entirely included in $\mathcal{C} \setminus \mathcal{O}_\mathcal{C}$ which passes through x. It is easy to see that for any $x \in \partial\mathcal{O} \cap \mathcal{C}$, the segments $[x, m_{i,\mathcal{C}}(x)]$ and $[x, m_{e,\mathcal{C}}(x)]$ are subsegments of $[x, m_i(x)]$ and $[x, m_e(x)]$ respectively, and lie on the line defined by the normal to $\partial\mathcal{O}$ at x.

Lemma 1 (Separation Condition Restricted to $\mathcal{C}$) If the Separation Condition is verified then $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \subset \mathcal{R}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C} \subseteq \mathcal{C} \setminus \mathrm{MA}_e(\partial\mathcal{O}_\mathcal{C})$.

Proof We prove the first part, $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \subset \mathcal{R}_\mathcal{C}$. A similar proof gives the second part.
Let m be a point in $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$, and $B(m)$ be the open medial ball centered at m. Two cases can happen:

- Either, the closest points to m in $\partial\mathcal{O}_\mathcal{C}$ are in $\partial\mathcal{O}$, in which case m is a point in $\mathrm{MA}_i(\partial\mathcal{O})$. The Separation Condition states that $\mathrm{MA}_i(\partial\mathcal{O}) \subset \mathcal{R}$, and so $m \in \mathcal{R}_\mathcal{C} = \mathcal{R} \cap \mathcal{C}$.

- Otherwise, one of the closest points to m in $\partial\mathcal{O}_\mathcal{C}$ is a point a in some section $A \in \mathcal{S}_\mathcal{C}$. If a is on the boundary of A, then since along the section-contours $\partial\mathcal{O}_\mathcal{C}$ is non-smooth, a lies in $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$ and coincides with m, and $m = a$ is trivially in $\mathcal{R}_\mathcal{C}$. Hence, we may assume that a lies in the interior of A. Therefore, the ball $B(m)$ is tangent to A at a, and the line segment $[a, m]$ is orthogonal to A. Since $B(m) \cap \partial\mathcal{C} = \emptyset$, m and a are in the same Voronoi cell of the Voronoi diagram of the faces of $\mathcal{C}$. Thus, $a \in \mathcal{S}_\mathcal{C}$ is the nearest point in $\partial\mathcal{C}$ to m. By the definition of $\mathcal{R}_\mathcal{C}$, we deduce that $m \in \mathcal{R}_\mathcal{C}$. □

Assume that the Separation Condition is verified. The first idea which comes into mind is to retract points of $\partial\mathcal{O}$ to $\partial\mathcal{R}$ by following the normal-directions. A point $x \in \partial\mathcal{O}$ which lies outside $\mathcal{R}$, can move towards $m_i(x) \in \mathcal{R}$ and stop when $\partial\mathcal{R}$ is reached. A point $x \in \partial\mathcal{O}$ which lies inside $\mathcal{R}$, can move toward $m_e(x)$ and stop when $\partial\mathcal{R}$ is reached. Since $\partial\mathcal{O}$ is assumed to be smooth, the normals form a continuous vector field. Hence, this deformation will be a continuous retraction if each normal intersects $\partial\mathcal{R}$ at a single point. In such a case, $\partial\mathcal{O}$ can be deformed to $\partial\mathcal{R}$ homeomorphically. But a major problem is that $\mathcal{R}$ may have a complex shape (with cavities), so that a normal to $\partial\mathcal{O}$ intersects $\partial\mathcal{R}$ in several points. In such a case, such a retraction is not continuous and does not provide a deformation retract of $\mathcal{O}$ onto $\mathcal{R}$. However, we will be essentially following this intuitive idea by looking for a similar deformation retract of $\mathcal{O}$ onto a subshape of $\mathcal{R}$ (the so-called medial shape).

In the next subsection and Section 5 (on the medial shape), we will obtain a set of consequences of the Separation Condition that we will need for the rest of the paper.

4.1 Guarantees on the Connections Between the Sections

We now show that if the sample of cutting planes verifies the Separation Condition, then in each cell $\mathcal{C}$ of the arrangement, the connection between the sections is the same in $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$.

Theorem 1 If the sample of cutting planes verifies the Separation Condition, $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ induce the same connectivity components on the sections of $\mathcal{C}$.

Proof The proof is given in two parts:

(I) **If two sections are connected in $\mathcal{R}_\mathcal{C}$, then they are connected in $\mathcal{O}_\mathcal{C}$.** Let A and A' be two sections connected in $\mathcal{R}_\mathcal{C}$. Let γ be a path in $\mathcal{R}_\mathcal{C}$ that connects a point $a \in A$ to a point $a' \in A'$. For the sake of a contradiction, suppose that a and a' are not in the same connected component of $\mathcal{O}_\mathcal{C}$. In this case, as γ joins two points in two different connected components of $\mathcal{O}_\mathcal{C}$, it intersects $\mathrm{MA}_e(\partial\mathcal{O}_\mathcal{C})$. This is a contradiction with the fact that $\gamma \subset \mathcal{R}_\mathcal{C}$, since according to Lemma 1 we have $\mathrm{MA}_e(\partial\mathcal{O}_\mathcal{C}) \cap \mathcal{R}_\mathcal{C} = \emptyset$.

(II) **If two sections are connected in $\mathcal{O}_\mathcal{C}$, then they are connected in $\mathcal{R}_\mathcal{C}$.** Let A and A' be two sections in a same connected component K of $\mathcal{O}_\mathcal{C}$. According to the non-smoothness of $\partial\mathcal{O}_\mathcal{C}$ at the boundary of the sections, ∂A and $\partial A'$ are contained in $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$. Thus, there is a path γ in $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \cap K$ that connects a point $a \in \partial A$ to a point $a' \in \partial A'$. According to Lemma 1, $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \subset \mathcal{R}_\mathcal{C}$. Thus, γ is a path in $\mathcal{R}_\mathcal{C}$ that connects A to A'.

$\square$

According to the above theorem, to prove the homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ under the Separation Condition, we may restrict to each of the corresponding connected components.

In the sequel, to simplify the notations and the presentation, we suppose that $\mathcal{O}_\mathcal{C}$ and thus $\mathcal{R}_\mathcal{C}$ are connected, and we show that $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ have the same homotopy type. It is clear that the same proofs can be applied to each corresponding connected components of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ to imply the homotopy equivalence in the general case of multiple connected components.

5 Medial Shape

In this section, we define an intermediate shape in each cell $\mathcal{C}$ of the arrangement called the *medial shape*. The medial shape enjoys a certain number of important properties, discussed in this section, which makes it playing an important role in obtaining the homotopy equivalence of the next sections.

Definition 7 (Medial Shape $\mathcal{M}_\mathcal{C}$) Let x be a point in $\mathcal{S}_\mathcal{C} \subset \partial\mathcal{O}_\mathcal{C}$. Let $w(x) = [x, m_{i,\mathcal{C}}(x)]$ be the segment in the direction of the normal to $\partial\mathcal{O}_\mathcal{C}$ at x which connects x to the point $m_{i,\mathcal{C}}(x) \in \mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$. We add to $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$ all the segment $w(x)$ for all the points $x \in \mathcal{S}_\mathcal{C}$. We call the resulting shape $\mathcal{M}_\mathcal{C}$, see Figure 2. More precisely, $\mathcal{M}_\mathcal{C} := \mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \cup (\bigcup_{x\in\mathcal{S}_\mathcal{C}} w(x))$.

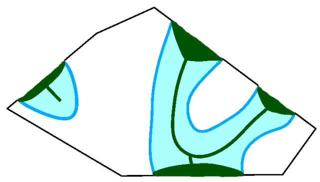

Figure 2: An illustration of the medial shape in green.

Proposition 1 The medial shape verifies the following set of properties:

(i) The medial shape contains the sections of $\mathcal{C}$, i.e., $\mathcal{S}_\mathcal{C} \subseteq \mathcal{M}_\mathcal{C}$.

(ii) There is a (strong) deformation retract r from $\mathcal{O}_\mathcal{C}$ to $\mathcal{M}_\mathcal{C}$. In particular, this map is a homotopy equivalence between $\mathcal{O}_\mathcal{C}$ and $\mathcal{M}_\mathcal{C}$. And its restriction to $\mathcal{S}_\mathcal{C}$ is the identity map.

(iii) Under the Separation Condition, $\mathcal{M}_\mathcal{C} \subseteq \mathcal{R}_\mathcal{C}$.

Proof

(i) This property is true by the definition of the medial shape.

(ii) This is easily obtained by deforming $\mathcal{O}_\mathcal{C}$ to $\mathcal{M}_\mathcal{C}$ in the direction of the normals to the boundary $\partial\mathcal{O}_\mathcal{C}$. Note that the boundary $\partial\mathcal{O}_\mathcal{C}$ is smooth except on the boundaries of sections in $\mathcal{S}_\mathcal{C}$, and the boundaries of the sections in $\mathcal{S}_\mathcal{C}$ are already in $\mathcal{M}_\mathcal{C}$, thus the deformation retract is well defined and easily seen to be continuous.

(*iii*) Since $\mathcal{M}_\mathcal{C} = \mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \cup \left(\bigcup_{x\in\mathcal{S}_\mathcal{C}} w(x)\right)$ and $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \subset \mathcal{R}_\mathcal{C}$, it will be sufficient to show that for any x in a section $A \in \mathcal{S}_\mathcal{C}$, $w(x) \subset \mathcal{R}_\mathcal{C}$. (Recall that $w(x)$ is the orthogonal segment to $\partial\mathcal{O}_\mathcal{C}$ at x that joins x to the corresponding medial point $m_{i,\mathcal{C}}(x)$ in $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$.) We will show that $w(x)$ is contained in the segment $[x, \mathrm{lift}(x)]$. The point x is the closest point in $\partial\mathcal{O}_\mathcal{C}$ to $m_{i,\mathcal{C}}(x)$. Thus, the ball centered at $m_{i,\mathcal{C}}(a)$ and passing through x is entirely contained in $\mathcal{O}$ and its interior is empty of points of $\partial\mathcal{C}$. Thus, in the Voronoi diagram of the faces of $\mathcal{C}$, $m_{i,\mathcal{C}}(a)$ is in the same Voronoi cell as x. On the other hand, x is the closest point in $\mathcal{S}_\mathcal{C} \subset \partial\mathcal{O}_\mathcal{C}$ to $\mathrm{lift}(x)$. It easily follows that $d(x, \mathrm{lift}(x)) \geq d(x, m_{i,\mathcal{C}}(x))$. It follows that the segment $[x, m_{i,\mathcal{C}}(x)] = w(x)$ is a subsegment of $[x, \mathrm{lift}(x)]$. Therefore, by the definition of $\mathcal{R}_\mathcal{C}$, $w(x) \subset \mathcal{R}_\mathcal{C}$.

□

We end this section with the following important remark and proposition which will be used in the next section. By replacing the shape $\mathcal{O}_\mathcal{C}$ with its complementary set we may define an *exterior medial shape* $\widetilde{\mathcal{M}_\mathcal{C}}$. This is more precisely defined as follows. Let $\widetilde{\mathcal{O}}$ be the closure of the complementary of $\mathcal{O}$ in $\mathbb{R}^3$. And let $\widetilde{\mathcal{O}_\mathcal{C}}$ be the intersection of $\widetilde{\mathcal{O}}$ with the cell $\mathcal{C}$. The medial shape of $\widetilde{\mathcal{O}_\mathcal{C}}$, denoted by $\widetilde{\mathcal{M}_\mathcal{C}}$, is the union of the medial shapes of the connected components of $\widetilde{\mathcal{O}_\mathcal{C}}$. Similarly, under the Separation Condition, the following proposition holds.

Proposition 2 Let $\widetilde{\mathcal{O}_\mathcal{C}}$ be the closure of the complementary of $\mathcal{O}_\mathcal{C}$ in $\mathcal{C}$ and $\widetilde{\mathcal{M}_\mathcal{C}}$ be the medial shape of $\widetilde{\mathcal{O}_\mathcal{C}}$. Under the Separation Condition: (*i*) There is a strong deformation retract from $\mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$ to $\mathcal{O}_\mathcal{C}$, and (*ii*) We have $\mathcal{R}_\mathcal{C} \subset \mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$.

Proof The proof of Property (*i*) is similar to the proof of Proposition 1 by deforming along the normal vectors to the boundary of $\widetilde{\mathcal{O}_\mathcal{C}}$. The second property (*ii*) is equivalent to $\widetilde{\mathcal{M}_\mathcal{C}} \subset \mathcal{C} \setminus \mathcal{R}_\mathcal{C}$. □

6 Topological Guarantees Implied by the Separation Condition

Throughout this section, we suppose that the Separation Condition holds. By the discussion at the end of Section 4.1, and without loss of generality, we may suppose that $\mathcal{O}_\mathcal{C}$ and hence $\mathcal{R}_\mathcal{C}$ are connected. Thus, $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are connected compact topological 3-manifolds embedded in $\mathbb{R}^3$.

In Section 5, we defined the medial shape $\mathcal{M}_\mathcal{C}$ and showed that $\mathcal{M}_\mathcal{C}$ is homotopy equivalent to $\mathcal{O}_\mathcal{C}$, by giving a (strong) deformation retract from $\mathcal{O}_\mathcal{C}$ onto $\mathcal{M}_\mathcal{C}$. We have also shown that under the Separation Condition, $\mathcal{M}_\mathcal{C} \subset \mathcal{R}_\mathcal{C}$. Using these properties, the goal will be to prove that the inclusion $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ is a homotopy equivalence. As the objects we are manipulating are all CW-complexes, homotopy equivalence is equivalent to weak homotopy equivalence according to Whitehead's theorem, see Appendix A. Hence, it will be enough to show that $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ induces isomorphism between the corresponding homotopy groups.

6.1 Injectivity on the Level of Homotopy Groups

We will make use of the lift function in $\mathcal{C}$, c.f. Section 2. We consider the restriction of the lift function to the reconstructed object $\mathcal{R}_\mathcal{C}$. According to the definition of $\mathcal{R}_\mathcal{C}$, $\mathcal{R}_\mathcal{C}$ is the union of all the segments $[a, \text{lift}(a)]$, for $a \in \mathcal{S}_\mathcal{C}$. On the other hand, the lift function retracts each segment $[a, \text{lift}(a)]$ to $\text{lift}(a)$ continuously. We infer the following simple observation.

Proposition 3 The lift function $\mathcal{L} : \mathcal{R}_\mathcal{C} \to \text{lift}(\mathcal{S}_\mathcal{C})$ is a homotopy equivalence.

According to the above proposition and the following diagram, to show that $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ is a homotopy equivalence, using Whitehead's theorem it will be sufficient to show that the restriction of the lift function to $\mathcal{M}_\mathcal{C}$ is a weak homotopy equivalence.

$$\begin{tikzcd} \mathcal{M}_\mathcal{C} \arrow[r, hook, "i"] \arrow[dr, "\mathcal{L}"'] & \mathcal{R}_\mathcal{C} \arrow[d, "\mathcal{L}"] \\ & \text{lift}(\mathcal{S}_\mathcal{C}) \end{tikzcd}$$

More precisely, if $\mathcal{L} : \mathcal{M}_\mathcal{C} \to \text{lift}(\mathcal{S}_\mathcal{C})$ is a weak homotopy equivalence, since $\mathcal{L} : \mathcal{R}_\mathcal{C} \to \text{lift}(\mathcal{S}_\mathcal{C})$ is also a homotopy equivalence and because of the commutativity of the above diagram, the inclusion $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ induces isomorphisms between the homotopy groups of $\mathcal{M}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$. Thus, Whitehead's theorem implies that i is a homotopy equivalence. We first show that under the Separation Condition, the restricted lift function $\mathcal{L}_{|\mathcal{M}_\mathcal{C}}$ induces injections on the level of homotopy groups.

Theorem 2 (Injectivity) Under the Separation Condition, the homomorphisms between the homotopy groups of $\mathcal{M}_\mathcal{C}$ and $\text{lift}(\mathcal{S}_\mathcal{C})$, induced by the lift function $\mathcal{L}$, are injective.

Proof Under the Separation Condition, we have $\mathcal{M}_\mathcal{C} \subset \mathcal{R}_\mathcal{C}$. Let $\widetilde{\mathcal{M}_\mathcal{C}}$ be the medial shape of the closure of the complementary of $\mathcal{O}_\mathcal{C}$ in $\mathcal{C}$. We refer to the discussion at the end of the previous section for more details. Recall that by Proposition 2, we have $\mathcal{R}_\mathcal{C} \subset \mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$, and there exists a deformation retract from $\mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$ to $\mathcal{O}_\mathcal{C}$ (in particular $\mathcal{O}_\mathcal{C}$ and $\mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$ are homotopy equivalent). We have now the following commutative diagram in which every map (except the lift function $\mathcal{L}$) is an injection (or an isomorphism) on the level of homotopy groups.

$$\begin{tikzcd} & \mathcal{O}_\mathcal{C} \arrow[dl, "\simeq"'] & \\ \mathcal{M}_\mathcal{C} \arrow[r, hook, "i"] \arrow[dr, "\mathcal{L}"'] & \mathcal{R}_\mathcal{C} \arrow[r, hook] \arrow[d, "\simeq"] & \mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}} \arrow[ul, "\simeq"'] \\ & \text{lift}(\mathcal{S}_\mathcal{C}) & \end{tikzcd}$$

Using this diagram, the injectivity on the level of homotopy groups is clear: For any integer $j \geq 1$, consider the induced homomorphism $\mathcal{L}_* : \pi_j(\mathcal{M}_\mathcal{C}) \to$

$\pi_j(\text{lift}(\mathcal{S}_\mathcal{C}))$. Let $x \in \pi_j(\mathcal{M}_\mathcal{C})$ be so that $\mathcal{L}_*(x)$ is the zero element of $\pi_j(\text{lift}(\mathcal{S}_\mathcal{C}))$. It is sufficient to show that x is the zero element of $\pi_j(\mathcal{M}_\mathcal{C})$. Following the maps of the diagram, and using the homotopy equivalence between $\text{lift}(\mathcal{S}_\mathcal{C})$ and $\mathcal{R}_\mathcal{C}$, we have that $i_*(x)$ is mapped to the zero element of $\pi_j(\mathcal{R}_\mathcal{C})$. Then, by the inclusion $\mathcal{R}_\mathcal{C} \hookrightarrow \mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$, it goes to the zero element of $\mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$, and by the two retractions, it will be mapped to the zero element of $\mathcal{M}_\mathcal{C}$. As this diagram is commutative, we infer that x is the zero element of $\mathcal{M}_\mathcal{C}$. Thus, $\mathcal{L}_* : \pi_j(\mathcal{M}_\mathcal{C}) \to \pi_j(\text{lift}(\mathcal{S}_\mathcal{C}))$ is injective for all $j \geq 1$. The injectivity for $j = 0$ is already proved in Theorem 1. □

We have shown that under the Separation Condition, the lift function $\mathcal{L} : \mathcal{M}_\mathcal{C} \to \text{lift}(\mathcal{S}_\mathcal{C})$ induces injective morphisms between the homotopy groups of $\mathcal{M}_\mathcal{C}$ and $\text{lift}(\mathcal{S}_\mathcal{C})$. If these induced morphisms were surjective, then $\mathcal{L}$ would be a homotopy equivalence (by Whitehead's theorem). We will show in Section 6.2 that the Separation Condition implies the surjectivity for all the homotopy groups except for dimension one (fundamental groups). Indeed, we will show that under the Separation Condition, all the i-dimensional homotopy groups of $\mathcal{M}_\mathcal{C}$ and $\text{lift}(\mathcal{S}_\mathcal{C})$ for $i \geq 2$ are trivial. Once this is proved, it will be sufficient to study the surjectivity of $\mathcal{L}_* : \pi_1(\mathcal{M}_\mathcal{C}) \to \pi_1(\text{lift}(\mathcal{S}_\mathcal{C}))$. Note that the injectivity in the general form above remains valid for the corresponding reconstruction problems in dimensions greater than three. However, the vanishing results on higher homotopy groups of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are only valid in dimensions two and three.

6.2 The topological structures of $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ are determined by their fundamental groups.

In this section, we show that if the Separation Condition is verified, then the topological structure of the portion of $\mathcal{O}$ in a cell $\mathcal{C}$ (i.e., $\mathcal{O}_\mathcal{C}$) is simple enough, in the sense that for all $i \geq 2$, the i-dimensional homotopy group of $\mathcal{O}_\mathcal{C}$ is trivial. We can easily show that $\mathcal{R}_\mathcal{C}$ has the same property.[1] As a consequence, the topological structures of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are determined by their fundamental group, $\pi_1(\mathcal{O}_\mathcal{C})$ and $\pi_1(\mathcal{R}_\mathcal{C})$.

We first state the following general theorem for an arbitrary embedded 3-manifold with connected boundary.

Theorem 3 Let K be a connected 3-manifold in $\mathbb{R}^3$ with a (non-empty) connected boundary. Then for all $i \geq 2$, $\pi_i(K) = \{0\}$.

We provide a proof of this theorem in Appendix B. This theorem can be also obtained from Corollary 3.9 of [Hat02]. From this theorem, we infer the two following theorems.

Theorem 4 Under the Separation Condition, $\pi_i(\mathcal{O}_\mathcal{C}) = \{0\}$, for all $i \geq 2$.

Proof We only make use of the fact that under the Separation Condition, any connected component of $\partial\mathcal{O}$ is cut by at least one cutting plane. In this case, every connected component of $\mathcal{O}_\mathcal{C}$ is a 3-manifold with connected boundary. The theorem follows as a corollary of Theorem 3. □

[1] Recall that for simplifying the presentation, we assume that $\mathcal{O}_\mathcal{C}$ and so $\mathcal{R}_\mathcal{C}$ are connected. The same proof shows that in the general case, the same property holds for each connected component of $\mathcal{O}_\mathcal{C}$ or $\mathcal{R}_\mathcal{C}$.

Theorem 5 $\pi_i(\mathcal{R}_\mathcal{C}) = \{0\}$, for all $i \geq 2$.

Proof Using Theorem 3, it will be sufficient to show that the boundary of any connected component of $\mathcal{R}_\mathcal{C}$ is connected. Let x and y be two points on this boundary, and let S and S' be two sections so that $x \in [a, \text{lift}(a)]$ for some $a \in S$ and $y \in [b, \text{lift}(b)]$ for some $b \in S'$. By the definition of $\mathcal{R}_\mathcal{C}$, x is connected to S in $\partial\mathcal{R}_\mathcal{C}$, and y is connected to S' in $\partial\mathcal{R}_\mathcal{C}$. On the other hand, S and S' are connected to each other in $\partial\mathcal{R}_\mathcal{C}$. Thus, x is connected to y in $\partial\mathcal{R}_\mathcal{C}$. □

7 Second Condition: Intersection Condition

In the previous section, we saw that under the Separation Condition, the topological structures of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are determined by their fundamental group $\pi_1(\mathcal{O}_\mathcal{C})$ and $\pi_1(\mathcal{R}_\mathcal{C})$, respectively. The goal of this section is to find a way to ensure an isomorphism between the fundamental groups of $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$. We recall that as $\mathcal{O}_\mathcal{C}$ and $\mathcal{M}_\mathcal{C}$ are homotopy equivalent, $\pi_1(\mathcal{O}_\mathcal{C})$ is isomorphic to $\pi_1(\mathcal{M}_\mathcal{C})$. On the other hand, $\mathcal{R}_\mathcal{C}$ and $\text{lift}(\mathcal{S}_\mathcal{C})$ are homotopy equivalent, and $\pi_1(\mathcal{R}_\mathcal{C})$ is isomorphic to $\pi_1(\text{lift}(\mathcal{S}_\mathcal{C}))$ (c.f. last diagram). Thus, it will be sufficient to compare $\pi_1(\mathcal{M}_\mathcal{C})$ and $\pi_1(\text{lift}(\mathcal{S}_\mathcal{C}))$.

We consider $\mathcal{L}_* : \pi_1(\mathcal{M}_\mathcal{C}) \to \pi_1(\text{lift}(\mathcal{S}_\mathcal{C}))$, the map induced by the lift function from $\mathcal{M}_\mathcal{C}$ to $\text{lift}(\mathcal{S}_\mathcal{C})$ on fundamental groups . We showed that $\mathcal{L}_*$ is injective. However, the map fails to be surjective in general. Figure 3 shows two shapes with different topologies, a torus and a (*twisted*) cylinder, that have same (inter)sections with a set of (two) cutting planes.

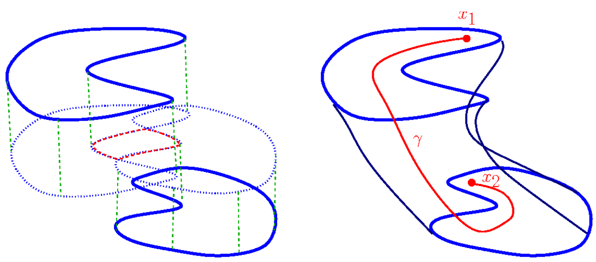

Figure 3: An example of the case where the lift function from $\mathcal{M}_\mathcal{C}$ to $\text{lift}(\mathcal{S}_\mathcal{C})$ fails to be surjective: x_1 and x_2 are two points with the same lift in $\text{lift}(\mathcal{S}_\mathcal{C})$. The lift of any curve γ connecting x_1 and x_2 in $\mathcal{M}_\mathcal{C}$ provides a non-zero element of $\pi_1(\text{lift}(\mathcal{S}_\mathcal{C}), x)$. The reconstructed object (at left) is a torus and is not homotopy equivalent to the original shape (at right) which is a twisted cylinder.

Hence, whatever is the reconstructed object from these sections, it would not be topologically consistent for at least one of these objects. In particular, the proposed reconstructed object ($\mathcal{R}$) is a torus which is not homotopy equivalent to the (*twisted*) cylinder ($\mathcal{O}$). In addition, we note that the Separation Condition may be verified for such a situation. Indeed, such a situation is exactly the case when the injective morphism between the fundamental groups of $\mathcal{O}$ and $\mathcal{R}$ is not surjective. This situation can be explained as follows: let x_1 and x_2 be two points in the sections S_1 and S_2 with the same lift x in $\text{lift}(\mathcal{S}_\mathcal{C})$. The lift of any curve γ connecting x_1 and x_2 in $\mathcal{M}_\mathcal{C}$ provides a non-zero element of $\pi_1(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x)$ which is not in the image of $\mathcal{L}_*$. We may avoid this situation with the following condition.

Definition 8 (Intersection Condition) We say that the set of cutting planes verifies the Intersection Condition if for any pair of sections S_i and S_j in $\mathcal{S}_\mathcal{C}$, and for any connected component X of $\mathrm{lift}(S_i) \cap \mathrm{lift}(S_j)$, the following holds: there is a path $\gamma \subset \mathcal{M}_\mathcal{C}$ from a point $a \in S_i$ to a point $b \in S_j$ with $\mathrm{lift}(a) = \mathrm{lift}(b) = x \in X$ so that $\mathcal{L}_*(\gamma)$ is the zero element of $\pi_1(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x)$, i.e., is contractible in $\mathrm{lift}(\mathcal{S}_\mathcal{C})$ with a homotopy respecting the base point x.

In Section 8, we will show that the Intersection Condition is verified if the set of cutting planes is sufficiently dense. Let us first prove the surjectivity of the map $\mathcal{L}_*$ under this Condition.

Theorem 6 Under the Intersection Condition, the induced map $\mathcal{L}_* : \pi_1(\mathcal{M}_\mathcal{C}) \to \pi_1(\mathrm{lift}(\mathcal{S}_\mathcal{C}))$ is surjective.

Proof Let y_0 be a fixed point for $\mathcal{M}_\mathcal{C}$ and $x_0 = \mathcal{L}(y_0)$. We show that $\mathcal{L}_* : \pi_1\big(\mathcal{M}_\mathcal{C}, y_0\big) \to \pi_1\big(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x_0\big)$ is surjective. Let α be a closed curve in $\mathrm{lift}(\mathcal{S}_\mathcal{C})$ which represents an element of $\pi_1\big(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x_0\big)$. We show the existence of an element $\beta \in \pi_1\big(\mathcal{M}_\mathcal{C}, y_0\big)$ such that $\mathcal{L}_*(\beta) = [\alpha]$, where $[\alpha]$ denotes the homotopy class of α in $\pi_1\big(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x_0\big)$.
We can divide α into subcurves $\alpha_1, \ldots, \alpha_m$ such that α_j joins two points x_{j-1} and x_j, and is entirely in the lift of one of the sections S_j, for $j = 1, \ldots, m$. We may assume $y_0 \in S_1 = S_m$. For each $j = 1, \ldots, m$, let β_j be the curve in S_j, joining two points z_j to w_j, which is mapped to α_j under $\mathcal{L}$. Note that w_j and z_{j+1} (possibly) live in two different sections, but have the same image (x_j) under the lift map $\mathcal{L}$. Let X_j be the connected component of $\mathrm{lift}(S_j) \cap \mathrm{lift}(S_{j+1})$ which contains x_j, see Figure 4. According to the Intersection Condition, there is a path $\gamma_j \subset \mathcal{M}_\mathcal{C}$ connecting a point $a_j \in S_j$ to a point $b_{j+1} \in S_{j+1}$ such that $\mathrm{lift}(a_j) = \mathrm{lift}(b_{j+1}) = x'_j \in X_j$ and the image of γ_j under $\mathcal{L}$ is the zero element of $\pi_1(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x'_j)$ (i.e., is contractible with a homotopy respecting the base point x'_j). Since X_j is connected, there is a path from x_j to x'_j in X_j, so lifting back this path to two paths from w_j to a_j in S_j and from b_{j+1} to z_{j+1} and taking the union of these two paths with γ_j, we infer the existence of a path $\gamma'_j \subset \mathcal{M}_\mathcal{C}$ connecting w_j to z_{j+1}, such that the image of γ'_j under $\mathcal{L}$ is contractible in $\mathrm{lift}(\mathcal{S}_\mathcal{C})$ with a homotopy respecting the base point x_j.

Let β be the path from x_0 to x_0 obtained by concatenating β_j and γ'_j alternatively, i.e., $\beta = \beta_1\gamma'_1\beta_2\gamma'_2 \ldots \beta_{m-1}\gamma'_m\beta_m\gamma'_m$. We claim that $\mathcal{L}_*([\beta]) = [\alpha]$. This is now easy to show: we have $\mathcal{L}_*(\beta) = \alpha_1\mathcal{L}_*(\gamma'_1)\alpha_2 \ldots \mathcal{L}_*(\gamma'_m)\alpha_m$, and all the paths $\mathcal{L}_*(\gamma'_j)$ are contractible to the constant path $[x_j]$ by a homotopy fixing x_j all the time. We deduce that under a homotopy fixing x_0, $\alpha_1\mathcal{L}_*(\gamma'_1) \ldots \mathcal{L}_*(\gamma'_m)\alpha_m$ is homotopic to $\alpha_1\alpha_2 \ldots \alpha_m = \alpha$, and this is exactly saying that $\mathcal{L}_*([\beta]) = [\alpha]$. And the surjectivity follows. $\square$

Putting together all the materials we have obtained, we infer the main theorem of this section.

Theorem 7 (Main Theorem-Part 0) Under the Separation and the Intersection Conditions, $\mathcal{R}_\mathcal{C}$ is homotopy equivalent to $\mathcal{O}_\mathcal{C}$.

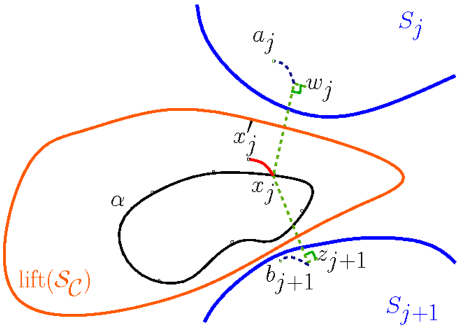

Figure 4: For the proof of Theorem 6.

7.1 Generalized Nerve Theorem and Homotopy Equivalence of $\mathcal{R}$ and $\mathcal{O}$

In this section, we extend the homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$, in each cell $\mathcal{C}$, to a global homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$. To this end, we make use of a generalization of the nerve theorem. This is a folklore theorem and has been observed and used by different authors. For a modern proof of a still more general result, we refer to Segal's paper [Seg68]. (See also [May03], for a survey of similar results.)

Theorem 8 (Generalized Nerve Theorem) Let $H : X \to Y$ be a continuous map. Suppose that Y has an open cover $\mathcal{K}$ with the following two properties:

- Finite intersections of sets in $\mathcal{K}$ are in $\mathcal{K}$.
- For each $U \in \mathcal{K}$, the restriction $H : H^{-1}(U) \to U$ is a weak homotopy equivalence.

Then H is a weak homotopy equivalence.

Let $F_\mathcal{C} : \mathcal{O}_\mathcal{C} \to \mathcal{R}_\mathcal{C}$ be the homotopy equivalence obtained in the previous sections between $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$. (So $F_\mathcal{C}$ is the composition of the retraction $\mathcal{O}_\mathcal{C} \to \mathcal{M}_\mathcal{C}$ and the inclusion $\mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$.) Let $H : \mathcal{O} \to \mathcal{R}$ be the map defined by $H(x) = H_\mathcal{C}(x)$ if $x \in \mathcal{O}_\mathcal{C}$ for a cell $\mathcal{C}$ of the arrangement of the cutting planes. Note that H is well-defined since $H_\mathcal{C}|_{\mathcal{S}_\mathcal{C}} = id_{\mathcal{S}_\mathcal{C}}$, for all $\mathcal{C}$. In addition, since for all cell $\mathcal{C}$, $H_\mathcal{C}$ is continuous, H is continuous as well.
We can now apply the generalized nerve theorem by the following simple trick. Let ϵ be an infinitesimal positive value. For any cell $\mathcal{C}$ of the arrangement of the cutting planes, we define $\mathcal{O}_\mathcal{C}^\epsilon = \{x \in \mathbb{R}^3,\ d(x, \mathcal{O}_\mathcal{C}) < \epsilon\ \}$. Let us now consider the open covering $\mathcal{K}$ of $\mathcal{O}$ by these open sets and all their finite intersections. It is straightforward to check that for ϵ small enough, the restriction of H to each element of $\mathcal{K}$ is a weak homotopy equivalence. Therefore, according to the generalized nerve theorem, H is a weak homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$. And by Whitehead's theorem, H is a homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$. Thus, we have proved:

Theorem 9 (Main Theorem-Part I) Under the Separation and the Intersection Conditions, the reconstructed object $\mathcal{R}$ is homotopy equivalent to the unknown original shape $\mathcal{O}$.

8 How to Ensure the Sampling Conditions?

In this section we provide sufficient conditions for ensuring the Separation and the Intersection Conditions, defined in the previous sections. For this, we need first some definitions.

Definition 9 (Reach) Let $\mathcal{O}$ be a connected compact 3-manifold with smooth boundary $\partial\mathcal{O}$ in $\mathbb{R}^3$. For $a \in \partial\mathcal{O}$, we define $\text{reach}(a) = \min\big(d(a, m_i(a)), d(a, m_e(a))\big)$. The quantity reach$(\mathcal{O})$ is defined as the minimum distance of $\partial\mathcal{O}$ from the medial axis of $\partial\mathcal{O}$:

$$\text{reach}(\mathcal{O}) := \min_{m\in \text{MA}(\partial\mathcal{O})} d(m, \partial\mathcal{O}) = \min_{a\in\partial\mathcal{O}} \text{reach}(a).$$

Note that as $\mathcal{O}$ is compact and $\partial\mathcal{O}$ is of class C^1, reach$(\mathcal{O})$ is strictly positive.

Definition 10 (Reach restricted to a cell of the arrangement) Given a cell $\mathcal{C}$ of the arrangement, we define $\text{reach}_{\mathcal{C}}(\mathcal{O}) = \min d(a, m(a))$, where either $a \in \partial\mathcal{O} \cap \mathcal{C}$ or $m(a) \in \text{MA}(\partial\mathcal{O}) \cap \mathcal{C}$. By definition, we have $\text{reach}(\mathcal{O}) = \min_{\mathcal{C}} (\text{reach}_{\mathcal{C}}(\mathcal{O}))$.

Definition 11 (Height of a Cell) Let $\mathcal{C}$ be a cell of the arrangement of the cutting planes and f be a face of $\mathcal{C}$. The *height* of f in $\mathcal{C}$, denoted by h_f, is defined as $h_f := \max_{x\in \text{Vor}_{\mathcal{C}}(f)} d(x, f)$. We also define the height of $\mathcal{C}$ as $h_{\mathcal{C}} := \max_{f\in\mathcal{F}_{\mathcal{C}}} h_f$.

As we will see, by upper-bounding the height of the cells by a factor related to the reach of the object, we can ensure the Separation Condition. Therefore, the Separation Condition can be ensured with a sufficiently dense sample of cutting planes. In order to ensure the Intersection Condition, we need a stronger condition on the height of the cells. As we will see, this condition is a *transversality* condition on the cutting planes that can be measured by the angle between the cutting planes and the normal to $\partial\mathcal{O}$ at contour-points.

Definition 12 (Angle α_a) Let a be a point on the boundary of a section $A \in \mathcal{S}_{\mathcal{C}}$ on the plane P_A. We define α_a as the angle between P_A and the normal to $\partial\mathcal{O}$ at a: $\alpha_a := \text{angle}(P_A, [a, m_i(a)])$.

Sufficient Conditions We now define the sampling conditions on the set of cutting planes.

(C1) For any cell $\mathcal{C}$ of the arrangement, $h_{\mathcal{C}} < \text{reach}_{\mathcal{C}}(\mathcal{O})$.

(C2) For any cell $\mathcal{C}$ of the arrangement, $h_{\mathcal{C}} < \frac{1}{2}\left(1 - \sin(\alpha_a)\right) \text{reach}(a), \forall a \in \partial\mathcal{S}_{\mathcal{C}}$.

Condition (C1) is based on the density of the sections and as we will see, it implies the Separation Condition. Together with the second condition, they imply the Intersection Condition. Condition (C2) is defined in a way that the *transversality* to $\partial\mathcal{O}$ and the *distance between the sections* are controlled simultaneously. (Indeed, $\sin(\alpha_a)$ is to control the transversality, and upperbounding $h_{\mathcal{C}}$ allows us to control the distance between the sections.) Therefore, by increasing the density of the sections of $\mathcal{O}$ (with preferably transversal cutting planes) we can ensure the required sampling conditions, and as a consequence, provide a topologically consistent reconstruction of $\mathcal{O}$. This is one of the main results of this paper:

Theorem 10 (Main Theorem-Part II) If the sample of cutting planes verifies Conditions (C1) and (C2), then the Separation and the Intersection Conditions are verified. Consequently, the proposed reconstructed object $\mathcal{R}$ is homotopy equivalent to the unknown original shape $\mathcal{O}$.

The two following sections are devoted to the proof of this theorem.

8.1 Separation Condition: A Sufficient Condition Based on the Density of Planes.

We now show that Condition (C1) implies the Separation Condition.

Lemma 2 (Condition (C1) implies the Separation Condition.) If for any cell $\mathcal{C}$ of the arrangement, $h_{\mathcal{C}} < \text{reach}_{\mathcal{C}}(\mathcal{O})$ then the Separation Condition is verified.

Proof The proof is straightforward. Let m be a point in $\text{MA}(\partial\mathcal{O})$ in a cell $\mathcal{C}$ of the arrangement. The point m belongs to the Voronoi cell of some face $f \in \mathcal{F}_{\mathcal{C}}$. We have $d(m, np(m)) \leq h_f \leq h_{\mathcal{C}} < \text{reach}_{\mathcal{C}}(\mathcal{O}) \leq d(m, \partial\mathcal{O})$. Therefore the Separation Condition is verified. □

Moreover, we can show that such a condition allows to control the Hausdorff distance between $\mathcal{O}$ and its approximation $\mathcal{R}$.

Theorem 11 (Approximation Guarantees) Let ϵ be a given positive constant such that for any cell $\mathcal{C}$ of the arrangement, $h_{\mathcal{C}} < \epsilon \ \text{reach}_{\mathcal{C}}(\mathcal{O})$. Then we have $d_H(\mathcal{O}, \mathcal{R}) < \epsilon \ \max_{\mathcal{C}} \text{reach}_{\mathcal{C}}(\mathcal{O})$.

Proof We claim that for any cell $\mathcal{C}$ of the arrangement $d_H(\mathcal{O}_{\mathcal{C}}, \mathcal{R}_{\mathcal{C}}) \leq h_{\mathcal{C}}$. The proof is given in two parts:

1. Let x be a point in $\mathcal{R}_{\mathcal{C}}$. According to the characterization of $\mathcal{R}_{\mathcal{C}}$, there exists a point $a \in \mathcal{S}_{\mathcal{C}}$ such that x belongs to the segment $[a, \text{lift}(a)]$. Since $a \in \mathcal{S}_{\mathcal{C}} \subset \mathcal{O}_{\mathcal{C}}$, we have $d(x, \mathcal{O}_{\mathcal{C}}) \leq d(x, a) \leq d(a, \text{lift}(a)) \leq h_{\mathcal{C}}$.

2. Let x be a point in $\mathcal{O}_{\mathcal{C}}$. If $x \in \mathcal{R}_{\mathcal{C}}$, $d(x, \mathcal{R}_{\mathcal{C}}) = 0$ and there is nothing to prove. Thus, we can suppose that x is not in $\mathcal{R}_{\mathcal{C}}$. Let $x' \in \partial\mathcal{O}_{\mathcal{C}}$ be such that x belongs to the segment $[x', m_{i,\mathcal{C}}(x')]$. According to Lemma 1 we have $m_{i,\mathcal{C}}(x') \in \mathcal{R}_{\mathcal{C}}$. Moreover, we have $x \notin \mathcal{R}_{\mathcal{C}}$. Thus the segment joining x to $m_{i,\mathcal{C}}(x')$ intersects $\partial\mathcal{R}_{\mathcal{C}}$. Let y be a point in $[x, m_{i,\mathcal{C}}(x')] \cap \partial\mathcal{R}_{\mathcal{C}}$. According to the characterization of $\mathcal{R}_{\mathcal{C}}$, there exists a point $a \in \partial\mathcal{S}_{\mathcal{C}}$ such that y belongs to the segment $[a, \text{lift}(a)]$, and so $d(y, a) \leq h_{\mathcal{C}}$.

   We claim that $d(y, x') \leq d(y, a)$. If $d(y, x') > d(y, a)$ then $d(x', m_{i,\mathcal{C}}(x')) = d(y, x') + d(y, m_{i,\mathcal{C}}(x')) > d(y, a) + d(y, m_{i,\mathcal{C}}(x')) \geq d(a, m_{i,\mathcal{C}}(x'))$. But $d(x', m_{i,\mathcal{C}}(x')) > d(a, m_{i,\mathcal{C}}(x'))$ contradicts the fact that x' is the nearest point of $\partial\mathcal{O}_{\mathcal{C}}$ to $m_i(x)$. Thus, $d(y, x') \leq d(y, a)$. On the other hand, since $y \in [x, m_{i,\mathcal{C}}(x')] \subseteq [x', m_{i,\mathcal{C}}(x')]$, $d(x, y) \leq d(x', y)$ and we have:

   $$d(x, \mathcal{R}_{\mathcal{C}}) \leq d(x, y) \leq d(x', y) \leq d(y, a) \leq h_{\mathcal{C}}.$$

Therefore, $d_H(\mathcal{O}_\mathcal{C}, \mathcal{R}) \leq h_\mathcal{C} < \epsilon\ \text{reach}_\mathcal{C}(\mathcal{O})$ for all $\mathcal{C}$ and the theorem is proved by taking the maximum over all the cells of the arrangement. $\square$

8.2 Intersection Condition: A Sufficient Condition Based on the Transversality.

In this section, we prove that under Conditions (C1) and (C2), the Intersection Condition is verified. We need the following notations.

Notation $\big(K_i(\mathcal{S}_\mathcal{C})$ and $K_e(\mathcal{S}_\mathcal{C})\big)$: Recall that $\text{VorDiag}(\mathcal{F}_\mathcal{C})$ is the locus of the points with more than one nearest point in $\partial\mathcal{C}$. We write $K_i(\mathcal{S}_\mathcal{C})$ for the set of points $x \in \text{VorDiag}(\mathcal{F}_\mathcal{C})$ such that all the nearest points of x in $\partial\mathcal{C}$ lie inside the sections. A point $x \in \text{VorDiag}(\mathcal{F}_\mathcal{C})$ is called to be in $K_e(\mathcal{S}_\mathcal{C})$ if all the nearest points of x in $\partial\mathcal{C}$ lie outside the sections.

Notation $\big(\bar{m}_i(a)$ and $\bar{m}_e(a)\big)$: Let a be a point on the boundary of a section $A \in \mathcal{S}_\mathcal{C}$ on the plane P_A. We write $\bar{m}_i(a)$ (resp. $\bar{m}_e(a)$) for the orthogonal projection of $m_i(a)$ (resp. $m_e(a)$) onto P_A. We have $d(m_i(a), \bar{m}_i(a)) = \sin(\alpha_a)\, d(a, m_i(a))$ and $d(m_e(a), \bar{m}_e(a)) = \sin(\alpha_a)\, d(a, m_e(a))$.

Lemma 3 If Condition (C2) is verified, for any $a \in \partial\mathcal{S}_\mathcal{C}$, we have $\text{lift}(\bar{m}_i(a)) \in K_i(\mathcal{S}_\mathcal{C})$ and $\text{lift}(\bar{m}_e(a)) \in K_e(\mathcal{S}_\mathcal{C})$. In addition, the two segments that join $\text{lift}(\bar{m}_i(a))$ to its nearest points in $\partial\mathcal{C}$ lie both entirely in $\mathcal{M}_\mathcal{C}$.

Proof We show that $\text{lift}(\bar{m}_i(a)) \in K_i(\mathcal{S}_\mathcal{C})$. The symmetric property for $\text{lift}(\bar{m}_e(a))$ can be proved similarly. Let us simplify the notation by writing x for $\bar{m}_i(a)$ in this proof. Let us also write $B(m_i(a))$ for the ball centered at $m_i(a)$ of radius $d(m_i(a), a)$. We have $d(m_i(a), x) \leq d(m_i(a), a)$. Thus, x lies in $B(m_i(a))$. As this ball is contained in $\mathcal{O}$, x is in $\mathcal{O}$. Consider now $\text{lift}(x)$ on the Voronoi diagram $\text{VorDiag}(\mathcal{S}_\mathcal{C})$, and call y the point distinct from x such that $\text{lift}(x) = \text{lift}(y)$. To have $\text{lift}(m_i(a)) \in K_i(\mathcal{S}_\mathcal{C})$, we need to show that y is in $\mathcal{O}$. We have

$$d(m_i(a), y) < d(m_i(a), x) + d(x, \text{lift}(x)) + d(\text{lift}(y), y) \leq$$

$$\sin(\alpha_a) d(a, m_i(a)) + 2\ h_\mathcal{C} \leq d(a, m_i(a)).$$

Thus, y belongs to $B(m_i(a))$, and we have $y \in \mathcal{O}$ and $\text{lift}(x) \in K_i(\mathcal{S}_\mathcal{C})$. In addition, the ball centered at $\text{lift}(x)$ which passes through x and y is entirely contained in $B(m_i(a)) \subset \mathcal{O}$. Thus, its interior is empty of points in $\partial\mathcal{O}_\mathcal{C}$ and it is a medial ball of $\mathcal{O}_\mathcal{C}$, and its center $\text{lift}(x)$ belongs to $\text{MA}_i(\partial\mathcal{O}_\mathcal{C})$. Since x and y are in $\mathcal{S}_\mathcal{C}$, according to the definition of $\mathcal{M}_\mathcal{C}$, the line-segments $[\text{lift}(x), x]$ and $[\text{lift}(x), y]$ lie entirely in $\mathcal{M}_\mathcal{C}$. $\square$

Lemma 4 Under Conditions (C1) and (C2), the Intersection Condition is verified.

Proof Let S_i and S_j be two sections in $\mathcal{S}_\mathcal{C}$ such that $\text{lift}(S_i) \cap \text{lift}(S_j)$ is non-empty. We will show that for any two points $a \in \partial S_i, b \in \partial S_j$ such that $\text{lift}(a) = \text{lift}(b)$, there exists a path $\gamma \subset \mathcal{M}_\mathcal{C}$ between a and b so that $\mathcal{L}_*(\gamma)$ is contractible in $\mathcal{L}_*(\pi_1(\mathcal{M}_\mathcal{C}))$. Let P_i be the cutting-plane of S_i. One of the two following cases happen:

• If the projection of the segment $[a, m_i(a)]$ onto P_i is not cut by any other cutting plane, then we claim that $\mathrm{lift}(a)$ is connected to $\mathrm{lift}(\bar{m}_i(a))$ in L_2.
Let $a \in \partial S_i$ be such that $\mathrm{lift}(a)$ is on the boundary of a connected component of $\mathrm{lift}(S_i) \cap \mathrm{lift}(S_j)$. According to Lemma 3, $\mathrm{lift}(\bar{m}_i(a))$ lies in $\mathrm{lift}(S_i) \cap \mathrm{lift}(S_j)$. We claim that $\mathrm{lift}(\bar{m}_i(a))$ is in the same connected component of $\mathrm{lift}(S_i) \cap \mathrm{lift}(S_j)$ as $\mathrm{lift}(a)$. Otherwise, the lift of the medial ball centered at $m_i(a)$ and passing through a (denoted by $\mathrm{lift}(B)$) intersects two different connected components of $\mathrm{lift}(S_i) \cap \mathrm{lift}(S_j)$, see Figure 6. Then, consider the maximal open ball B' contained in $\mathrm{lift}(B)$ which is empty of points of $\mathrm{lift}(S_j)$. B' is tangent to $\mathrm{lift}(\partial S_j)$ at two points $\mathrm{lift}(x)$ and $\mathrm{lift}(x')$. It is easy to see that $\mathrm{lift}(\bar{m}_i(x))$ (and also $\mathrm{lift}(\bar{m}_i(x')))$ lies in B', which is itself contained in $\mathrm{lift}(B)$. Since $\mathrm{lift}(B)$ is entirely contained in $\mathrm{lift}(S_i)$, and $\mathrm{lift}(\bar{m}_i(x)) \in B' \subseteq \mathrm{lift}(B)$, at least one of the nearest points of $\mathrm{lift}(\bar{m}_i(x))$ in $\partial\mathcal{C}$ is not in $\mathcal{S}_\mathcal{C}$. This is a contradiction with Lemma 3.

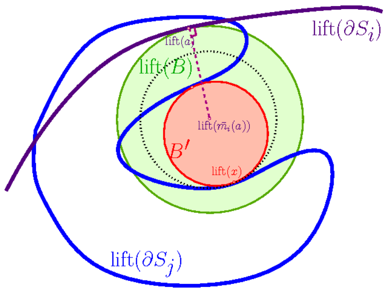

Figure 5: In Lemma 4: to prove that the lift of the segment $[a, \bar{m}_i(a)]$ lies in $\mathrm{lift}(S_i) \cap \mathrm{lift}(S_j)$.

Therefore, $\mathrm{lift}(a)$ is connected to $\mathrm{lift}(\bar{m}_i(a))$ in L_2. Let us call a' and b' the nearest points of $\mathrm{lift}(\bar{m}_i(a))$ in S_i and S_j respectively, see Figure 6-left. According to Lemma 3, the line-segments $[a', \mathrm{lift}(\bar{m}_i(a))]$ and $[b', \mathrm{lift}(\bar{m}_i(a))]$ lie inside $\mathcal{M}_\mathcal{C}$. We now define a path γ as the concatenation of four line-segments: $[a, a'] \subset S_i$, $[a', \mathrm{lift}(\bar{m}_i(a))]$, $[b', \mathrm{lift}(\bar{m}_i(a))]$ and $[b', b] \subset S_j$. We know that $[a', \mathrm{lift}(\bar{m}_i(a))]$ and $[b', \mathrm{lift}(\bar{m}_i(a))]$ are mapped to $\mathrm{lift}(\bar{m}_i(a))$ by the lift function. Thus, the image of γ under the lift function is the line-segment $[\mathrm{lift}(a), \mathrm{lift}(\bar{m}_i(a))]$, which is trivially contractible in $\mathcal{L}_*(\pi_1(\mathcal{M}_\mathcal{C}))$.

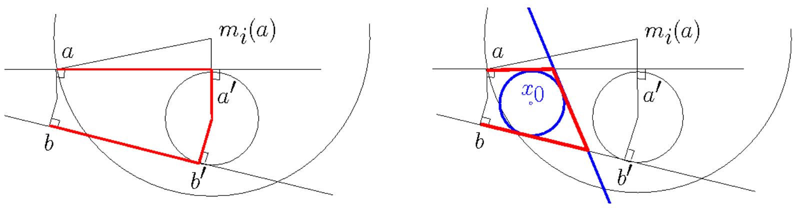

Figure 6: Two cases considered in the proof of Lemma 4: in red a path between a and b in $\mathcal{M}_\mathcal{C}$.

• If the projection of the segment $[a, m_i(a)]$ onto P_i, denoted by $\overline{[a, m_i(a)]}$, is cut by a cutting plane P, then we claim that a is connected to b in $\mathcal{S}_\mathcal{C}$. Let us call l the lift of $\overline{[a, m_i(a)]}$. For any point $x \in l$, we write $B(x)$ for the open ball centered at x and tangent to the planes of sections S_i and S_j. Since $\mathrm{lift}(a) \in \mathrm{lift}(S_i) \cap \mathrm{lift}(S_j)$, $B(\mathrm{lift}(a))$ is empty of points in $\partial\mathcal{C}$. In particular, it does not intersect P. Hence, there exists a point $x_0 \in l$, such that P is tangent to $B(x_0)$. See Figure 6-right. We observe also that P intersects the segments $\overline{[a, m_i(a)]}$ and the corresponding segment in S_j which is also mapped to l by the lift function. In this case, there is a path in $\mathcal{S}_\mathcal{C}$ (and so in $\mathcal{M}_\mathcal{C}$) between a and b so that its image by the lift function is the concatenation of $[\mathrm{lift}(a), x_0]$ with a contractible path from x_0 to x_0. □

9 Deforming the Homotopy Equivalence to a Homeomorphism

Using the homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$, we can show that they are indeed homeomorphic.

Theorem 12 (Main Theorem-Part III) Under the Separation and the Intersection Conditions, the two topological manifolds $\mathcal{R}$ and $\mathcal{O}$ are homeomorphic (in addition, they are isotopic).

Although, this result is stronger than the homotopy equivalence, the way our proof works makes essentially use of the topological study of the previous sections. And so is based on the homotopy equivalence of the previous sections.

Proof Again, we do first a reasoning in each cell of the arrangement and show the existence of a homeomorphism between $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ inducing identity on each section of $\mathcal{S}_\mathcal{C}$. Gluing these homeomorphisms together, one obtains a global homeomorphism between $\mathcal{R}$ and $\mathcal{O}$. Let $\mathcal{C}$ be a cell of the arrangement of the cutting planes. A similar method used to prove the homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ shows that $\partial\mathcal{R} \cap \mathcal{C}$ and $\partial\mathcal{O} \cap \mathcal{C}$ are homotopy equivalent and are indeed homeomorphic, and in addition there exists a homeomorphism $\beta_\mathcal{C} : \partial\mathcal{O} \cap \mathcal{C} \to \partial\mathcal{R} \cap \mathcal{C}$ which induces identity on the boundary of sections in $\mathcal{S}_\mathcal{C}$. We showed that the topology of $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ is completely determined by their fundamental groups, i.e., all the higher homotopy groups of $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ are trivial. Moreover, there is an isomorphism between $\pi_1(\mathcal{O}_\mathcal{C})$ and $\pi_1(\mathcal{R}_\mathcal{C})$, and the induced map $(\beta_\mathcal{C})_* : \pi_1(\partial\mathcal{O} \cap \mathcal{C}) \to \pi_1(\partial\mathcal{R} \cap \mathcal{C})$ on first homotopy groups is consistent with this isomorphism (in the sense that there exists a commutative diagram of first homotopy groups). This shows that there is no obstruction in extending $\beta_\mathcal{C}$ to a map $\alpha_\mathcal{C} : \mathcal{O}_\mathcal{C} \to \mathcal{R}_\mathcal{C}$, inducing the corresponding isomorphism between $\pi_1(\mathcal{O}_\mathcal{C})$ and $\pi_1(\mathcal{R}_\mathcal{C})$, and such that the restriction of $\alpha_\mathcal{C}$ to $\mathcal{S}_\mathcal{C}$ remains identity. Since all the higher homotopy groups of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are trivial, it follows that $\alpha_\mathcal{C}$ is a homotopy equivalence. We are now in order to apply the following theorem due to Waldhausen, which shows that α can be deformed to homeomorphism between $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$, by a deformation which does not change the homeomorphism $\alpha_\mathcal{C}$ between the boundaries. A compact 3-manifold M is called *irreducible* if $\pi_2(M)$ is trivial. Remark that $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are irreducible.

Theorem 13 (Waldhausen) Let $f : M \to M'$ be a homotopy equivalence between orientable irreducible 3-manifolds with boundaries such that f takes the boundary of M onto the boundary of M' homeomorphically. Then f can be deformed to a homeomorphism $M \to M'$ by a homotopy which is fixed all the time on the boundary of M. (See [Mat03], page 220, for a proof.)

Applying Waldhausen's theorem, one obtains a homeomorphism $\tilde{\alpha}_{\mathcal{C}}$ from $\mathcal{O}_{\mathcal{C}}$ to $\mathcal{R}_{\mathcal{C}}$ which is identity on the sections in $\mathcal{S}_{\mathcal{C}}$. Gluing $\tilde{\alpha}_C$, one obtain a global homeomorphism form $\mathcal{O}$ to $\mathcal{R}$. Moreover, according to Chazal and Cohen-Steiner's work [CCS05] (Corollary 3.1), since $\mathcal{R}$ and $\mathcal{O}$ are homeomorphic and $\mathcal{R}$ contains the medial axis of $\mathcal{O}$, $\mathcal{R}$ is isotopic to $\mathcal{O}$. □

Conclusion

We have presented one of the first topological studies in shape reconstruction from cross-sectional data. We showed that the generalization of the classical overlapping criterion to solve the *correspondence problem* between unorganized cross-sections, proposed by Liu et al. in [LBD+08], preserves the homotopy type of the shape under some appropriate sampling conditions. In addition, we proved that in this case, the homotopy equivalence between the reconstructed object and the original shape can be deformed to a homeomorphism. Even, more strongly, the two objects are isotopic. Moreover, we can easily extend the same topological guarantees to a related algorithm based on the Voronoi diagram of the cross-sections (which can be seen as the dual of the Delaunay-based algorithm presented in [BM07]).

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A Homotopy Preliminaries

In this section, we briefly recall some concepts and notations that are used in the paper.

Notation For any set $X \subset R^n$, the boundary of X is denoted be ∂X.

Definition 13 (Homotopy) A *homotopy* between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0,1] \to Y$ such that for all points $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. f is said to be *homotopic* to g if there exists a homotopy between f and g.

Definition 14 (Homotopy Equivalence) Two topological spaces X and Y are *homotopy equivalent* or of the *same homotopy type* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y.

Definition 15 (Homotopy Groups, Fundamental Group) Let X be a space with a base point $x_0 \in X$. Let S^i denote the i-sphere for a given $i \geq 1$, in which we fixed a base point b. The i-dimensional homotopy group of X at the base point x_0, denoted by $\pi_i(X, x_0)$, is defined to be the set of homotopy classes of maps $f : S^i \to X$ that map the base point b to the base point x_0.

Thus if X is path-connected, the group $\pi_i(X, x_0)$ is, up to isomorphism, independent of the choice of base point x_0. In this case the notation $\pi_i(X, x_0)$ is often abbreviated to $\pi_i(X)$.

Let X be a path-connected space. The first homotopy group of X, $\pi_1(X)$, is called the *fundamental group* of X.

Definition 16 (Simply-Connected Spaces) The path-connected space X is called *simply-connected* if it has trivial fundamental group.

Definition 17 (Weak Homotopy Equivalence) A map $f : X \to Y$ is called a weak homotopy equivalence if the induced group homomorphisms by f on the corresponding homotopy groups, $f_* : \pi_i(X) \to \pi_i(Y)$, for $i \geq 0$, are all isomorphism. (It is easy to see that any homotopy equivalence is a weak homotopy equivalence, but the inverse is not necessarily true. However, Whitehead's Theorem states that the inverse is true for maps between CW-complexes.)

Theorem 14 (Whitehead's Theorem) If a map $f : X \to Y$ between connected CW-complexes induces isomorphisms $f_* : \pi_i(X) \to \pi_i(Y)$ for all $i \geq 0$, then f is a homotopy equivalence.

Definition 18 ((Strong) Deformation Retract) Let X be a subspace of Y. A homotopy $H : Y \times [0,1] \to Y$ is said to be a (strong) deformation retract of Y to X if:

- For all $y \in Y$, $H(y,0) = y$ and $H(y,1) \in X$.
- For all $x \in X, H(x,1) = x$.
- (and for all $x \in X, H(x,t) = x$.)

Definition 19 (Homeomorphism) Two topological spaces X and Y are *homeomorphic* if there exists a continuous and bijective map $h : X \to Y$ such that h^{-1} is continuous. the map h is called a homeomorphism from X to Y.

Definition 20 (Isotopy) Two topological spaces X and Y embedded in $\mathbb{R}^3$ are *isotopic* if there exists a continuous map $i : [0,1] \times X \to \mathbb{R}^3$ such that $i(0,.)$ is the identity over X, $i(1, X) = Y$ and for any $t \in [0,1]$, $i(t,.)$ is an homeomorphism from X onto its image. The map i is called an isotopy from X to Y.

Let us recall the definition of the universal cover, and refer to classical books in topology for more details.

Definition 21 (Universal Cover) Let X be a topological space. A *covering space* of X is a space C together with a continuous surjective map $\phi : C \to X$ such that for every $x \in X$, there exists an open neighborhood U of x, such that $\phi^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by ϕ. A connected covering space is called a *universal cover* if it is simply connected.

The universal cover exists and is unique up to homeomorphism.

Lemma 5 (Lifting Property of the Universal Cover) Let X be a (path-) connected topological space and $\widetilde{X}$ be its universal cover, and $\phi : \widetilde{X} \to X$ be the map given by the covering. Let Y be any simply connected space, and $f : Y \to X$ be a continuous map. Given two points $\tilde{x} \in \tilde{X}$ and $y \in Y$ with $\phi(\tilde{x}) = f(y)$, there exists a unique continuous map $g : Y \to \widetilde{X}$ so that $\phi \circ g = f$ and $\phi(y) = \tilde{x}$.This is called the *lifting property* of $\widetilde{X}$.

Since all the spheres S_i of dimension $i \geq 2$ are simply connected, we may easily deduce.

Corollary 1 For any (path-) connected space X with the universal cover $\tilde{X}$, we have $\pi_i(\tilde{X}) = \pi_i(X)$ for all $i \geq 2$.

We now recall the definition of the Hurewicz map $h_i : \pi_i(X) \to H_i(X)$. For an element $[\alpha] \in \pi_i(X)$ presented by $\alpha : S^i \to X$, $h_i([\alpha])$ is defined as the image of the fondumental class of S^i in $H_i(S^i)$ under the map $\alpha_* : H_i(S^i) \to H_i(X)$, i.e., $h_i([\alpha]) = \alpha_*(1)$.

Theorem 15 (Hurewicz Isomorphism Theorem) The first non-trivial homotopy and homology groups of a simply-connected space occur in the same dimension and are isomorphic. In other words, for X simply connected, the Hurewicz map $h_i : \pi_i(X) \to H_i(X)$ is an isomorphism for the first i with π_i (or equivalently H_i) non-trivial.

B Proof of Theorem 3

In this section, we prove that for any connected 3-manifold K in $\mathbb{R}^3$ with a (non-empty) connected boundary, we have $\pi_i(K) = \{0\}$, for all $i \geq 2$. We use the continuity of the boundary of K to show that the two dimensional homotopy group of K is trivial. To this end, we need the following theorem called the Sphere Theorem (for 3-manifolds), see [Bat71] for a proof.

Theorem 16 (Sphere Theorem) Let K be an orientable 3-manifold such that $\pi_2(K)$ is not the trivial group. Then there exists an **embedding** $e : S^2 \to K$ which represents a non-zero element of $\pi_2(K)$.

Using the Sphere theorem, we prove Theorem 3 in two parts.

- We claim that if K is a connected 3-manifold in $\mathbb{R}^3$ with (non-empty) connected boundary, then $\pi_2(K) = \{0\}$.

  For the sake of a contradiction, suppose that $\pi_2(K)$ is non-trivial. According to the Sphere theorem, there exists an embedding $e : S^2 \to K$ which represents a non-zero element of $\pi_2(K)$. The closed surface $e(S^2)$ separates $\mathbb{R}^3$ into two connected components, one bounded (the interior of $e(S^2)$) and the other unbounded (the exterior of $e(S^2)$). Since the boundary of K is connected, the complementary of K, denoted by K^c, is connected. K^c being connected and disjoint from $e(S^2)$, lies in the exterior of $e(S^2)$. Hence, the interior of $e(S^2)$ is contained in K; and e can be extended to the interior of $e(S^2)$ (which is a 2-disk). This contradicts with the fact that e represents a non-zero element of $\pi_2(K)$.

- We now prove that the i-dimensional homotopy groups of K, for $i \geq 3$, are all trivial. Using the dimension of the manifold K, we will be able to prove that the corresponding homology groups are trivial. Then, in order to relate the homology and homotopy groups, one may think of applying Hurewicz Theorem. However, Hurewicz Theorem holds only for simply-connected spaces. Thus, we consider the universal cover of K, and apply Hurewicz Theorem on it. Afterwards, we make use of the relation between K and its universal cover to prove the purposed statement for K.

  Let $\widetilde{K}$ be the universal cover of K. By Corollary 1, it will be enough to show that $\pi_i(\tilde{K}) = \{0\}$ for all i (since $\widetilde{K}$ is simply connected, so we have $\pi_1(\widetilde{K}) = \{0\}$). We already now that $\pi_2(\widetilde{K}) = \pi_2(K) = \{0\}$.

  The three dimensional homology group of any connected non-compact 3-manifold is trivial. As $\widetilde{K} \backslash \partial \widetilde{K}$ (the interior of $\widetilde{K}$) is a non-compact 3-manifold, its three-dimensional homology group is trivial. On the other hand, the homology groups of $\widetilde{K}$ and its interior are the same. Thus, we have $H_3(\widetilde{K}) = H_3(\widetilde{K} \backslash \partial \widetilde{K}) = \{0\}$. By Hurewicz Theorem, we infer that $\pi_3(\widetilde{K}) = \{0\}$ as well. Also, $\widetilde{K}$ being a 3-manifold, all the higher homology groups $H_i(\widetilde{K})$ are trivial, for all $i \geq 4$. Reasoning by induction, again by Hurewicz Theorem, we obtain $\pi_i(\widetilde{K}) = \{0\}$, for all $i \geq 4$. And the theorem follows.

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