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Manifold Reconstruction using Tangential Delaunay Complexes

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Abstract: We give a provably correct algorithm to reconstruct a k-dimensional manifold embedded in d-dimensional Euclidean space. Input to our algorithm is a point sample coming from an unknown manifold. Our approach is based on two main ideas : the notion of tangential Delaunay complex defined in [6, 19, 20], and the technique of sliver removal by weighting the sample points [13]. Differently from previous methods, we do not construct any subdivision of the embedding d-dimensional space. As a result, the running time of our algorithm depends only linearly on the extrinsic dimension d while it depends quadratically on the size of the input sample, and exponentially on the intrinsic dimension k. To the best of our knowledge, this is the first certified algorithm for manifold reconstruction whose complexity depends linearly on the ambient dimension. We also prove that for a dense enough sample the output of our algorithm is isotopic to the manifold and a close geometric approximation of the manifold.

Key-words: Tangential Delaunay complex, manifold learning, manifold reconstruction, sampling conditions, sliver exudation.

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Reconstruction de variétés avec le complexe tangent

Résumé : Nous proposons un algorithme certifié permettant de reconstruire une variété de dimension k plongée dans un espace euclidien de dimension d. L'entrée de l'algorithme est un ensemble fini de points échantillonnant une variété. La sortie est une approximation de cette variété. Notre approche utilise deux idées principales : la notion de complexe tangent et la technique de suppression des slivers par pondération des points de l'échantillon. Au contraire des méthodes développées auparavant, notre algorithme ne construit aucune subdivision de l'espace ambient, ce qui a pour conséquence que sa complexité ne dépend que linéairement de la dimension extrinsèque d; elle dépend de manière quadratique de la taille de l'échantillon et de manière exponentielle de la dimension intrinsèque k. A notre connaissance, c'est le premier algorithme de reconstruction dont la complexité ne dépende pas exponentiellement de la dimension extrinsèque. Nous prouvons également que si l'échantillon est suffisamment dense, la sortie de l'agorithme est une variété triangulée isotope à la variété mesurée.

Mots-clés : Complexe de Delaunay tangent, Apprentissage de variétés, Reconstruction de variétés, Conditions d'échantillonnage, Suppression des slivers

1 Introduction

Manifold reconstruction consists in computing a PL approximation of an unknown manifold $\mathbb{M} \subset \mathbb{R}^d$ from a finite sample of unorganized points $\mathcal{P}$ lying on $\mathbb{M}$ or close to $\mathbb{M}$. When the manifold is a two-dimensional surface embedded in $\mathbb{R}^3$, the problem is known as the surface reconstruction problem. Surface reconstruction is a problem of major practical interest which has been extensively studied in the fields of Computational Geometry, Computer Graphics and Computer Vision. In the last decade, solid foundations have been established and the problem is now pretty well understood. Refer to Dey's book [17], and the survey by Cazals and Giesen in [9] for recent results. The output of those methods is a triangulated surface that approximates $\mathbb{M}$. This triangulated surface is usually extracted from a 3-dimensional of the ambient space (typically a grid or a triangulation). Although rather inoffensive in 3-dimensional space, such data structures depend exponentially on the dimension of the ambient space, and all attempts to extend those geometric approaches to more general manifolds has led to algorithms whose complexities depend exponentially on d [27, 11, 14].

The problem in higher dimensions is also of great practical interest in data analysis and machine learning. In those fields, the general assumption is that, even if the data are represented as points in a very high dimensional space $\mathbb{R}^d$, they in fact live on a manifold of much smaller intrinsic dimension [29]. If the manifold is linear, well-known global techniques like principal component analysis (PCA) or multi-dimensional scaling (MDS) can be efficiently applied. When the manifold is highly nonlinear, several more local techniques have attracted much attention in visual perception and many other areas of science. Among the prominent algorithms are Isomap [30], LLE [28], Laplacian eigenmaps [3], Hessian eigenmaps [18], diffusion maps [24, 26], principal manifolds [31]. Most of those methods reduces to computing an eigendecomposition of some connection matrix. In all cases, the output is a mapping of the original data points into $\mathbb{R}^k$ where k is the estimated intrinsic dimension of $\mathbb{M}$. Those methods come with guarantees only in very restricted cases (if any). For example, Isomap provides a correct embedding only if $\mathbb{M}$ is isometric to a convex open set of $\mathbb{R}^k$. To be able to better approximate the sampled manifold, another route is to extend the work on surface reconstruction and to construct a PL approximation of $\mathbb{M}$ from the sample in such a way that, under appropriate sampling conditions, the quality of the approximation can be guaranteed. First investigations along this line can be found in the work of Cheng, Dey and Ramos [14], and Boissonnat, Guibas and Oudot [7]. In both cases, however, the complexity of the algorithms is exponential in the ambient dimension d, which highly reduces their practical relevance.

In this paper, we extend the geometric techniques developped in small dimensions and propose a way to avoid computing data structures in the ambient space. We assume that $\mathbb{M}$ is a smooth manifold of known dimension k and that we can compute the tangent space to $\mathbb{M}$ at any sample point. Under those conditions, we propose a provably correct algorithm that allows to construct a simplicial complex of dimension k that approximates $\mathbb{M}$. The complexity of the algorithm is linear in d, quadratic in the size n of the sample, and exponential in k. Our work builds on [14] and [7] but dramatically reduces the dependance on d, which is exponential in the cited works. To the best of our knowledge, this is the first certified algorithm for manifold reconstruction whose complexity depends only linearly on the ambient dimension. In the same spirit, Chazal and Oudot [12] have devised an algorithm of intrinsic complexity to solve the easier problem of computing the homology of a manifold from a sample.

Our approach is based on two main ideas : the notion of *tangential Delaunay complex* defined in [20, 6, 19], and the technique of sliver removal by weighting the sample points [13]. The tangential complex is obtained by gluing local (Delaunay) triangulations around each sample point. The tangential complex is a subcomplex of the d-dimensional Delaunay triangulation of the sample points but it can be computed using mostly operations in the k-dimensional tangent spaces at the sample points. Hence the dependence on k rather than d in the complexity. However, due to the presence of so-called inconsistencies, the local triangulations may not form a triangulated manifold. Although this problem has been reported [20], no solution was known except for the case of curves ($k = 1$) [19]. We show that we can remove inconsistencies by weighting the sample points under appropriate sample conditions. We can then prove that the approximation returned by our algorithm is isotopic to $\mathbb{M}$, and a close geometric approximation of $\mathbb{M}$.

Our algorithm can be seen as a *local* version of the cocone algorithm of Cheng et al. [14]. By local, we mean that we do not compute any d-dimensional data structure like a grid or a triangulation of the ambient space. Still, the tangential complex is a subcomplex of the d-dimensional Delaunay triangulation of the data points and therefore implicitly relies on a global partition of the ambient space. This is key to our analysis and makes our method depart from other local algorithms that have been proposed in the surface reconstruction literature [16, 22]. We can also forsee applications of the tangential complex and of our construction each time computations in the tangent space of a manifold are required, e.g. for dimensionality reduction and approximating the Laplace Beltrami operator [4].

Notations. In the rest of the paper, we assume that $\mathbb{M}$ is a smooth manifold of dimension k embedded in $\mathbb{R}^d$. We call $\mathcal{P} = \{p_1, \ldots, p_n\}$ a finite sample of points from $\mathbb{M}$. We denote by T_p the k-dimensional tangent space at point $p \in \mathbb{M}$.

We write $B(c, r)$ for the d-dimensional ball centered at c of radius r.

We define the angle between two vector spaces U and V as

$$\angle UV = \max_{u \in U} \min_{v \in V} \angle uv.$$

The simplex $\tau = \text{conv}(p_1, \ldots, p_i)$ is also denoted by $[p_1, \ldots, p_i]$ and we identify τ with the set of its vertices when there is no ambiguity. Hence, we write $p_j \in \tau$ if p_j is a vertex of τ. The affine hull of a τ is denoted by $\text{aff}(\tau)$, and τ is a j-simplex if j is the dimension of $\text{aff}(\tau)$. The normal space of $\text{aff}(\tau)$ is denoted by N_τ.

Radius and center of the smallest sphere encircling a simplex τ is denoted by R_τ and c_τ respectively .

2 Definitions and preliminaries

2.1 Weighted Delaunay triangulation

Weighted points. A weighted point is a pair consisting of a point p of $\mathbb{R}^d$, called the *center* of the weighted point, and a non-negative real number $\omega(p)$, called the *weight* of the weighted point. It might be convenient to visualize the weighted point $(p, \omega(p))$ as the sphere[1] centered at p of radius $\omega(p)$.

Two weighted points (or spheres) $(p, \omega(p))$ and $(q, \omega(q))$ are called *orthogonal* when $\|p-q\|^2 = \omega(p)^2 + \omega(q)^2$, *further than orthogonal* when $\|p-q\|^2 > \omega(p)^2 + \omega(q)^2$, and *closer than orthogonal* when $\|p-q\|^2 < \omega(p)^2 + \omega(q)^2$.

Given a point set $\mathcal{P} = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$, a *weight function* on $\mathcal{P}$ is a non-negative real-valued function $\omega : \mathcal{P} \to [0, \infty)$. Write $p_i^\omega = (p_i, \omega(p_i))$ and $\mathcal{P}^\omega = \{p_1^\omega, \ldots, p_n^\omega\}$.

We define the *relative amplitude* of ω, denoted as $\tilde{\omega}$, as $\max_{p \in \mathcal{P}, q \in \mathcal{P} \setminus \{p\}} \frac{\omega(p)}{||p-q||}$. In the paper, we assume that $\tilde{\omega} \leq \omega_0 < 1/2$, for some constant ω_0 to be fixed later.

Given a subset τ of $d+1$ weighted points whose centers are affinely independent, there exists a unique sphere orthogonal to the weighted points of τ. The sphere is called the *orthosphere* of τ and its center and radius are called the orthocenter and the orthoradius of τ. If τ is a j-sliver, $j < d$, the orthosphere of τ is the minimal sphere that is orthogonal to the (weighted) vertices of τ. Plainly, its center o_τ lies in $\mathrm{aff}(\tau)$. Radius of the orthosphere of τ is denoted by R'_τ.

A finite set of weighted points $\mathcal{P}^\omega$ is said to be in *general position* if there exists no sphere orthogonal to $d+2$ weighted points of $\mathcal{P}^\omega$.

Weighted Voronoi diagram and Delaunay triangulation. Let ω be a weight function defined over $\mathcal{P}$. We define the weighted Voronoi cell of $p \in \mathcal{P}$ as

$$\mathrm{Vor}^\omega(p) = \{x \in \mathbb{R}^d : ||p-x||^2 - \omega^2(p) \leq ||q-x||^2 - \omega^2(q), \forall\, q \in \mathcal{P}\}.$$

The weighted Voronoi cells and their k-dimensional faces, $0 \leq k \leq d$, form a cell complex, called the weighted Voronoi diagram of $\mathcal{P}$, that decomposes $\mathbb{R}^d$ into convex polyhedral cells.

Let τ be a subset of points of $\mathcal{P}$ and write $\mathrm{Vor}^\omega(\tau) = \cap_{x \in \tau} \mathrm{Vor}^\omega(x)$. If the points of $\mathcal{P}$ are in general position, $\mathrm{Vor}^\omega(\tau) = \emptyset$ when $|\tau| > d+1$. The collection of all simplices $\mathrm{conv}(\tau)$ such that $\mathrm{Vor}^\omega(\tau) \neq \emptyset$ constitutes the weighted Delaunay triangulation $\mathrm{Del}^\omega(\mathcal{P})$. The mapping that associates to the face $\mathrm{Vor}^\omega(\tau)$ of $\mathrm{Vor}^\omega(\mathcal{P})$ the face $\mathrm{conv}(\tau)$ of $\mathrm{Del}^\omega(\mathcal{P})$ is a *duality*, i.e. a bijection that reverses the inclusion relation. For simplicity, we will use in the sequel the same notation τ to denote the simplex $\mathrm{conv}(\tau)$ and the set of its vertices.

Alternatively, a d-simplex τ is in $\mathrm{Del}^\omega(\mathcal{P})$ if the orthosphere of τ is further than orthogonal from all weighted points in $\mathcal{P}^\omega \setminus \{\tau^\omega\}$.

The weighted Delaunay triangulation of a set of weighted points can be computed efficiently in small dimensions and has found many applications, see e.g. [14, 7]. In this paper, we use weighted Delaunay triangulations for two main reasons. The first one is that the restriction of a d-dimensional weighted

[1] A hypersphere is simply called a sphere when there is no ambiguity.

Voronoi diagram to an affine space of dimension k is a k-dimensional weighted Voronoi diagram that can be computed without computing the d-dimensional diagram (see Lemma 1). The other main reason is that some flat simplices named slivers can be removed from a Delaunay triangulation by weighting the vertices (see [14, 7, 13] and Subsections 2.3 and 4.1).

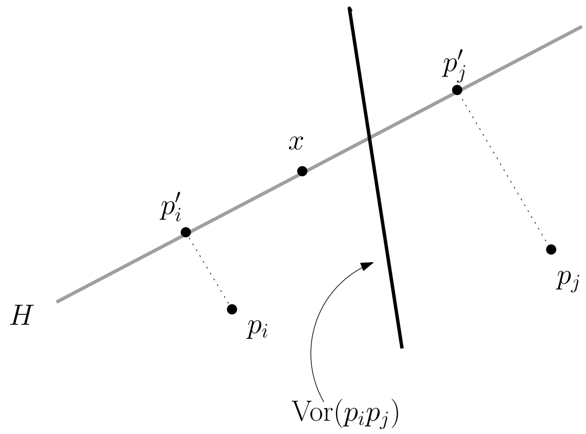

Figure 1: Refer to Lemma 1. The red line denotes the k-dimensional plane H and the black line denotes $\mathrm{Vor}^{\omega}(p_i p_j)$.

Lemma 1 *Let H be a k-dimensional affine space of $\mathbb{R}^d$. The restriction of the weighted Voronoi diagram of $\mathcal{P}$ to H is the k-dimensional weighted Voronoi diagram of $\mathcal{P}'$ where $\mathcal{P}'$ is the orthogonal projection of $\mathcal{P}$ onto H and the squared weight of p'_i is $\omega^2(p_i) - \|p_i - p'_i\|^2$.*

Proof. By Pythagoras theorem, we have $\forall x \in H \cap \mathrm{Vor}^{\omega}(p_i)$, $\|x - p_i\|^2 - \omega^2(p_i) \leq \|x - p_j\|^2 - \omega^2(p_j) \Leftrightarrow \|x - p'_i\|^2 + \|p_i - p'_i\|^2 - \omega^2(p_i) \leq \|x - p'_j\|^2 + \|p_j - p'_j\|^2 - \omega^2(p_j)$, where p'_i denotes the orthogonal projection of $p_i \in \mathcal{P}$ onto H. Hence the restriction of $\mathrm{Vor}^{\omega}(\mathcal{P})$ to H is the weighted Voronoi diagram of the weighted points $(p'_i, \omega_i) \in H$ where $\omega_i^2 = -\|p_i - p'_i\|^2 + \omega^2(p_i)$. $\square$

2.2 Sampling conditions

Local feature size. The *medial axis* of $\mathbb{M}$ is the closure of the sets of points of $\mathbb{R}^d$ that have more than one nearest neighbor on $\mathbb{M}$. The *local feature size* of $x \in \mathbb{M}$, $\mathrm{lfs}(x)$, is the distance of x to the medial axis of $\mathbb{M}$. As is well known and can be easily proved, lfs is *Lipschitz continuous* i.e, $\mathrm{lfs}(x) \leq \mathrm{lfs}(y) + \|x - y\|$.

(ε, δ)-sample. The point sample $\mathcal{P}$ is said to be a (ε, δ)-sample (where $0 < \delta < \varepsilon < 1$) if (1) for any point $x \in \mathbb{M}$ there exist a point $p \in \mathcal{P}$ such that $||x-p|| \leq \varepsilon\, \mathrm{lfs}(x)$, and (2) for any two distinct points $p, q \in \mathcal{P}$, $||p-q|| \geq \delta\, \mathrm{lfs}(p)$.[2] The ratio ε/δ is called the *sparsity ratio* of $\mathcal{P}$.

We will use the following results from [21]. We write l_p for the distance between $p \in \mathcal{P}$ and its nearest neighbor in $\mathcal{P} \setminus \{p\}$.

[2] Observe that the sparsity condition (2) is mandatory if one wants to infer the dimension of $\mathbb{M}$ from a sample [21].

Lemma 2 ((ε, δ)−SAMPLING PROPERTIES) *Given an (ε, δ)-sample $\mathcal{P}$ of $\mathbb{M}$, we have*

1. $\delta \, \mathrm{lfs}(p) \leq l_p \leq \frac{2\varepsilon}{1-\varepsilon} \mathrm{lfs}(p)$.
2. *For any two points $p, q \in \mathbb{M}$ such that $||p - q|| = t \, \mathrm{lfs}(p)$, $0 < t < 1$, $\sin \angle(pq, T_p) \leq t/2$.*
3. *Let p be a point in $\mathbb{M}$. Let x be a point in T_p such that $||p - x|| \leq t \, \mathrm{lfs}(p)$ for some $0 < t \leq 1/4$. Let x' be the point on $\mathbb{M}$ closest to x. Then $||x - x'|| \leq 2t^2 \, \mathrm{lfs}(p)$.*

2.3 Slivers and good simplices

Consider a j-simplex τ, where $1 \leq j \leq k$. We denote by R_τ, L_τ, V_τ and $\rho(\tau) = R_\tau / L_\tau$ the circumradius, the shortest edge length, the volume, and the radius-edge ratio of τ respectively. The property of a simplex to have a good radius edge ratio is measured in terms of parameter ρ_0, i.e a simplex τ has a good radius-edge ratio implies $\rho(\tau) \geq \rho_0$. We define $\sigma(\tau) = V_\tau / L_\tau^j$, as the *quality measure* of τ. The orthocenter of τ is denoted by o_τ and its orthoradius by R'_τ.

If $p \in \tau$, we define $\tau_p = \tau \setminus \{p\}$ to be the $(j-1)$-face of τ opposite to p. Assuming there is no ambiguity on τ, we also write D_p for the distance from p to the affine hull of τ_p, and H_p for the distance from o_τ to $\mathrm{aff}(\tau_p)$.

Lemma 3 ([13]) *Let τ be a simplex of $\mathrm{Del}^\omega(\mathcal{P})$. Let p be any vertex of τ and write $H(\omega(p))$ (instead of H_p) for the signed distance of the orthocenter of τ to $\mathrm{aff}(\tau_p)$ parametrized by the weight of p. We have $H(\omega(p)) = H(0) - \frac{\omega^2(p)}{2D_p}$.*

Slivers are a special type of flat simplices. The property of being a sliver is measured in terms of a parameter σ_0, called the *sliverity bound*, to be fixed later in Section 4.

Slivers are defined by induction on the dimension as in [25]: (1) a simplex of dimension less than 3 is not a sliver, and (2) for $j \geq 3$, a j-simplex τ is a j-sliver if none of its boundary simplices is a sliver and $\sigma(\tau) < \sigma_0^j$. We have the following result for j-slivers.

Lemma 4 *If τ is a j-sliver then $D_p < j\sigma_0 L_\tau$ for all vertices $p \in \tau$.*

Proof. The volume of τ is $V_\tau = V_{\tau_p} \, . \, D_p/j = \sigma(\tau_p) L_{\tau_p}^{j-1} \, . \, D_p/j$ and it is also equal to $\sigma(\tau) L_\tau^j$. Since τ is a j-sliver we have $\sigma(\tau) < \sigma_0^j$ and $\sigma(\tau_p) \geq \sigma_0^{j-1}$. Therefore we get

$$D_p = j \, \frac{\sigma(\tau)}{\sigma(\tau_p)} \times \frac{L_\tau^j}{L_{\tau_p}^{j-1}} < j\sigma_0 L_\tau .$$

□

A simplex τ is called a *good simplex* if the radius-edge ratio of $\tau \geq \rho_0$ and τ nor its subsimplices are slivers.

Lemma 5 (NORMAL APPROXIMATION) *Let τ be a good j-simplex for $j \leq k$ with vertices on a k-dimensional smooth manifold $\mathbb{M}$, and $p \in \tau$ s.t. the lengths of the edges of τ that are incident to p are less than $b\varepsilon \mathrm{lfs}(p)$ for $b\varepsilon < 1/4$. Then, for any normal vector n_p of $\mathbb{M}$ at p, τ has a normal n_τ such that $\angle n_p n_\tau \leq a_j \varepsilon$, where a_j depends on j, σ_0, ρ_0 and b.*

Proof. The proof is a variant of a proof of [14]. The proof is by induction on j. For $j = 0$, the claim is trivial and $a_0 = 0$. Let τ be a j-simplex, p and q two vertices of τ. We write τ_q for the $(j-1)$-simplex $\tau \setminus \{q\}$, n_p for a normal vector to $\mathbb{M}$ at p, and q' for the orthogonal projection of q onto $\text{aff}(\tau_q)$.

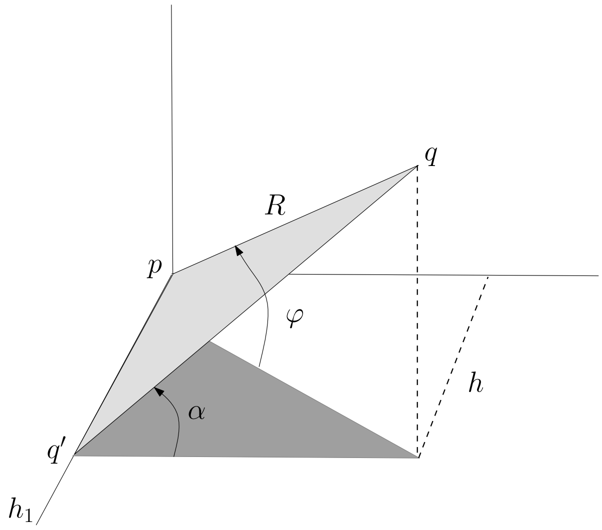

Figure 2: For the proof of Lemma 5.

We assume (induction hypothesis) that τ_q has a normal n_{τ_q} s.t $\sin \angle(n_{\tau_q}, n_p) \leq a_{j-1}\varepsilon$. Let h be the $(d-1)$-hyperplane containing τ_q and normal to n_{τ_q} (Refer to Fig. 7). We denote by h_1 the $(d-2)$-hyperplane of h orthogonal to qq'. Assume that $q \notin h$. Moreover, h_1 contains τ_q since τ_q is contained in h and is normal to qq'. Consider now the hyperplane h_2 that contains h_1 and q. Observe that h_2 contains τ and write n_τ for its normal.

We want to bound $\sin \angle(n_\tau, n_p) \leq \sin \angle(n_{\tau_q}, n_p) + \sin \angle(n_{\tau_q}, n_\tau)$. The first term is bounded by the induction hypothesis. Let us bound the second one. Writing $\alpha = \angle(n_{\tau_q}, n_\tau)$ and ϕ the angle between pq and h (see Fig. 7), we have

$$\sin\alpha = \frac{||p-q||}{||q-q'||}\sin\phi = \frac{||p-q||}{D_q}\sin\phi \leq \frac{2R_\tau}{D_q}\sin\phi \leq \frac{2\rho_0 L_\tau}{D_q}\sin\phi \leq \frac{2^j}{j!\,\rho_0^{j-2}\sigma_0^j}\sin\phi,$$

as $D_p \geq \frac{j!\,\rho(\tau)^{j-1}\sigma(\tau)}{2^{j-1}}$ by Lemma 30, $L_\tau, \rho(\tau) \geq \rho_0$ and $\sigma(\tau) \geq \sigma_0^j$.

Since $\phi \leq \angle(n_{\tau_q}, n_p) + \angle(T_p, pq)$, we have using the induction hypothesis and Lemma 2 (2)

$$\sin\phi \leq \sin\angle(n_{\tau_q}, n_p) + \sin\angle(T_p, pq) \leq \left(a_{j-1} + \frac{b}{2}\right)\varepsilon.$$

We conclude that

$$\sin\angle(n_\tau, n_p) \leq \frac{2^j}{j!\,\rho_0^{j-2}\sigma_0^j}\left(a_{j-1} + \frac{b}{2}\right)\varepsilon + a_{j-1}\varepsilon \stackrel{\text{def}}{=} a_j\varepsilon.$$

□

2.4 Tangential Delaunay complex and inconsistent configurations

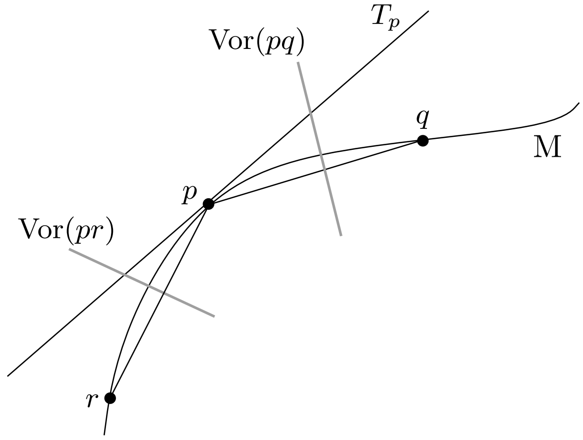

Figure 3: $[pq]$ and $[pr]$ are edges of the star of p in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ since their dual Voronoi edges intersect the tangent space T_p at p.

Let $\mathrm{Del}^{\omega}_{p_i}(\mathcal{P})$ be the weighted Delaunay triangulation of $\mathcal{P}$ restricted to the tangent space T_{p_i}. Equivalently, the simplices of $\mathrm{Del}^{\omega}_{p_i}(\mathcal{P})$ are the simplices of $\mathrm{Del}^{\omega}(\mathcal{P})$ whose Voronoi dual faces intersect T_{p_i}, i.e. $\tau \in \mathrm{Del}^{\omega}_{p_i}(\mathcal{P})$ iff $\mathrm{Vor}^{\omega}(\tau) \cap T_{p_i} \neq \emptyset$. Observe that $\mathrm{Del}^{\omega}_{p_i}(\mathcal{P})$ is in general a k-dimensional triangulation. Since this situation can always be ensured by applying some infinitesimal perturbation on $\mathcal{P}$, we will assume, in the rest of the paper, that the points of $\mathcal{P}$ are in *general position*, meaning that all $\mathrm{Del}^{\omega}_{p_i}(\mathcal{P})$ are k-dimensional triangulations.[3] Finally, write star(p_i) for the star of p_i in $\mathrm{Del}^{\omega}_{p_i}(\mathcal{P})$, i.e. the set of simplices that are incident to p_i in $\mathrm{Del}^{\omega}_{p_i}(\mathcal{P})$.

We call *tangential Delaunay complex* or tangential complex for short, the simplicial complex $\{\tau, \tau \in \mathrm{star}(p), p \in \mathcal{P}\}$. We denoted it by $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$. By our general position assumption, $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is a k-dimensional complex contained in $\mathrm{Del}^{\omega}(\mathcal{P})$.

By duality, computing star(p_i) is equivalent to computing the restriction of the (weighted) Voronoi cell of p_i to T_{p_i}, which, by Lemma 1, reduces to computing a cell in a k-dimensional weighted Voronoi diagram embedded in T_{p_i}. It follows that the tangential complex can be computed without constructing any data structure of dimension higher than k, the intrinsic dimension of $\mathbb{M}$.

The tangential Delaunay complex is *not* in general a triangulated manifold and therefore not a good approximation of $\mathbb{M}$. This is due to the presence of so-called inconsistencies. Consider a k-simplex τ of $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ with two vertices p_i and p_j such that τ is in star(p_i) but not in star(p_j). We write $B_i(\tau)$ for the open ball centered on T_{p_i} that is orthogonal to the (weighted) vertices of τ^{ω}, and denote by c_{p_i} and r_{p_i} its center and its radius. According to our definition, τ is inconsistent iff $B_i(\tau)$ is further than orthogonal from all weighted points in $\mathcal{P}^{\omega} \setminus \tau^{\omega}$ while there exists a weighted point of $\mathcal{P}^{\omega} \setminus \tau^{\omega}$, say p_l^{ω}, that is closer

[3] We will take care of the general position assumption in Section 7.2.

than orthogonal from $B_j(\tau)$. We deduce from the above discussion that the line segment $[c_i c_j]$ has to penetrate the interior of $\mathrm{Vor}^\omega(p_l)$.

We formally define an inconsistent configuration as follows.

Definition 1 (Inconsistent configuration) *$\phi = [p_1, \ldots, p_{k+2}]$ is called an inconsistent configuration of $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$ witnessed by p_i, p_j, p_l if*

- *The k-simplex $\tau = \phi \setminus \{p_l\}$ is in $\mathrm{star}(p_i)$ but not in $\mathrm{star}(p_j)$.*
- *τ is a good simplex.*
- *$\mathrm{Vor}^\omega(p_l)$ is the first cell of $\mathrm{Vor}^\omega(\mathcal{P})$ whose interior is intersected by $[c_i c_j]$, where $c_i = T_{p_i} \cap \mathrm{Vor}^\omega(\tau)$ and $c_j = T_{p_j} \cap \mathrm{aff}(\mathrm{Vor}^\omega(\tau))$, and $[c_i c_j]$ is oriented from c_i to c_j.*

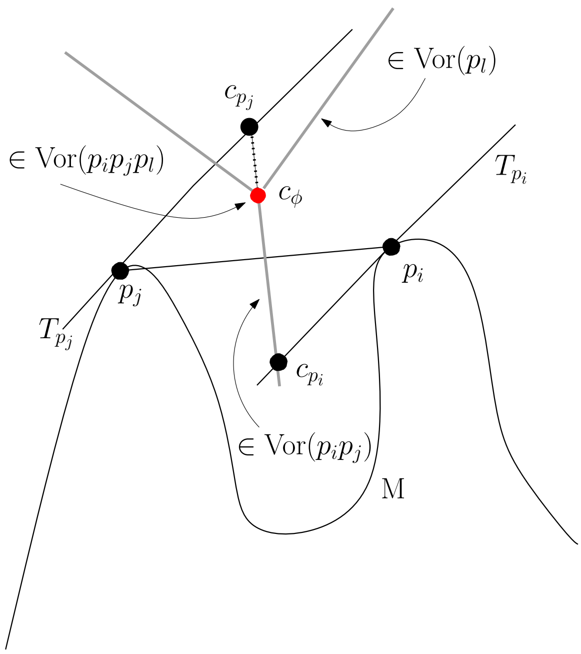

Figure 4: An inconsistent configuration in the unweighted case. Edge $[p_i p_j]$ is in $\mathrm{Del}_{p_i}(\mathcal{P})$ but not in $\mathrm{Del}_{p_j}(\mathcal{P})$ since $\mathrm{Vor}(p_i p_j)$ intersects T_{p_i} but not T_{p_j}. This happens because $[c_{p_i} c_{p_j}]$ penetrates (at c_ϕ) the Voronoi cell of a point $p_l \neq p_i, p_j$, therefore creating an inconsistent configuration $\phi = [p_i, p_j, p_l]$.

For an inconsistent configuration ϕ, we denote by c_ϕ an itersection point as defined in the definition. Note that c_ϕ is the center of a sphere that is orthogonal to the weighted vertices of τ and also to p_l, and further than orthogonal from all the other weighted points of $\mathcal{P}^\omega$. Equivalenty, c_ϕ is the point on $[c_{p_i} c_{p_j}]$ that belongs to $\mathrm{Vor}^\omega(\phi)$. We call c_ϕ a *witness center* of ϕ.

Hence, an inconsistent configuration is a $(k+1)$-simplex of $\mathrm{Del}^\omega(\mathcal{P})$. However, the subfaces of an inconsistent configuration may not belong to the tangential complex. We write $\mathrm{Faces}^\omega(\mathcal{P})$ for the set of all k-subfaces of the inconsistent configurations of $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$.

3 Structural results

In the rest of the paper, we will make the following hypothesis

Hypothesis 1 *$\mathcal{P}$ is an (ε, δ)-sample of $\mathbb{M}$ where $\varepsilon < 0.09$ and whose sparsity ratio ε/δ is at most some positive constant c_0. We assume further that $\tilde{\omega} \leq \omega_0$ where $\tilde{\omega}$ is the relative amplitude of the weight assignement ω and ω_0 is a positive constant less than $1/2$.*

3.1 Properties of the simplices of the tangential Delaunay complex

We state now two lemmas which are slight variants of results of [14]. The proofs are given in Appendix C.

Lemma 6 *For all $x \in T_p \cap \mathrm{Vor}^\omega(p)$, $||p - x|| \leq c_1 \varepsilon \mathrm{lfs}(p)$ if $c_1 > 2$ and $\varepsilon < \frac{1}{2c_1}(-c_1 - 1 + \sqrt{c_1^2 + 6c_1 - 7})$. In particular we can take $c_1 = 4.41$ and $\varepsilon < 0.09$.*

Lemma 7 *Let $c_2 = c_1(1 + 1/\sqrt{1 - 4\omega_0^2})$ (note that $c_2 > 2c_1$). For ε sufficiently small*

1. *If pq is an edge of $\mathrm{Del}_p^\omega(\mathcal{P})$, then $||p - q|| \leq c_2 \varepsilon \mathrm{lfs}(p)$.*
2. *Assume that $3c_2\varepsilon < 1$. If pq is an edge of $\mathrm{Del}_{T\mathbb{M}}^\omega(\mathcal{P})$, then $||p - q|| \leq 3c_2\varepsilon \mathrm{lfs}(p)$.*
3. *Let τ be a simplex in $\mathrm{Del}_{T\mathbb{M}}^\omega(\mathcal{P})$ and R'_τ be the orthoradius of τ. Then $R'_\tau \leq \beta_0 L_\tau$ and radius-edge ratio of τ is β'_0, where $\beta_0 = \frac{3c_1\varepsilon}{2\delta}$ and $\beta'_0 = \frac{\beta_0}{\sqrt{1-4\omega_0^2}}$.*

3.2 Properties of inconsistent configurations

Refer to Appendix C for the proofs of the lemmas. We only give below the proof of Lemma 8 which can serve as an illustrative example.

Lemma 8 *Let $c_4 = \frac{c_1}{\sqrt{1-4\omega_0^2}}$, $c_5 = \frac{2c_4}{\sqrt{1-4\omega_0^2}}$ and assume that $\min\{12c_2, 2a_k\}\varepsilon < 1$. Let $\phi = [p_1, \ldots, p_{k+2}]$ be an inconsistent configuration witnessed by p_i, p_j, and p_l. Then 1. $dist(c_\phi, \tau) \leq 2a_k c_4 \varepsilon^2 \mathrm{lfs}(p_i)$, 2. $\|p_i - c_\phi\| \leq 2c_4 \varepsilon \mathrm{lfs}(p_i)$, and 3. $\|x - c_\phi\| \leq c_5 \varepsilon \mathrm{lfs}(p_i)$ for all vertices x of ϕ.*

Proof. From the definition of inconsistent configurations, $\tau = \phi \setminus \{p_l\}$ belongs to $\mathrm{Del}_{p_i}^\omega(\mathcal{P})$. By Lemma 6, $\|p_i - c_{p_i}\| \leq c_1 \varepsilon \mathrm{lfs}(p_i)$ and, by Lemma 5, there exists a constant a_k such that $\angle(\mathrm{aff}(\tau), T_{p_i}) \leq a_k \varepsilon$, which implies that $\tan^2 \angle(\mathrm{aff}(\tau), T_{p_i}) \leq \frac{a_k^2\varepsilon^2}{1 - a_k^2\varepsilon^2} < 4a_k^2\varepsilon^2$. Observing that $||p_i - o_\tau|| \leq ||p_i - c_{p_i}||$ since o_τ is the closest point to p_i in $\mathrm{aff}(\mathrm{Vor}^\omega(\tau))$, we then deduce

$$\|c_{p_i} - o_\tau\| \leq \|c_{p_i} - p_i\| \sin \angle(\mathrm{aff}(\tau), T_{p_i}) \leq a_k c_1 \varepsilon^2 \mathrm{lfs}(p_i).$$

We bound now $\|p_j - c_{p_i}\|$. From Lemma 31 (Appendix B), we have

$$\|p_j - c_{p_i}\| \leq \frac{\|p_i - c_{p_i}\|}{\sqrt{1 - 4\omega_0^2}} \leq \frac{c_1 \varepsilon \, \mathrm{lfs}(p_i)}{\sqrt{1 - 4\omega_0^2}}.$$

Moreover, $\sin\angle(\text{aff}(\tau), T_{p_j}) \leq a_k\varepsilon$ (Lemma 5), and $||p_j - o_\tau|| \leq ||p_j - c_{p_i}||$ as o_τ is the closest point to p_j in $\text{aff}(\text{Vor}^\omega(\tau))$. Hence we have,

$$\|c_{p_j} - o_\tau\| \leq \|p_j - o_\tau\| \tan\angle(\text{aff}(\tau), T_{p_j}) < \frac{2a_k c_1 \varepsilon^2}{\sqrt{1-4\omega_0^2}} \text{lfs}(p_i)$$

As $c_\phi \in [c_{p_i} c_{p_j}]$, we conclude that $\|o_\tau - c_\phi\| \leq \frac{2a_k c_1 \varepsilon^2}{\sqrt{1-4\omega_0^2}} \text{lfs}(p_i) = 2a_k c_4 \varepsilon^2 \text{lfs}(p_i)$. Therefore

$$\begin{aligned}
\|p_i - c_\phi\| &\leq \|p_i - o_\tau\| + \|o_\tau - c_\phi\| \\
&\leq \|p_i - c_{p_i}\| + \|o_\tau - c_\phi\| \quad (\text{as } \|p_i - o_\tau\| \leq \|p_i - c_\phi\|) \\
&\leq c_1\varepsilon\text{lfs}(p_i) + 2a_k c_4 \varepsilon^2 \text{lfs}(p_i) \leq 2c_4\varepsilon\text{lfs}(p_i) \quad (\text{as } 2a_k\varepsilon < 1 \text{ and } c_4 \geq c_1)
\end{aligned}$$

From Lemma 31, we have $\|x - c_\phi\| \leq \frac{\|p_i - c_\phi\|}{\sqrt{1-4\omega_0^2}}$ for all vertices x of ϕ. □

The following lemma is the analogous of Lemma 7 for simplices of $\text{Faces}^\omega(\mathcal{P})$ (which are not necessarily simplices of $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$).

Lemma 9 *Assume that* $\max\{2a_k, 16c_5\}\varepsilon \leq 1$. *Let τ be a j-dimensional simplex of* $\text{Faces}^\omega(\mathcal{P})$ *incident to p.*

1. *If pq is an edge of* $\text{Faces}^\omega(\mathcal{P})$, *then* $||p - q|| \leq 4c_5\varepsilon\text{lfs}(p)$.
2. *Let R'_τ denotes the radius of the smallest orthosphere of τ. Then $R'_\tau \leq \beta_1 L_\tau$ and the radius-edge ratio of τ is β'_1, where $\beta_1 = \left(\frac{5c_4\varepsilon}{2\delta}\right)$ and $\beta'_1 = \frac{\beta_1}{\sqrt{1-4\omega_0^2}}$.*

Lemma 10 *Assume* $\max\{2a_k,\, 16c_5\}\varepsilon < 1$. *Let τ be a j-simplex in* $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P}) \cup \text{Faces}^\omega(\mathcal{P})$. *For all vertices x of τ, the distance between the orthocenter of τ and* $\text{aff}(\tau_x)$ *is at most* $c_6\varepsilon\text{lfs}(x)$, *where* $c_6 = c_5\,(10 + 8\beta_1 + 8\omega_0)$.

The following lemma is analogous to Lemma 10. Instead of simplices in $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P}) \cup \text{Faces}^\omega(\mathcal{P})$, we consider inconsistent configurations with good faces.

Lemma 11 *Assume that* $\max\{16c_5,\, 2a_k\}\,\varepsilon < 1$. *Let* $\phi = [p_1, \ldots, p_{k+2}]$ *be an inconsistent configuration s.t. all k-subfaces are good simplices. Then the distance between c_ϕ of ϕ to the affine hull of any k-subface of ϕ is less than* $4(c_1 + c_2 + c_4)a_k\varepsilon^2\text{lfs}(v)$ *for any vertex v of ϕ.*

The following lemma is the analogous of Lemma 4 for inconsistent configurations, and generalizes Lemma 8. It shows that an inconsistent configuration ϕ has a small D_p for all vertices p of ϕ and therefore behave like slivers. Moreover the inconsistent configurations lie close to $\mathbb{M}$.

Lemma 12 *Assume that* $\max\{16c_5,\, 2a_k\}\,\varepsilon < 1$. *Let* $\phi = [p_1, \ldots, p_{k+2}]$ *be an inconsistent configuration whose k-subfaces are good simplices. Then the distance D_{p_i} between p_i and the k-subface ϕ_{p_i} opposite to p_i is less than* $4(2c_5 + a_k)c_5\varepsilon^2\text{lfs}(p_i)$.

Lemma 9 shows that, in order to construct star(p) and search for inconsistencies involving p, it is enough to consider the points of $\mathcal{P}$ that lie in ball $B_p = B(p, 4c_5\varepsilon \mathrm{lfs}(p))$. Since ε and lfs(p) are not known in practice, we will consider instead the ball $B'_p = B(p, 4c_5c_0\, l_p)$ where $l_x = \min_{q\in\mathcal{P}, q\neq x} \|x - q\|$. It is easily seen that $l_x : \mathbb{M} \to \mathbb{R}$ is 1-Lipschitz and, by Lemma 2, we have $\delta \mathrm{lfs}(p) \leq l_p \leq \frac{2\varepsilon}{1-\varepsilon} \mathrm{lfs}(p)$. It follows that B'_p contains B_p if $\varepsilon/\delta \leq c_0$. We call $LN_p = B'_p \cap \mathcal{P}$ the *local neighborhood* of p. The sparsity of $\mathcal{P}$, Corollary 1 (in Appendix A) and a packing argument imply the following lemma.

Lemma 13 *If* $\max(2,\, 32c_5c_0,\, 2a_k,\, 16c_5)\,\varepsilon \leq 1$*, the number of points of* LN_p *is less than a cosntant* $N = (128c_5c_0 + 2)^k\, c_0^k = 2^{O(k)}$.

3.3 Range of weights for slivers and inconsistencies

In the rest of this section, we assume that the weights of all the points except one, say p, are fixed. The following two lemmas bound the measure of the set of squared weights $\omega^2(p)$ that may create slivers incident to p in $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P}) \cup \mathrm{Faces}^\omega(\mathcal{P})$ or inconsistent configurations incident to p.

Definition 2 (Sliverity range) *Let* ω *be a weight assignment of relative amplitude at most* ω_0 *we keep fixed except for* $\omega(p)$*. The sliverity range of a simplex* τ *incident on a point* $p \in \mathcal{P}$ *is the measure of the set of all squared weights* $\omega^2(p)$ *for which* τ *is a sliver of* $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$ *or* $\mathrm{Faces}^\omega(\mathcal{P})$.

Lemma 14 (Sliverity range) *Assume that* $\max\{2a_k,\, 16c_5\}\varepsilon < 1$*. Let* τ *be a* j*-sliver in* $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P}) \cup \mathrm{Faces}^\omega(\mathcal{P})$ *incident on* p*. Then the sliverity range of* τ *w.r.t* p *is at most* $\Delta_s(j) = A_j\sigma_0\, l_p^2$ *with* $A_j = 16jc_5c_6c_0^2$.

Proof. Let $\omega(p)$ be the weight of p and let $H(\omega(p))$ be the signed distance of the orthocenter of τ to aff(τ_p). From Lemma 10 we have $|H(\omega(p))| \leq c_6\varepsilon \mathrm{lfs}(p)$, for all $\tau \in \mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P}) \cup \mathrm{Faces}^\omega(\mathcal{P})$. Moreover, using Lemma 3 and the fact that τ is a candidate j-sliver, $H(\omega(p)) = H(0) - \frac{\omega^2(p)}{2D_p} \leq H(0) - \frac{\omega^2(p)}{2j\sigma_0 L_\tau}$. It follows that the sliverity range of τ is at most $4j\sigma_0 L_\tau c_4\varepsilon \mathrm{lfs}(p)$. Using the facts that $L_\tau \leq 4c_5\varepsilon \mathrm{lfs}(p)$ (from Lemma 7), $\mathrm{lfs}(p) \leq l_p/\delta$ and $\varepsilon/\delta \leq \rho_0$, the sliverity range is less than $16jc_5c_6\sigma_0c_0^2\, l_p^2$. □

The inconsistency range of a configuration is defined in a way similar to the inconsistency range of a sliver.

Definition 3 (Inconsistency range) *Let* ω *be a weight assignment of relative amplitude at most* ω_0 *we keep fixed except for* $\omega(p_i)$*. The inconsistency range of an inconsistent configuration* $\phi = [p_1, \ldots, p_{k+2}]$ *w.r.t* p_i *is the measure of the set of all squared weights* $\omega^2(p_i)$ *for which all the* j*-dimensional subfaces of* ϕ *are not slivers, for* $j = 3, \ldots, k$*, and* ϕ *is an inconsistent configuration of* $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$.

Lemma 15 (Inconsistency range) *Assume that* $\max\{2a_k,\, 16c_5\}\varepsilon < 1$*. The inconsistency range of any configuration* $\phi = [p_1, \ldots, p_{k+2}]$ *is at most* $\Delta_i = B\,\varepsilon^2\, l_{p_i}^2$ *with* $B = (c_1 + c_2 + c_4)(2c_5 + a_k)a_kc_5c_0^2$.

Proof. Since ϕ is an inconsistent configuration, hence it implies that ϕ belongs to $\mathrm{Del}^\omega(\mathcal{P})$. Let p_i be any vertex of ϕ and $H(\omega(p_i))$ denotes the signed distance of the orthocenter of ϕ from ϕ_{p_i}. $H(\omega(p_i))$ is positive if p_i and the orthocenter of ϕ, denoted by o, lie on the same side. From Lemma 3, we have

$$H(\omega(p_i)) = H(0) - \frac{\omega^2(p_i)}{2D_{p_i}}. \tag{1}$$

Since ϕ is an inconsistent configuration with no faces being a slivers , we have from Lemma 11 the distance between c_ϕ and $\text{aff}(\phi_{p_i})$ is at most $h = 4(c_1 + c_2 + c_4)a_k\varepsilon^2 \text{lfs}(p_i)$. And $H(\omega(p_i)) = \text{dist}(o_\phi, \text{aff}(\phi_{p_i}))$[4] is less than the distance of c_ϕ from $\text{aff}(\phi_{p_i})$. With $\text{lfs}(p_i) \leq l_{p_i}/\delta$ and $\varepsilon/\delta \leq c_0$, we get $H(\omega(p_i)) \leq h \leq 4(c_1 + c_2 + c_4)a_k c_0 \varepsilon l_{p_i}$.

Moreover, since ϕ is an inconsistent configuration, hence the distance D_{p_i} between p_i and $\text{aff}(\phi_{p_i})$ is at most $4(2c_5 + a_k)c_5 c_0 \varepsilon l_{p_i}$.

Hence, if $\omega(p)$ and $\omega'(p)$ are two weights for which ϕ is an inconsistent configuration, we have $|H(\omega(p_i) - H(\omega'(p_i)| = \frac{1}{2D_{p_i}}|\omega^2(p_i) - \omega'^2(p)| \leq 2h$, from which we deduce that the inconsistency range of ϕ w.r.t to p_i is at most

$$4hD \leq 64(c_1 + c_2 + c_4)(2c_5 + a_k)a_k c_5 c_0^2 \times \varepsilon^2\, l_{p_i}^2 = B\,\varepsilon^2\, l_{p_i}^2.$$

□

4 Inconsistencies removal

In this section, we will show how to find a weight assignment for the points of $\mathcal{P}$ so as to remove all inconsistent configurations. Once this is done, all stars become coherent and the resulting tangential complex is a simplicial k-manifold.

This idea has already been successfully applied to remove slivers in 3 and higher dimensions [14, 13, 7]. The basic reasons why this approach works in our context are the following. If $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$ nor $\text{Faces}^\omega(\mathcal{P})$ contain slivers, inconsistencies in $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$ can only happen as inconsistent configurations. Thanks to Lemmas 14 and 15, the sliverity ranges of the slivers of $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$ and $\text{Faces}^\omega(\mathcal{P})$ are small, as well as the inconsistency ranges of the inconsistent configurations. It follows that we can remove any sliver in $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$ or $\text{Faces}^\omega(\mathcal{P})$, and any inconsistent configuration by weighting one of their vertices. Lastly, we observe that, thanks to Lemma 13, there are only a constant number of simplices of dimensions at most $k+1$ that may appear in $\text{Del}^\omega(\mathcal{P})$ and be incident to a given point of $\mathcal{P}$. Hence, we can remove all slivers of $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$ and $\text{Faces}^\omega(\mathcal{P})$ and all inconsistent configurations that are incident to a given point p by weighting p.

As before, we assume that k is given as well as an upper bound c_0 on the sparsity of $\mathcal{P}$. We choose for the bound ω_0 on the relative amplitude $\tilde{\omega}$ of the weight assignment ω any value in the interval $[0, 1/2)$. The constants c_1, c_2, c_3, c_4 and the number N can be then computed. Assuming that all those constants have been fixed, it will remain to fix σ_0, which will also fix a_k (that depends on c_5, σ_0 and $\rho_0 = \beta'_1$).

4.1 Algorithm: removing slivers and inconsistencies

The algorithm consists of computing a weight assignment ω such that no inconsistent configuration remain in $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$. The overall algorithm consists of

[4] o_ϕ is the orthocenter of ϕ.

two main steps. First, we compute a set of weight assignments ω so that no j-slivers appear in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}) \cup \mathrm{Faces}^{\omega}(\mathcal{P})$ for $3 \leq j \leq k$. We proceed by increasing dimensions of the slivers to be removed and weight each point p_i of $\mathcal{P}$ in turn. Initially all weights are set to 0 and they can only be increased during the course of the algorithm.

for $j = 3$ to k **do** (Compute ω so that $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}) \cup \mathrm{Faces}^{\omega}(\mathcal{P})$ contain no slivers)
  for $i = 1$ to n **do**
    $\omega(p_i)$ = **weight**$(j, p_i, \omega(p_i))$; (see below)
    update$(\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}), p_i)$; (see below)
  endfor
endfor

We weight each point in turn a second time so as to remove all inconsistent configurations in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$. The initial weights are the weights provided by the previous procedure. As before, weights can only increase.

for $i = 1$ to n **do** (Compute ω so that $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ contains no inconsistent configurations)
  $\omega(p_i)$ = **weight2**$(p_i, \omega(p_i))$; (see below)
  update$(\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}), p_i)$;
endfor

Upon termination, $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ contains no inconsistencies, i.e. any simplex appears in the stars of all its vertices.

Function update$(\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}), p)$

- Update the stars of all points $x \in LN_p$ by modifying $\mathrm{Del}^{\omega}_{x}(LN_x)$ if $p \in LN_x$.

Function weight$(j, p, \omega(p))$

1. Call function **detect_candidate-slivers**(j', p) for all $3 \leq j' \leq j$, to find all the so-called candidate j'-slivers, i.e. j'-slivers incident to p that appear in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ or in $\mathrm{Faces}^{\omega}(\mathcal{P})$ when $\omega(p)$ varies (the other weights remaining fixed).

2. Select the minimum weight in the interval $[\omega(p), \omega_0\, l_p]$ for which $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and $\mathrm{Faces}^{\omega}(\mathcal{P})$ contain no j'-slivers for $3 \leq j' \leq j$ calculated in step 1 are incident on p.

Function detect_candidate-slivers(j, p)

1. We first detect all possible j-simplices of $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ incident on p for all possible $\omega(p)$. This is done in the following way: (1) we vary the weight of p from 0 to its maximum, which is $\omega_0 l_p$, keeping the weights of the other points constant (2) for each new weight assignment to p we modify the stars of the points in LN_p [5] and detect from the stars the new j-simplices

[5] We don't have to calculate the stars of the points outside LN_p as they won't contain any simplices incident to p.

incident to p that have not been detected thus far. The weight of point p changes only in a finite number of instances

$$0 = P_0 < P_1 < \cdots < P_{n-1} < P_n = \omega_0 l_p.$$

We determine the next weight assignment of p in the following way. For each new simplex τ currently incident to p, we keep it in a priority queue ordered by the weight of p at which τ will be destroyed for the first time. Hence the minimum weight in the priority queue gives the next weight assignment for p. Since the number of points in LN_p is bounded, the number of simplices incident to p is also bounded, as well as the number of times we have to change the weight of p.

2. As the new j-simplices are generated, we detect the ones that are j-slivers.

3. Once we have detected all possible j-slivers incident on p, we find out all possible inconsistent configurations incident to p, by calling the function **detect_inconsistent-configuration**(p). From the inconsistent configurations, we extract the j-slivers of $\text{Faces}^{\omega}(\mathcal{P})$ that are incident to p.

Function detect_inconsistent-configuration(p)

1. We vary the weight of p from 0 to $\omega_0 l_p$, keeping the weight of the rest of the points constant. Once we have assigned a new weight to p we modify the stars of the points in LN_p.

2. Detecting the inconsistent configurations incident to p once we have modified the stars of the points in LN_p is more complicated than detecting the simplices incident to p. We consider all points p_i in LN_p. Let τ be a k-simplex in the star of p_i, and let p_j be a vertex of τ such that τ is not in the star of p_j. We calculate the Voronoi diagram of the points in LN_p restricted to the line segment $[c_{p_i} c_{p_j}]$, where $c_{p_i} = T_{p_i} \cap \text{Vor}^{\omega}(\tau)$ and $c_{p_j} = T_{p_j} \cap \text{aff}(\text{Vor}^{\omega}(\tau))$. From the restricted Voronoi diagram, we find a point r whose Voronoi cell intersects for the first time the line segment $[c_{p_i} c_{p_j}]$ oriented from c_{p_i} to c_{p_j}. If $p \in \phi = \tau \cup \{r\}$, then we report ϕ.

3. As in the **detect_candidate-slivers** function the weight of p is changed only a finite number of times. For each current inconsistent configuration ϕ incident to p, we keep it in a priority queue the weight of p for which ϕ will be destroyed for the first time. The minimum weight in the priority queue gives the next weight assignment of p.

Function weight2$(p, \omega(p))$

1. Call functions **detect_candidate-slivers**(j, p) for $3 \leq j \leq k$, and **detect_inconsistent-configuration**(p) to find all the slivers and inconsistent configurations incident on p.

2. Select a weight in the interval $[\omega(p), \omega_0\, l_p]$ for which no j-slivers for $3 \leq j \leq k$ nor any inconsistent configurations calculated in step 1 are incident to p.

Lemma 16 *Function weight returns a weight for p_i such that there is no candidate j'-sliver nor any inconsistent configuration incident to $p_{i'}$ for all $j' \leq k$ and $i' \leq i$.*

Proof. We call *new* simplex a candidate simplex that appears in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ after changing the weight of p_i. Since the weight are always increasing, a new simplex is either a k-simplex incident to p_i or a subface of such a k-simplex.

The proof is by induction. The assertion is trivially true for $j < 3$. Assume that the assertion holds for weight$(j', p_{i'})$ for all $j' < j$ and all $i' < i$. We now prove that after applying Function weight(j, p_i), the assertion holds for all $j' \leq j$ and all $i' \leq i$. In fact, it is sufficient to prove the assertion for the new simplices by the remark above.

Function weight(j, p_i) does not create any candidate j'-sliver incident to p_i for all $j' \leq j$ (Step 1). It cannot create j''-slivers, $j'' < j$, that do not contain p_i either. Indeed, such a j''-sliver should have existed before applying weight(j, p_i), which would contradict the induction hypothesis. □

Lemma 17 *Function weight2 returns a weight for p_i such that there is no candidate j'-sliver in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ nor in $\mathrm{Faces}^{\omega}(\mathcal{P})$, and no inconsistent configuration incident to $p_{i'}$ for all $j' \leq k$ and $i' \leq i$.*

Proof. The proof that $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and $\mathrm{Faces}^{\omega}(\mathcal{P})$ contain no candidate j'-slivers, $j' \leq k$ and $i' \leq i$ is the same as for function weight.

Consider the case of inconsistent configurations. We first observe that, since a new inconsistent configuration must contain a new k-simplex, it must be incident to p_i. As weight2(p_i) does not create any inconsistent configuration incident to p_i (Step 2), the lemma is proved. □

We conclude from Lemma 16 and 17 that the tangential Delaunay complex for the weight assignment ω output by the overall algorithm does not contain any inconsistent configuration.

Let us summarize our result in the following theorem.

Theorem 1 (Termination conditions) *If $\mathcal{P}$ is an (ε, δ)-sample of $\mathbb{M}$ and if the following properties are satisfied,*

1. $\omega_0 \in [0, 1/2)$;
2. $\varepsilon > \delta \geq c_0\varepsilon > 0$ *for some known* c_0;
3. $\varepsilon < \varepsilon_0 = \min\left(\frac{1}{2c_1}(-c_1 - 1 + \sqrt{c_1^2 + 6c_1 - 7}),\ \frac{1}{2a_k},\ \frac{1}{16c_5},\ \frac{1}{32c_5c_0}\right)$;
4. $A_k\sigma_0 + B\,\varepsilon^2 \leq \frac{\omega_0^2}{N^{k+2}}$ *(Pumping equation), where A_k and N are absolute constants, and B depends on σ_0 $\rho_0 = \max\{\beta_0', \beta_1'\}$ and k;*

then, the above algorithm outputs $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ without any candidate slivers or inconsistent configuration.

Proof. Hypotheses $(1) - (3)$ are needed for the structural results proved earlier in Sections 2, 3 to hold.

We have seen that the algorithm modifies the weight of a given point $k - 1$ times. Let $\omega_2(p) = 0, \omega_3(p), \ldots, \omega_{k+1}(p)$ be the sequence of weights assigned to

p. For all $3 \leq i \leq k$, $\omega_i(p)$ is the weight assigned to the point p s.t no j'-slivers are incident on the point p for the $3 \leq j' \leq i$. From the **weight** function, we know that for all $3 \leq i \leq k$, we have the following (1) $\omega_i(p) \in [\omega_{i-1}(p), \omega_0\, l_p]$, and (2) $\omega_i(p)$ is the least weight s.t no j'-slivers are incident on the point p for the $3 \leq j' \leq i$.

From Lemmas 14, 13 we have

$$\begin{aligned} |\omega_i^2(p) - \omega_{i-1}^2(p)| &\leq \sum_{3\leq j\leq i} (N^{j+1} A_j\, \sigma_0 l_p^2) \\ &\leq (\sum_{3\leq j\leq i} N^{j+1}) A_i \sigma_0 l_p^2 < 2N^{i+1}\, A_i \sigma_0 l_p^2, \end{aligned}$$

the last inequality follows from the fact that $\sum_{3\leq j\leq i} N^{j+1} < 2N^{i+1}$. Therefore

$$\begin{aligned} |\omega_k^2(p) - \omega_2^2(p)| &= \sum_{3\leq i\leq k} |\omega_i^2(p) - \omega_{i-1}^2(p)| \\ &< \sum_{3\leq i\leq k} 2N^{i+1} A_i \sigma_0 l_p^2 < 4N^{k+1} A_k \sigma_0 l_p^2. \end{aligned} \tag{2}$$

From the **weight2** function, we have (1) $\omega_{k+1}(p) \in [\omega_k(p), \omega_0\, l_p]$, and (2) $\omega_{k+1}(p)$ is a weight s.t no j'-slivers for $3 \leq j' \leq k$ and inconsistent configurations incident on p.

From Lemmas 14, 15 and 13, measure of the sets of squared weights of p that will create a j-sliver for $3 \leq j \leq k$ or an inconsistent configuration incident to p is less than

$$\gamma = \sum_{3\leq i\leq k} N^{i+1} \times A_i \sigma_0 l_p^2 \,+\, N^{k+2} B\varepsilon^2 l_p^2 < 2N^{k+1} A_k \sigma_0 l_p^2 + N^{k+2} B \varepsilon^2 l_p^2. \tag{3}$$

Combining Equations 2 and 3 we get

$$\begin{aligned} |\omega_k^2(p) - \omega_2^2(p)| + \gamma &< 6N^{k+1} A_k \sigma_0 l_p^2 + N^{k+2} B \varepsilon^2 l_p^2 \\ &\leq N^{k+2} A_k \sigma_0 l_p^2 + N^{k+2} B \varepsilon^2 l_p^2. \;\; \text{(as } N \geq 6) \end{aligned}$$

If the above quantity is less than $\omega_0^2\, l_p^2$, the above algorithm will terminate and find a weight assignment that removes all slivers of dimensions up to k from $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and $\mathrm{Faces}^{\omega}(\mathcal{P})$, and all inconsistent configurations from $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$. □

4.2 Time complexity

Before the total time complexity of the algorithm is calculated we first calculate the individual time complexity for each function.

At each point $p \in \mathcal{P}$ we maintain $\mathrm{Del}^{\omega}_p(LN_p)$. We need to calculate LN_p for all points $p \in \mathcal{P}$ once. The total time complexity to do this operation is $O(d|\mathcal{P}|^2)$.

From Lemma 1, the time complexity of updating $\mathrm{Del}^{\omega}_p(LN_p)$ is

$$(d\,|LN_p| + k^3\, O(|LN_p|^{\lceil \frac{k}{2} \rceil})) = d\, 2^{O(k^2)}. \tag{4}$$

The last equality follows from the fact that $|LN_p| \leq N = 2^{O(k)}$ from Lemma 13. The constant in the big-O is independent of d (see Lemma 13).

Function **update**$(\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}), p)$ modifies $\mathrm{Del}^{\omega}_p(LN_p)$ as well as $\mathrm{Del}^{\omega}_x(LN_x)$ for all the points $x \in LN_p$. Since we assume that $\mathcal{P}$ is a dense sample satisfying the first three hypothesis of Theorem 1, we have $\mathrm{Del}^{\omega}_p(\mathcal{P}) = \mathrm{Del}^{\omega}_p(LN_p)$, and $|LN_p| \leq N = 2^{O(k)}$ for all $p \in \mathcal{P}$. Hence for calculating $\mathrm{Del}^{\omega}_p(\mathcal{P})$, we only need to calculate the restricted weighted Delaunay triangulation of LN_p restricted to T_p. Since we modify $\mathrm{Del}^{\omega}_x(LN_p)$ for all the points in LN_p, we have the total time complexity of **update**$(\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}), p)$ is less than

$$d2^{O(k^2)} \times |LN_p| = d2^{O(k^2)} \times N = d\, 2^{O(k^2)}. \tag{5}$$

Function **detect_inconsistent-configuration**(p) detects all the possible inconsistent configurations incident to p. We vary the weight of p in increasing order from 0 to $\omega_0 l_p$. The number of times $\omega(p)$ is changed is equal to number of possible inconsistent-configuration incident to p, which is atmost $|LN_p|^{k+1} = 2^{O(k^2)}$. We also need to calculate for each inconsistent configuration ϕ the minimum weight $\omega(p)$ of p for which ϕ will be destroyed for the first time. The time complexity of this operation is $O(dk^3)$, since we need to calculate the sign of determinant of $k \times k$ matrix. Once the weight of p is fixed, we calculate the stars of all the points in LN_p, which takes less than $d2^{O(k^2)}$ time. Then all k-simplices in the stars of all points in LN_p are checked for inconsistencies. If a given simplex τ is inconsistent then the time complexity to find the inconsistent configuration ϕ corresponding to τ is $dLN_p = d2^{O(k)}$. The total number of k-simplices we need to check is less $|LN_p| \times 2^{O(k^2)} = 2^{O(k^2)}$. Hence the total time complexity of **detect_inconsistent-configuration**(p) is less than

$$2^{O(k^2)}(O(dk^3) + d2^{O(k^2)} + d2^{O(k)} \times 2^{O(k^2)}) = d\, 2^{O(k^2)}. \tag{6}$$

Similarly the time complexity for **detect_candidate-slivers**(j, p) is $d\, 2^{O(k^2)}$.

For the function **weight**$(j, p, \omega(p))$ we first need to calculate all the possible j'-slivers that can be incident on p, for all $3 \leq j' \leq j$ by calling **detect_candidate-slivers**(j', p). The time complexity of this step is $(j-3) \times d\, 2^{O(k^2)} = d2^{O(k^2)}$, as time complexity of **detect_candidate-slivers**(j, p) is $d\, 2^{O(k^2)}$. Then we find the minimum weight of p in the range $[\omega(p), \omega_0 l_p]$ that will remove all the slivers calculated in the first step. This operation takes $j^3 2^{O(k^2)} = 2^{O(k^2)}$ time. Hence the total time complexity of **weight**$(j, p, \omega(p))$ is less than $d2^{O(k^2)}$.

Similarly, the time complexity of the function **weight**$(p_i, \omega(p_i))$ is $d\, 2^{O(k^2)}$.

In the course of the algorithm for each point $p \in \mathcal{P}$, we call **weight**$(j, p, \omega(p))$ once for all $3 \leq j \leq k$ and **weight2**$(p, \omega(p))$ once. Adding everything up, the time compelxity for our algorithm is

$$d|\mathcal{P}|^2 + (k-2) \times d\, 2^{O(k^2)}|\mathcal{P}| = d|\mathcal{P}|^2 + d\, 2^{O(k^2)}|\mathcal{P}|. \tag{7}$$

4.3 Space complexity

Storing the point sample $\mathcal{P}$ which lies in d-dimensional space requires $d|\mathcal{P}|$ space.

Storing $\mathrm{Del}^{\omega}_p(LN_p)$ at each point p of $\mathcal{P}$ requires

$$\sum_{p \in \mathcal{P}} d\, O(k|LN_p|^{\lceil \frac{k}{2} \rceil}) \leq d\, 2^{O(k^2)}|\mathcal{P}|, \tag{8}$$

space. The last inequality follows from the fact that $|LN_p| \leq N = 2^{O(k)}$ for all points $p \in \mathcal{P}$ (refer to Lemma 13).

Therefore the total space complexity of the algorithm is

$$d|\mathcal{P}| + d\, 2^{O(k^2)}|\mathcal{P}| = d\, 2^{O(k^2)}|\mathcal{P}|. \tag{9}$$

5 Topological and geometric guarantees

In this section it is assumed that $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ has no slivers and inconsistent configurations. These assumptions are justified as the $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ returned by the algorithm given in Subsection 4.1 has these properties, refer to Theorem 1.

5.1 Local Properties

Let $\pi : \mathbb{R}^d \to \mathbb{M}$ maps each point of $\mathbb{R}^d$ to its closest point of $\mathbb{M}$. PL manifold $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is has the following properties if ε is sufficiently small.

Lemma 18 (PROPERTIES OF $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$)

1. (CONTINUITY OF π ON $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$) *The map π restricted to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is well defined and continuous.*

2. (PIECEWISE-LINEAR MANIFOLD) $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ *is a picewise-linear manifold without boundaries.*

3. $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ *has a vertex in each connected component of* $\mathbb{M}$.

4. (SMALL SIMPLEX CONDITION) *Every simplex τ of $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ has a circumsphere of radius $\leq c_2\varepsilon \mathrm{lfs}(x)$, where x is any vertex of τ.*

5. (FLAT SIMPLEX CONDITION) *Given a k-simplex τ and a vertex p of the simplex, then $\angle N_\tau N_p \leq a_k \varepsilon$.*

Proof.

1. Let τ be a simplex in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and p a vertex of τ. Since $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ does not contain any inconsistent configuration, hence τ belongs to $\mathrm{Del}^{\omega}_{p}(\mathcal{P})$. From Lemma 7 we know that $\tau \subseteq B(p, c_2\varepsilon\mathrm{lfs}(p)) \subseteq B(p, 0.5\mathrm{lfs}(p))$. Therefore $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ does not contain any point of the medial axis of $\mathbb{M}$ and π is continuous at all points except the medial axis of $\mathbb{M}$. Hence the restriction of π to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is continuous.

2. To prove that $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is a piecewise linear manifold without boundary we have to show that for all point $p \in \mathcal{P}$, p does not lie on the boundary of $\mathrm{star}(p)$. Above sufficiency condition follows if we can show that $\mathrm{Vor}^{\omega}(p) \cap T_p$ is bounded. From Lemma 6 we have $\mathrm{Vor}^{\omega}(p) \cap T_p \subseteq B(p, c_1\varepsilon\mathrm{lfs}(p))$. Hence for all point $p \in \mathcal{P}$, p does not lie on the boundary of $\mathrm{star}(p)$, which implies $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is a piecewise linear manifold without boundary.

3. Every point $p \in \mathcal{P}$ is a vertex of $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and we assume point sample $\mathcal{P}$ has one point from every connected component of $\mathbb{M}$.

4. Let τ be a simplex of $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$. Since $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ has no inconsistencies hence τ is in $\mathrm{star}(x) \subseteq \mathrm{Del}^{\omega}_{x}(\mathcal{P})$ for all vertex x of τ. Since τ is in $\mathrm{Del}^{\omega}_{x}(\mathcal{P})$, hence from Lemma 7 we have τ is contained in $B(x, c_2\varepsilon \mathrm{lfs}(x))$.

5. Let τ be a k-simplex of $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$. From Lemma 18 (4) we know that for any vertex p of τ, τ is contained in $B(p, c_2\varepsilon \mathrm{lfs}(p))$. We know that sliverity constant and radius-edge ratio of $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is σ_0 and β_0 respectively. From Lemma 5 we have $\angle N_p N_\tau \leq a_k \varepsilon$.

□

Using ideas from [2, 8] and the above properties, we show $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is isotopic to $\mathbb{M}$.

5.2 Isotopy

Now we prove the final lemmas that will prove the isotopy between $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and $\mathbb{M}$. We prove the isotopy between $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and $\mathbb{M}$ using the map π restricted to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$. We first show in Lemma 19 (extension of Lemma 16 in [1] to higher dimension and weighted case) that π is injective on points of $\mathcal{P}$, i.e for all points $p \in \mathcal{P}$ and $x \in \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$, $\pi(x) = p \implies x = p$.

Definition 4 (Adjacent j-simplices) *Two j-simplices are said to be adjacent iff they share a common $(j-1)$-dimensional face.*

Then in Lemma 20 (extension of Lemma 18 in [1] to higher dimension), we show that π restricted to *adjacent simplices* is also injective. In Lemma 21 we show that the restriction of π to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is injective using Lemmas 19 and 20. In Lemma 22 we show that the restriction of π to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is surjective using the facts that the restriction of π to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is injective, $\mathbb{M}$ is a manifold without boundary and $\mathcal{P}$ contains points from each connected components of $\mathbb{M}$. Finally in Theorem 2 we prove the isotopy using the previous results.

Lemma 19 *Let $p \in \mathcal{P}$, $n_p \in N_p$ be a unit vector and l_p the normal fiber $[p, p + \mathrm{lfs}(p) \,.\, n_p]$, then no k-simplex intersects the interior of the segment l_p.*

Proof. Let N denote the ball centered at $m = p + \mathrm{lfs}(p) \,.\, n_p$ and tangent to the manifold at p. In order to intersect segment pm, a k-simplex $s \in \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ would have to intersect N, and so would the smallest orthogonal Delaunay ball B of s, centered at x with radius r. Let H denote the $k-1$-flat space passing through the intersection of N and B.

The $(d-1)$ flat space H decomposes the ball N into two sections $N^+ = N \cap H^+$, $N^- = N \cap H^-$ and the $(k-1)$ ball $N \cap H$, where H^+ and H^- are the two open half spaces formed by H. Similarly, B is also decomposed by H into B^+, B^- and $B \cap H$. It follows from geometry that if $N^+ \subset B^+$ then B^+ and N^- lie on opposite sides of H. Without loss of generality we can assume that $N^+ \subset B^+$.

Since the vertices of s lie on $\mathbb{M}$, and hence not in the interior of N, $s \subset B^+$. Since s belongs to weighted Delaunay triangulation, hence

$$||p - x|| \geq \sqrt{r^2 + \omega^2(p)} \geq r.$$

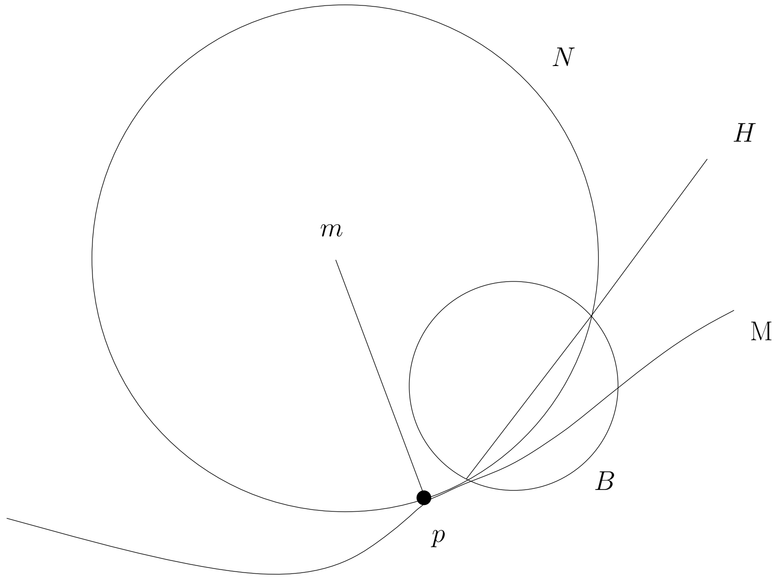

Figure 5: Refer to Lemma 19. Figure is taken from [2] and shows the case when the points are not weighted.

So p cannot lie in the interior of B hence $p \notin H^+$. We will show that $m \notin H^+$. If m was inside $H^+ \subset B^+$, the radius of B would be at least $1/2\ \text{lfs}(v)$ for any vertex $v \in s$, but if we choose ε small enough to make sure it is less that $1/2\ \text{lfs}(v)$, then we will reach a contradiction. Therefore p, m and hence the fiber l_p lies in H^- proving that H separates s and l_p. $\Box$

We assume that ε small enough such that $\arcsin(a_k\varepsilon) \le \pi/8$.

Lemma 20 *Let τ_1 and τ_2 be two k-dimensional adjacent simplices of $\text{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$. Then the map π restricted to $\tau_1 \cup \tau_2$ is injective.*

Proof. Assume, for a contradiction that there exist two points y, z in $\tau_1 \cup \tau_2$ such that $\pi(y) = \pi(z) = x$. Let p be a common vertex of τ_1 and τ_2. Since we assume that $\text{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ has no inconsistencies hence both τ_1 and τ_2 belong to $\text{Del}^{\omega}_{p}(\mathcal{P})$. Therefore from Lemma 2 (3) and Lemma 7 (1) we have

$$\|p - x\| \le (c_2\varepsilon + 2c_2^2\varepsilon^2)\text{lfs}(p). \tag{10}$$

For ε sufficiently small we get $\|p - x\| \le 2c_2\varepsilon$. Hence from Normal variation Lemma 26, we get $\angle N_p\, N_x \le \arcsin(8c_2\varepsilon)$. We consider the following two cases.

Case 1: W.l.o.g assume that two points lie on the same k-simplex τ_1. The line segments connecting y, x (denoted by l_{yx}) and z, x (denoted by l_{zx}) are normals at x. This implies the vector $\overrightarrow{yz} \in N_x$ and we have reached a contradiction as N_x and N_τ are almost parallel (as $\angle N_x N_p \le \arcsin(8c_2\varepsilon)$).

Case 2: Now assume that y and z lie on different simplices.

Claim 1 *Two adjacent k-simplex in $\text{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ meet at an angle greater than $\pi - 2\arcsin(a_k\varepsilon)$.*

Proof. Let τ_1 and τ_2 be two adjacent k-simplex in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$, let τ be the shared $(k-1)$-simplex shared by the two simplex, and let p be a vertex of τ. As $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ has no inconsistencies hence both τ_1, τ_2 belong to $\mathrm{Del}^{\omega}_{p}(\mathcal{P})$. Therefore we have two empty ortho-spheres B_1, B_2 for τ_1 , τ_2 respectively, centered at point v_1, v_2 of T_p. The boundaries of the ortho-spheres B_1 and B_2 intersect in a $(d-1)$-dimensional ortho-sphere C contained in a hyperplane H, with $\tau \subseteq H$. The line passing through v_1 and v_2 is perpendicular to H, hence $\angle HN_p = 0$. Since $\omega_0 < 1/2$ implies $p \in \mathrm{Vor}^{\omega}(p)$, we have v_1 and v_2 lying on the opposite side of H. Since τ_1 and τ_2 are not slivers, hence $\angle N_{\tau_i} N_p \leq \arcsin(a_k\,\varepsilon)$ for $i = 1, 2$, from Lemma 5. Hence smaller angle at which τ_1 and τ_2 meet is greater than $\pi - 2\arcsin(a_k\varepsilon)$. □

Hence from above Claim 1, the smaller angle at which τ_1 and τ_2 meet is greater than $\pi - 2\arcsin(a_k\,\varepsilon)$. Since we have already shown that y and z cannot lie on the same simplex hence y and z cannot lie on the common $(k-1)$-dimensional face of both τ_1 and τ_2. The lines l_{yx} and l_{zx} are normal at the point x, this implies vector $\overrightarrow{yz} \in N_x$. This implies the vector $\overrightarrow{yz}$ intersect both τ_1, τ_2 at almost right angles but that contradicts the Claim 1 which says τ_1, τ_2 meet at an angle greater than $\pi - 2\arcsin(a_k\varepsilon) \geq 3\pi/4$. □

Lemma 21 (Injectivity lemma) *For $\varepsilon \leq \varepsilon_0$, the map $\pi : \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}) \rightarrow \mathbb{M}$ is injective.*

Proof. We will first show that ($\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}), \pi$) is a *covering space* of $\pi($ $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$). Lets denote $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$, $\pi($ $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$) by N, N' respectively. The (N, π) is a covering space of N' if, for every $x \in N'$, there is a path-connected *elementary neighborhood* V_x around x such that each path-connected component of $\pi^{-1}(V_x)$ is mapped homeomorphically onto V_x by π.

To construct V_x, we note that the set of points $|\pi^{-1}(x)|$ corresponding to a point $x \in N'$ is non-zero and finite, since π is injective on each triangle of N and there are only a finite a triangles. For each point $q \in \pi^{-1}(x)$ take an open neighborhood U_q of around q, homeomorphic to a disk and small enough so that U_q is contained only in k-simplex that contain q. We claim that π maps each U_q homeomorphically onto $\pi(U_q)$. This is because it is continuous, it is onto $\pi(U_q)$ by definition, and, since any two points x and y in U_q are in adjacent triangles, it is one-to-one by Lemma 20. Let $U'(x) = \cap_{q\,\in\,\pi^{-1}(x)}\ \pi(U_q)$, the intersection of the maps of each of the U_q. $U'(x)$ is the intersection of a finite number of open neighborhood, each containing x, so we can find an open disk V_x around x. V_x is path connected, and each component of $\pi^{-1}(V_x)$ is a subset of some U_q and hence is mapped homeomorphically onto V_x by π. Therefore it follows that (N, π) is a covering space for N'.

We can show that π defines a homeomorphism between N and N'. Since N is onto N' by definition, we need only show that π is one-to-one. Consider one connected component C of N'. From properties covering space [10, 23], we know that the sets $\pi^{-1}(x)$ for all $x \in G$ have the same cardinality. We know from lemma 19 that π is one-to-one at every sample point. Since each connected component of M contains some samples, hence it follows directly the π is one-to-one, and N and N' are homeomorphic. □

Lemma 22 (Surjectivity lemma) *Let $\mathcal{P}$ be a T-loose ε-sample and $\varepsilon \leq \varepsilon_0$. The map $\pi : \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}) \rightarrow \mathbb{M}$ is surjective.*

Proof. This follows directly from the fact that π is injective, and $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ has no boundaries hence it cannot contain a part of the connected component of $\mathbb{M}$, and hence $\pi(\ \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})\ )$ must consist of a subset of a connected components of $\mathbb{M}$. Since $\mathcal{P}$ contains at least one point from every connected component hence every connected components will be covered by the map π. □

Theorem 2 (Isotopy Lemma) *For $\varepsilon < \varepsilon_0$, $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and $\mathbb{M}$ are isotopic.*

Proof. The fact that $\mathbb{M}$ is homeomorphic to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ follows directly from Lemmas 21 and 22. The map π restricted to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ provides the isotopy to $\mathbb{M}$. □

5.3 Hausdorff Distance

In this section we bound the distance from well-formed $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ to $\mathbb{M}$, and then use the surjectivity of π to show that the bound also holds for the distance from $\mathbb{M}$ to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$.

With Lemmas 2 and 18, we can bound the distance between $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ to $\mathbb{M}$.

Lemma 23 *Every point $q \in \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is at most $2.5(c_2\varepsilon)^2\mathrm{lfs}(p) \leq 2.5(c_2\varepsilon)^2 \times \max_{x\in\mathcal{P}}\{\mathrm{lfs}(x)\}$ from $\mathbb{M}$, where p is a vertex of the simplex τ containing q in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$, and $\varepsilon \leq \varepsilon_0$.*

Proof. Since $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ has no inconsistent configuration therefore τ belongs to star(p) in $\mathrm{Del}^{\omega}_{p}(\mathcal{P})$. For all vertex x of τ we have from Lemmas 7, $||p-x|| \leq c_2\varepsilon\mathrm{lfs}(p)$. From Lemma 2 we have $\sin\angle(px, T_p) \leq c_2\varepsilon/2$. Therefore the distance between x and T_p, denoted by $\mathrm{dist}(x, T_p)$,

$$\mathrm{dist}(x, T_p) = ||p-x|| \sin\angle(px, T_p) \leq (c_2\varepsilon)^2/2\ .\ \mathrm{lfs}(p).$$

Since q is a point in the simplex τ, we have from above calculation, the distance between q and T_p is less than $(c_2\varepsilon)^2/2\ .\ \mathrm{lfs}(p)$.

Let q' be the point on T_p closest q. Since $||p-q|| \leq c_2\varepsilon\mathrm{lfs}(p)$, so $||p-q'|| \leq c_2\varepsilon\mathrm{lfs}(p)$. Let q'' be the point closest to q' on $\mathbb{M}$. From Lemma 2, we have $||q'-q''|| \leq 2(c_2\varepsilon)^2\mathrm{lfs}(p)$.

So from triangle inequality we get

$$||q-q''|| \leq ||q-q'|| + ||q'-q''|| \leq 2.5(c_2\varepsilon)^2\mathrm{lfs}(p) \leq 2.5(c_2\varepsilon)^2\ .\ \max_{x\in\mathcal{P}}\{\mathrm{lfs}(x)\},$$

as $||q-q'|| = \mathrm{dist}(q, T_p) \leq 0.5(c_2\varepsilon)^2\mathrm{lfs}(p)$ and $||q'-q''|| \leq 2(c_2\varepsilon)^2\mathrm{lfs}(p)$. Hence distance between x and $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$, denoted by $\mathrm{dist}(x, \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}))$,

$$\mathrm{dist}(q, \mathbb{M}) \leq ||q-q''|| \leq 2.5(c_2\varepsilon)^2\mathrm{lfs}(p) \leq 2.5(c_2\varepsilon)^2\ .\ \max_{x\in\mathcal{P}}\{\mathrm{lfs}(x)\}.$$

□

Lemma 24 *Every point $x \in \mathbb{M}$ is at distance at most $3(c_2\varepsilon)^2\mathrm{lfs}(x) \leq 3(c_2\varepsilon)^2 \times \max_{y\in\mathbb{M}}\{\mathrm{lfs}(y)\}$ from $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and $\varepsilon \leq \varepsilon_0$.*

Proof. Let $x \in \mathbb{M}$. Since the restriction of π to $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is surjective, we have $x' \in \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ such that $\pi(x') = x$. According to Theorem 23, $\|x' - x\| \leq 2.5\varepsilon^2 \mathrm{lfs}(p) \leq 2.5\varepsilon^2 \ . \ \max_{x\in\mathcal{P}} \mathrm{lfs}(x)$, where p is any vertex of the simplex containing x' in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$.

Therefore we have

$$\begin{aligned} ||x-p|| &\leq ||x'-p|| + ||x-x'|| \\ &\leq (c_2\varepsilon + 2.5(c_2\varepsilon)^2)\mathrm{lfs}(p) \\ &\leq (c_2\varepsilon + 2.5(c_2\varepsilon)^2)(\mathrm{lfs}(x) + ||x-p||) \ \ \text{(as lfs has 1-Lipschitz property)}. \end{aligned}$$

Arranging the inequality we get

$$\|x-p\| \leq \frac{c_2\varepsilon + 2.5(c_2\varepsilon)^2}{1 - c_2\varepsilon - 2.5(c_2\varepsilon)^2}\mathrm{lfs}(x). \tag{11}$$

We can choose the $\varepsilon \leq \varepsilon_0$ (a constant) such that $\|x-p\| \leq 0.2\mathrm{lfs}(x)$. Hence,

$$\|x-x'\| \leq 2.5(c_2\varepsilon)^2\mathrm{lfs}(p) \leq 2.5(c_2\varepsilon)^2(\mathrm{lfs}(x) + \|x-p\|) \leq 3(c_2\varepsilon)^2\mathrm{lfs}(x).$$

Hence distance between x and $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$, denoted by $\mathrm{dist}(x, \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}))$, is

$$\mathrm{dist}(x, \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})) \leq ||x-x'|| \leq 3(c_2\varepsilon)^2\mathrm{lfs}(x) \leq 3(c_2\varepsilon)^2 \ . \ \max_{y\in\mathbb{M}} \mathrm{lfs}(y).$$

□

From Lemmas 23 and 24, we get the following bound on the Hausdroff distance between $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ and $\mathbb{M}$:

Theorem 3 (Hausdorff distance**)** *Let $\mathbb{M}$ be a manifold without boundary and $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ be well-formed, with $\varepsilon \leq \varepsilon_0$. Then the Hausdorff distance between $\mathbb{M}$ and $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is at most $3(c_2\varepsilon)^2 \ . \ \max_{x\in\mathbb{M}} \ \mathrm{lfs}(x)$.*

6 Summary of results

Combining Theorem 1 and results from Section 5 we get the following theorem.

Theorem 4 (Final theorem**)** *If $\mathcal{P}$ is an (ε, δ)-sample of $\mathbb{M}$ and if the following conditions are satisfied,*

- $\omega_0 \in [0, 1/2)$;
- $\varepsilon > \delta \geq c_0\varepsilon > 0$ *for some known* c_0;
- $\varepsilon < \varepsilon_0 = \min\left(\frac{1}{2\,c_1}(-c_1 - 1 + \sqrt{c_1^2 + 6c_1 - 7}),\ \frac{1}{2a_k},\ \frac{1}{16c_5},\ \frac{1}{32c_5c_0}\right)$;
- $A_k\sigma_0 + B\,\varepsilon^2 \leq \frac{\omega_0^2}{N^{k+2}}$ *(* Pumping equation*), where A_k and N are absolute constants, and B depends on σ_0, $\rho_0 = \max\{\beta_0', \beta_1'\}$ and k;*
- $\arcsin(a_k\varepsilon) \leq \pi/8$.

then, the algorithm outputs $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ without any sliver nor inconsistent configuration, and $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ has the following properties

- Bijection
  The projection $\pi : \mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P}) \to \mathbb{M}$ *is a bijection;*

- Pointwise approximation
  For all $x \in \mathbb{M}$, $dist(x, \pi^{-1}(x)) = O(\varepsilon^2 \mathrm{lfs}(x))$*;*

- Normal approximation
  For all $x \in \mathbb{M}$, $\angle N_x N_\tau = O(\varepsilon)$, *where* τ *is a* k*-simplex of* $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ *containing the point* $\pi^{-1}(x)$*;*

- Topological correctness
  $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ *is isotopic to* $\mathbb{M}$*.*

7 Extensions

7.1 Dimension and tangent space estimation.

The reconstruction algorithm given here works if the dimension of $\mathbb{M}$ is known as well as the tangent space at each point of $\mathcal{P}$. Algorithms are known to compute the dimension of a manifold from a finite sample. Some of them can be used to estimate tangent spaces [15, 21]. Our algorithm is robust to small perturbation of the tangent spaces.

7.2 Removing the general position assumption

It is assumed that the points are in general position. This section will show how this assumption can be removed. The idea behind the calculations done here is similar to one done in Section 8 of [14].

The following properties of $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ will be used

1. The radius-edge ratio of any simplex in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ is more than equal to β'_0, refer to Lemma 6.

2. Let τ be a k-dimensional simplex in $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ such that it is not a sliver. Then $\angle(T_p, \mathrm{aff}(\tau)) \le a_k\varepsilon\ \forall p \in \tau$, refer to Lemmas 6 and 5.

Lemma 25 *Let* $k \ge 2$, τ *be a* $(k+1)$*-dimensional* $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$ *and*

$$\varepsilon < \frac{\sigma_0(k+1)}{\beta'_0(3c_2 + 2a_k)}.$$

If the subfaces of τ *are not slivers, then* τ *is a sliver.*

Proof. Let pq be the smallest edge of the simplex τ and let r be a vertex in $\tau \setminus \{p, q\}$. Since τ_r is not a sliver (as all subfaces of τ are not slivers), we have as $\sin\angle(\mathrm{aff}(\tau_r), T_x) \le a_k\varepsilon$ for all vertices x of τ_r, refer Lemma 5. We have from Lemma 7 $\|p - r\| \le 3c_2\mathrm{lfs}(p)$, and $\sin\angle(pr, T_p) \le 3c_2\varepsilon/2$ from Lemma 2. Therefore

$$\begin{aligned}
D_p &= \sin\angle(pr, \mathrm{aff}(\tau_r))\|p - r\| \\
&\le \sin(\angle(pr, T_p) + \angle(\mathrm{aff}(\tau_r), T_p))\|p - r\| \\
&\le (\sin\angle(pr, T_p) + \sin\angle(\mathrm{aff}(\tau_r), T_p))\,\|p - r\| \\
&\le (3c_2/2 + a_k)\varepsilon\|p - r\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{vol}(\tau) &= D_r \times \frac{\mathrm{vol}(\tau_r)}{(k+1)} \\
&\leq (a_k + 3c_2/2)\varepsilon\|p-r\| \times \frac{\sigma(\tau_p)L_\tau^k}{(k+1)} \quad (\text{as } \mathrm{vol}(\tau_r) = \sigma(\tau_r)L_{\tau_r}^k,\ L_\tau = L_{\tau_r}) \\
&\leq (2a_k + 3c_2)\varepsilon R_\tau \times \frac{\sigma_0^k L_\tau^k}{(k+1)} \qquad (\text{as } \sigma(\tau_r) \geq \sigma_0^k,\ \|p-r\| \leq 2R_\tau) \\
&\leq \frac{\beta_0'(2a_k + 3c_2)\varepsilon}{(k+1)} \times \sigma_0^k\, L_\tau^{k+1} \qquad (\text{as the edge-radius ratio is } \beta_0') \\
&< \sigma_0^{k+1}\, L_\tau^{k+1}.
\end{aligned}$$

□

In our algorithm we pump all j-slivers for $j \in \{3, \ldots, k\}$ from $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P}) \cup \mathrm{Faces}^\omega(\mathcal{P})$. From Lemma 25, we can see that, if we remove also $(k+1)$-slivers from $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$, then, if $\mathcal{P}$ is dense enough wrt the sliverity bound σ_0, specifically if

$$\varepsilon < \frac{\sigma_0(k+1)}{\beta_0'(3c_2 + 2a_k)}$$

then there will be no $(k+1)$-slivers in $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$. The proof of the above claim is the following. If a $(k+1)$-simplex τ remains in the final complex $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$ returned by the algorithm, it cannot be a sliver nor its subfaces. But as the sample is dense (i.e $\varepsilon < \frac{\sigma_0(k+1)}{\beta_0'(3c_2+2a_k)}$) and none of its subfaces are slivers, Lemma 25 implies that τ is a $(k+1)$-sliver. Hence we have reached a contradiction. So we will have no $(k+1)$-simplices in $\mathrm{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$.

8 Conclusion

We have given the first algorithm that is able to reconstruct a smooth manifold in a time that depends only linearly on the dimension of the ambient space. We believe that our algorithm is of great interest when the dimension of the manifold is small, even if it is embedded in a space of very high dimension. This situation is quite common in practical applications in machine learning.

The algorithm is rather simple. The basic ingredients we need are data structures for constructing weighted Delaunay triangulations in k-flats. We intend to implement the algorithm in the near future, using an extension of the code developped for constructing Delaunay triangulations in any fixed dimension [5].

In practice, the dimension of $\mathbb{M}$ is usually not known. We can use the algorithms given in [21, 15] to estimate the dimension k of the manifold and the tangent space at each sample point. One interesting feature of our approach is that it is pretty robust and still works if we only have approximate tangent spaces at the sample points.

We forsee other applications of the tangential complex and of our construction each time computations in the tangent space of a manifold are required, e.g. for dimensionality reduction and approximating the Laplace Beltrami operator [4].

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A Volume Lemma

We want to bound the volume of $B(p, \epsilon\text{lfs}(p)) \cap \mathbb{M}$. We assume that $\mathbb{M}$ be a smooth k-manifold embedded in $\mathbb{R}^d$. From [27], we have

Lemma 26 (Normal variation) *Let p, q be two points in $\mathbb{M}$ with $\|p-q\| \leq \alpha\,\text{lfs}(p)$, $\alpha < \frac{1}{2}$, and $\theta = \angle N_p N_q$. We then have $2\sin\frac{\theta}{2} \leq 1 - \sqrt{1-4\alpha}$, and the weaker bound $\sin\theta \leq 4\alpha$.*

Proof. It is proved in [27] (Propositions 6.2 and 6.3)[6] that $2\sin\frac{\theta}{2} \leq 1 - \sqrt{1 - \frac{4\|p-q\|}{\text{lfs}(p)}} \leq 1 - \sqrt{1-4\alpha}$. $\square$

For all point $x \in \mathbb{M}$, we define the following projection map $f_x : B(x, \varepsilon\text{lfs}(x)) \cap \mathbb{M} \to T_x$.

Lemma 27 *For all points $p \in B(x, \varepsilon\text{lfs}(x))$, the derivative df_x is nonsingular, where $\varepsilon < \frac{1}{2}$.*

Proof. Let df_x be singular for some point $p \in B(x, \varepsilon\text{lfs}(x)) \cap \mathbb{M}$. This implies that vector $\overrightarrow{pf_x(p)}$ is in T_p. But $\overrightarrow{pf_x(p)}$ is perpendicular to T_x, i.e, vector $\overrightarrow{pf_x(p)}$ is in N_p. Hence we have a contradiction from Lemma 26, as $\angle N_p N_x \leq \arcsin 2\varepsilon < \pi/2$. $\square$

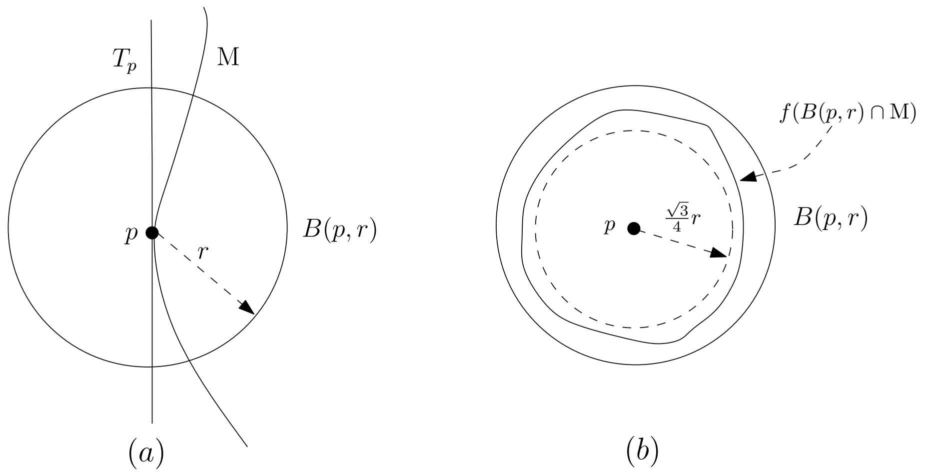

Figure 6: Refer to Lemma 28. In the figure $r = \varepsilon\text{lfs}(p)$ and f is the projection map of $\text{vol}(B(p,r) \cap \mathbb{M})$ on T_p.

Lemma 28 *Let $A = B(p,r) \cap \mathbb{M}$ where $r \leq \varepsilon\text{lfs}(p)$. Then, $\phi_k r^k \geq \text{vol}(A) \geq \phi_k r^k \cos^k\theta$, where ϕ_k is the volume of k-dimensional unit ball , $\sin\theta = \varepsilon/2$ and $\varepsilon < \frac{1}{2}$.*

[6] Although the tighter bound given here is (surprisingly) not explicitly stated in [27].

Proof. Let $B = B(p,r) \cap T_p$ be the k-dimensional ball radius $r\cos\theta$ centered at p lying in T_p. Let $f_p^A = \{f_p(x) | \forall x \in A\}$. We will show that $B \subseteq f_p^A$. Since f_p is a projection we have

$$\mathrm{vol}(B) \geq \mathrm{vol}(A) \geq \mathrm{vol}(f_p^A) \geq \mathrm{vol}(B^k(p, r\cos\theta)). \tag{12}$$

To prove that $B \subseteq f_p^A$, notice that f_p is an open map whose derivative is nonsingular for all $x \in A$, from Lemma 27. Therefore f_p is locally invertable and there exists a ball $B_s^k = B(p,s) \cap T_p$ of radius s s.t $f_p^{-1}(B_s^k) \subseteq A$. Let $s_{\sup} = \sup\{s \,|\, f_p^{-1}(B_s^k) \subseteq A\}$. Then we have $f_p^{-1}(B_\delta^k) \subseteq A$, where $\delta < s_{\sup}$. For all $x \in \mathrm{int}(B_{s_{\sup}}^k)$, then $f_p^{-1}(x) \in A$. Let q in the closure of A such that either (i) f_p is singular at q or (ii) $q \notin A$. From Lemma 27 we know that case (i) is not possible , therefore $q \notin A$, then $||p-q|| = r \leq \varepsilon \mathrm{lfs}(p)$. The angle between the line pq and $pf_p(q)$ is less than θ. □

From, Lemma 28 we have

Corollary 1 *Let $A = B(p,r) \cap \mathbb{M}$ where $r \leq \varepsilon \mathrm{lfs}(\mathbb{M})$. Then, $\phi_k r^k \geq \mathrm{vol}(A) \geq \phi_k r^k \cos^k\theta > \phi_k r^k/2^k$, where ϕ_k is volume of k-dimensional unit ball , $\sin\theta = \varepsilon/2$ and $\varepsilon < \frac{1}{2}$.*

B Properties of simplices

Lemma 29 (Volume of a simplex) *The volume of a j-simplex whose edges have length at most L is at most $\frac{L^j}{j!}$.*

Proof. $\mathrm{vol}(\tau) = \frac{1}{j!}\det|p_1 - p_0, \ldots, p_j - p_0| \leq \frac{1}{j!}\|p_1 - p_0\| \times \ldots \times \|p_j - p_0\| \leq \frac{L^j}{j!}$. □

Lemma 30 (Height of a simplex) *Let τ be a j-simplex. The distance D_p between a vertex p of τ and the affine hull of $\tau_p = \tau \backslash \{p\}$ is at least $\frac{j!\,\rho(\tau)^{j-1}\sigma(\tau)}{2^{j-1}} L_\tau$.*

Proof. Using the previous lemma and the definition of the sliver measure, we get

$$D_p = \frac{j\,\mathrm{vol}(\tau)}{\mathrm{vol}(\tau_p)} \geq \frac{j\,\sigma(\tau) L_\tau^j}{\frac{(2R_\tau)^{j-1}}{(j-1)!}} \geq \frac{j!\,\rho(\tau)^{j-1}\sigma(\tau)}{2^{j-1}} L_\tau.$$

from which the claim follows. □

Lemma 31 *Let τ be a j-dimensional simplex of $\mathrm{Del}^\omega(\mathcal{P})$ then*

1. *$\forall z \in \mathrm{aff}(\mathrm{Vor}^\omega(\tau))$ and $\forall\, p, x \in \mathcal{P}$ we have $\|x - z\| \leq \frac{\|p-z\|}{\sqrt{1-4\omega_0^2}}$.*
2. *$R_\tau \leq \frac{R'_\tau}{\sqrt{1-4\omega_0^2}}$.*
3. *$\forall z \in \mathrm{aff}(\mathrm{Vor}^\omega(\tau))$ and $x \in \tau$, $Z = \sqrt{\|x-z\|^2 - \omega^2(x)} \geq R'_\tau$.*

Proof.

1. Observe that o_τ is the closest point to x in $\mathrm{aff}(\mathrm{Vor}^\omega(\tau))$. If $\|z-x\| \le \|z-p\|$ then the lemma is proved. Hence assume that $\|z-x\| > \|z-p\|$. Since $z \in \mathrm{aff}(\mathrm{Vor}^\omega(\tau))$

$$\begin{aligned}\|p-z\|^2 &= \|x-z\|^2 + \omega^2(p) - \omega^2(x)\\ &\ge \|x-z\|^2 - \omega^2(x)\\ &\ge \|x-z\|^2 - \omega_0^2\|p-x\|^2\\ &\ge \|x-z\|^2 - \omega_0^2(\|p-z\| + \|z-x\|)^2\\ &> \|x-z\|^2 - 4\omega_0^2\|x-z\|^2\\ &= (1-4\omega_0^2)\,\|x-z\|^2.\end{aligned}$$

2. For all vertices x of τ, $l_x \le 2R_\tau$ and $\omega(x) \le 2\omega_0 R_\tau$. From definition we have

$$\begin{aligned}R'_\tau &\ge \min_{x\in\tau}\sqrt{\|c_\tau - x\| - \omega^2(x)}\\ &\ge \sqrt{1-4\omega_0^2}\,R_\tau.\end{aligned}$$

3. We know that $o_\tau = \mathrm{aff}(\mathrm{Vor}^\omega(\tau)) \cap \mathrm{aff}(\tau)$. Therefore

$$Z^2 = \|x-z\|^2 - \omega^2(x) = \|x-o_\tau\|^2 + \|o_\tau - z\|^2 - \omega^2(x) = {R'_\tau}^2 + \|o_\tau - z\|^2 \ge {R'_\tau}^2.$$

□

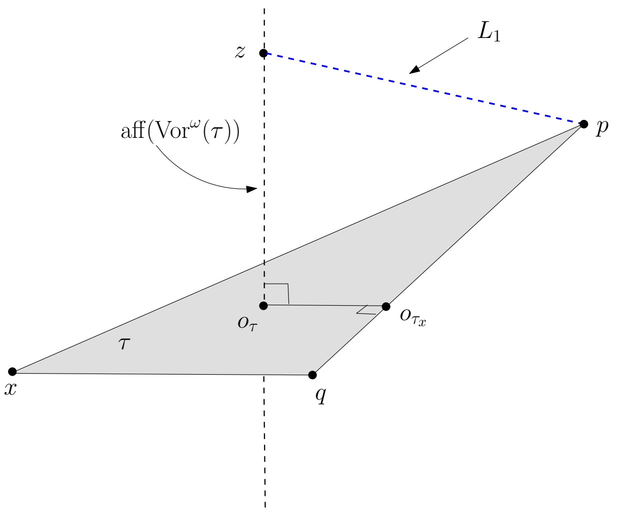

Figure 7: For the proofs of Lemmas 31 and 32.

Lemma 32 *Let τ be a j-dimensional simplex of $\mathrm{Del}^\omega(\mathcal{P})$ and $p \in \tau$ s.t.*

1. *There exists $z \in \mathrm{aff}(\mathrm{Vor}^\omega(\tau)$ s.t. $\|z-p\| \le L_1$.*
2. *$\|p-x\| \le L_2$ for all vertices x of τ.*
3. *$R'_\tau \le \gamma_0 L_\tau$.*

$H_x = dist(o_\tau, \text{aff}(\tau_x)) \le L_1 + (1+\gamma_0+\omega_0)L_2$ *for all vertices* x *of* τ.

Proof. Let $x \in \tau$ and $\tau_x = \tau \setminus \{x\}$. H_x is equal to $\|o_\tau - o_{\tau_x}\|$. By Lemma 31 $\|x - z\| \le \frac{L_1}{\sqrt{1-4\omega_0^2}}$. Let q be a vertex of τ_x. We have

$$\begin{aligned} \|o_\tau - o_{\tau_x}\| &\le \|o_\tau - p\| + \|p - q\| + \|q - o_{\tau_x}\| \\ &\le \|z - p\| + \|p - q\| + \|q - o_{\tau_x}\| \\ &\le L_1 + L_2 + \sqrt{R'^2_{\tau_x} + \omega^2(q)} \\ &\le L_1 + L_2 + \sqrt{\gamma_0^2 L_2^2 + \omega_0^2 L_2^2} \\ &\le L_1 + (1+\gamma_0+\omega_0)L_2 \end{aligned}$$

□

C Proofs

Proof. (Lemma 6) Assume for a contradiction that there exist a point $x \in \text{Vor}^\omega(p) \cap T_p$ s.t $||p - x|| > c_1\varepsilon\text{lfs}(p)$. Let q be a point on the segment px such that $||p-q|| = c_1\varepsilon\text{lfs}(p)/2$. Let q' be the point nearest to q on $\mathbb{M}$. From Lemma 2, we have $||q - q'|| \le c_1^2\varepsilon^2\text{lfs}(p)/2$.

Hence, $||p - q'|| \le ||p - q|| + ||q - q'|| < \frac{c_1}{2}\varepsilon(1 + c_1\varepsilon)\text{lfs}(p)$. From the 1-Lipschitz property, $\text{lfs}(q') \le \text{lfs}(p) + ||p - q'|| < (1 + \frac{c_1}{2}\varepsilon(1+c_1\varepsilon))\,\text{lfs}(p)$, which yields

$$||p-q'|| \ge ||p-q|| - ||q-q'|| > \frac{c_1}{2}\varepsilon(1-c_1\varepsilon)\,\text{lfs}(p) > \frac{c_1(1-c_1\varepsilon)}{2+c_1\varepsilon(1+c_1\varepsilon)}\,\varepsilon\text{lfs}(q') > \varepsilon\text{lfs}(q'),$$

if $c_1(1 - c_1\varepsilon) > 2 + c_1\varepsilon(1 + c_1\varepsilon)$ (*). Hence there exist a point $t \in \mathcal{P}$, s.t $||q' - t|| \le \varepsilon\text{lfs}(q') < \varepsilon(1 + \frac{c_1}{2}\varepsilon(1 + c_1\varepsilon))\text{lfs}(p)$. We thus have (using again (*))

$$||q - t|| \le ||q - q'|| + ||q' - t|| < \frac{c_1}{2}\,\varepsilon\text{lfs}(p).$$

From Fig. 8, we can see that $\angle ptx > \pi/2$. This implies that $||p - x||^2 - ||x - t||^2 - ||p - t||^2 > 0$. So,

$$\begin{aligned} ||x-p||^2 - ||x-t||^2 - \omega^2(p) + \omega^2(t) &\ge ||x-p||^2 - ||x-t||^2 - \omega^2(p) \\ &\ge ||x-p||^2 - ||x-t||^2 - \omega_0^2.||p-t||^2 \\ &\ge ||x-p||^2 - ||x-t||^2 - ||p-t||^2 \quad (\text{as } \omega_0 \in [0,1/2)) \\ &> 0. \end{aligned}$$

This implies $x \notin \text{Vor}^\omega(p)$, which contradicts our initial assumption. We conclude that $\text{Vor}^\omega(p) \cap T_p \subseteq B(p, c_1\varepsilon\text{lfs}(p))$.

Inequality (*) is satisfied for $\varepsilon < \frac{1}{2c_1}(-c_1 - 1 + \sqrt{c_1^2 + 6c_1 - 7})$. The quantity on the right hand side is positive when $c_1 > 2$. We maximize the bound on ε by taking $c_1 = 3 + \sqrt{2} \approx 4.41$. Inequality (*) then holds for $\varepsilon < 0.09$. □

Proof. (Lemma 7) 1. Since $pq \in \text{Del}_p^\omega(\mathcal{P})$ therefore $T_p \cap \text{Vor}^\omega(pq) \ne \emptyset$. Let $x \in T_p \cap \text{Vor}^\omega(pq)$. From Lemma 6 we have $||p - x|| \le c_1\varepsilon\text{lfs}(p)$. If $||q - x|| \le$

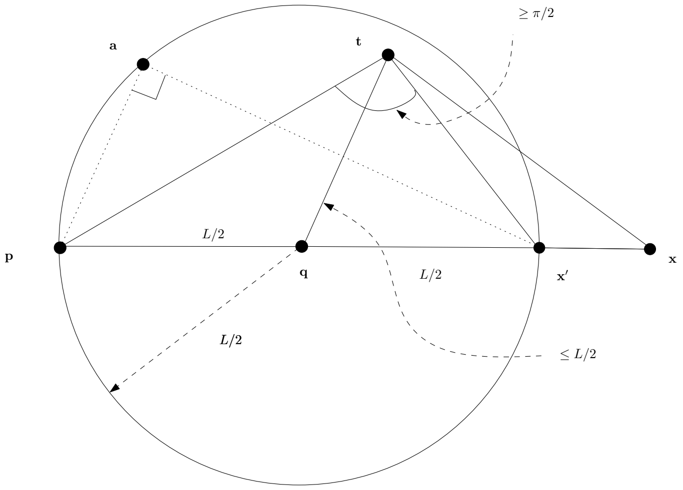

Figure 8: Refer to Lemma 6. x' is a point on the line segment such that $||p-x'|| = c_1\varepsilon\,\mathrm{lfs}(p)$, $L = c_1\varepsilon\,\mathrm{lfs}(p)$, $\angle pax' = \pi/2$ and $\angle ptx \geq \angle ptx' > \pi/2$.

$||p-x||$ then the inequality holds. Otherwise, $||p-q|| \leq ||p-x|| + ||q-x|| < 2||q-x||$. Since $x \in T_p \cap \mathrm{Vor}^{\omega}(\{p,q\})$, we have from Lemma 31 (Appendix B), $\|q-x\| \leq \frac{\|p-x\|}{\sqrt{1-4\omega_0^2}}$.

Hence we get $||p-q|| \leq ||p-x|| + ||q-x|| \leq c_1(1+1/\sqrt{1-4\omega_0^2})\varepsilon\mathrm{lfs}(p) = c_2\varepsilon\mathrm{lfs}(p)$.

2. From the definition of $\mathrm{Del}^{\omega}_{T\mathbb{M}}(\mathcal{P})$, there exist a vertex r in τ such that $[pq] \in \mathrm{Del}^{\omega}_{r}(\mathcal{P})$. From 1, $\|r-p\|$ and $\|r-q\|$ are at most $c_2\varepsilon\mathrm{lfs}(r)$. From the 1-Lipschitz property of lfs and $3c_2\varepsilon < 1$, we have $\mathrm{lfs}(r) \leq \mathrm{lfs}(p) + \|p-r\| \leq \frac{\mathrm{lfs}(p)}{1-c_2\varepsilon} \leq \frac{3}{2}\,\mathrm{lfs}(p)$. We conclude that $\|p-q\| \leq \|p-r\| + \|r-q\| \leq 3c_2\varepsilon\mathrm{lfs}(p)$.

3. Let $z \in \mathrm{Vor}^{\omega}(\tau) \cap T_p$, and $Z = \sqrt{||z-p||^2 - \omega^2(p)}$. By definition the ball centered at z with radius Z is orthogonal to the weighted vertices of τ. From Lemma 31 (Appendix B), we have $Z \geq R'_{\tau}$. Hence it suffices to prove $Z \leq \beta_0 L_{\tau}$. Since $z \in \mathrm{Vor}^{\omega}(\tau) \cap T_p$, we deduce from Lemma 6 that $||z-p|| \leq c_1\varepsilon\mathrm{lfs}(p)$. Therefore

$$Z = \sqrt{||z-p||^2 - \omega^2(p)} \leq ||z-p|| \leq c_1\varepsilon\mathrm{lfs}(p) \leq c_1\left(\frac{\varepsilon}{\delta}\right)\delta\mathrm{lfs}(p).$$

For any vertex x of τ, we have $\|p-x\| \leq c_2\varepsilon\mathrm{lfs}(p)$ (By 1.). Since $2c_2\varepsilon \leq 1$ and lfs is 1-Lipschitz, $\mathrm{lfs}(p) \leq 3\mathrm{lfs}(x)/2$. Therefore, taking for x a vertex of the shortest edge of τ, we have

$$Z \leq c_1\left(\frac{\varepsilon}{\delta}\right)\delta\mathrm{lfs}(p) \leq c_1\left(\frac{\varepsilon}{\delta}\right)\delta \times \frac{3\mathrm{lfs}(x)}{2} \leq \left(\frac{3c_1\varepsilon}{2\delta}\right) L_{\tau} = \beta_0 L_{\tau}.$$

From Lemma 31 we have $R_{\tau} \leq \frac{R'_{\tau}}{\sqrt{1-4\omega_0^2}}$. There fore radius-edge ratio of τ is $\beta'_0 = \frac{\beta_0}{\sqrt{1-4\omega_0^2}}$. □

Proof. (Lemma 9) Let $\phi = [p_1, \ldots, p_{k+2}]$ be an inconsistent configuration witnessed by p_i, p_j, p_l.

1. If pq is an edge of ϕ, Lemma 8 implies that $\|p_i - c_\phi\| \leq 2c_4\varepsilon\text{lfs}(p_i)$ and that $\|x - c_\phi\| \leq c_5\varepsilon\text{lfs}(x)$, $\forall x \in \phi$. Let p, q be two vertices of ϕ, then

$$\begin{aligned}\|p-q\| &\leq \|p-c_\phi\| + \|q-c_\phi\| \\ &\leq 2c_5\varepsilon\text{lfs}(p_i) \\ &\leq 4c_5\varepsilon\text{lfs}(p)\end{aligned}$$

the last inequality follows from the fact that lfs is 1-Lipschitz function and $2c_5\varepsilon < 1$.

2. Let $Z = \sqrt{\|c_\phi - p_i\|^2 - \omega^2(p_i)}$. Since $c_\phi \in \text{Vor}^\omega(\tau)$, the ball centered at c_ϕ with radius Z is orthogonal to weighted vertices of τ. From Lemma 31, we have $Z \geq R'_\tau$. Hence if we can show that $Z \leq \beta_1 L_\tau$ then we are done. Using $\|p_i - c_\phi\| \leq 2c_4\varepsilon\text{lfs}(p_i)$ we get

$$Z = \sqrt{\|c_\phi - p_i\| - \omega^2(p_i)} \leq \|c_\phi - p_1\| \leq 2c_4\varepsilon\text{lfs}(p_i).$$

For all vertices x in τ we have from Lemma 9(1) $\|p_i - x\| \leq 4c_5\varepsilon\text{lfs}(p_i)$. Since $16c_5\varepsilon \leq 1$ we get from the 1-Lipschitz property of lfs, $\text{lfs}(p_i) \leq 5\text{lfs}(x)/4$. Therefore

$$Z \leq 2c_4\left(\frac{\varepsilon}{\delta}\right)\delta\text{lfs}(p_i) \leq 2c_4\left(\frac{\varepsilon}{\delta}\right)\delta \times \frac{5\text{lfs}(x)}{4} \leq \frac{5c_4\varepsilon\,\text{lfs}(x)}{2} \leq \left(\frac{5c_4\varepsilon}{2\delta}\right)L_\tau = \beta_1 L_\tau.$$

From Lemma 31, we have $R_\tau \leq \frac{R'_\tau}{\sqrt{1-4\omega_0^2}}$. Therefore the radius-edge ratio for τ is $\beta'_1 = \frac{\beta_1}{\sqrt{1-4\omega_0^2}}$. □

Proof. (Lemma 10) Let τ be a simplex of $\text{Del}^\omega_{T\mathbb{M}}(\mathcal{P})$. Then there exists a vertex p s.t $p \in \tau_1$, $\tau \subseteq \tau_1$ and τ_1 belongs to $\text{Del}^\omega_p(\mathcal{P})$. From Lemma 6 we have $\|p - c_p\| \leq c_1\varepsilon\text{lfs}(p) \leq 2c_1\varepsilon\text{lfs}(x)$ where $c_p = \text{Vor}^\omega(\tau_1)\cap$ for all vertices x of τ. The last inequality follows from the facts that lfs is 1-Lipschitz, $\|p - x\| \leq 3c_2\varepsilon\text{lfs}(p)$ (refer to Lemma 7) and $3c_2\varepsilon < 16c_2\varepsilon < 1$. We have from Lemma 31, $\|x - c_p\| \leq \frac{2c_1\varepsilon\text{lfs}(x)}{\sqrt{1-4\omega_0^2}}$ as $c_p \in \text{Vor}^\omega(\tau) \subseteq \text{aff}(\text{Vor}^\omega(\tau))$. Therefore using Lemma 32 we get

$$\begin{aligned}\text{dist}(o_\tau, \tau_x) &\leq \frac{2c_1\varepsilon\text{lfs}(x)}{\sqrt{1-4\omega_0^2}} + 3(1+\beta_0+\omega_0)c_2\varepsilon\text{lfs}(x) \\ &\leq c_6\varepsilon\text{lfs}(x),\end{aligned}$$

as L_1, L_2, γ_0 in Lemma 32 (Appendix B) corresponds to $\frac{2c_1\varepsilon\text{lfs}(x)}{\sqrt{1-4\omega_0^2}}$, $3c_2\varepsilon\text{lfs}(x)$ (from Lemma 7 (1)) and β_0 (Lemma 7 (2))respectively. The proof is similar when τ belongs to $\text{Faces}^\omega(\mathcal{P})$. □

Proof. (Lemma 11) Assume w.l.o.g. that ϕ is witnessed by p_1, p_2 and p_{k+2}. Let $\tau = \phi \setminus p_{k+2}$ and p_i be any vertex of τ (i.e. $i \in [1, k+1]$). Write, as usual, ϕ_{p_i} for the k-simplex $\phi \setminus \{p_i\}$. Let o_{ϕ_i} and o_τ be the orthocenters of ϕ_{p_i} and τ respectively. By Lemma 8, we have that $\text{dist}(c_\phi, \text{aff}(\tau)) = \|o_\tau - c_\phi\| \leq 2c_4\varepsilon^2\text{lfs}(p_1)$. This proves the lemma for τ (one of the k-subfaces of ϕ).

To complete the proof, it is enough to consider the case of ϕ_i. We have $\|o_\tau - p_1\| \leq \|c_{p_1} - p_1\| \leq c_1\varepsilon\text{lfs}(p_1)$ and $\|p_1 - p_i\| \leq c_2\varepsilon\text{lfs}(p_1)$ (from Lemmas 6 and 7 respectively). If q denotes a common vertex of ϕ_{p_i} and τ, we get, using Lemma 5,

$$\begin{aligned}\mathrm{dist}(\mathrm{aff}(\phi_{p_i}), o_\tau) &\leq \|o_\tau - q\| \sin\angle(\mathrm{aff}(\tau), \mathrm{aff}(\phi_{p_i}) \\ &\leq \|o_\tau - q\|(\sin\angle(\mathrm{aff}(\tau), T_q) + \sin\angle(\phi_{p_i}, T_q)) \\ &\leq (\|o_\tau - p_1\| + \|p_1 - q\|)2a_k\varepsilon \\ &\leq 2(c_1 + c_2)a_k\varepsilon^2 \mathrm{lfs}(p_1).\end{aligned}$$

Since $\mathrm{dist}(c_\phi, \mathrm{aff}(\phi_{p_i}) = \|o_{\phi_i} - c_\phi\| \leq \mathrm{dist}(\mathrm{aff}(\phi_{p_i}), o_\tau) + \|o_\tau - c_\phi\| \leq 2(c_1 + c_2 + c_4)a_k\varepsilon^2 \mathrm{lfs}(p_1)$.

To conclude, we observe that, for any vertex v of ϕ, $\mathrm{lfs}(p_1) \leq \mathrm{lfs}(v) + \|p_1 - v\|$ (Lemma 9), and use $\|p_1 - v\| \leq 4c_5 \mathrm{lfs}(p_1)$ and $8c_5\varepsilon \leq 1$. □

Proof. (Lemma 12) Let p be a vertex of ϕ_{p_i}. By Lemma 9(2) and Lemma 5, we have $\sin\angle(\mathrm{aff}(\tau), T_p) \leq a_k\varepsilon$, and, by Lemma 2(2), we have $\sin\angle(pp_i, T_p) \leq 2c_5\varepsilon$. We therefore have, $\sin\angle(pp_i, \mathrm{aff}(\tau)) \leq \sin\angle(pp_i, T_p) + \sin\angle(T_p, \mathrm{aff}(\tau)) \leq (2c_5 + a_k)\varepsilon$. Hence

$$D_{p_i} = \|p - p_i\| \sin\angle(pp_i, \mathrm{aff}(\tau)) < 4(2c_5 + a_k)c_5\varepsilon^2 \mathrm{lfs}(p_i), \tag{13}$$

since $\|p - p_i\| \leq 4c_5\varepsilon \mathrm{lfs}(p_i)$ from Lemma 9. □

Proof. (Lemma 13) For convenience, write $\nu = 16c_5c_0\varepsilon \leq 1/2$ and observe that $LN_p \subset B''_p = B(p, \nu \mathrm{lfs}(p))$ since, by Lemma 2, $l_p \leq \frac{2\varepsilon}{1-\varepsilon} \mathrm{lfs}(p) \leq 4\varepsilon \mathrm{lfs}(p)$. We will count the number of points in $B''_p \cap \mathcal{P}$. Let x and y be two points of $B''_p \cap \mathcal{P}$. We have $\mathrm{lfs}(x), \mathrm{lfs}(y) \geq \mathrm{lfs}(p)(1 - \nu) \geq \mathrm{lfs}(p)/2$, since lfs is 1-Lipschitz. The balls $B(x, l_x/2)$ and $B(y, l_y/2)$ are disjoint, and, since $l_x \geq \delta\, \mathrm{lfs}(x) \geq \frac{\delta}{2} \mathrm{lfs}(p)$ (and similarly for l_y), the balls $B_x = B(x, \frac{\delta}{4} \mathrm{lfs}(p))$ and $B_y = B(y, \frac{\delta}{4} \mathrm{lfs}(p))$ are also disjoint. Observe that both balls B_x and B_y are contained in $B_p^+ = B(p, \mu\, \varepsilon \mathrm{lfs}(p))$ where $\mu = \frac{\nu}{\varepsilon} + \frac{\delta}{4\varepsilon} \leq 16c_5c_0 + \frac{1}{4}$.

A packing argument now allows to conclude. Specifically, by Corollary 1 (see Appendix A), we have that $\mathrm{vol}(B_x \cap \mathbb{M}) > \phi_k \left(\frac{\delta\, \mathrm{lfs}(p)}{8}\right)^k$ and $\mathrm{vol}(B_p^+ \cap \mathbb{M}) < \phi_k\, (\mu\varepsilon \mathrm{lfs}(p))^k$, where ϕ_k is the volume of the k-dimensional unit ball. We conclude that the number of points of $\mathcal{P} \cap B''_p$ is less than $\left(\frac{8\mu\varepsilon}{\delta}\right)^k \leq (128c_5c_0 + 2)^k\, c_0^k = 2^{O(k)}$. □

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